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OPERATOR ALGEBRAS WITH CONTRACTIVE APPROXIMATE IDENTITIES

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ABSTRACT. We study operator algebras with contractive approximate identities and their double centralizer algebras. These operator algebras can be characterized as L^∞-Banach algebras with contractive approximate identities. We provide two examples, which show that given a non-unital operator algebra A with a contractive approximate identity, its double centralizer algebra $M(A)$ may admit different operator algebra matrix norms, with which $M(A)$ contains A as an M-ideal. On the other hand, we show that there is a unique operator algebra matrix norm on the unitalization algebra A^1 of A such that A^1 contains A as an M-ideal.

1. **Introduction.** Given a Hilbert space H, we let $B(H)$ denote the space of all bounded linear operators on H with the operator norm. For each $n \in \mathbb{N}$, the set of natural numbers, there is a natural norm $\|\cdot\|_n$ on the space $M_n\big(B(H)\big)$ of all $n \times n$ matrices on $B(H)$ obtained by identifying $M_n\big(B(H)\big)$ with $B(H^n)$, where H^n is the direct sum of n copies of H. We call this family of norms $\{\|\cdot\|_n\}$ the *operator matrix norm* on $B(H)$. A norm closed subspace (resp., a norm closed subalgebra) of $B(H)$ together with the operator matrix norm is called a *concrete operator space* (resp., a *concrete operator algebra*). A concrete operator algebra is called *unital* if it contains the identity operator on the Hilbert space.

Operator matrix norms play an important role in the study of operator spaces and operator algebras. In [Ru1] and [BRS], we have succeeded in characterizing concrete operator spaces and concrete unital operator algebras as L^∞-matricially normed spaces and unital L^∞-Banach algebras (or equivalently, unital L^∞-Banach pseudo-algebras), respectively. For the convenience of the reader, let us recall these results.

An L^∞*-matricially normed space* is a vector space V over the complex numbers $\mathbb{C}$ together with a norm $\|\cdot\|_n$ on each matrix space $M_n(V)$ such that the following conditions are satisfied:

(M1) $$\|x \oplus y\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$$

(M2) $$\|\alpha x \beta\|_n \leq \|\alpha\|\,\|x\|_n\|\beta\|$$

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for all $x \in M_n(V)$, $y \in M_m(V)$ and $\alpha, \beta \in M_n(\mathbb{C})$ (*cf.* [Ef] and [Ru1]). In this paper, we assume that all L^∞-matricially normed spaces are norm complete, *i.e.*, $M_n(V)$ is a Banach space for each $n \in \mathbb{N}$.

Given L^∞-matricially normed spaces V and W and a linear map $\varphi: V \longrightarrow W$, there is a natural map $\varphi_n: M_n(V) \longrightarrow M_n(W)$ defined by

$$\varphi_n([x_{ij}]) = [\varphi(x_{ij})].$$

We let $\|\varphi\|_{\text{cb}} = \sup\{\|\varphi_n\| : n \in \mathbb{N}\}$. The map φ is called *completely bounded* (resp., *completely contractive*, *completely isometric*) if $\|\varphi\|_{\text{cb}} < +\infty$ (resp., $\|\varphi\|_{\text{cb}} \leq 1$, each φ_n is an isometry). We will use $B(V, W)$ (resp., $\text{CB}(V, W)$) to denote the space of all bounded maps (resp., the space of all completely bounded maps) from V into W.

It is easy to see that every concrete operator space is an L^∞-matricially normed space. On the other hand, we have

THEOREM 1.1 [RU1]. *Every L^∞-matricially normed space is completely isometrically linearly isomorphic to a concrete operator space.*

Theorem 1.1 characterizes operator spaces as L^∞-matricially normed spaces. Recently, we have found an extremely simple proof of this theorem in [ER6].

An *L^∞-Banach algebra* is an associative algebra A over the complex number $\mathbb{C}$ such that

1) A is an L^∞-matricially normed space, *i.e.*, an (abstract) operator space
2) the associative multiplication $m: A \times A \longrightarrow A$ is a completely contractive bilinear map.

We recall that a bilinear map $m: A \times A \longrightarrow A$ is *completely contractive* (*cf.* [CS]) if it satisfies

$$\|m_n([a_{ij}], [b_{jk}])\|_n = \left\| \left[\sum_{j=1}^{n} m(a_{ij}, b_{jk}) \right] \right\|_n \leq \|[a_{ij}]\|_n \|[b_{jk}]\|_n$$

for all $[a_{ij}]$, $[b_{jk}] \in M_n(A)$ and $n \in \mathbb{N}$. We note that condition 2) is equivalent to the following: for each $n \in \mathbb{N}$, the matrix algebra $M_n(A)$ is a Banach algebra with respect to the multiplication m_n and the norm $\| \cdot \|_n$.

An L^∞-Banach algebra A is called *unital* if it has a multiplicative unit e of norm one, *i.e.*, it has a distinguished element e of norm one such that

$$m(a, e) = a = m(e, a)$$

for all $a \in A$. An L^∞-Banach algebra A is said to have a *contractive approximate identity* if there is a net $\{a_\alpha : \alpha \in \Lambda\}$ of contractive elements in A such that

$$\|m(a_\alpha, a) - a\| \longrightarrow 0, \quad \|m(a, a_\alpha) - a\| \longrightarrow 0$$

for all $a \in A$.

In the above definition of L^∞-Banach algebras, we assume that the algebras are associative. If we drop the associativity, we get the definition of L^∞-Banach pseudo-algebras. It is easy to see that every concrete unital operator algebra is a unital L^∞-Banach algebra, and thus a unital L^∞-Banach pseudo-algebra. Conversely, we have

THEOREM 1.2 [BRS]. *Every unital L^∞-Banach pseudo-algebra is completely isometrically unital isomorphic to a concrete unital operator algebra.*

Theorem 1.2 charaterizes concrete unital operator algebras as unital L^∞-Banach pseudo-algebras. It also shows that unital L^∞-Banach pseudo-algebras are automatically associative and thus can be identified with unital L^∞-Banach algebras. We can simply regard these algebras as (abstract) *unital operator algebras*, and call their matrix norms *unital operator algebra matrix norms*.

There is no doubt that unital operator algebras are the most important operator algebras in our study. But many interesting operator algebras fail to have units. For example, norm closed two-sided ideals of operator algebras usually have no units. But some "nice" ideals, *i.e.*, M-ideals in unital operator algebras have contractive approximate identities (*cf.* [ER2] and [PR]). This motivated us to study a more general class of operator algebras, *i.e.*, operator algebras with contractive approximate identities.

In §2, we begin by studying the double centralizer algebras of L^∞-Banach algebras. We show that for any L^∞-Banach algebra A with a contractive approximate identity, there is a canonical matrix norm $\{|||\cdot|||_n\}$ on its double centralizer algebra $M(A)$. With this matrix norm, $M(A)$ is a unital L^∞-Banach algebra containing A as an M-ideal (Theorem 2.3). It follows from Theorem 1.2 that $M(A)$ is completely isometrically unital isomorphic to a unital concrete operator algebra on a Hilbert space. As a consequence, A is completely isometrically isomorphic to a concrete operator algebra (Theorem 2.4). Furthermore, we show that A can be identified with a concrete non-degenerate operator algebra on a Hilbert space (Proposition 2.5).

In §3, we study some properties of the double centralizer algebras of operator algebras with contractive approximate identities. We show in Theorem 3.1 that if A is a non-degenerate operator algebra on a Hilbert space H, and if we let $B = \{x \in B(H) : xA \subseteq A \text{ and } Ax \subseteq A\}$ be the double multiplier algebra of A on H, then $M(A)$ is completely isometrically unital isomorphic to B. Furthermore, we show in Proposition 3.2 that $M(A)$ is completely isometrically unital isomorphic to a unital subalgebra of A^{**}, where the latter is the second dual of A.

It is known that for any C^*-algebra A, there is a natural *C^*-algebra matrix norm* on A. With this C^*-algebra matrix norm, A is an operator algebra with a contractive approximate identity and its double centralizer algebra $M(A)$ has a unique unital C^*-algebra matrix norm such that $M(A)$ contains A as an M-ideal. But this is not necessarily true for operator algebras. Given an operator algebra A with a contractive approximate identity, there might exist different unital operator algebra matrix norms on $M(A)$, with which $M(A)$ contains A as an M-ideal.

In §4, we study the unital operator algebra matrix norm structure on the double centralizer algebras of operator algebras with contractive approximate identities. We will discuss two examples, Example 4.3 and Example 4.4. In Example 4.3, we show that there is a unital operator algebra matrix norm $\{|||\cdot|||'_n\}$ on $\ell^\infty(\mathbb{N})$, the double centralizer algebra of $c_0(\mathbb{N})$, which is different from the canonical unital operator algebra (in fact, C^*-algebra) matrix norm $\{|||\cdot|||_n\}$ on $\ell^\infty(\mathbb{N})$. In this case, we acturally get $|||\cdot|||'_1 \neq |||\cdot|||_1$.

For non-self adjoint operator algebras, it is possible to construct different unital operator algebra matrix norms $\{|||\cdot|||'_n\}$ on their double centralizer algebras such that $|||\cdot|||'_1 = |||\cdot|||_1$ (see Example 4.4).

In §5, we study the unitalization algebras of non-unital operator algebras with contractive approximate identities. The main result, Theorem 5.2, in this section shows that for every such operator algebra A, there is a unique unital operator algebra matrix norm on the unitalization algebra A^1 of A such that A^1 contains A as an M-ideal.

2. **Double centralizer algebras.**

The theory of double centralizers, or double multipliers was first studied by G. Hochschild in [Ho] for associative algebras over a field k. It was also studied by S. Helgason in [He] for certain commutative Banach algebras, by B. Johnson in [Jo] for Banach algebras with bounded approximate identities, and by R. Busby in [Bu] for C^*-algebras. We begin this section by recalling the definition of double centralizers for associative algebras in [Ho]. We will delete the multiplication m in our notation unless it is necessary.

Let A be an associative algebra over a field k. A *double centralizer* of A is a pair (S, T) of linear maps S and T on A which satifies the following conditions:

(DC1) $$aS(b) = T(a)b$$

(DC2) $$S(ab) = S(a)b$$

(DC3) $$T(ab) = aT(b)$$

for all $a, b \in A$.

If A is an L^∞-Banach algebra with a contractive approximate identity, any maps S and T satisfying (DC1) are automatically linear and bounded, and satisfy conditions (DC2) and (DC3) (*cf.* [Jo]). In this case, the double centralizers of A can be defined as follows.

DEFINITION 2.1. *Let A be an L^∞-Banach algebra with a contractive approximate identity. A double centralizer of A is a pair (S, T) of maps S and T on A which satisfies condition (DC1).*

We let $M(A)$ denote the set of all double centralizers of A. With the operations defined by

$$(S_1, T_1) + (S_2, T_2) = (S_1 + S_2, T_1 + T_2)$$
$$\alpha(S, T) = (\alpha S, \alpha T)$$
$$(S_1, T_1) \circ (S_2, T_2) = (S_1 \circ S_2, T_2 \circ T_1)$$

for all (S, T) and $(S_i, T_i) \in M(A)$ and scalars α, $M(A)$ is an associative unital algebra with unit $(\mathrm{id}_A, \mathrm{id}_A)$, which is called the *double centralizer algebra* of A. In the following, we will show that there is a natural operator matrix norm on $M(A)$ such that $M(A)$ is a unital L^∞-Banach algebra.

LEMMA 2.2. *Let A be an L^∞-Banach algebra with a contractive approximate identity. For any $(S,T) \in M(A)$, the linear maps S and T are completely bounded with*

$$\|S\|_{\mathrm{cb}} = \|S\| = \|T\| = \|T\|_{\mathrm{cb}}.$$

Furthermore, for any $[(S_{ij}, T_{ij})] \in M_n\big(M(A)\big)$, we have

$$\|[S_{ij}]\|_{\mathrm{cb}} = \|[S_{ij}]\| = \|[T_{ij}]\| = \|[T_{ij}]\|_{\mathrm{cb}}.$$

PROOF. Let $\{a_\alpha\}_{\alpha\in\Lambda}$ be a contractive approximate identity for A. For each $\alpha \in \Lambda$, $n \in \mathbb{N}$, define $a_\alpha^n = \begin{bmatrix} a_\alpha & & 0 \\ & \ddots & \\ 0 & & a_\alpha \end{bmatrix}$. Then $\{a_\alpha^n\}_{\alpha\in\Lambda}$ is clearly a contractive approximate identity for $M_n(A)$. Given any $(S,T) \in M(A)$, we have

$$\begin{aligned}\|S\|_{\mathrm{cb}} &= \sup\{\|[S(a_{ij})]\|_n : \|[a_{ij}]\|_n \le 1\} \\ &= \sup\{\|a_\alpha^n[S(a_{ij})]\|_n : \|[a_{ij}]\|_n \le 1, \alpha \in \Lambda\} \\ &= \sup\{\|[a_\alpha S(a_{ij})]\|_n : \|[a_{ij}]\|_n \le 1, \alpha \in \Lambda\} \\ &= \sup\{\|[T(a_\alpha)a_{ij}]\|_n : \|[a_{ij}]\|_n \le 1, \alpha \in \Lambda\} \\ &\le \|T\|.\end{aligned}$$

Similarly, we can show that $\|T\|_{\mathrm{cb}} \le \|S\|$. Thus S and T are completely bounded and

$$\|S\|_{\mathrm{cb}} = \|S\| = \|T\| = \|T\|_{\mathrm{cb}}.$$

Given any element $[(S_{ij}, T_{ij})] \in M_n\big(M(A)\big)$, we have

$$[S_{ij}] \text{ and } [T_{ij}] \in M_n\big(B(A,A)\big) \cong B\big(A, M_n(A)\big).$$

A similar argument shows that

$$\|[S_{ij}]\|_{\mathrm{cb}} = \|[S_{ij}]\| = \|[T_{ij}]\| = \|[T_{ij}]\|_{\mathrm{cb}}. \qquad \blacksquare$$

By Lemma 2.2, we can define a norm $|||\cdot|||_n$ on each $M_n\big(M(A)\big)$ by

$$|||[(S_{ij}, T_{ij})]|||_n = \|[S_{ij}]\| \quad (= \|[T_{ij}]\|)$$

for all $[(S_{ij}, T_{ij})] \in M_n\big(M(A)\big)$. We call $\{|||\cdot|||_n\}$ the *canonical matrix norm* on $M(A)$.

THEOREM 2.3. *Let A be an L^∞-Banach algebra with a contractive approximate identity. The double centralizer algebra $M(A)$ with the canonical matrix norm is a unital L^∞-Banach algebra containing A as an M-ideal.*

PROOF. First we recall that $B(A,A)$ is an L^∞-matricially normed space with the matrix norm obtained by identifying $M_n\big(B(A,A)\big)$ with $B\big(A, M_n(A)\big)$ (*cf.* [ER1]). Then $M(A)$ with the canonical matrix norm $\{|||\cdot|||_n\}$ is an L^∞-matricially normed space.

The multiplication $m: M(A) \times M(A) \longrightarrow M(A)$ defined by

$$m\big((S_1, T_1), (S_2, T_2)\big) = (S_1 \circ S_2, T_2 \circ T_1)$$

for all $(S_i, T_i) \in M(A)$ is unital and associative. The unit element $(\mathrm{id}_A, \mathrm{id}_A)$ has norm one. It remains to show that m is a completely contractive bilinear map.

For any $[(S_{ij}, T_{ij})], [(S'_{jk}, T'_{jk})] \in M_n\big(M(A)\big)$, we have

$$\begin{aligned}
\Big|\Big|\Big| m_n\big([(S_{ij}, T_{ij})], [(S'_{jk}, T'_{jk})]\big)\Big|\Big|\Big|_n &= \Big|\Big|\Big|\Big[\Big(\sum_{j=1}^{n} S_{ij} \circ S'_{jk}, \sum_{j=1}^{n} T'_{jk} \circ T_{ij}\Big)\Big]\Big|\Big|\Big|_n \\
&= \Big|\Big|\Big[\sum_{j=1}^{n} S_{ij} \circ S'_{jk}\Big]\Big|\Big| \\
&= \sup\Big\{\Big|\Big|\Big[\sum_{j=1}^{n} a_\alpha S_{ij} \circ S'_{jk}(a)\Big]\Big|\Big|_n : \|a\| \leq 1, \alpha \in \Lambda\Big\} \\
&= \sup\{\|[T_{ij}(a_\alpha)][S'_{jk}(a)]\|_n : \|a\| \leq 1, \alpha \in \Lambda\} \\
&\leq \|[T_{ij}]\| \, \|[S'_{jk}]\| = |||[(S_{ij}, T_{ij})]|||_n \, |||[(S'_{jk}, T'_{jk})]|||_n.
\end{aligned}$$

Thus $M(A)$ is a unital L^∞-Banach algebra.

Next we show that the algebra A can be identified with an M-ideal in $M(A)$. For any $a \in A$, we let L_a (resp., R_a) be the left (resp., right) multiplication map defined by $L_a(b) = ab$ (resp., $R_a(b) = ba$) for all $b \in A$. Since A is an associative algebra, the pair (L_a, R_a) of bounded maps belongs to $M(A)$. Thus $\mu(a) = (L_a, R_a)$ for $a \in A$ defines a map $\mu: A \longrightarrow M(A)$. Clearly μ is an algebraic homomorphism such that $\mu(A)$ is a two-sided ideal in $M(A)$. Furthermore, we can show that μ is a complete isometry.

To see this, for any $[a_{ij}] \in M_n(A)$, we have

$$|||[(L_{a_{ij}}, R_{a_{ij}})]|||_n = \|[L_{a_{ij}}]\| = \sup\{\|[a_{ij}a]\|_n : \|a\| \leq 1\} \leq \|[a_{ij}]\|_n.$$

On the other hand, we have

$$\begin{aligned}
\|[a_{ij}]\|_n &= \sup\{\|[a_\alpha a_{ij}]\|_n : \alpha \in \Lambda\} \\
&= \sup\{\|[R_{a_{ij}}(a_\alpha)]\|_n : \alpha \in \Lambda\} \\
&\leq \|[R_{a_{ij}}]\| = |||[(L_{a_{ij}}, R_{a_{ij}})]|||_n.
\end{aligned}$$

Hence, $\mu: A \longrightarrow M(A)$ is a complete isometric injection, and A can be identified with an M-ideal, *i.e.*, a norm closed two-sided ideal with a contractive approximate identity, in $M(A)$ (*cf.* [ER2]). ■

It follows from Theorem 1.2 and Theorem 2.3 that $M(A)$ is completely isometrically unital isomorphic to a concrete unital operator algebra, which contains A as an M-ideal. As an immediate consequence, we have

THEOREM 2.4. *Every L^∞-Banach algebra with a contractive approximate identity is completely isometrically isomorphic to a concrete operator algebra.*

Owing to Theorem 2.4, every L^∞-Banach algebra A with a contractive approximate identity can be identified with a concrete operator algebra on a Hilbert space H. Let $[AH]$

be the linear subspace spanned by $\{a\xi : a \in A,\ \xi \in H\}$ and $H_0 = \overline{[AH]}$ the closure of $[AH]$ in H. It is clear that H_0 is an A-invariant subspace of H. If $H_0 = H$, we call A *non-degenerate* on H. If $H_0 \neq H$, we can restrict A to H_0 and get a new operator algebra matrix norm $\{\|\cdot\|_{H_0,n}\}$ on A.

PROPOSITION 2.5. *The operator algebra matrix norm $\{\|\cdot\|_{H_0,n}\}$ coincides with the original operator algebra matrix norm $\{\|\cdot\|_n\}$ on A. Thus every L^∞-Banach algebra with a contractive approximate identity can be identified with a concrete non-degenerate operator algebra on a Hilbert space.*

PROOF. For every (fixed) $[a_{ij}] \in M_n(A)$ $(n \in \mathbb{N})$, it is clear that

$$\|[a_{ij}]\|_{H_0,n} \leq \|[a_{ij}]\|_n.$$

On the other hand, we have

$$\begin{aligned}
\|[a_{ij}]\|_n &= \sup\{\|[a_{ij}a_\alpha]\|_n : \alpha \in \Lambda\} \\
&= \sup\left\{\left\|[a_{ij}a_\alpha]\begin{bmatrix}\xi_1\\ \vdots\\ \xi_n\end{bmatrix}\right\| : \left\|\begin{bmatrix}\xi_1\\ \vdots\\ \xi_n\end{bmatrix}\right\| = 1, \xi_i \in H, \alpha \in \Lambda\right\} \\
&= \sup\left\{\left\|[a_{ij}]\begin{bmatrix}a_\alpha\xi_1\\ \vdots\\ a_\alpha\xi_n\end{bmatrix}\right\| : \left\|\begin{bmatrix}\xi_1\\ \vdots\\ \xi_n\end{bmatrix}\right\| = 1, \xi_i \in H, \alpha \in \Lambda\right\} \\
&\leq \|[a_{ij}]\|_{H_0,n},
\end{aligned}$$

where $\{a_\alpha\}$ is a contractive approximate identity for A. This shows that $\{\|\cdot\|_{H_0,n}\} = \{\|\cdot\|_n\}$ on A, and thus we can identify A with a non-degenerate operator algebra on the Hilbert space H_0. ∎

We note that, in our proof of Theorem 2.3, the associativity of the algebras is essential. It allows us to identify any given algebra A with an ideal in its double centralizer algebra $M(A)$. Recently, we have found a different proof of the characterization theorem (Theorem 2.4 and Proposition 2.5) by using the second dual approach in [Ru2]. In that case, it is not necessary to assume the associativity of the algebras. The result is still true for L^∞-Banach pseudo-algebras.

In the rest of this paper, we assume that all algebras are L^∞-Banach algebras with contractive approximate identites. We identify these algebras with operator algebras with contractive approximate identities acting non-degenerately on Hilbert spaces.

3. **Some properties of double centralizer algebras.** Given a non-degenerate operator algebra A on a Hilbert space H, we let B be the *double multiplier algebra* of A defined by

$$B = \{x \in B(H) : xA \subseteq A \text{ and } Ax \subseteq A\}.$$

It is clear that B is a unital operator algebra containing A as an M-ideal. For any $x \in B$, the pair (L_x, R_x) of left and right multiplication maps is belong to $M(A)$. This defines a map $\bar{\mu}: B \longrightarrow M(A)$ by letting

$$\bar{\mu}(x) = (L_x, R_x).$$

THEOREM 3.1. *$\bar{\mu}$ is a completely isometric unital isomorphism from B onto $M(A)$.*

PROOF. It is easy to see that $\bar{\mu}$ is a completely contractive unital homomorphism from B into $M(A)$. Given any $[x_{ij}] \in M_n(B)$, we have

$$\begin{aligned}\|[x_{ij}]\|_n &= \sup\Big\{\Big\|[x_{ij}][a_{jk}][\xi_k]\Big\| : \Big\|[a_{jk}][\xi_k]\Big\| \leq 1, a_{jk} \in A, \xi_k \in H\Big\} \\ &= \sup\Big\{\Big\|\Big[\sum a_\alpha x_{ij} a_{jk}\Big][\xi_k]\Big\| : \|[a_{jk}][\xi_k]\| \leq 1, a_{jk} \in A, \xi_k \in H, \alpha \in \Lambda\Big\} \\ &= \sup\{\|[R_{x_{ij}}(a_\alpha)][a_{jk}][\xi_k]\| : \|[a_{jk}][\xi_k]\| \leq 1, a_{jk} \in A, \xi_k \in H, \alpha \in \Lambda\} \\ &\leq \|[R_{x_{ij}}]\| = \|\|[(L_{x_{ij}}, R_{x_{ij}})]\|\|_n.\end{aligned}$$

Hence, $\mu: B \longrightarrow M(A)$ is a complete isometry.

Next we show that μ is onto. Given any $(S,T) \in M(A)$, we define an operator x on $[AH]$ by

$$x\Big(\sum_i a_i\xi_i\Big) = \sum_i S(a_i)\xi_i$$

for all $\sum_i a_i\xi_i \in [AH]$. If $\sum_i a_i\xi_i = \sum b_j\zeta_j \in [AH]$, then

$$\begin{aligned}\sum_i S(a_i)\xi_i &= \lim_\alpha \sum_i a_\alpha S(a_i)\xi_i = \lim_\alpha \sum_i T(a_\alpha)a_i\xi_i = \lim_\alpha \sum_j T(a_\alpha)b_j\zeta_j \\ &= \lim_\alpha \sum_j a_\alpha S(b_j)\zeta_j = \sum_j S(b_j)\zeta_j.\end{aligned}$$

Therefore, x is a well-defined linear map, and we have

$$\begin{aligned}\Big\|x\Big(\sum_i a_i\xi_i\Big)\Big\| &= \Big\|\sum_i S(a_i)\xi_i\Big\| = \lim_\alpha\Big\|\sum_i a_\alpha S(a_i)\xi_i\Big\| \\ &= \lim_\alpha\Big\|\sum_i T(a_\alpha)(a_i)\xi_i\Big\| \\ &= \lim_\alpha\Big\|T(a_\alpha)\Big(\sum_i a_i\xi_i\Big)\Big\| \leq \|T\|\,\Big\|\sum_i a_i\xi_i\Big\|.\end{aligned}$$

Hence, x extends to a bounded linear map on $H = \overline{[AH]}$, still denoted by x, with

$$\|x\| \leq \|T\| = \|\|(S,T)\|\|_1.$$

Finally for any $a \in A$ and $\xi \in H$, we have $(xa)(\xi) = x(a\xi) = S(a)\xi$. This shows that $S = L_x$. Similarly, $T = R_x$. Thus we have $x \in B$ and $\bar{\mu}(x) = (S,T)$. Hence, $\bar{\mu}$ is onto. ∎

If A is an operator algebra acting non-degenerately on a Hilbert space H, then the matrix algebra $M_n(A)$ is an operator algebra acting non-degenerately on H^n. Identifying with $M(A) = \{x \in B(H) : xA \subseteq A \text{ and } Ax \subseteq A\}$ (Theorem 3.1), we can easily get the completely isometric unital isomorphism

$$M_n\big(M(A)\big) \cong M\big(M_n(A)\big).$$

We remark that one can directly prove this result for double centralizer algebras without using Theorem 3.1. To see this, let $[(S_{ij}, T_{ij})] \in M_n\big(M(A)\big)$. Define $(\mathbf{S}, \mathbf{T}) \in M\big(M_n(A)\big)$ by

$$\mathbf{S}([a_{ij}]) = [s_{ij}], \quad \text{where } s_{ij} = \sum_{k=1}^{n} S_{ik}(a_{kj})$$

$$\mathbf{T}([a_{ij}]) = [t_{ij}], \quad \text{where } t_{ij} = \sum_{k=1}^{n} T_{kj}(a_{ik}).$$

Then $[(S_{ij}, T_{ij})] \longrightarrow (\mathbf{S}, \mathbf{T})$ is a completely isometric unital isomorphism from $M_n\big(M(A)\big)$ onto $M\big(M_n(A)\big)$. We leave the details for the reader.

For any operator algebra A with a contractive approximate identity, it is known that both $M(A)$ and A^{**} are unital operator algebras containing A as an M-ideal (*cf.* Theorem 2.3 and [ER2]). In the following proposition, we study the relation between $M(A)$ and A^{**}.

PROPOSITION 3.2. *Let A be an operator algebra with a contractive approximate identity. Then $M(A)$ is completely isometrically unital isomorphic to a unital subalgebra of A^{**}.*

PROOF. We identified A with an operator subalgebra of a C^*-algebra B. Then A^{**} is a weak* closed subalgebra of B^{**}. Letting $\{\pi, H\}$ be the universal representation of B, we have $B^{**} = \bar{B}^{\sigma}$, and thus $A^{**} = \bar{A}^{\sigma}$ on H. Since A has a contractive approximate identity $\{a_\alpha\}$, A^{**} has a unit $e = \lim a_\alpha$, which is a projection in $B(H)$. Without loss of generality, we can assume that $e = 1_H$ (replace H by eH if necessary). Thus A is non-degenerate on H and $A^{**} = \bar{A}^{\sigma}$.

By Theorem 3.1, we have

$$M(A) = \{x \in B(H) : xA \subseteq A \text{ and } Ax \subseteq A\}.$$

Since A is dense in $M(A)$ with respect to the strictly strong topology (*cf.* [Bu]), and the strictly strong topology is stronger than the σ-weak topology when A is non-degenerate on H, we have

$$M(A) \subseteq \bar{A}^{\sigma} = A^{**}. \qquad \blacksquare$$

We remark that if $K_\infty(H)$ is the C^*-algebra of all compact operators on a separable Hilbert space H, then $M\big(K_\infty(H)\big) = B(H) = K_\infty(H)^{**}$. It is known among the people in this area that the similar result is true for nest algebras. Indeed, if R is a nest algebra on a separable Hilbert space H, then $A = R \cap K_\infty(H)$ is an M-ideal in R and $A^{**} = R$ (*cf.* K. Davidson [Da]). Thus A is a non-degenerate operator algebra on H. Easy calculation shows that $M(A) = R = A^{**}$.

4. **Unital operator algebra matrix norms on** $M(A)$**.** Let A be an operator algebra with a contractive approximate identity. It has been shown in §2 that there is a canonical unital operator algebra matrix norm $\{|||\cdot|||_n\}$ on its double centralizer algebra $M(A)$ such that $M(A)$ contains A as an M-ideal. In this section, we show that there might exist different unital operator algebra matrix norms on $M(A)$ such that $M(A)$ still contains A as an M-ideal.

PROPOSITION 4.1. *If $\{|||\cdot|||'_n\}$ is any unital operator algebra matrix norm on $M(A)$ such that $\{|||\cdot|||'_n\} = \{|||\cdot|||_n\}$ on A, then we have*

$$|||\cdot|||_n \leq |||\cdot|||'_n \quad \text{for all } n \in \mathbb{N}.$$

PROOF. Let $A \subseteq B(H)$ non-degenerately. By Theorem 3.1, we have

$$\big(M(A), \{|||\cdot|||_n\}\big) = \{x \in B(H) : xa \in A,\ ax \in A \text{ for all } a \in A\}.$$

Hence, for any $[x_{ij}] \in M_n\big(M(A)\big) \subseteq M_n\big(B(H)\big)$, we have

$$\begin{aligned}
|||[x_{ij}]|||_n &= \sup\{\|[x_{ij}a]\|_n : a \in A, \|a\| \leq 1\}\\
&= \sup\left\{ \left|\left|\left| [x_{ij}] \begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} \right|\right|\right|'_n : a \in A, \|a\| \leq 1 \right\}\\
&\leq \sup\left\{ |||[x_{ij}]|||'_n \left|\left|\left| \begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} \right|\right|\right|'_n : a \in A, \|a\| \leq 1 \right\}\\
&\leq |||[x_{ij}]|||'_n.
\end{aligned}$$

■

LEMMA 4.2. *Let A be a unital algebra with unit 1, and let $\big\{\{\|\cdot\|_n^k\}_{n=1}^\infty\big\}_{k=1}^\infty$ be a sequence of unital operator algebra matrix norms on A. If for every non-zero element $[a_{ij}] \in M_n(A)$, there exist positive numbers $0 < \alpha < \beta < \infty$ such that*

$$\alpha \leq \|[a_{ij}]\|_n^k \leq \beta$$

for all $k \in \mathbb{N}$, then

$$\|[a_{ij}]\|_n = \overline{\lim_{k\to\infty}} \|[a_{ij}]\|_n^k$$

determines a unital operator algebra matrix norm on A.

PROOF. 1) It is clear from the hypothesis that for any $[a_{ij}] \in M_n(A)$,

$$\|[a_{ij}]\|_n = \overline{\lim_{k\to\infty}} \|[a_{ij}]\|_n^k$$

is well defined and non-negative, and $\|[a_{ij}]\|_n = 0$ if and only if $[a_{ij}] = 0$.

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2) For every $\alpha, \beta \in M_n(\mathbb{C})$ and $a = [a_{ij}] \in M_n(A)$,

$$\begin{aligned}\|\alpha a\beta\|_n &= \overline{\lim_{k\to\infty}} \|\alpha a\beta\|_n^k \le \overline{\lim_{k\to\infty}} \|\alpha\|\,\|a\|_n^k\|\beta\| \\ &= \|\alpha\|(\overline{\lim_{k\to\infty}} \|a\|_n^k)\|\beta\| = \|\alpha\|\,\|a\|_n\|\beta\|.\end{aligned}$$

3) For every $a, b \in M_n(A)$,

$$\begin{aligned}\|a+b\|_n &= \overline{\lim_{k\to\infty}} \|a+b\|_n^k \le \overline{\lim_{k\to\infty}} (\|a\|_n^k + \|b\|_n^k) \\ &\le \overline{\lim_{k\to\infty}} \|a\|_n^k + \overline{\lim_{k\to\infty}} \|b\|_n^k = \|a\|_n + \|b\|_n.\end{aligned}$$

4) For every $a \in M_n(A)$, $b \in M_m(A)$,

$$\begin{aligned}\|a\oplus b\|_{n+m} &= \overline{\lim_{k\to\infty}} \|a\oplus b\|_{n+m}^k = \overline{\lim_{k\to\infty}} \max\{\|a\|_n^k, \|b\|_m^k\} \\ &= \max\{\overline{\lim_{k\to\infty}} \|a\|_n^k, \overline{\lim_{k\to\infty}} \|b\|_m^k\} \\ &= \max\{\|a\|_n, \|b\|_m\}.\end{aligned}$$

5) $\|1\|_1 = \overline{\lim}_{k\to\infty} \|1\|_1^k = 1$.

6) For every $a, b \in M_n(A)$,

$$\begin{aligned}\|ab\|_n &= \overline{\lim_{k\to\infty}} \|ab\|_n^k \le \overline{\lim_{k\to\infty}} \|a\|_n^k\|b\|_n^k \le \overline{\lim_{k\to\infty}} \|a\|_n^k \cdot \overline{\lim_{k\to\infty}} \|b\|_n^k \\ &= \|a\|_n\|b\|_n.\end{aligned}$$

Hence, $\{\|\cdot\|_n\}$ is a unital operator algebra matrix norm on A. ∎

EXAMPLE 4.3. Let $\ell^\infty(\mathbb{N})$ (resp., $c_0(\mathbb{N})$) be the space of all bounded sequences (resp., the space of all sequences converging to 0). Thus $\ell^\infty(\mathbb{N})$ with the canonical unital operator algebra (in fact, the C^*-algebra) matrix norm $\{\|\cdot\|_n\}$ is the double centralizer algebra of $c_0(\mathbb{N})$. In this example, we construct a different unital operator algebra matrix norm $\{|||\cdot|||_n'\}$ on $\ell^\infty(\mathbb{N})$ such that $\ell^\infty(\mathbb{N})$ still contains $c_0(\mathbb{N})$ as an M-ideal.

Let $\{e_i : i \in \mathbb{N}\}$ be the canonical orthonormal basis of $\ell^2(\mathbb{N})$. Identifying elements $\mathbf{a} = \{a_i\} \in \ell^\infty(\mathbb{N})$ with the bounded operators $\mathbf{a}$ on $\ell^2(\mathbb{N})$ determined by $\mathbf{a}(e_i) = a_i e_i$, we have

$$c_0(\mathbb{N}) \subseteq \ell^\infty(\mathbb{N}) \subseteq B\big(\ell^2(\mathbb{N})\big).$$

For $k, l \in \mathbb{N}$, we let $E_{k,l}$ be the standard matrix elements in $B\big(\ell^2(\mathbb{N})\big)$, *i.e.*,

$$E_{k,l}(e_i) = \begin{cases} e_k & \text{if } l = i \\ 0 & \text{otherwise.}\end{cases}$$

For each $k \in \mathbb{N}$, we define $T_k \colon \ell^\infty(\mathbb{N}) \longrightarrow B\big(\ell^2(\mathbb{N})\big)$ by

$$T_k(\mathbf{a}) = \mathbf{a} + (a_k - a_{k+1})E_{k,k+1},$$

for all $\mathbf{a} = \{a_i\} \in \ell^\infty(\mathbb{N})$, and define unital operator algebra matrix norms $\{\|\cdot\|_n^k\}$ on $M_n\big(\ell^\infty(\mathbb{N})\big)$ by

$$\|[\mathbf{a}^{pq}]\|_n^k = \|[T_k(\mathbf{a}^{pq})]\|_n = \|[\mathbf{a}^{pq} + (a_k^{pq} - a_{k+1}^{pq})E_{k,k+1}]\|_n,$$

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for all $[\mathbf{a}^{pq}] = [\{a_i^{pq}\}] \in M_n\big(\ell^\infty(\mathbb{N})\big)$. It is clear that

$$\|[\mathbf{a}^{pq}]\|_n \le \|[\mathbf{a}^{pq}]\|_n^k \le 3\|[\mathbf{a}^{pq}]\|_n.$$

It follows from Lemma 4.2 that the matrix norm defined by

$$|||[\mathbf{a}^{pq}]|||'_n = \overline{\lim_{k\to\infty}}\, \|[\mathbf{a}^{pq}]\|_n^k$$

is a unital operator algebra matrix norm on $\ell^\infty(\mathbb{N})$.

If we let $\mathbf{a} = \{1, -1, 1, -1, \ldots\} \in \ell^\infty(\mathbb{N})$,

$$|||\mathbf{a}|||_1 = \overline{\lim}\{\|\mathbf{a} \pm 2E_{k,k+1}\|\} > \|\mathbf{a}\|_1.$$

Hence, the unital operator algebra matrix norm $\{|||\cdot|||'_n\}$ is different from the canonical unital operator algebra matrix norm $\{\|\cdot\|_n\}$ on $\ell^\infty(\mathbb{N})$.

Finally for any $\mathbf{a} = \{a_i\} \neq 0 \in c_0(\mathbb{N})$, there exists a sufficient large integer N such that for all $i > N$, $3|a_i| < \|\mathbf{a}\|_1$. It follows that $\|\mathbf{a}\|_1^k = \|\mathbf{a} + (a_k - a_{k+1})E_{k,k+1}\|_1 = \|\mathbf{a}\|_1$ for all $k > N$, and thus we have $|||\mathbf{a}|||'_1 = \|\mathbf{a}\|_1$. Similar argument shows that $|||[\mathbf{a}^{pq}]|||'_n = \|[\mathbf{a}^{pq}]\|_n$ for all $[\mathbf{a}^{pq}] \in M_n\big(c_0(\mathbb{N})\big)$. Therefore, we have $\{|||\cdot|||'_n\} = \{|||\cdot|||_n\}$ on $c_0(\mathbb{N})$. ■

EXAMPLE 4.4. Let A be the commutative unital operator subalgebra of $M_3(\mathbb{C})$ generated by I, E_{12}, and E_{13}, *i.e.*,

$$A = \left\{ \begin{bmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\} \subseteq M_3(\mathbb{C}).$$

Let $\{\|\cdot\|_n\}$ be the natural unital operator algebra matrix norm on A, and let $\{\|\cdot\|'_n\}$ be a matrix norm on A defined by

$$\|[a_{ij}]\|'_n = \|[a_{ij}]^{\mathrm{tr}}\|_n = \|[a_{ji}]\|_n$$

for $[a_{ij}] \in M_n(A)$. Clearly, A with this matrix norm $\{\|\cdot\|'_n\}$ is an L^∞-matricially normed space and $\|1\|'_1 = \|1\|_1 = 1$. Since A is commutative, we have

$$\begin{aligned}\|[a_{ij}][b_{jk}]\|'_n &= \|[\Sigma a_{ij}b_{jk}]^{\mathrm{tr}}\|_n \\ &= \|[\Sigma b_{jk}a_{ij}]^{\mathrm{tr}}\|_n \\ &= \|[b_{jk}]^{\mathrm{tr}}[a_{ij}]^{\mathrm{tr}}\|_n \\ &\le \|[b_{jk}]^{\mathrm{tr}}\|_n\, \|[a_{ij}]^{\mathrm{tr}}\|_n = \|[a_{ij}]\|'_n\, \|[b_{ij}]\|'_n.\end{aligned}$$

Hence, $\{\|\cdot\|'_n\}$ is a unital operator algebra matrix norm on A.

Now we consider the commutative operator algebra

$$A_\infty = \left\{ \begin{bmatrix} a_1 & & & & \\ & a_2 & & 0 & \\ & & \ddots & & \\ & 0 & & a_k & \\ & & & & \ddots \end{bmatrix} : a_k \in A, \|a_k\| \to 0 \right\} \subseteq B\big(\mathbb{C}^3 \otimes \ell^2(\mathbb{N})\big).$$

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It is easy to see that

$$M(A_\infty)=\left\{\begin{bmatrix} x_1 & & & & \\ & x_2 & & 0 & \\ & & \ddots & & \\ & 0 & & x_k & \\ & & & & \ddots \end{bmatrix} : x_k\in A, \sup\{\|x_k\|\}<\infty\right\}.$$

For every $[x_{ij}]\in M_n\big(M(A_\infty)\big)$, where $x_{ij}=\begin{bmatrix} x_1^{ij} & & & & \\ & x_2^{ij} & & 0 & \\ & & \ddots & & \\ & 0 & & x_k^{ij} & \\ & & & & \ddots \end{bmatrix}$ with $x_k^{ij}\in A$, we can write (up to a unitary permutation)

$$[x_{ij}]\cong\begin{bmatrix} [x_1^{ij}] & & & & \\ & [x_2^{ij}] & & & \\ & & \ddots & & \\ & & & [x_k^{ij}] & \\ & & & & \ddots \end{bmatrix}$$

with $[x_k^{ij}]\in M_n(A)$ and $\sup\{\|[x_k^{ij}]\|_n : k\in N\}<+\infty$. Letting $\{|||\cdot|||_n\}$ be the canonical unital operator algebra matrix norm on $M(A_\infty)$, we have

$$|||[x_{ij}]|||_n=\sup\{\|[x_k^{ij}]\|_n : k\in\mathbb{N}\}.$$

Define a new matrix norm $\{|||\cdot|||_n'\}$ on $M(A_\infty)$ by

$$|||[x_{ij}]|||_n'=\max\{|||[x_{ij}]|||_n, \overline{\lim_{k\to\infty}}\|[x_k^{ij}]\|_n'\}.$$

It follows from Lemma 4.2 that $\{|||\cdot|||_n'\}$ is a unital operator algebra matrix norm on $M(A_\infty)$.

For every $x=\begin{bmatrix} a_1 & & & \\ & \ddots & & \\ & & a_k & \\ & & & \ddots \end{bmatrix}\in M(A_\infty)$, with $a_k\in A$, we have

$$\overline{\lim_{k\to\infty}}\|a_k\|_1'=\overline{\lim_{k\to\infty}}\|a_k\|_1\le\sup\{\|a_k\|_1\}=|||x|||_1.$$

Hence, $|||x|||_1'=|||x|||_1$ for all $x\in M(A_\infty)$. This shows that $|||\cdot|||_1'=|||\cdot|||_1$ on $M(A_\infty)$.

Let $[x_{ij}]$ be an element in $M_2\big(M(A_\infty)\big)$ given by

$$x_{11}=\begin{bmatrix} E_{12} & & & \\ & \ddots & & \\ & & E_{12} & \\ & & & \ddots \end{bmatrix},\qquad x_{21}=\begin{bmatrix} E_{13} & & & \\ & \ddots & & \\ & & E_{13} & \\ & & & \ddots \end{bmatrix},$$

$$x_{12}=x_{22}=0.$$

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We can write (up to a unitary permutation)

$$[x_{ij}] \cong \begin{bmatrix} \begin{bmatrix} E_{12} & 0 \\ E_{13} & 0 \end{bmatrix} & & \\ & \ddots & \\ & & \ddots \end{bmatrix}.$$

It is clear that $|||[x_{ij}]|||_2 = 1$. On the other hand, since

$$\left\| \begin{bmatrix} E_{12} & 0 \\ E_{13} & 0 \end{bmatrix}^{\mathrm{tr}} \right\|_2 = \left\| \begin{bmatrix} E_{12} & E_{13} \\ 0 & 0 \end{bmatrix} \right\|_2 = \sqrt{2},$$

we get

$$|||[x_{ij}]|||_2' = \max\{\|[x_{ij}]\|_2, \overline{\lim_{k\to\infty}}\, \|[x_k^{ij}]\|_2'\} = \sqrt{2} > |||[x_{ij}]|||_2.$$

This shows that the matrix norm $\{|||\cdot|||_n'\}$ is different from the canonical matrix norm $\{|||\cdot|||_n\}$ on $M(A_\infty)$.

Finally for any $[x_{ij}] \in M_n(A_\infty)$, we can write (up to a unitary permutation)

$$[x_{ij}] \cong \begin{bmatrix} [x_1^{ij}] & & & \\ & \ddots & & \\ & & [x_k^{ij}] & \\ & & & \ddots \end{bmatrix}$$

with $\|[x_k^{ij}]\|_n' \longrightarrow 0$ as $k \longrightarrow \infty$. It follows that $|||[x_{ij}]|||_n' = |||[x_{ij}]|||_n$. This shows that $\{|||\cdot|||_n'\}$ coincides with $\{|||\cdot|||_n\}$ on A_∞. ∎

5. **Unital operator algebra matrix norms on unitalization algebras.** Throughout this section, we assume that A is a non-unital operator algebra with a contractive approximate identity, and its double centralizer algebra $M(A)$ has the canonical unital operator algebra matrix norm. Let $A^1 = A \oplus \mathbb{C}$ be the *unitalization algebra* of A with multiplication defined by

$$(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)$$

for all $(a, \alpha), (b, \beta) \in A \oplus \mathbb{C}$. In this case, the unit element $e = (\mathrm{id}_A, \mathrm{id}_A) \in M(A)$ is not contained in A. Thus we can identify A^1 with the unital subalgebra $A \oplus \mathbb{C}e$ of $M(A)$ and get a unital operator algebra matrix norm on A^1. With this matrix norm, A^1 is a unital operator algebra containing A as an M-ideal. The main result in this section is to show that this is the only such unital operator algebra matrix norm on A^1. In our argument, we need some terminologies from the theory of operator spaces.

Given an operator space V, there is a natural operator space structure on the dual space V^* of V obtained by identifying $M_n(V^*)$ with $\mathrm{CB}(V, M_n)$, the space of all completely bounded maps from V into M_n (*cf.* [BP], [ER3]). The second dual V^{**} of V is also an operator space which contains V as a weak*-dense subspace. Given two operator spaces

V and W, we will assume that V and W are norm closed subspaces of $B(H)$ and $B(K)$ for some Hilbert spaces H and K, respectively. Define the *CM-direct sum* of V and W as the operator space direct sum

$$V \oplus W \subseteq B(H \oplus K).$$

We note that this direct sum is independent of the choice of Hilbert spaces, and we denote this direct sum as $V \oplus_{\mathrm{CM}} W$ (*cf.* [ER4]). A direct sum of operator subspaces V and W is called a *CL-direct sum*, denoted by $V \oplus_{\mathrm{CL}} W$, if we have a complete isometry

$$(V \oplus_{\mathrm{CL}} W)^* \cong V^* \oplus_{\mathrm{CM}} W^*.$$

This is equivalent to the universal property that for any complete contractions $\phi\colon V \longrightarrow X$ and $\psi\colon W \longrightarrow X$, the corresponding map $\phi \oplus \psi\colon V \oplus_{\mathrm{CL}} W \longrightarrow X$ defined by

$$\phi \oplus \psi(v, w) = \phi(v) + \psi(w)$$

is also a complete contraction (*cf.* [ER5]).

Let V be an operator subspace of an operator space W. We call V a *CL-summand* in W if there is an operator subspace L in W such that $W = L \oplus_{\mathrm{CL}} V$. Let $P\colon W \longrightarrow V$ be the projection from W onto V given by $P(x + y) = y$ for all $x \in L$ and $y \in V$. We have $P = \theta \oplus \mathrm{id}_V$, where $\theta\colon L \longrightarrow W$ is the zero map and $\mathrm{id}_V\colon V \hookrightarrow W$ is the embedding map. It follows that P is a complete contraction. We call P the *CL-projection* from W onto V. In this case, the subspace L is also a CL-summand in W and the CL-projection $\mathrm{id}_W - P$ from W onto L is also completely contractive.

An operator subspace V of an operator space W is called a *CM-ideal* if $V^{\perp\perp} = \bar{V}^{w^*}$ is a *CM-summand* in W^{**}. This is equivalent to say that $V^{\perp}$ is a CL-summand in W^*, *i.e.*, there is an operator subspace L in W^* such that

$$W^* = L \oplus_{\mathrm{CL}} V^{\perp}.$$

Since any CM-ideal (resp., CL-direct sum) is an M-ideal (resp., L-direct sum) when we regard the spaces as Banach spaces, it is known by K. Davidson [Da] that the Banach space L is isometrically isomorphic to V^*. In the following theorem, we generalize K. Davidson's result to operator space case.

THEOREM 5.1. *Let V be a* CM-*ideal in an operator space W and let L be an operator subspace in W^* such that $W^* = L \oplus_{\mathrm{CL}} V^{\perp}$. Then there is a complete isometry from L onto V^*.*

PROOF. Let $T\colon L \longrightarrow V^*$ be the restriction map given by

$$T(\varphi) = \varphi|_V$$

for all $\varphi \in L$. It is clear that T is a well-defined complete contraction and T is one-to-one since $\varphi_1, \varphi_2 \in L$ such that $T(\varphi_1) = T(\varphi_2)$ implies $\varphi_1 - \varphi_2 \in L \cap V^{\perp} = \{0\}$.

Given any $[\psi_{ij}] \in M_n(V^*) = \mathrm{CB}(V, M_n)$, it follows from the Wittstock-Arveson Hahn-Banach extension theorem [Wi1] [Wi2] that there exists an element $[\varphi_{ij}] \in$

$M_n(W^*) = \mathrm{CB}(W, M_n)$ such that $\|[\varphi_{ij}]\|_{\mathrm{cb}} = \|[\psi_{ij}]\|_{\mathrm{cb}}$ and $[\varphi_{ij}|_V] = [\psi_{ij}]$. Let Q be the CL-projection from W^* onto its CL-summand L. We have

$$[Q(\varphi_{ij})|_V] = [\varphi_{ij}|_V] = [\psi_{ij}]$$

and

$$\|[\psi_{ij}]\|_{\mathrm{cb}} \leq \|[Q(\varphi_{ij})]\|_{\mathrm{cb}} \leq \|[\varphi_{ij}]\|_{\mathrm{cb}} = \|[\psi_{ij}]\|_{\mathrm{cb}}.$$

Thus $[Q(\varphi_{ij})]$ is a norm preserving extension of $[\psi_{ij}]$ in $M_n(L)$ such that $T_n\big([Q(\varphi_{ij})]\big) = [\psi_{ij}]$. Finally, since T is one-to-one, it gives a complete isometry from L onto V^*. ∎

Notice that for any $\psi \in V^*$, there is, in fact, a unique norm preserving extension $\varphi \in W^*$ of ψ (*cf.* [Da]). In this case, one must have $\varphi = Q(\varphi)$. But we do not know if this is true or not for general $[\psi_{ij}] \in M_n(V^*)$. The difficulty is that we do not have "good" L-summand property for the norms on $M_n(W^*)$ when $n > 1$.

Let A be an operator algebra with a contractive approximate identity and let A^1 be its unitalization algebra with a fixed unital operator algebra matrix norm such that A^1 contains A as an M-ideal. Consider a linear functional $\tau: A^1 \longrightarrow \mathbb{C}$ defined by

$$\tau(a, \alpha) = \alpha$$

for all $(a, \alpha) \in A^1$. The linear functional τ is a unital homomorphism since $\tau(0, 1) = 1$ and

$$\tau\big((a, \alpha)(b, \beta)\big) = \tau(ab + \alpha b + \beta a, \alpha\beta) = \alpha\beta = \tau(a, \alpha)\tau(b, \beta)$$

for all $(a, \alpha), (b, \beta) \in A^1$. As $\ker \tau = A$ has co-dimension one in A^1, τ is bounded on A^1. Furthermore, it follows from Banach algebra theory that τ is contractive, and thus has norm one. Therefore, we have

$$A^\perp = \mathbb{C}\tau.$$

Since A is an M-ideal in A^1, it is also a CM-ideal in A^1 (*cf.* [ER4]), and thus there is a norm closed subspace L in $(A^1)^*$ such that

$$(A^1)^* = L \oplus_{\mathrm{CL}} A^\perp = L \oplus_{\mathrm{CL}} \mathbb{C}\tau.$$

Owing to this fact and Theorem 5.1, we can show

THEOREM 5.2. *There is a unique unital operator algebra matrix norm on A^1 such that A^1 contains A as an M-ideal.*

PROOF. It is clear that A^1 with the canonical unital operator algebra matrix norm $\{||| \cdot |||_n\}$ obtained from the double centralizer algebra $M(A)$ is a unital operator algebra containing A as an M-ideal. If $\{||| \cdot |||'_n\}$ is another such unital operator algebra matrix norm on A^1, an argument similar to that in Proposition 4.1 shows that $||| \cdot |||_n \leq ||| \cdot |||'_n$ for all $n \in \mathbb{N}$.

If we let $A^1_A = (A^1, \{|||\cdot|||'_n\})$ and $A^1_M = (A^1, \{|||\cdot|||_n\})$, the identity map $\mathrm{id}_A\colon A^1_A \longrightarrow A^1_M$ is a completely contractive unital isomorphism. By duality,

$$\Phi = \mathrm{id}^*_A\colon (A^1_M)^* = L_M \oplus_{\mathrm{CL}} \mathbb{C}\tau \longrightarrow (A^1_A)^* = L_A \oplus_{\mathrm{CL}} \mathbb{C}\tau$$

is a complete contraction. It suffices to show that Φ^{-1} is a complete contraction.

By Theorem 5.1, both L_M and L_A are completely isometric to A^*. Thus $\Phi|_{L_M}\colon L_M \longrightarrow L_A$ is a complete isometry. It follows that both $\Phi|^{-1}_{L_M}\colon L_A \longrightarrow L_M \subseteq (A^1_M)^*$ and $\mathrm{id}_{\mathbb{C}\tau}\colon \mathbb{C}\tau \longrightarrow \mathbb{C}\tau \subseteq (A^1_M)^*$ are complete isometries. Thus

$$\Phi^{-1} = \Phi|^{-1}_{L_M} \oplus \mathrm{id}_{\mathbb{C}\tau}\colon L_A \oplus_{\mathrm{CL}} \mathbb{C}\tau \longrightarrow (A^1_M)^*$$

is a complete contraction. ∎

Using Theorem 5.2, we can easily get the following result.

PROPOSITION 5.3. *Let A and B be two operator algebras with contractive approximate identities, and let $\tau\colon A \longrightarrow B$ be a completely isometric isomorphism. The (unique) extension $\tilde{\tau}$ of τ defined by*

$$\tilde{\tau}(a,\alpha) = \big(\tau(a),\alpha\big)$$

is a completely isometric unital isomorphism from A^1 onto B^1.

PROOF. Given $S \in \mathrm{CB}(A,A)$, we define $\tilde{\tau}(S) \in \mathrm{CB}(B,B)$ by

$$\tilde{\tau}(S)\big(\tau(a)\big) = \tau\big(S(a)\big) \quad \text{for all } a \in A.$$

Thus if $(S,T) \in M(A)$, we have $\tilde{\tau}(S,T) = \big(\tilde{\tau}(S),\tilde{\tau}(T)\big) \in M(B)$. It is easy to see that $\tilde{\tau}\colon M(A) \longrightarrow M(B)$ is a completely isometric unital isomorphism. Since A^1 (resp., B^1) with the unique unital operator algebra matrix norm is a unital operator subalgebra of $M(A)$ (resp., $M(B)$), $\tilde{\tau}$ restricted to A^1 determines a completely isometric unital isomorphism from A^1 onto B^1 such that

$$\tilde{\tau}(a,\alpha) = \big(\tau(a),\alpha\big)$$

for all $(a,\alpha) \in A^1$. ∎

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