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ON THE ERGODIC AVERAGES AND THE ERGODIC HILBERT TRANSFORM

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ABSTRACT. Let T be an invertible measure-preserving transformation on a σ-finite measure space (X,μ) and let $1 < p < \infty$. This paper uses an abstract method developed by José Luis Rubio de Francia which allows us to give a unified approach to the problems of characterizing the positive measurable functions v such that the limit of the ergodic averages or the ergodic Hilbert transform exist for all $f \in L^p(v\,d\mu)$. As a corollary, we obtain that both problems are equivalent, extending to this setting some results of R. Jajte, I. Berkson, J. Bourgain and A. Gillespie. We do not assume the boundedness of the operator $Tf(x) = f(Tx)$ on $L^p(v\,d\mu)$. However, the method of Rubio de Francia shows that the problems of convergence are equivalent to the existence of some measurable positive function u such that the ergodic maximal operator and the ergodic Hilbert transform are bounded from $L^p(v\,d\mu)$ into $L^p(u\,d\mu)$. We also study and solve the dual problem.

Introduction and results. Let $(X,\mathcal{F},\mu)$ be a σ-finite measure space and let $T: X \longrightarrow X$ denote an invertible measure-preserving transformation. This transformation defines an operator acting on measurable functions, denoted by the same letter T, and defined by

$$Tf(x) = f(Tx). \tag{0.1}$$

For each nonnegative integer, n, we consider the averages

$$T_nf = (n+1)^{-1}\sum_{k=0}^{n} T^kf. \tag{0.2}$$

Associated to these averages we have the following maximal operator:

$$Mf = M_Tf = \sup_{n\geq 0}|T_nf|. \tag{0.3}$$

We also consider the ergodic Hilbert transform and the ergodic maximal Hilbert transform, associated to T, defined by

$$Hf = \lim_{n\to\infty}\sum_{k=-n}^{n}{}' \frac{1}{k}T^kf \tag{0.4}$$

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and

$$H^*f = \sup_{n\geq 0}\Big|\sum_{k=-n}^{n}{}' \frac{1}{k}T^k f\Big| \tag{0.5}$$

where the prime denotes omission of the 0-th term. The limit will be understood in the pointwise sense through all the paper.

The maximal ergodic theorem asserts that Mf satisfies the inequality

$$\int_X |Mf|^p\, d\mu \leq C_p \int_X |f|^p\, d\mu \tag{0.6}$$

for $1 < p < \infty$. We also have a.e. convergence of the averages defined in (0.2) for functions in $L^p(d\mu)$.

On the other hand, in [C], Cotlar studied the ergodic Hilbert transform. In particular it was shown that if $1 < p < \infty$ and $f \in L^p(d\mu)$ then Hf exists a.e. and

$$\int_X |Hf|^p\, d\mu \leq \int_X |H^*f|^p\, d\mu \leq C_p \int_X |f|^p\, d\mu \tag{0.7}$$

(easier proofs can be found in [P1] and [P2]).

Several weighted versions of the inequalities (0.6) and (0.7) can be found in the literature (see [AT], [AM], [M1]).

Let us forget the measure μ for a while and let us consider a new measure ν such that T is a nonsingular transformation with respect to ν, *i.e.* if $\nu(E) = 0$ then $\nu(T^{-1}E) = 0$. Then we could ask the following question: for fixed p, $1 < p < \infty$, which are the finite measures ν such that the averages $T_n f$ converge a.e. for all $f \in L^p(d\nu)$? This problem was treated in [MT] (see also [S]) for finite measures in the following way: since T is invertible, these measures are necessarily equivalent to a finite invariant measure, *i.e*, there exists a finite measure μ such that T preserves the measure μ and $\nu = v\, d\mu$ for some measurable function v, $0 < v < \infty$ a.e. (see [MT]). Therefore the problem could be reduced to the following: for a finite invariant measure μ, characterize those positive, measurable functions v for which the averages defined in (0.2) converge a.e. for all the functions $f \in L^p(v\, d\mu)$, $1 < p < \infty$.

Analogously, it is considered in [GM] the problem of characterizing those positive measurable functions v for which there exists $Hf(x)$ a.e. for every $f \in L^p(v\, d\mu)$, $1 < p \leq 2$.

The characterization found is the same in both papers, namely $M(v^{-\frac{1}{p-1}})(x) < \infty$ a.e. Moreover, it is shown in [MT] that the condition is equivalent to the existence of a certain positive weight u such that M maps $L^p(v\, d\mu)$ into $L^p(u\, d\mu)$, obtaining in this way that the existence of the limit of the averages implies a dominated ergodic theorem with a change of measure. However the analogue result for H and H^* could not be proved in [GM] although it was established the equivalence with a weighted weak type inequality.

One of the main tools in the above mentioned papers is Nikishin's theorem (see [GR]), in fact this is the reason of the restriction $p \leq 2$ for H in [GM]. The purpose of this paper is to review the problems studied in [GM] and [MT] and extend the results in both papers under the scope of an abstract method developed by Professor Rubio de Francia. Our results include the study of the so called *helical Hilbert transform*, a more general operator than the ergodic Hilbert transform (see [AP] for its connection with Harmonic Analysis). The definition of the helical Hilbert transform $H_\theta, \theta \in \mathbf{R}$, is

$$H_\theta f = \lim_{n\to\infty} \sum_{k=-n}^{n}{}' \frac{e^{ik\theta}}{k} T^k f, \tag{0.8}$$

and

$$H_\theta^* f = \sup_{n\geq 0} \Big| \sum_{k=-n}^{n}{}' \frac{e^{ik\theta}}{k} T^k f \Big| \tag{0.9}$$

is the corresponding maximal operator, *i.e.* the maximal helical Hilbert transform.

The first main theorem we shall prove is the following.

THEOREM A. *Let $(X, \mathcal{F}, \mu)$ be a finite measure space, $1 < p < \infty$, $T: X \longrightarrow X$ a measure-preserving transformation and v a positive measurable function. For each nonnegative integer n, let $R_n f$ denote either $\frac{1}{n+1}\sum_{k=0}^{n} T^k f$ or $\sum_{k=-n}^{\prime n} \frac{e^{ik\theta}}{k} T^k f$ where $\theta \in \mathbf{R}$. Let $R^* f = \sup_n |R_n f|$ and $Rf = \lim_{n\to\infty} R_n f$ in the pointwise sense. (Therefore R^* is either the ergodic maximal operator or the maximal helical Hilbert transform, and R is the limit of the averages or the helical Hilbert transform.) The following are equivalent:*

(a) For every $f \in L^p(v\,d\mu)$, $Rf(x)$ exists almost everywhere.

(b) For every $f \in L^p(v\,d\mu)$, $R^* f(x) < \infty$ a.e.

(c) There exist a positive measurable function u and a positive constant C such that for every $f \in L^p(\nu\,d\mu)$ and all $\lambda > 0$

$$\int_{\{x\in X: R^* f(x) > \lambda\}} u\,d\mu \leq \frac{C}{\lambda^p} \int_X |f|^p v\,d\mu.$$

(d) There exist a positive measurable function u and a positive constant C such that for every $f \in L^p(\nu\,d\mu)$

$$\int_X |R^* f|^p u\,d\mu \leq C \int_X |f|^p v\,d\mu.$$

(e) There exist a positive measurable function u and a positive constant C such that for every $f \in L^p(\nu\,d\mu)$ and all $\lambda > 0$

$$\int_{\{x\in X: |Rf(x)| > \lambda\}} u\,d\mu \leq \frac{C}{\lambda^p} \int_X |f|^p v\,d\mu.$$

(f) There exist a positive measurable function u and a positive constant C such that for every $f \in L^p(\nu\,d\mu)$

$$\int_X |Rf|^p u\,d\mu \leq C \int_X |f|^p v\,d\mu.$$

(g) There exist a positive measurable function u and a positive constant C such that for every $f \in L^p(\nu\,d\mu)$ and all $\lambda > 0$

$$\sup_{n\geq 0} \int_{\{x\in X: |R_n f(x)| > \lambda\}} u\,d\mu \leq \frac{C}{\lambda^p} \int_X |f|^p v\,d\mu.$$

(h) *There exist a positive measurable function u and a positive constant C such that for every $f \in L^p(\nu\, d\mu)$*

$$\sup_{n\geq 0} \int_X |R_n f|^p u\, d\mu \leq C \int_X |f|^p v\, d\mu.$$

(i) $Mv^{-\frac{1}{p-1}}(x) < \infty$ *a.e.*

We should mention that Theorem A was known for the ergodic maximal operator M defined in (0.3) (see [MT]), and partially for the Hilbert transform if $1 < p \leq 2$ (see [GM]). We shall give a different and unified approach to the proof that allows to conclude the result for $1 < p < \infty$ for the ergodic maximal operator and the ergodic Hilbert transform. We should also remark that the equivalence with the statements (c) and (f) was not shown in [GM], even in the case $1 < p \leq 2$. Therefore, the approach of this paper gives that the existence of the ergodic Hilbert transform Hf (or the helical Hilbert transform) for all $f \in L^p(v\, d\mu)$, implies that H and H^* are of strong type (p,p) with a change of measure.

One of the interesting points of Theorem A is the equivalence of (i), (a) applied to $R_n f = \sum'^{n}_{k=-n} \frac{e^{ik\theta}}{k} T^k f$ and (a) applied to $R_n f = \frac{1}{n+1} \sum_{k=0}^{n} T^k f$. This gives the following corollary.

COROLLARY. *Let $(X, \mathcal{F}, \mu)$, p, T and v be as in Theorem A. Let θ be a real number. then, the limit of the sequence $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} T^k f(x)$ exists a.e. for all $f \in L^p(v\, d\mu)$ if and only if the limit of the sequence $\lim_{n\to\infty} \sum'^{n}_{k=-n} \frac{e^{ik\theta}}{k} T^k f(x)$ exists a.e. for all $f \in L^p(v d\mu)$.*

The equivalence of the corollary has been studied recently in the setting of operators T not necessarily induced by pointwise transformations. In [J] R. Jajte proved that the equivalence holds for unitary operators on L^p, and in [BBG] this result is extended for invertible operators on L^p, $1 < p < \infty$ with $\sup_{n\in\mathbf{Z}} \|T^n\| < \infty$. We should remark that the result about pointwise transformations studied in the present paper is not a particular case of the corresponding in [BBG] because the operator $Tf(x) = f(Tx)$ induced by a measure-preserving transformation does not need to satisfy that $\sup_{n\in\mathbf{Z}} \|T^n\| < \infty$, where the norms are taken in our spaces $L^p(v\, d\mu)$. In fact, it is possible to give an example of a measure-preserving transformation T such that the averages are not uniformly bounded on $L^p(v\, d\mu)$ but the limit of the averages $\frac{1}{n+1} \sum_{k=0}^{n} T^k f$ and then the ergodic Hilbert transform Hf exists a.e. for all $f \in L^p(v\, d\mu)$ (see [M2]). Moreover it can be proved that if $T: L^p(v\, d\mu) \longrightarrow L^p(v\, d\mu)$ is induced by a pointwise ergodic transformation such that $\sup_{n\in\mathbf{Z}} \|T^n\| < \infty$ then v is essentially constant, and therefore, this case reduces to the classical one.

Let us point out here that the case $p = 1$, $\theta = 0$ of the corollary has been obtained in [GM, Corollary 2.11].

Theorem A can also be viewed as the answer to the following question: Given a positive weight v, under which conditions on v, does there exist a positive measurable function u such that for all $f \in L^p(v d\mu)$

$$\int_X |Rf|^p u\, d\mu \leq C \int_X |f|^p v\, d\mu,$$

where R is M, H_θ or H_θ^*? As Theorem A shows, the answer is the same for all these operators. The second main result of this paper answers the dual problem, *i.e.*, given a positive weight u, under which condition on u does there exist a positive weight v such that for all $f \in L^p(v\,d\mu)$

$$\int_X |Rf|^p u\,d\mu \leq C \int_X |f|^p v\,d\mu,$$

where R is as above?

THEOREM B. *Let $(X, \mathcal{F}, \mu)$ be a finite measure space, $1 < p < \infty$, $T: X \to X$ a measure-preserving transformation and u a positive measurable function. For each nonnegative integer n, let $R_n f$ denote either $\frac{1}{n+1}\sum_{k=0}^{n} T^k f$ or $\sum_{k=-n}^{\prime n} \frac{e^{ik\theta}}{k} T^k f$ where $\theta \in R$. Let $R^* f = \sup_n |R_n f|$ and $Rf = \lim_{n\to\infty} R_n f$ in the pointwise sense. The following are equivalent.*

(a) There exist a positive measurable function v and a positive constant C such that for every $f \in L^p(\nu\,d\mu)$

$$\int_X |Rf|^p u\,d\mu \leq C \int_X |f|^p v\,d\mu.$$

(b) There exist a positive measurable function v and a positive constant C such that for every $f \in L^p(\nu\,d\mu)$

$$\int_X |R^* f|^p u\,d\mu \leq C \int_X |f|^p v\,d\mu.$$

(c) There exist a positive measurable function v and a positive constant C such that for every $f \in L^p(\nu\,d\mu)$

$$\sup_{n\geq 0} \int_X |R_n f|^p u\,d\mu \leq C \int_X |f|^p v\,d\mu.$$

(d) $Mu(x) < \infty$ a.e.

The organization of the paper is as follows. In Section 1 we collect the technical results that we shall need for the proofs of Theorems A and B that we shall give in Sections 2 and 3. In what follows, the letter C will always mean a positive constant not necessarily the same at each occurrence, and if $1 < p < \infty$ then p' will be the conjugate exponent, *i.e.*, the number p' such that $p + p' = pp'$.

1. **Technical lemmas.** We shall need the following definitions and lemmas about the maximal function and the helical Hilbert transform on the integers.

DEFINITIONS 1.1. If a is a real-valued function on $\mathbf{Z}$ (the set of all integers) we define the one-sided Hardy-Littlewood maximal function ma on $\mathbf{Z}$ by

$$ma(j) = \sup_{n\geq 0} \frac{1}{n+1}\Big|\sum_{k=0}^{n} a(j+k)\Big|, \quad (j \in \mathbf{Z}).$$

On the other hand, we define for $\theta \in \mathbf{R}$ the helical Hilbert transform $h_\theta a$ and the maximal rotated helical Hilbert transform $h_\theta^* a$ on $\mathbf{Z}$ by

$$h_\theta a(j) = \lim_{n\to\infty} \sum_{k=-n}^{n}{}' \frac{e^{ik\theta}}{k} a(j+k), \quad (j \in \mathbf{Z})$$

and

$$h_\theta^* a(j) = \sup_{n\geq 0} \Big| \sum_{k=-n}^{n}{}' \frac{e^{ik\theta}}{k} a(j+k) \Big|, \quad (j \in \mathbf{Z}).$$

respectively, where the prime denotes omission of the 0-th term. Therefore, m, h_θ and h_θ^* are, respectively, the ergodic maximal operator, the helical Hilbert transform and the maximal helical Hilbert transform associated to the transformation on $\mathbf{Z}$ given by $k \longrightarrow k+1$ which preserves the counting measure. In what follows $\gamma(S)$ will denote the counting measure of S.

LEMMA 1.2. *The following inequalities are true for* $1 < p < \infty$.

(1.3) There exists C such that for any $\lambda > 0$ *and any sequence* $\{a_n\}_n$ *of functions on* $\mathbf{Z}$

$$\gamma\Big(\Big\{j : \Big(\sum_{n=1}^{\infty} |ma_n(j)|^p\Big)^{1/p} > \lambda\Big\}\Big) \leq \frac{C}{\lambda} \sum_{j=-\infty}^{\infty} \Big(\sum_{n=1}^{\infty} |a_n(j)|^p\Big)^{1/p}.$$

(1.4) There exists C such that for any $\lambda > 0$ *and any sequence* $\{a_n\}_n$ *of functions on* $\mathbf{Z}$

$$\gamma\Big(\Big\{j : \Big(\sum_{n=1}^{\infty} |h_\theta^* a_n(j)|^p\Big)^{1/p} > \lambda\Big\}\Big) \leq \frac{C}{\lambda} \sum_{j=-\infty}^{\infty} \Big(\sum_{n=1}^{\infty} |a_n(j)|^p\Big)^{1/p}.$$

PROOF. We observe that the space $\mathbf{Z}$ endowed with the distance $d(i,j) = |i-j|$ and with the counting measure γ is an space of homogeneous type and therefore, (1.3) is nothing but saying that the Hardy-Littlewood maximal operator m maps $L^1_{l^p}(\mathbf{Z})$ into weak $-L^1_{l^p}(\mathbf{Z})$, $1 < p < \infty$, see [RT].

In order to prove (1.4) it will suffice to establish the inequality for $\theta = 0$ (the case of the Hilbert transform on the integers). It is known that h_0^* is bounded from $L^2(\mathbf{Z})$ into $L^2(\mathbf{Z})$ (see [HMW] for a proof). This says that h_0 is a Calderón-Zygmund operator in the space of homogeneous type $(\mathbf{Z}, d, \gamma)$. Therefore, h_0 and h_0^*, see [RT], are bounded from $L^1_{l^p}(\mathbf{Z})$ into weak $-L^1_{l^p}(\mathbf{Z})$.

DEFINITION 1.5. If $\{a_n\}_n$ is a sequence of real valued functions on $\mathbf{Z}$ we define $m^t(a_n)$ on $\mathbf{Z}$ by

$$m^t(a_n)(j) = \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=-n}^{0} a_n(j+k), \quad (j \in \mathbf{Z}),$$

and $h_\theta^{*t}(a_n)$ by

$$h_\theta^{*t}(a_n)(j) = \sum_{n=0}^{\infty} \sum_{k=-n}^{n}{}' \frac{e^{ik\theta}}{k} a_n(j+k), \quad (j \in \mathbf{Z}).$$

REMARK 1.6. We observe that the operator m (respectively h_θ^*) defined in (1.1) is bounded from $L^p_{l^q}(\mathbf{Z})$ into $L^p_{l^q}(\mathbf{Z})$ for $1 < p, q < \infty$ if and only if the sequence valued operator

$$\tilde{m}a(j) = \left\{ \frac{1}{n+1} \sum_{k=0}^{n} a(j+k) \right\}_n$$

(respectively the sequence valued operator $\tilde{h}_\theta^* a(j) = \{ \sum_{k=-n}^{\prime n} \frac{e^{ik\theta}}{k} a(j+k) \}_n$) is bounded from $L^p_{l^q}(\mathbf{Z})$ into $L^p_{l^q(l^\infty)}(\mathbf{Z})$. To see this, we observe that

$$ma(j) = \|\tilde{m}a(j)\|_{l^\infty} \text{ and } h_\theta^* a(j) = \|\tilde{h}_\theta^* a(j)\|_{l^\infty}$$

On the other hand, it is easy to check that the operators m^t and h_θ^{*t} are the transpose operators of $\tilde{m}$ and $\tilde{h}_\theta^*$. Since $\tilde{m}$ and $\tilde{h}_0^*$ are Calderón-Zygmund operators on the space $(\mathbf{Z}, d, \gamma)$ of homogeneous type then $\tilde{m}$ and $\tilde{h}_0^*$ are bounded from $L^q_{l^p}(\mathbf{Z})$ into $L^q_{l^p(l^\infty)}(\mathbf{Z})$, and therefore, by duality, m^t and h_0^{*t} are bounded from $L^q_{l^p(l^1)}(\mathbf{Z})$ to $L^q_{l^p}(\mathbf{Z})$. But m^t and h_0^{*t} are again Calderón-Zygmund operators and therefore they map $L^1_{l^p(l^1)}(\mathbf{Z})$ into weak $-L^1_{l^p}(\mathbf{Z})$, $1 < p < \infty$; in other words the following inequalities are true for $1 < p < \infty$ and $\lambda > 0$:

$$\gamma\Big(\Big\{j : \Big(\sum_{n=1}^{\infty} |m^t(a_{k,n})(j)|^p\Big)^{1/p} > \lambda\Big\}\Big) \leq \frac{C}{\lambda} \sum_{j=-\infty}^{\infty} \Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty} |a_{k,n}(j)|\Big)^p\Big)^{1/p}, \tag{1.7}$$

$$\gamma\Big(\Big\{j : \Big(\sum_{n=1}^{\infty} |h_0^{*t}(a_{k,n})(j)|^p\Big)^{1/p} > \lambda\Big\}\Big) \leq \frac{C}{\lambda} \sum_{j=-\infty}^{\infty} \Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty} |a_{k,n}(j)|\Big)^p\Big)^{1/p}. \tag{1.8}$$

It is also clear that (1.8) holds with h_θ instead of h_0, *i.e.*

$$\gamma\Big(\Big\{j : \Big(\sum_{n=1}^{\infty} |h_\theta^{*t}(a_{k,n})(j)|^p\Big)^{1/p} > \lambda\Big\}\Big) \leq \frac{C}{\lambda} \sum_{j=-\infty}^{\infty} \Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty} |a_{k,n}(j)|\Big)^p\Big)^{1/p}. \tag{1.9}$$

DEFINITION 1.10. Assume that s is an operator defined on functions $a: \mathbf{Z} \longrightarrow \mathbf{R}$ as

$$sa(j) = \sup_n |t_n a(j)|$$

where t_n is a family of linear operators acting over functions $a: \mathbf{Z} \longrightarrow \mathbf{R}$. We shall denote by s_{2^r} the "truncated" operator

$$s_{2^r} a(j) = \sup_{0 \leq n \leq 2^r} |t_n a(j)|$$

Analogously, if s^t is defined over a sequence $(a_n): \mathbf{Z} \longrightarrow \mathbf{R}^{\mathbf{N}}$ as

$$s^t(a_n)(j) = \sum_{n=0}^{\infty} t_n a_n(j),$$

we shall denote by $s^t_{2^r}$ the operator

$$s^t_{2^r}(a_n)(j) = \sum_{n=0}^{2^r} t_n a_n(j).$$

Observe that if $t_n a(j) = \frac{1}{n+1} \sum_{k=0}^{n} a(k+j)$ then s is the maximal operator m, and if $t_n a(j) = \frac{1}{n+1} \sum_{k=-n}^{0} a(j+k)$ then s^t is m^t. Analogously, if $t_n a(j) = \sum_{k=-n}^{\prime n} \frac{e^{ik\theta}}{k} a(j+k)$ then s is h_θ^* and s^t is h_θ^{*t}.

PROPOSITION 1.11. *Assume that* $w\colon \mathbf{Z} \longrightarrow [0,\infty]$ *is a positive function,* $0 < s < 1 < p < \infty$, $k \in \mathbf{N}$.

(i) If we denote by s_{2^r} *either the truncated maximal function or the truncated maximal helical Hilbert transform on the integers then the following inequality holds*

(1.12)
$$\sum_{2^{r-1}\le|j|<2^r}\Big(\sum_n |s_{2^r}(a_n\chi_{[-2^{r+1},2^{r+1}]})(j)|^p\Big)^{s/p} \le C2^{r(1-\frac{s}{p})}\Big(\frac{1}{1+2^{r+2}}\sum_{l=-2^{r+1}}^{2^{r+1}}\big(w(l)\big)^{-p'/p}\Big)^{s/p'}\Big(\sum_n\sum_{l=-2^{r+1}}^{2^{r+1}}|a_n(l)|^p w(l)\Big)^{s/p}$$

(ii) If we denote by $S^t_{2^r}$ *the truncation of any of the operators defined in (1.5) then the following inequality holds*

(1.13)
$$\sum_{2^{r-1}\le|j|<2^r}\Big(\sum_k |s^t_{2^r}(a_{nk}\chi_{[-2^{r+1},2^{r+1}]})(j)|^p\Big)^{s/p} \le C2^{r(1-\frac{s}{p})}\Big(\frac{1}{1+2^{r+2}}\sum_{l=-2^{r+1}}^{2^{r+1}}\big(w(l)\big)^{-p'/p}\Big)^{s/p'}\Big(\sum_k\sum_{l=-2^{r+1}}^{2^{r+1}}\Big(\sum_n|a_{nk}(l)|\Big)^p w(l)\Big)^{s/p}$$

PROOF. The proofs of (1.12) and (1.13) are the same. They use as the main step Kolmogorov condition, see [GR,V.2.8], and afterwards inequalities (1.3), (1.4), (1.7) and (1.8). We shall prove (1.12) in the case that s is the maximal function defined in (1.1).

As we said above, by Kolmogorov condition and (1.3), we have

$$\begin{aligned}
&\sum_{2^{r-1}\le|j|<2^r}\Big(\sum_n |m_{2^r}(a_n\chi_{[-2^{r+1},2^{r+1}]})(j)|^p\Big)^{s/p}\\
&\le C\big(\gamma(\{j: 2^{r-1}\le|j|<2^r\})\big)^{1-s}\Big(\sum_{l=-\infty}^{\infty}\Big(\sum_n|a_n(l)|^p\chi_{[-2^{r+1},2^{r+1}]}(l)\Big)^{1/p}\Big)^s\\
&= C2^{r(1-\frac{s}{p})}\Big(2^{-\frac{r}{p'}}\sum_{l=-\infty}^{\infty}\Big(\sum_n|a_n(l)|^p\chi_{[-2^{r+1},2^{r+1}]}(l)\Big)^{1/p}\big(\omega(l)\big)^{1/p}\big(\omega(l)\big)^{-1/p}\Big)^s\\
&\le C2^{r(1-\frac{s}{p})}\Big(2^{-\frac{r}{p'}}\Big(\sum_{l=-2^{r+1}}^{2^{r+1}}\sum_n|a_n(l)|^p w(l)\Big)^{1/p}\Big(\sum_{l=-2^{r+1}}^{2^{r+1}}\big(w(l)\big)^{-p'/p}\Big)^{1/p'}\Big)^s\\
&\le C2^{r(1-\frac{s}{p})}\Big(\sum_n\sum_{l=-2^{r+1}}^{2^{r+1}}|a_n(l)|^p w(l)\Big)^{s/p}\Big(\frac{1}{1+2^{r+2}}\sum_{l=-2^{r+1}}^{2^{r+1}}\big(w(l)\big)^{-p'/p}\Big)^{s/p'}
\end{aligned}$$

where in the two last inequalities we have used Hölder's inequality and the definition of m.

Now we introduce the concept of ergodic rectangle and state a lemma proved in [M1].

DEFINITION 1.14. Let $T\colon X \longrightarrow X$ be a invertible measure preserving transformation and let s be a positive integer. The measurable set $B \subset X$ is the base (with respect to T) of an ergodic rectangle of length s if $T^iB \cap T^jB = \emptyset$, $i \neq j$, $0 \le i,j \le s-1$. In such case the set $R = \bigcup_{0\le i\le s-1} T^iB$ will be called *ergodic rectangle* with base B and length s.

LEMMA 1.15. *Let Y be a measurable subset of X and let k be a positive integer. Then there exists a countable family $\{B_i : i \in \mathbf{Z}^+\}$ of sets of finite measure such that*

(i) $Y = \bigcup_{i=0}^{\infty} B_i$

(ii) $B_i \cap B_j = \emptyset$ *if* $i \neq j$

(iii) *For every i, B_i is the base of an ergodic rectangle of length $s(i) \leq k$ and such that if $s(i) < k$ then $T^{s(i)}A = A$ for every measurable set $A \subset B_i$.*

Now we state the result of Rubio de Francia in the version that we shall use, see [FT].

THEOREM 1.16. *Let (Y, d) be a measure space, F and G Banach spaces and let $\{A_r\}_{r=-\infty}^{\infty}$ be a sequence of disjoint sets in Y such that $Y = \bigcup_{r=-\infty}^{\infty} A_r$. Assume that $0 < s < p < \infty$ and T is a sublinear operator which satisfies*

$$\Big\| \Big(\sum_j \|Tf_j\|_F^p\Big)^{1/p} \Big\|_{L^s(A_r, d\nu)} \leq C_r \Big(\sum_j \|f_j\|_G^p\Big)^{1/p} \quad (r \in \mathbf{Z}). \tag{1.17}$$

where for each $r \in \mathbf{Z}$, C_r is a constant depending on G, F, p and s. Then there exists a positive function u on Y such that

$$\int_Y \|Tf(x)\|_F^p u(x)\, d\nu(x) \leq C\|f\|_G$$

holds.

2. **Proof of Theorem A.** We shall prove this theorem as follows: (g) $\Rightarrow$ (i), (a) $\Rightarrow$ (b) $\Rightarrow$ (i) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (a), (d) $\Rightarrow$ (f) $\Rightarrow$ (e) $\Rightarrow$ (a), (d) $\Rightarrow$ (h) $\Rightarrow$ (g).

We first observe that the implications (a) $\Rightarrow$ (b), (d) $\Rightarrow$ (c), (f) $\Rightarrow$ (e), (d) $\Rightarrow$ (h), (h) $\Rightarrow$ (g) are obvious. Second, (c) implies (a) because of the weak type inequality (c) and the Banach's principle since Rf exists a.e. for all $f \in L^p(v\,d\mu) \cap L^1(d\mu)$ which is a dense class in $L^p(v\,d\mu)$. In the same way, (d) implies the existence of Rf for all $f \in L^p(v\,d\mu)$ and then it is obvious that (f) holds. It is also clear (e) $\Rightarrow$ (a) because statement (a) is included in statement (e).

In order to complete the proof we have to prove (g) $\Rightarrow$ (i), (b) $\Rightarrow$ (i) and (i) $\Rightarrow$ (d).

(g) $\Rightarrow$ (i). For every positive integer k, let $\{B_{i,k} : i \in \mathbf{Z}^+\}$ be the sequence given by Lemma 1.15 for X and an integer sufficiently large, for example $4k$. Let $s(i,k)$ be the length of the rectangle with base $B_{i,k}$. We consider $X = Y \cup Z$ where

$$Y = \bigcap_{k=1}^{\infty} \Big(\bigcup_{\{i : s(i,k) = 4k\}} B_{i,k} \Big)$$

and

$$Z = \bigcup_{k=1}^{\infty} \bigcup_{\{i : s(i,k) < 4k\}} B_{i,k}$$

It is clear that $Mv^{-\frac{1}{p-1}}(x) < \infty$ a.e. in Z (see [GM]). Now we will prove that (i) holds for almost every x in Y. Let us fix a positive integer k and let $B_{i,k}$ be a base of a rectangle with length $s(i,k) = 4k$. For each integer r we define

$$E_{i,k,r} = \Big\{ x \in B_{i,k} : 2^{r+1} < \frac{1}{k} \sum_{j=0}^{k-1} v^{-\frac{1}{p-1}}(T^j x) \leq 2^{r+2} \Big\}$$

and for a measurable set $E \subset E_{i,k,r}$ with $\mu(E) > 0$, let $\mathcal{R}$ be the rectangle with base E and length k, *i.e.* $\mathcal{R} = \bigcup_{0\le j\le k-1} T^jE$.

Let g be the function with support on $\mathcal{R}$ defined by

$$g(T^lx) = e^{-il\theta}v^{-1/p-1}(T^lx) \quad \text{if } 0 \le l \le k-1 \text{ and } x \in E.$$

Then if $R_nf(x) = \sum_{l=-n}^{n} \frac{e^{il\theta}}{l} f(T^lx)$ we have for $-k \le j < 0$ and $x \in E$,

$$\begin{aligned} |R_{2k}g(T^jx)| &= \Big|\sum_{l=-2k}^{2k} \frac{e^{il\theta}}{l} g(T^{l+j}x)\Big| = \Big|\sum_{l=0}^{k-1} \frac{e^{i(l-j)\theta}}{l-j} g(T^lx)\Big| \\ &= \Big|\sum_{l=0}^{k-1} \frac{e^{-ij\theta}}{l-j} v^{-\frac{1}{p-1}}(T^lx)\Big| \ge \frac{1}{2k}\sum_{l=0}^{k-1} v^{-\frac{1}{p-1}}(T^lx). \end{aligned}$$

Now, if $R_nf(x) = \frac{1}{n}\sum_{l=0}^{n-1} f(T^lx)$ we take $g = v^{-1/p-1}\chi_{\mathcal{R}}$ and then for $-k \le j < 0$ and $x \in E$

$$|R_{2k}g(T^jx)| \ge \frac{1}{2k}\sum_{l=0}^{k-1} v^{-\frac{1}{p-1}}(T^lx).$$

Therefore, in both cases we have a function g such that $|g| = v^{-\frac{1}{p-1}}\chi_{\mathcal{R}}$ and

$$|R_{2k}g(T^jx)| \ge \frac{1}{2k}\sum_{l=0}^{k-1} v^{-\frac{1}{p-1}}(T^lx) \quad \text{for } x \in E \text{ and } -k \le j < 0.$$

Then, by the definition of $E_{i,k,r}$

$$|R_{2k}g(T^jx)| \ge 2^r \quad \text{for } x \in E \text{ and } -k \le j < 0.$$

Consequently, it follows from (g) that

$$\int_{\bigcup_{j=-k}^{-1} T^jE} u\, d\mu \le 2^{-rp}C\int_{\bigcup_{j=0}^{k-1} T^jE} v^{-\frac{1}{p-1}}\, d\mu$$

that is

$$2^{rp}\int_E \sum_{j=-k}^{-1} u(T^jx)\, d\mu(x) \le C\int_E \sum_{j=0}^{k-1} v^{-\frac{1}{p-1}}(T^jx)\, d\mu(x)$$

Therefore, by the definition of $E_{i,k,r}$ we have

$$\int_E \Big(\frac{1}{k}\sum_{l=0}^{k-1} v^{-\frac{1}{p-1}}(T^lx)\Big)^p \frac{1}{k}\sum_{j=-k}^{-1} u(T^jx)\, d\mu(x) \le 4^pC\int_E \frac{1}{k}\sum_{j=0}^{k-1} v^{-\frac{1}{p-1}}(T^jx)\, d\mu(x)$$

and, thus, we obtain that there exists a constant C such that for almost every x in B_{ik}

$$\Big(\frac{1}{k}\sum_{j=0}^{k-1} v^{-\frac{1}{p-1}}(T^jx)\Big)^{p-1} \frac{1}{k}\sum_{j=-k}^{-1} u(T^jx) \le C$$

and, therefore, for a.e. x in Y and every positive integer k.

On the other hand, assuming without loss of generality that $u \in L^1(d\mu)$ and $u \le v$, we get by Birkhoff's ergodic theorem that

$$0 < \lim_{k\to\infty} \frac{1}{k} \sum_{j=-k}^{-1} u(T^j x) = \alpha(x) < \infty$$

Therefore

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} v^{-\frac{1}{p-1}}(T^j x) \le \left(\frac{C}{\alpha(x)}\right)^{\frac{1}{p-1}} \quad \text{a.e. } x \text{ in } Y$$

and then

$$Mv^{-\frac{1}{p-1}}(x) < \infty \quad \text{a.e. } x \text{ in } Y.$$

(b) $\Rightarrow$ (i). If $1 < p \le 2$ we can use Nikishin Theorem, see [GR], and then we obtain (g) for $1 < p \le 2$ and therefore by the above proof we get (i).

If $2 < p$ we have

$$L^2(w\,d\mu) \subset L^p(v\,d\mu) + L^1(d\mu)$$

where $w = v^{\frac{1}{p-1}}$. In order to prove this inclusion we take a nonnegative $f \in L^2(w\,d\mu)$ and define

$$f = g + h$$

where $g(x) = f(x)$ if $f(x) \le v^{-\frac{1}{p-1}}(x)$ and 0 if $f(x) > v^{-\frac{1}{p-1}}(x)$. The following inequalities prove that $g \in L^p(v\,d\mu)$:

$$\int_X g^p v\,d\mu = \int_X g^2 v^{\frac{1}{p-1}} g^{p-2} v^{\frac{p-2}{p-1}}\,d\mu \le \int_X g^2 v^{\frac{1}{p-1}}\,d\mu \le \int_X f^2 w\,d\mu < \infty.$$

In order to see that $h \in L^1(d\mu)$ we apply Hölder inequality and we set

$$\begin{aligned}\int_X h\,d\mu &\le \left(\int_X h^2 v^{\frac{1}{p-1}}\,d\mu\right)^{1/2} \left(\int_{\{x:h(x)\neq 0\}} v^{-\frac{1}{p-1}}\,d\mu\right)^{1/2} \\ &\le \left(\int_X f^2 w\,d\mu\right)^{1/2} \left(\int_{\{x:h(x)\neq 0\}} v^{-\frac{2}{p-1}} \cdot v^{\frac{1}{p-1}}\,d\mu\right)^{1/2} \\ &\le \left(\int_X f^2 w\,d\mu\right)^{1/2} \left(\int_{\{x:h(x)\neq 0\}} h^2 \cdot v^{\frac{1}{p-1}}\,d\mu\right)^{1/2} \le \int_X f^2 w < \infty.\end{aligned}$$

Once we have proved $L^2(w\,d\mu) \subset L^p(v\,d\mu) + L^1(d\mu)$ with $w = v^{1/p-1}$ we proceed as follows. By (b), the inclusion and the classical fact that $R^*f(x) < \infty$ a.e. for every $f \in L^1(d\mu)$ we obtain $R^*f(x) < \infty$ a.e. for every $f \in L^2(w)$, *i.e.*, we have (b) for $p = 2$ and the weight w. Then, by the case $p = 2$ of the implication (b) $\Rightarrow$ (i), we have $Mw^{-1}(x) = Mv^{-\frac{1}{p-1}}(x) < \infty$ a.e.

(i) $\Rightarrow$ (d). For each $r \in \mathbf{Z}$ we consider the invariant set

$$X_r = \left\{x : 2^r \le \limsup_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} v^{-\frac{1}{p-1}}(T^k x) < 2^{r+1}\right\}.$$

It is clear that $X = \bigcup_{r=-\infty}^{\infty} X_r$ and the sets X_r are pairwise disjoint.

By using Theorem 1.16 it is clear that in order to prove (d) it is enough to prove that

$$\Big\|\Big(\sum_j |R^* f_j|^p\Big)^{1/p}\Big\|_{L^s(X_r, d_\mu)} \le C_r \sum_j \int_X |f_j(x)|^p v(x)\, d\mu(x) \tag{2.1}$$

in other words we take $A_r = X_r, F = \mathbf{C}$ (the complex numbers), $G = L^p(v\,d\mu)$ and $T = R^*$.

By Fatou's Lemma it is clear that in order to have (2.1) it is enough to show

$$\Big\|\Big(\sum_j |R^*_{2^k} f_j|^p\Big)^{1/p}\Big\|_{L^s(X_r, d\mu)} \le C_r \sum_j \int_X |f_j(x)|^p v(x)\, d\mu(x) \tag{2.2}$$

where we denote by $R^*_{2^k} f$, $k = 0, 1, 2, \ldots$, the truncated operator

$$R^*_{2^k} f(x) = \sup_{n \le 2^k} |R_n f(x)|,$$

and C_r is a constant independent of k.

We now prove (2.2). Given a function $g: X \longrightarrow \mathbf{C}$ and $x \in X$ we define the function $g^x: \mathbf{Z} \longrightarrow \mathbf{C}$ by $g^x(i) = g(T^i x)$. Given $k = 0, 1, 2, 3, \ldots$ we consider $S_k = \{i \in \mathbf{Z} : 2^{k-1} < |i| \le 2^k\}$. Since T is a measure preserving transformation and X_r is invariant, we have

$$\int_{X_r} \Big(\sum_j |R^*_{2^k} f_j(x)|^p\Big)^{s/p} d\mu(x) = \frac{1}{2^k} \sum_{2^{k-1} < |i| \le 2^k} \int_{X_r} \Big(\sum_j |R^*_{2^k} f_j(T^i x)|^p\Big)^{s/p} d\mu(x).$$

Assume now that $R_n f(x) = \frac{1}{n+1} \sum_{j=0}^n f(T^j x)$; therefore $R^*_{2^k} = M_{2^k}$ (the proof is the same in the other cases). Observe that if $i \in S_k$ then

$$M_{2^k} f_j(T^i x) \le m_{2^k}(f_j^x \chi_{[-2^{k+1}, 2^{k+1}]})(i)$$

Therefore, by using Proposition 1.11 with $w = v^x$ and Hölder inequality, we have.

$$\begin{aligned}
&\int_{X_r} \frac{1}{2^k} \sum_{2^{k-1} < |i| \le 2^k} \Big(\sum_j |M_{2^k} f_j(T^i x)|^p\Big)^{s/p} d\mu \\
&\le C 2^{-k\frac{s}{p}} \int_{X_r} \Big((M_T + M_{T^{-1}}) v^{-\frac{1}{p-1}}(x)\Big)^{s/p'} \Big(\sum_{j,l} |f_j^x(l)|^p \chi_{[-2^{k+1}, 2^{k+1}]}(l) v^x(l)\Big)^{s/p} d\mu(x) \\
&\le C \Big(\int_{X_r} \Big((M_T + M_{T^{-1}}) v^{-\frac{1}{p-1}}(x)\Big)^{\frac{s}{p'}(\frac{p}{s})'} d\mu(x)\Big)^{1/(p/s)'} \\
&\quad \cdot \Big[\frac{1}{2^k} \int_{X_r} \sum_j \sum_{l=-2^{k+1}}^{2^{k+1}} |f_j(T^l x)|^p v(T^l x)\, d\mu(x)\Big]^{s/p} \\
&= C \Big(\int_{X_r} \Big((M_T + M_{T^{-1}}) v^{-\frac{1}{p-1}}(x)\Big)^{\frac{s}{p'}(\frac{p}{s})'} d\mu(x)\Big)^{\frac{1}{(p/s)'}} \Big(\int_{X_r} \sum_j |f_j(x)|^p v(x)\, d\mu(x)\Big)^{s/p}
\end{aligned}$$

where in the last equality we have used that T is a measure-preserving transformation.

Finally, we observe that as $\beta = \frac{s}{p'}(\frac{p}{s})' < 1$ we have

$$\Big(\int_{X_r} [(M_T v^{-\frac{1}{p-1}} + M_{T^{-1}} v^{-\frac{1}{p-1}})(x)]^\beta\, d\mu(x)\Big)^{1/\beta} \le \big(\mu(X_r)\big)^{\frac{1}{\beta}-1} \|M_T v^{-\frac{1}{p-1}} + M_{T^{-1}} v^{-\frac{1}{p-1}}\|_{L^1_*(d\mu, X_r)}$$

where $\|f\|_{L^1_*}(d\mu)$ is the weak $-L^1$ norm of w. Then since $M_T + M_{T^{-1}}$ is of weak type $(1,1)$

$$\Big(\int_{X_r} [(M_T v^{-\frac{1}{p-1}} + M_{T^{-1}} v^{-\frac{1}{p-1}})(x)]^\beta \, d\mu(x)\Big)^{1/\beta} \leq \big(\mu(X_r)\big)^{\frac{1}{\beta}-1} \|v^{-\frac{1}{p-1}}\|_{L^1(X_r)} \leq C_r$$

For the last inequality see [GM]. This ends the proof of (2.2) and therefore the proof of (i) $\Rightarrow$ (d).

3. **Proof of Theorem B.** We shall make the proof as follows: (a) $\Rightarrow$ (d) $\Rightarrow$ (b) $\Rightarrow$ (c), (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (d). First, we observe that (b) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a) are obvious.

On the other hand, by duality, (a) is equivalent to

$$\int_X |R_{T^{-1}} f(x)|^{p'} \big(v(x)\big)^{-p'/p} \, d\mu(x) \leq C \int_X |f(x)|^{p'} u^{-\frac{p'}{p}}(x) \, d\mu(x)$$

where $R_{T^{-1}}$ is the same operator as R but defined with respect to the transformation T^{-1}. Therefore, by Theorem A, the weight $u^{-p'/p}$ must satisfy

$$M_{T^{-1}}[(u^{-p'/p})^{-\frac{1}{p'-1}}](x) < \infty \quad \text{a.e.}$$

i.e.

$$\sup_{n\geq 0} \frac{1}{n+1} \sum_{k=0}^{n} u(T^{-k}x) < \infty \quad \text{a.e.}$$

but this is equivalent to $Mu(x) < \infty$ a.e. which is (d). Therefore (a) $\Rightarrow$ (d). In the same way it is proved (c) $\Rightarrow$ (d).

Now we shall prove (d) $\Rightarrow$ (b). We define the operator R^{*t} acting on sequences of functions (f_n) by

$$R^{*t}(f_n)(x) = \sum_{n}^{\infty} R_n f_n(x).$$

R^* maps $L^p(v\,d\mu)$ into $L^p(u\,d\mu)$ if and only if the operator

$$f \longmapsto \{R_n f\}_n$$

maps $L^p(v\,d\mu)$ into $L^p_{l^\infty}(u\,d\mu)$, and therefore, by duality, if and only if R^{*t} maps $L^{p'}_{l^1}(u^{-p'/p}\,d\mu)$ into $L^{p'}(v^{-p'/p}\,d\mu)$. We can apply to the operator R^{*t} the techniques developed in the proof of Theorem A and by using (1.12), (1.13) and Theorem 1.16 we can conclude that R^{*t} maps $L^{p'}_{l^1}(u^{-p'/p}\,d\mu)$ into $L^{p'}(v^{-p'/p}\,d\mu)$ if $u^{-p'/p}$ satisfies $M\big((u^{-p'/p})^{-1/(p'-1)}\big)(x) < \infty$ a.e., but this is exactly condition (d).

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