Canad. J. Math. Vol. **54** (4), 2002 pp. 673–693

Local L-Functions for Split Spinor Groups

Mahdi Asgari

Abstract. We study the local L-functions for Levi subgroups in split spinor groups defined via the Langlands-Shahidi method and prove a conjecture on their holomorphy in a half plane. These results have been used in the work of Kim and Shahidi on the functorial product for $\mathrm{GL}_2 \times \mathrm{GL}_3$.

1 Introduction

The purpose of this work is to prove a conjecture on the holomorphy of local Langlands L-functions defined via the Langlands-Shahidi method in split spinor groups. These local factors appear in the Euler products of global automorphic L-functions and information about their holomorphy is frequently exploited in order to prove results about the analytic properties of global objects. In particular, in a recent important work, H. Kim and F. Shahidi have used some cases of our result here in order to handle some local problems in their long-awaited result on the existence of symmetric cube cusp forms on GL_2 (*cf.* [11], [12]).

Apart from trace formula methods, two methods have been suggested to study these factors: the Rankin-Selberg method which uses "zeta integrals" and the Langlands-Shahidi method which uses "Eisenstein series". Our focus in this work is on the latter [13], [16], [18], [20].

Let $\mathbf{M}$ be a (quasi) split connected reductive linear algebraic group defined over a non-archimedean local field F of characteristic zero. Let $\widehat{M} = {}^L M^0$ denote the connected component of its L-group. Let σ be an irreducible admissible unramified representation of $M = \mathbf{M}(F)$. The Satake isomorphism attaches a unique semisimple conjugacy class A in $\widehat{M}$ to each such σ. (When $\mathbf{M}$ is quasi-split, one needs ${}^L M$.) When the above data comes from a global automorphic representation, these conditions are satisfied for all but finitely many places.

If r is a (finite dimensional) complex analytic representation of $\widehat{M}$, one defines the local Langlands L-function by

$$L(s, \sigma, r) = \det\bigl(I - r(A) q^{-s}\bigr)^{-1}, \quad s \in \mathbb{C},$$

where q is the cardinality of the residue class field of F.

However, it remains to define such local L-functions for the remaining places and the resulting completed L-functions should satisfy some global properties such as analytic continuation (with a finite number of poles) and functional equation.

The idea of the Langlands-Shahidi method is to use intertwining operators to define the γ-factors, which are essentially quotients of the local L-functions. These

Received by the editors February 1, 2001; revised May 25, 2001.
AMS subject classification: 11F70.

factors are defined in terms of the harmonic analysis on the group. In this method one assumes $\mathbf{M}$ to be the Levi component of a parabolic subgroup $\mathbf{P} = \mathbf{MN}$ in a (quasi) split connected reductive group $\mathbf{G}$ with $\mathbf{N}$ the unipotent radical of $\mathbf{P}$. Also, one assumes that σ is generic, *i.e.*, it has a Whittaker model. (One could define the L-functions for any irreducible admissible tempered σ to be the same as that of the unique generic one that conjecturally exists in its L-packet.)

The representation r is a representation of the L-group whose restriction to the connected component, $\widehat{M}$, is assumed to be an irreducible constituent of the adjoint representation of $\widehat{M}$ on the Lie algebra of the dual of $N = \mathbf{N}(F)$. Such r's turn out to cover most of the interesting L-functions studied so far.

Defining these L-functions, Shahidi in [20] set forth the following:

Conjecture If σ is tempered, then $L(s, \sigma, r)$ is holomorphic for $\Re(s) > 0$.

This conjecture was subsequently proved in a series of papers for the cases where $\mathbf{G}$ is a general linear, symplectic, special orthogonal, or unitary group (*cf.* [17], [19], [21], [6]). Given that it is enough to prove the conjecture for $\mathbf{G}$ simple, it remained to prove the conjecture for the so-called spinor groups, the simply-connected double coverings of special orthogonal groups, as well as exceptional groups.

The conjecture is well-known for an exceptional group of type G_2 (*cf.* p. 284 of [19]). In this work, we prove it for split spinor groups. We expect the same methods to work for quasi-split spinor groups as well and we hope to address those cases elsewhere.

H. Kim has recently proved the conjecture for Levi subgroups of E-type exceptional groups except for the following four Levi subgroups: Cases E_7-3, E_8-3, and E_8-4 of [18] as well as Case (xxviii) of [13] ($D_7 \subset E_8$) (*cf.* [10]). The above four Levi subgroups all involve a spin group or an exceptional group of type E_6 for which we do not have enough information about the discrete series representations (*e.g.*, how to obtain them from supercuspidals). The author has proved the conjecture for an exceptional group of type F_4 (*cf.* [1]).

The general idea of the proof, both in our work and in Kim's, is based on the construction of discrete series representations out of supercuspidal ones. This has been worked on by many people including I. Bernstein and A. Zelevinsky [3], [29] for general linear groups and D. Ban, C. Jantzen, C. Mœglin, G. Muić, and M. Tadić for other classical groups (*cf.* [2], [8], [9], [14], [15], [25], [26], [27]). In Section 4 we will imitate Tadić's method for GSpin groups, defined in Section 2. We then follow the computations of Section 4 of [6] to prove the holomorphy in our cases in Section 5.

The fact that the spinor groups have complicated Levi factors (*cf.* 2.1) is an obstacle in applying the same methods that have been applied in other classical groups in order to prove the conjecture. Our idea is to consider another class of groups, GSpin, whose derived groups are spinor groups while having much "easier" Levi subgroups. This is similar to the case of Levi subgroups of GL_n versus those of SL_n.

The author would like to offer his deep gratitude to Professor F. Shahidi for his constant help and supervision throughout this work which was part of the author's Ph.D. dissertation at Purdue University. The author is also grateful to D. Ban, D. Goldberg, C. Jantzen, H. Kim, A. Roche, R. Schmidt, J.-K. Yu, and Y. Zhang for many helpful discussions at various stages of this work. Thanks are also due to the referee for making a number of comments and suggestions which improved this paper.

2 The Structure of GSpin Groups

We define the reductive group GSpin_m whose derived group will be Spin_m and study the structure of its Levi components in detail. This section will, in fact, present the main idea of the paper which is to replace the study of the L-functions for Spin groups with those of GSpin groups since the latter have much nicer Levi subgroups.

Throughout this work, we will be dealing with split spinor groups. The same methods are expected to go through for quasi-split spinor groups which we hope to address elsewhere.

To study the L-functions via the Langlands-Shahidi method which is the subject of current work one looks at parabolic induction. Hence, we have to know what the Levi subgroups in Spin_m look like. In other classical groups such as SO or Sp the Levi subgroups are isomorphic to a product of general linear groups and another SO or Sp of a smaller rank.

Question Are the Levi subgroups in Spin_m isomorphic to

$$\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \cdots \times \mathrm{GL}_{n_k} \times \mathrm{Spin}_{m'},$$

with $2(n_1 + n_2 + \cdots + n_k) + m' = m$?

The answer turns out to be negative.

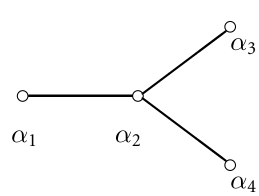
Example 2.1 The group Spin_5 is simply-connected of type B_2. The Siegel Levi (generated by the long root) is isomorphic to $\mathrm{GL}_1 \times \mathrm{SL}_2$ since it has to be the same as the Levi generated by the long root in Sp_4. Similarly, the non-Siegel Levi in Spin_5 is isomorphic to GL_2. This is exactly the opposite of what one would expect if the answer to the above question were positive.

As a more complicated example, one could look at the Siegel Levi in Spin_{2n+1} when $n = 2k$ is even. As in [22], one can verify that the Levi subgroup is isomorphic to

$$\frac{\mathrm{GL}_1 \times \mathrm{SL}_n}{\{(\lambda, \lambda \cdot I_n) : \lambda^k = 1\}},$$

while one would expect to get GL_n if the answer to the above question were positive.

As another example, consider Spin_8. This is a group of type D_4 with the following Dynkin diagram.



Now let $\mathbf{M}$ be the standard Levi generated by $\{\alpha_2, \alpha_3, \alpha_4\}$. If the answer to the above question were positive, we would expect $\mathbf{M}$ to be isomorphic to $\mathrm{GL}_1 \times \mathrm{Spin}_6 = \mathrm{GL}_1 \times \mathrm{SL}_4$. However, we show that

$$\mathbf{M} \simeq \frac{\mathrm{GL}_1 \times \mathrm{SL}_4}{\{(1, I_4), (-1, -I_4)\}}.$$

To see this note that $\mathbf{M} = \mathbf{A} \cdot \mathbf{M}_D$ (almost direct product), where $\mathbf{M}_D$ is the derived group of $\mathbf{M}$ and

$$\mathbf{A} = \Big(\bigcap_{i=2}^{4} \ker \alpha_i\Big)^0.$$

Since Spin_8 is simply-connected, so is the derived group of its Levis. Hence, $\mathbf{M}_D \simeq \mathrm{SL}_4$. To compute $\mathbf{A}$ note that it lies in the maximal torus $\mathbf{T}$ of Spin_8 and again simply-connectedness implies that each $t \in \mathbf{T}$ can be written uniquely as

$$t = \prod_{j=1}^{4} \alpha_j^\vee(t_j), \quad t_j \in \mathrm{GL}_1 .$$

Thus,

$$\alpha_i(t) = \prod_{j=1}^{4} t_j^{\langle \alpha_i, \alpha_j^\vee \rangle},$$

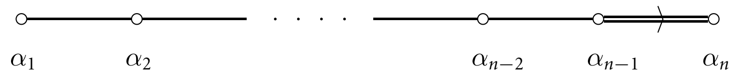
where $\langle \cdot, \cdot \rangle$ denotes the $\mathbb{Z}$-pairing between the roots and coroots. Computing the kernels one can describe $\mathbf{A}$ as

$$\{\alpha_1^\vee(\lambda^2)\alpha_2^\vee(\lambda^2)\alpha_3^\vee(\lambda)\alpha_4^\vee(\lambda) \mid \lambda \in \mathrm{GL}_1\}.$$

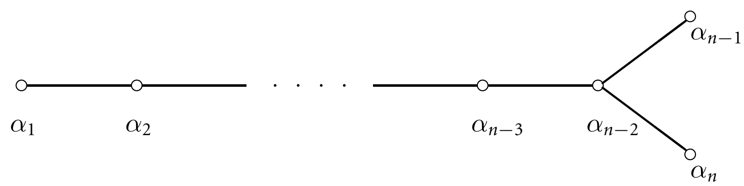
All that remains now is to find the intersection of $\mathbf{A}$ and $\mathbf{M}_D$ which consists of the elements in $\mathbf{A}$ for which $\lambda^2 = 1$. Therefore, $\mathbf{A} \cap \mathbf{M}_D = \{1, \alpha_3^\vee(-1)\alpha_4^\vee(-1)\}$. This proves our assertion.

We will eventually need to study tempered (in fact, discrete series) representations of the F-points of such Levi subgroups, where F is a non-archimedean local field of characteristic zero. Clearly such complicated Levi components don't help at all. To remedy this we will look instead at another class of groups, namely GSpin groups, which have Spin groups as their derived groups. These groups turn out to have simpler Levi factors. Since the local L-functions we are interested in depend only on the derived group of the group in question, the L-functions are the same for both classes of groups.

The group Spin_m is the split simple simply-connected algebraic group of type B_n if $m = 2n + 1$ with Dynkin diagram



and of type D_n if $m = 2n$ with the following Dynkin diagram.



This group is a double covering, as algebraic groups, of the group SO_m.

Fix a maximal torus $\mathbf{T}$ contained in a fixed Borel subgroup $\mathbf{B}$ in Spin_m. This corresponds to a choice of a system of simple roots $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ as above. We denote the associated coroots by $\Delta^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_n^\vee\}$. We then have $\alpha_i \colon \mathbf{T} \longrightarrow \mathrm{GL}_1$ and $\alpha_i^\vee \colon \mathrm{GL}_1 \longrightarrow \mathbf{T}$ such that $\alpha_i \circ \alpha_j^\vee(x) = x^{\langle \alpha_i, \alpha_j^\vee \rangle}$ where $\langle \cdot, \cdot \rangle \colon X \times X^\vee \longrightarrow \mathbb{Z}$ is the $\mathbb{Z}$-pairing between characters and cocharacters. Following the same computations as in 2.1, one immediately gets the following well-known fact.

Proposition 2.2 *The center of* Spin_m *is equal to*

$$\{1, c\} \simeq \mathbb{Z}/2\mathbb{Z}$$

if $m = 2n + 1$*, where* $c = \alpha_n^\vee(-1)$*. If* $m = 2n$*, then it is equal to*

$$\{1, c, z, cz\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

provided that n is even and

$$\{1, z, c, z^3\} \simeq \mathbb{Z}/4\mathbb{Z}$$

otherwise, where

$$c = \alpha_{n-1}^\vee(-1)\alpha_n^\vee(-1)$$

and

$$z = \prod_{j=1}^{n-2} \alpha_j^\vee\big((-1)^j\big) \cdot \alpha_{n-1}^\vee(-1)$$

if n is even and

$$z = \prod_{j=1}^{n-2} \alpha_j^\vee\big((-1)^j\big) \cdot \alpha_{n-1}^\vee(\sqrt{-1})\alpha_n^\vee(\sqrt{-1})$$

if n is odd. Observe that $c = z^2$.

We can now define the group GSpin_m.

Definition 2.3 For $m \geq 3$ define

$$\mathbf{G} = \frac{\mathrm{GL}_1 \times \mathrm{Spin}_m}{\{(1,1), (-1,c)\}}$$

where c is defined as in 2.2. We will call this group GSpin_m. Write $\mathbf{A}$ for the image of $\{(\lambda,1) \mid \lambda \in \mathrm{GL}_1\}$ and $\mathbf{H}$ for the image of $\{(1,g) \mid g \in \mathrm{Spin}_m\}$ in $\mathbf{G}$. Then, $\mathbf{G} = \mathbf{A} \cdot \mathbf{H}$ with $\mathbf{S} = \mathbf{A} \cap \mathbf{H} = \{1, c\}$. Clearly, the derived group of GSpin_m is Spin_m.

We also define GSpin_0 and GSpin_1 to be GL_1. (The group GSpin_2 is not defined here and will not come up in our discussions.)

We now give an alternative description of GSpin_m in terms of root datum which will be helpful later on.

Proposition 2.4 *The reductive group* GSpin_m *can be given by the root datum* $(X, R, X^\vee, R^\vee)$ *where*

$$X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n,$$

and

$$X^\vee = \mathbb{Z}e_0^* \oplus \mathbb{Z}e_1^* \oplus \cdots \oplus \mathbb{Z}e_n^*,$$

equipped with the standard $\mathbb{Z}$*-pairing while* R *and* $R^\vee$ *are defined as follows.*

(a) *If* $m = 2n+1$*, then*

$$R = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n\},$$
$$R^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \alpha_2^\vee = e_2^* - e_3^*, \ldots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = 2e_n^* - e_0^*\}.$$

(b) *If* $m = 2n$*, then*

$$R = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\},$$
$$R^\vee = \{\alpha_1^\vee = e_1^* - e_2^*, \ldots, \alpha_{n-1}^\vee = e_{n-1}^* - e_n^*, \alpha_n^\vee = e_{n_1}^* + e_n^* - e_0^*\}.$$

Proof We give the proof for the case of $m = 2n+1$. The other case is similar. We compute the root datum of GSpin_{2n+1} from that of Spin_{2n+1} and verify that it can be written as above. Start with the character lattice of Spin_{2n+1}. Since Spin_{2n+1} is simply-connected, this lattice is the same as the weight lattice of the Lie algebra $\mathfrak{so}_{2n+1}$ which is well-known. Choose $f_1, f_2, \ldots, f_n$ (*cf.* [7]) such that the $\mathbb{Z}$-span of $f_1, f_2, \ldots, f_n$, $(f_1 + \cdots + f_n)/2$ is the character lattice of Spin_{2n+1}. The character lattice of $\mathrm{GL}_1 \times \mathrm{Spin}_{2n+1}$ can then be written in the same way as the $\mathbb{Z}$-span of $f_0, f_1, f_2, \ldots, f_n$, $(f_1 + \cdots + f_n)/2$. Now the characters of GSpin_{2n+1} are those that have trivial value at the element $(-1, c)$. Note that $f_i(-1,c) = 1$ for $1 \leq i \leq n$ and $\big(f_0 + (f_1 + \cdots + f_n)/2\big)(-1,c) = 1$. In fact, the character lattice of GSpin_{2n+1} can be written as the $\mathbb{Z}$-span of $f_1, f_2, \ldots, f_n$, $f_0 + (f_1 + \cdots + f_n)/2$. Using the $\mathbb{Z}$-pairing of the root datum, one can also compute the cocharacter lattice which turns out to be spanned by f_0^* and $f_i^* + (f_0^*/2)$, $1 \le i \le n$. Now for $1 \le i \le n$ set $e_i = f_i$ and set $e_0 = f_0 + (f_1 + \cdots + f_n)/2$. Also set $e_i^* = f_i^* + (f_0^*)/2$ and $e_0^* = f_0^*$. Notice that the roots and coroots which were primarily given in terms of f_i's can now be written in terms of e_i's as in the proposition. For example, the coroot $2f_n^*$ can be written as $2(e_n^* - f_0^*/2) = 2e_n^* - f_0^* = 2e_n^* - e_0^*$. We chose to write them in terms of e_i's since they become more similar to the way root datum of other classical groups, *e.g.*, GSp_{2n} are usually written (*cf.* 2.6). ∎

Remark 2.5 The fact that $(f_1 + \cdots + f_n)/2$ lies in the character lattice of Spin_{2n+1} is what keeps this group from having Levi subgroups similar to those of classical groups. On the other hand, there is no such element in the character lattice of GSpin_{2n+1}.

Remark 2.6 The root datum of GSpin_{2n+1} is the dual root datum to the one for the group GSp_{2n} while that of GSpin_{2n} is dual to root datum of GSO_{2n} (*cf.* [25] for the root datum of GSp_{2n}, for example).

We are now ready to compute the Levi subgroups of $\mathbf{G}$.

Theorem 2.7 *Let* $\mathbf{L}$ *be a Levi subgroup in* $\mathbf{G}$*. Then,* $\mathbf{L}$ *is isomorphic to*

$$\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \cdots \times \mathrm{GL}_{n_k} \times \mathrm{GSpin}_{m'}$$

with $2(n_1 + n_2 + \cdots + n_k) + m' = m$ *and* $m' \neq 2$.

Remark 2.8 Note that if $m' = 2$, then m is automatically even and $\mathbf{G}$ is a group of type $D_{m/2}$ in which case leaving any of the last two roots out will produce isomorphic Levi subgroups. Since we are only interested in L-functions, we will not deal with the maximal Levi corresponding to the simple root that is one to the last in our ordering. This will allow us to avoid some minor changes that would be otherwise necessary later on when we study irreducibility of induced representations in this case (*cf.* [2]).

Proof By 2.6 we know that the dual of GSpin_m is GSp_{2n} if $m = 2n + 1$ and GSO_{2n} if $m = 2n$. Now notice that the Levi subgroups in GSpin are just duals of those in GSp or GSO which are well-known to be products of general linear groups and another smaller rank group of the same type. This will complete the proof. ∎

3 Local L-Functions and the Langlands-Shahidi Method

What we will review in this section is available in the generality of quasi-split connected reductive groups. However, we will be working with a split simple group. Since our results concern split spinor groups this does not impose a real restriction while allowing us to avoid some technicalities such as in the definition of L-group. In fact, we will only deal with the connected component of what is usually called the L-group which involves Galois group or, more precisely, the Weil-Deligne group (*cf.* [28] and [4]). We refer to the connected component of the L-group as the complex dual or simply the dual of the group in question.

Let F be a non-archimedean local field of characteristic zero. Let $\mathcal{O}$ be the ring of integers in F and $\mathcal{P}$ denote its unique maximal ideal. Then $\mathcal{O}/\mathcal{P}$ is a finite field whose cardinality we will denote by q.

Let $\mathbf{G}$ be a split connected reductive linear algebraic group defined over F. Fix a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ and write $\mathbf{B} = \mathbf{TU}$ where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$ and $\mathbf{T}$ is a fixed maximal torus in $\mathbf{G}$. Let Δ be the corresponding system of simple roots of $\mathbf{G}$ with respect to $\mathbf{T}$. Let $\mathbf{P}$ be a maximal standard parabolic subgroup in $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{MN}$, ($\mathbf{N} \subset \mathbf{U}$) and $\mathbf{T} \subset \mathbf{M}$. Then $\mathbf{M} = \mathbf{M}_\theta$, where $\theta = \Delta \setminus \{\alpha\}$ for a unique simple root $\alpha \in \Delta$.

Let $\mathfrak{a} = \operatorname{Hom}\big(X(\mathbf{M})_F, \mathbb{R}\big)$, where $X(\mathbf{M})_F$ denotes the F-rational characters of $\mathbf{M}$. We define the dual of $\mathfrak{a}$ and its complexification as follows:

$$\mathfrak{a}^* := X(\mathbf{M})_F \otimes_{\mathbb{Z}} \mathbb{R},$$

$$\mathfrak{a}^*_{\mathbb{C}} := \mathfrak{a}^* \otimes \mathbb{C}.$$

Recall that since $\mathbf{M}$ is assumed to be maximal, $\mathfrak{a}^*_{\mathbb{C}}/\mathfrak{z}^*_{\mathbb{C}}$ is one-dimensional and we will shortly choose an identification of it with $\mathbb{C}$. Here $\mathfrak{z}$ denotes the Lie algebra of the split torus in the center of $\mathbf{G}$.

Let $\rho_{\mathbf{P}}$ be half of the sum of positive roots of $\mathbf{G}$ whose root vectors generate $\mathbf{N}$ and define $\widetilde{\alpha}$ to be

$$\frac{1}{\langle \rho_{\mathbf{P}}, \alpha^\vee \rangle} \rho_{\mathbf{P}} \in \mathfrak{a}^*,$$

where $\langle \cdot, \cdot \rangle$ is the $\mathbb{Z}$-pairing between the roots and coroots of $\mathbf{G}$. We now identify $s \in \mathbb{C}$ with $s\tilde{\alpha} \in \mathfrak{a}^*_{\mathbb{C}}$. Also, define a homomorphism

$$H_P \colon M \longrightarrow \mathfrak{a}$$

via

$$q^{\langle \chi, H_P(\cdot) \rangle} = |\chi(\cdot)|_F, \quad \chi \in X(\mathbf{M})_F.$$

We also have

$$q^{\langle s\tilde{\alpha}, H_P(\cdot) \rangle} = |\tilde{\alpha}(\cdot)|_F^s, \tag{1}$$

where $\tilde{\alpha}(\cdot)$ is the value of $\tilde{\alpha}$ on an element of M and $s \in \mathbb{C}$.

Let $\mathbf{G}$ have the root datum $(X, R, X^\vee, R^\vee)$, where X and $X^\vee$ denote the root lattice and coroot lattice of $\mathbf{G}$ equipped with a $\mathbb{Z}$-pairing

$$\langle \cdot, \cdot \rangle \colon X \times X^\vee \to \mathbb{Z},$$

while R and $R^\vee$ are the set of roots and coroots. The root datum obtained by interchanging the roots and coroots, $(X^\vee, R^\vee, X, R)$, determines another connected reductive group called the dual of $\mathbf{G}$ (*cf.* [23]). When considered as a group over complex numbers, we refer to it as the complex dual of G and denote it by $\widehat{G}$. It is the connected component of the L-group of $\mathbf{G}$.

Now consider $\widehat{M}$, the complex dual of $\mathbf{M}$ which can be realized as a Levi subgroup in $\widehat{G}$. We now define the local Langlands L-function.

Definition 3.1 Let σ be an *unramified* representation of $M = \mathbf{M}(F)$ (*i.e.*, having an $\mathbf{M}(\mathcal{O})$-fixed vector). The Satake isomorphism then associates to it a semisimple conjugacy class, A_σ, in $\widehat{M}$ (which we will normally identify with a representative A_σ in $\widehat{T}$, the maximal torus of $\widehat{M}$). For a finite-dimensional complex representation r of $\widehat{M}$ define the local Langlands L-function as follows.

$$L(s,\sigma,r) := \det\bigl(I - r(A_\sigma)q^{-s}\bigr)^{-1}, \quad s \in \mathbb{C}.$$

Clearly, in the above definition, we did not have to assume $\mathbf{M}$ to be a Levi subgroup in another group. However, we will need this for the Langlands-Shahidi method.

That method uses specific r's which turn out to include many important examples of L-functions studied so far. To describe them let $\widehat{P}$ be a standard parabolic in $\widehat{G}$ having $\widehat{M}$ as its Levi factor and write $\widehat{P} = \widehat{M}\widehat{N}$ (*cf.* [4]). Define $\widehat{n}$ to be the Lie algebra of $\widehat{N}$ on which $\widehat{M}$ acts via the adjoint action, r. Then, there exists a positive integer m such that

$$\widehat{n} = \bigoplus_{i=1}^{m} V_i,$$

where V_i, spanned by root vectors $X_{\beta^\vee} \in \widehat{n}$ with $\langle \tilde{\alpha}, \beta^\vee \rangle = i$, is an irreducible subrepresentation of $r = \bigoplus_{i=1}^{m} r_i$ (*cf.* [18]). Hence, one gets the L-functions $L(s,\sigma,r_i)$ for each $1 \le i \le m$.

How do we define $L(s,\sigma,r_i)$ if σ is ramified? The Langlands-Shahidi method does this assuming that σ is an irreducible admissible generic representation.

Shahidi studied intertwining operators and local coefficients in a series of papers. These in turn lead to the definition of γ-factors which we recall below.

We remark that these results cover both global and local (archimedean and non-archimedean) cases. However, since in this work we are only concerned with non-archimedean local places, we present the part of existence of γ-factors that we will be using.

We should also remark that throughout this work a multiplicative character χ of $U = \mathbf{U}(F)$ is fixed which one needs in the definition of a generic representation (*i.e.*, having Whittaker functions with respect to χ or being χ-generic). We do this by fixing an additive character ψ_F of F on which the γ-factors as well as ϵ-factors, to be introduced later, depend. The L-functions, however, do not depend on this additive character. We also assume that χ and ψ_F are compatible in the sense of Section 3 of [20]. For simplicity, we will not repeat them throughout this work.

Shahidi proved in Theorem 3.5 of [20] that there exist m complex functions

$$\gamma_i(s,\sigma,r,\psi_F) = \gamma(s,\sigma,r_i),$$

each a rational function in q^{-s}, satisfying certain local and global properties.

Below we recall some of these properties that will be helpful to us. We should remark that Theorem 3.5 of [20] in its complete generality is the main result of [20] which also establishes uniqueness of these γ-factors.

Proposition 3.2 *For each $1 \leq i \leq m$ we have,*

$$\gamma(s, \sigma, r_i, \psi_F)\gamma(1 - s, \tilde{\sigma}, r_i, \bar{\psi}_F) = 1.$$

Moreover, $\gamma(s, \sigma, \tilde{r}_i) = \gamma(s, \tilde{\sigma}, r_i)$.

Definition 3.3 Let $P_{\sigma,i} \in \mathbb{C}[X]$ with $P_{\sigma,i}(0) = 1$ be such that $P_{\sigma,i}(q^{-s})$ is the normalized numerator of $\gamma(s, \sigma, r_i)$. If σ is tempered, define the local Langlands L-function $L(s, \sigma, r_i)$ to be the inverse of this polynomial, $P_{\sigma,i}(q^{-s})^{-1}$. One then defines $L(s, \sigma, r_i)$ for arbitrary irreducible admissible generic σ via the Langlands classification (*cf.* [20, p. 308]).

Proposition 3.4 *There exists a monomial in q^{-s}, denoted by $\epsilon(s, \sigma, r_i, \psi_F)$, such that*

$$\gamma(s, \sigma, r_i, \psi_F) = \epsilon(s, \sigma, r_i, \psi_F)L(1 - s, \sigma, \tilde{r}_i)/L(s, \sigma, r_i).$$

Remark 3.5 Note that the L-functions on the right hand side of the above proposition do not depend on ψ_F while the ϵ-factor does. Also, by arguments on p. 287 of [19] and Proposition 7.8 of [20], we know that if σ is unitary supercuspidal, then $L(s, \sigma, r)$ is, up to a monomial in q^{-s}, equal to $L(-s, \sigma, \tilde{r})$.

Another important property of the γ-factors is their multiplicativity which is essential in our work. Let $\mathbf{M} = \mathbf{M}_\theta$ be a maximal standard Levi subgroup in $\mathbf{G}$ and let σ be an irreducible admissible χ-generic representation of $\mathbf{M}(F)$. Assume $\mathbf{P}' = \mathbf{M}' \cdot \mathbf{N}'$ to be a standard parabolic subgroup inside $\mathbf{M}$ and $\mathbf{M}' = \mathbf{M}_{\theta'}$ with $\theta' \subset \theta$. Also assume that $\sigma \subset \mathrm{Ind}_{P' \uparrow M}\, \sigma' \otimes 1$ where σ' is an irreducible admissible generic representation of M'. Let $w = w_{\ell,\Delta} w_{\ell,\theta}^{-1}$ where $w_{\ell,\Delta} = w_{\ell,\Delta}^{-1}$ is the unique longest element in the Weyl group of $\mathbf{G}$ and $w_{\ell,\theta} = w_{\ell,\theta}^{-1}$ is the unique longest element in that of $\mathbf{M}$. Set $\theta_1 = \theta'$ and $w_1' = w$. There exists a simple root $\alpha_1 \in \Delta$ such that $w_1'(\alpha_1) < 0$. Define $\Omega_1 = \theta_1 \cup \{\alpha_1\}$. Now $\mathbf{M}_{\Omega_1}$ contains $\mathbf{M}_{\theta_1}$ as a maximal Levi. Let $w_1 = w_{\ell,\Omega_1} w_{\ell,\theta_1}^{-1}$ and $w_2' = w_1' w_1^{-1}$. Let $\theta_2 = w_1(\theta_1)$ and continue this process to define successive θ_j's and Ω_j's. There exists n such that $w_n' = 1$, *i.e.*, $w = w_{n-1} w_{n-2} \cdots w_1$.

Proposition 3.6 *If we set $\overline{w}_j = w_{j-1} \cdots w_1$ for $2 \leq j \leq n-1$ and $\overline{w}_1 = 1$, then $\overline{w}_j(\sigma')$ is a representation of $\mathbf{M}_{\theta_j}(F)$ and we have*

$$\gamma(s, \sigma, r_i, \psi_F) = \prod_{j \in S_i} \gamma\big(s, \overline{w}_j(\sigma'), R_{ij}, \psi_F\big).$$

Each r_i, $1 \leq i \leq m$, is an irreducible constituent of the adjoint action of the L-group of $\mathbf{M}$ on the Lie algebra of the L-group of $\mathbf{N}$ and R_j, $1 \leq j \leq n-1$, is the same action for $\mathbf{M}_{\theta_j}$ in $\mathbf{M}_{\Omega_j}$. Every irreducible component of the restriction of r_i to the L-group of $\mathbf{M}'$ is equivalent, under $\overline{w}_j$, to an irreducible constituent R_{ij} of some R_j for a unique j. The set S_i consists of all such j for a given i.

To explain the sets S_i more clearly, note that in the above product, i is fixed and out of (possibly) several j's we take the appropriate constituent of appropriate R_j's on the right hand side for the product corresponding to the fixed i on the left hand side. All the constituents of all the j's would then cover all the r_i's, $i = 1, 2, \ldots, m$. In other words, $R_j = \bigoplus R_{ij}$ with the sum over all i's such that $j \in S_i$.

Remark 3.7 The γ-factors are defined via "local coefficients" (*cf.* part 3 of Theorem 3.5 of [20] and its proof). We refer to [16] for discussion about these coefficients. Local coefficients only depend on the derived group of the reductive group in question. This implies that we only need to study the γ-factors and L-functions for semisimple groups. In fact, it is enough to prove the holomorphy for these L-functions, which is our purpose in this work, for Levi subgroups of simple groups.

Another property of γ-factors that we will need is that twisting the representation on the Levi by certain characters translates into a shift in the complex parameter s. More precisely,

Proposition 3.8 *Let σ_0 be an irreducible admissible representation of M and let $\sigma = \sigma_0 \otimes q^{\langle s_0\widetilde{\alpha}, H_P(\cdot)\rangle}$, where H_P is as in equation (1) of Section 3. Then $L(s, \sigma, r) = L(s + s_0, \sigma_0, r)$.*

This property turns out to be essential in proving that the L-functions for discrete series (and tempered) representations, which are obtained as products of those for supercuspidal representations, have their possible poles for $\Re(s) > 0$ cancelled among different terms, *i.e.*, they are holomorphic.

4 Discrete Series Representations in GSpin Groups

In this section we describe the parabolically induced representations on $\mathrm{GSpin}_m(F)$ that contain discrete series subrepresentations. We apply a method similar to the one used by M. Tadić. His method is based on Casselman's square integrability criteria for which we need some preparation. Since Proposition 3.6 was formulated in terms of subrepresentations, we will state the following results using subrepresentations as opposed to subquotients although the latter might seem more natural.

As before, let Δ be the set of simple roots of $\mathbf{G} = \mathrm{GSpin}_m$ with respect to some maximal torus $\mathbf{T}$. For any $\theta \subset \Delta$, define

$$A_\theta = \Big(\bigcap_{\alpha\in\theta} \ker\alpha\Big)^0.$$

Standard Levi components in $\mathbf{G}$ are centralizers in $\mathbf{G}$ of these A_θ's, hence in one-one correspondence with the subsets of Δ. We denote such Levi subgroups by $\mathbf{M}_\theta$ and a standard parabolic containing them by $\mathbf{P}_\theta$. We also define

$$A_\theta^- = \{a \in A_\theta : |\alpha(a)| \le 1, \forall \alpha \in \Delta \setminus \theta\}.$$

Moreover, two subsets θ and Ω of Δ are said to be associate if there is an element w in the Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}$ such that $w(\theta) = \Omega$.

https://doi.org/10.4153/CJM-2002-025-8 Published online by Cambridge University Press

For an irreducible admissible representation σ of $M = \mathbf{M}(F)$, we let $i_{G,M}(\sigma)$ denote the normalized parabolically induced representation from $\sigma \otimes 1$ on the parabolic subgroup $P = MN$ to $G = \mathbf{G}(F)$. Similarly, if π is an irreducible representation of G, then we let $r_{M,G}(\pi)$ denote the normalized Jacquet module of π with respect to P (*cf.* [5] for more detailed explanation of these notations). We can now state Casselman's square integrability criteria for our case.

Theorem 4.1 *Let $\mathbf{M}_\theta$ be the Levi component of a parabolic subgroup $\mathbf{P}_\theta = \mathbf{M}_\theta \mathbf{N}_\theta$ in $\mathbf{G} = \mathrm{GSpin}_m$. Assume that σ is a supercuspidal representation of $M = \mathbf{M}_\theta(F)$. An irreducible admissible representation $\pi \hookrightarrow i_{G,M}(\sigma)$ is square integrable if and only if*

(a) *π restricted to A_Δ is unitary.*
(b) *for every $\Omega \subset \Delta$ associate to θ and every central character χ of $r_{M_\Omega,G}(\pi)$ we have $|\chi(a)| < 1$ for all $a \in A_\Omega^- \setminus A_\varnothing(\mathcal{O})A_\Delta$.*

Proof This is a version of Theorem 6.5.1 of [5]. ■

We now give another version of the above that is more suitable to use for our purposes.

Proposition 4.2 *Define*

$$\beta_i = (\underbrace{1, \dots, 1}_{i \text{ times}}, 0, \dots, 0) \in \mathbb{R}^n.$$

Let $\pi \hookrightarrow i_{G,M}(\sigma)$ be an irreducible smooth representation of $\mathrm{GSpin}_m(F)$ and assume that $\sigma = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k \otimes \tau$, where ρ_i is a supercuspidal representation of $\mathrm{GL}_{n_i}(F)$ and τ is one of $\mathrm{GSpin}_{m'}(F)$, is such that the corresponding Levi is minimal among the ones with $r_{M,G}(\pi) \neq 0$. Write each ρ_i as $\rho_i = \nu^{e(\rho_i)} \rho_i^u$ with $e(\rho_i) \in \mathbb{R}$ and ρ_i^u unitarizable. Here, ν denotes $|\det(\cdot)|_F$. Define

$$e_*(\sigma) = \big(\underbrace{e(\rho_1), \dots, e(\rho_1)}_{n_1 \text{ times}}, \dots, \underbrace{e(\rho_k), \dots, e(\rho_k)}_{n_k \text{ times}}, \underbrace{0, \dots, 0}_{m' \text{ times}}\big).$$

If π is square integrable, then

$$\begin{gathered} \big(e_*(\sigma), \beta_{n_1}\big) > 0, \\ \big(e_*(\sigma), \beta_{n_1+n_2}\big) > 0, \\ \vdots \\ \big(e_*(\sigma), \beta_{n_1+\cdots+n_k}\big) > 0. \end{gathered}$$

Conversely, if the above inequalities hold for all such partitions and all such σ's, then π is square integrable.

Proof The proof of this statement is straight forward from 4.1. ■

We will need the following result of Zelevinsky [29].

Proposition 4.3 **(Zelevinsky)** *Let π_i, $i = 1, 2$ be irreducible supercuspidal representations of $\mathrm{GL}_{n_i}(F)$. If $\pi_1 \not\simeq \pi_2\nu^{\pm 1}$, then $\pi_1 \times \pi_2$ is irreducible, where $\pi_1 \times \pi_2$ denotes the induced representation from $\pi_1 \otimes \pi_2$ via parabolic induction.*

We now present the main statements of this section. Let G denote $\mathrm{GSpin}_m(F)$ and M denote the F-points of the standard Levi corresponding to the partition $m = 2(n_1 + n_2 + \cdots + n_k) + m'$. Assume that ρ_i is a supercuspidal representation of $\mathrm{GL}_{n_i}(F)$ and τ is one of $\mathrm{GSpin}_{m'}(F)$ and let ρ be the representation on M defined by

$$\rho = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k \otimes \tau.$$

Also, set

$$\rho_1 \times \rho_2 \times \cdots \times \rho_k \rtimes \tau = i_{G,M}(\rho).$$

Write $\rho_i = \nu^{e_i} \rho_i^u$ with $e_i \in \mathbb{R}$ and ρ_i^u unitary.

Theorem 4.4 *If $\rho_1 \times \rho_2 \times \cdots \times \rho_k \rtimes \tau$ has a discrete series subrepresentation, then $\rho_i^u \simeq \widetilde{\rho_i^u}$ for $i = 1, 2, \ldots, k$.*

Theorem 4.5 *If $\rho_1 \times \rho_2 \times \cdots \times \rho_k \rtimes \tau$ has a discrete series subrepresentation, then $2e_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, k$.*

Proof We use the notation and techniques used in [26], [25], [24], [2]. Let π be the discrete series subrepresentation in the above theorems. Then ρ is a quotient of the Jacquet module $r_{M,G}(\pi)$ by Frobenius reciprocity.

Fix $i_0 \in \{1, 2, \ldots, k\}$ and set

$$Y_{i_0}^0 = \left\{ i \in \{1, \ldots, k\} \mid \exists \alpha \in \mathbb{Z} \text{ such that } \rho_{i_0} \simeq \nu^\alpha \rho_i \right\},$$
$$Y_{i_0}^1 = \left\{ i \in \{1, \ldots, k\} \mid \exists \alpha \in \mathbb{Z} \text{ such that } \widetilde{\rho_{i_0}} \simeq \nu^\alpha \rho_i \right\},$$
$$Y_{i_0} = Y_{i_0}^0 \cup Y_{i_0}^1,$$
$$Y_{i_0}^c = \{1, \ldots, k\} \setminus Y_{i_0}.$$

We first prove Theorem 4.4. Assume $\rho_{i_0}^u \not\simeq \tilde{\rho}_{i_0}^u$. For $j_0, j_0' \in Y_{i_0}^0$, $j_1, j_1' \in Y_{i_0}^1$, and $j_c \in Y_{i_0}^c$ we have all the following relations (*cf.* 4.3).

$$\begin{aligned}
&\rho_{j_0} \times \tilde{\rho}_{j_0'} \simeq \tilde{\rho}_{j_0'} \times \rho_{j_0}, &&\rho_{j_1} \times \tilde{\rho}_{j_1'} \simeq \tilde{\rho}_{j_1'} \times \rho_{j_1},\\
&\rho_{j_0} \times \rho_{j_1} \simeq \rho_{j_1} \times \rho_{j_0}, &&\tilde{\rho}_{j_1} \times \tilde{\rho}_{j_1} \simeq \tilde{\rho}_{j_1} \times \tilde{\rho}_{j_0},\\
&\rho_{j_0} \times \rho_{j_c} \simeq \rho_{j_c} \times \rho_{j_0}, &&\tilde{\rho}_{j_0} \times \rho_{j_c} \simeq \rho_{j_c} \times \tilde{\rho}_{j_0},\\
&\rho_{j_1} \times \rho_{j_c} \simeq \rho_{j_c} \times \rho_{j_1}, &&\tilde{\rho}_{j_1} \times \rho_{j_c} \simeq \rho_{j_c} \times \tilde{\rho}_{j_1}.
\end{aligned}$$

If $m' \neq 0$ or 1, then

$$\rho_{j_0} \rtimes \tau \simeq \tilde{\rho}_{j_0} \rtimes \tau, \quad \rho_{j_1} \rtimes \tau \simeq \tilde{\rho}_{j_1} \rtimes \tau$$

and if $m' = 0$ or 1, then

$$\rho_{j_0} \rtimes 1 \simeq \tilde{\rho}_{j_0} \rtimes 1, \quad \rho_{j_1} \rtimes 1 \simeq \tilde{\rho}_{j_1} \rtimes 1.$$

Write

$$\begin{aligned} Y_{i_0}^0 &= \{a_1, \dots, a_{k_0}\}, \\ Y_{i_0}^1 &= \{b_1, \dots, b_{k_1}\}, \\ Y_{i_0}^c &= \{d_1, \dots, d_{k_c}\}, \end{aligned}$$

with $a_i < a_j$, $b_i < b_j$, and $d_i < d_j$ for $i < j$. We use the above relations to get

$$\begin{aligned} &\rho_1 \times \cdots \times \rho_k \rtimes \tau \\ &\quad \simeq \rho_{a_1} \times \cdots \times \rho_{a_{k_0}} \times \rho_{d_1} \times \cdots \times \rho_{d_{k_c}} \times \rho_{b_1} \times \cdots \times \rho_{b_{k_1}} \rtimes \tau \\ &\quad \simeq \rho_{a_1} \times \cdots \times \rho_{a_{k_0}} \times \rho_{d_1} \times \cdots \times \rho_{d_{k_c}} \times \rho_{b_1} \times \cdots \times \rho_{b_{k_1-1}} \times \tilde{\rho}_{b_{k_1}} \rtimes \tau \\ &\quad \simeq \rho_{a_1} \times \cdots \times \rho_{a_{k_0}} \times \rho_{d_1} \times \cdots \times \rho_{d_{k_c}} \times \tilde{\rho}_{b_{k_1}} \times \rho_{b_1} \times \cdots \times \rho_{b_{k_1-1}} \rtimes \tau \\ &\quad \simeq \cdots \\ &\quad \simeq \rho_{a_1} \times \cdots \times \rho_{a_{k_0}} \times \tilde{\rho}_{b_{k_1}} \times \cdots \tilde{\rho}_{b_1} \times \rho_{d_1} \times \cdots \times \rho_{d_{k_c}} \rtimes \tau. \end{aligned}$$

In the same way, we get

$$\rho_1 \times \cdots \times \rho_k \rtimes \tau \simeq \rho_{b_1} \times \cdots \times \rho_{b_{k_1}} \times \tilde{\rho}_{a_{k_0}} \times \cdots \tilde{\rho}_{a_1} \times \rho_{d_1} \times \cdots \times \rho_{d_{k_c}} \rtimes \tau.$$

We also get similar relations in the case of $m' = 0$ or 1 with τ in the above replaced by 1. Now Frobenius reciprocity implies that the representations

$$\rho' = \rho_{a_1} \otimes \cdots \otimes \rho_{a_{k_0}} \otimes \tilde{\rho}_{b_{k_1}} \otimes \cdots \tilde{\rho}_{b_1} \otimes \rho_{d_1} \otimes \cdots \otimes \rho_{d_{k_c}} \rtimes \tau$$

and

$$\rho'' = \rho_{b_1} \otimes \cdots \otimes \rho_{b_{k_1}} \otimes \tilde{\rho}_{a_{k_0}} \otimes \cdots \tilde{\rho}_{a_1} \otimes \rho_{d_1} \otimes \cdots \otimes \rho_{d_{k_c}} \rtimes \tau$$

are the quotients of the corresponding Jacquet modules. Consider the representation $\rho_{a_1} \times \cdots \times \rho_{a_{k_0}} \times \rho_{b_1} \times \cdots \times \rho_{b_{k_1}}$ of $\mathrm{GL}_u(F)$ for some integer u. Now we have $\big(\beta_u, e_*(\rho')\big) = -\big(\beta_u, e_*(\rho'')\big)$ which is a contradiction with Proposition 4.2 since not both of them could be strictly positive.

Theorem 4.5 is proved similarly. Note that since ρ_i is in particular a tempered representation of $\mathrm{GL}_{n_i}(F)$ it is generic which implies that if $\nu^\alpha \rtimes \tau$ reduces, then $2\alpha \in \mathbb{Z}$. One will use this to finish the proof. ■

5 Holomorphy of Local L-Functions in Split Spinor Groups

The following is Conjecture 7.1 of [20]. Our main result will be a proof of this conjecture for Levi subgroups in GSpin groups. Notation is the same as in Section 3.

Conjecture 5.1 Assume that σ is tempered. Then for $1 \le i \le m$

$$L(s, \sigma, r_i)$$

is holomorphic for $\Re(s) > 0$.

Remark 5.2 It is enough to prove the conjecture for σ in discrete series since the L-functions for tempered σ are products of those for discrete series σ by Proposition 3.6.

This conjecture is a theorem in the following cases (*cf.* [20]).

Proposition 5.3 *If $m = 1$, then $L(s, \sigma, r_1)$ is holomorphic for $\Re(s) > 0$.*

Proposition 5.4 *If $m = 2$ and the second L-function is of the form*

$$L(s, \sigma, r_2) = \prod_j (1 - a_j q^{-s})^{-1},$$

(possibly an empty product) with all $a_j \in \mathbb{C}$ of absolute value one, then the first L-function $L(s, \sigma, r_1)$ is holomorphic for $\Re(s) > 0$.

The assumption of the above proposition holds in particular when r_2 is one-dimensional.

Proposition 5.5 *Assume σ to be unitary supercuspidal. Then $L(s, \sigma, r_i)$ is holomorphic for $\Re(s) > 0$, $1 \le i \le m$. In fact, each $L(s, \sigma, r_i)$ is a product (possibly empty) as in Proposition 5.4 with $|a_j| = 1$. Moreover, $L(s, \sigma, r_i) = 1$ if $i \ge 3$.*

We now look at GSpin groups. The complex dual of a Levi subgroup of the form described in 2.7 is isomorphic to

$$\mathrm{GL}_{n_1}(\mathbb{C}) \times \mathrm{GL}_{n_2}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_k}(\mathbb{C}) \times \mathrm{GSp}_{2l}(\mathbb{C})$$

if $m' = 2l + 1$ and

$$\mathrm{GL}_{n_1}(\mathbb{C}) \times \mathrm{GL}_{n_2}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{n_k}(\mathbb{C}) \times \mathrm{GSO}_{2l}(\mathbb{C})$$

if $m' = 2l$. This is a Levi subgroup in $\mathrm{GSp}_{2(n_1+\cdots+n_k+l)}(\mathbb{C})$ if $m' = 2l + 1$ or in $\mathrm{GSO}_{2(n_1+\cdots+n_k+l)}(\mathbb{C})$ if $m' = 2l$, respectively.

By looking at matrices we get the following description of r in the case of a maximal Levi.

https://doi.org/10.4153/CJM-2002-025-8 Published online by Cambridge University Press

Proposition 5.6 *Let* $\mathbf{M}$ *be a maximal Levi subgroup in* GSpin_m *of the form* $\mathrm{GL}_n \times \mathrm{GSpin}_{m'}$*, where* $m = 2n + m'$ *and* $m' \neq 2$*. Denote the standard representations of the groups* $\mathrm{GL}_n(\mathbb{C})$*,* $\mathrm{GSp}_{2l}(\mathbb{C})$*, and* $\mathrm{GSO}_{2l}(\mathbb{C})$ *by* ρ_n*,* R^1_{2l}*, and* R^2_{2l}*, respectively and let* μ *be the multiplicative character defining* GSp_{2l} *or* $\mathrm{GSO}\ 2l$ *in each case. Then the adjoint action of the complex dual of this Levi on the Lie algebra of the unipotent radical of a parabolic containing this dual Levi may be written as follows.*

(a) *Assume* $m' = 2l + 1$ *and* $l \geq 1$*. Then,* $r = r_1 \oplus r_2$ *and* $r_1 = \rho_n \otimes \widetilde{R^1_{2l}}$ *while* $r_2 = \mathrm{Sym}^2 \rho_n \otimes \mu^{-1}$*. If* $l = 0$*, then* $r = r_1$ *and* $r_1 = \mathrm{Sym}^2 \rho_n \otimes \mu^{-1}$*. (If* $l = 1$ *note that* GSpin_3 *is* GL_2 *and its dual is* $\mathrm{GSp}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})$*.)*

(b) *Assume* $m' = 2l$ *and* $l \geq 2$*. Then,* $r = r_1 \oplus r_2$ *and* $r_1 = \rho_n \otimes \widetilde{R^2_{2l}}$ *while* $r_2 = \wedge^2 \rho_n \otimes \mu^{-1}$*. If* $l = 0$*, then* $r = r_1$ *and* $r_1 = \wedge^2 \rho_n \otimes \mu^{-1}$*. Note that we are not allowing the case* $l = 1$ *(cf. 2.8).*

We now state our final result.

Theorem 5.7 *Let* $M = \mathrm{GL}_n(F) \times \mathrm{GSpin}_m(F)$ *where* F *is a non-archimedean local field of characteristic zero. Let* $\pi_1 \otimes \pi_2$ *be an irreducible admissible generic representation of* M *which is in the discrete series. For* $i = 1, 2$*, let* r_i*'s be as in the above proposition and let* $L(s, \pi_1 \otimes \pi_2, r_i)$ *denote the local L-functions defined via the Langlands-Shahidi method as in Section 3.*

(a) *The local L-function* $L(s, \pi_1 \otimes \pi_2, r_1)$ *is holomorphic for* $\Re(s) > 0$*.*

(b) *The local L-function* $L(s, \pi_1 \otimes \pi_2, r_2)$ *is holomorphic for* $\Re(s) > 0$*.*

Remark 5.8 Note that these L-functions for non-maximal Levi subgroups M are defined as products of the maximal cases (*cf.* 3.6). Also, the holomorphy of the above L-functions for Levi subgroups of GSpin groups imply the same about the L-functions for arbitrary Levi subgroups in the split spinor groups by Remark 3.7 since they have the same derived group.

Moreover, we may take the representation of M to be tempered since the L-functions are then products of those for discrete series by Proposition 3.6.

Proof Note that the second L-function in either of the two cases of Proposition 5.6 is independent of π_2 and only yields symmetric square or exterior square L-function for π_1 (twisted by a character which is unitary) on GL_n. Both of these L-functions can be obtained from certain Levi subgroups in other classical groups in which case we already know the result: the Siegel Levi in SO_{2n+1} has $m = 1$ (*cf.* Proposition 5.3) and gives the symmetric square L-function while the Siegel Levi in Sp_{2n} has $m = 2$ and its second L-function gives the exterior square L-function. See [21] for more details about these L-functions.

As for the first L-function in each case the same arguments as in Section 4 of [6] work. Theorems 4.4 and 4.5 of our Section 4 are two of the ingredients one needs in order for those arguments to work out. We now give a summary of those arguments for completeness and refer to Section 4 of [6] for the details.

Replacing π_2 by its contragredient, denote the first L-function by $L(s, \pi_1 \times \pi_2)$. (This is the Rankin-Selberg product of π_1 and π_2.) We also denote the γ-factor defining $L(s, \pi_1 \times \pi_2)$ by $\gamma(s, \pi_1 \times \pi_2)$ (*cf.* Definition 3.3).

Both π_1 and π_2 are discrete series representations. Assume that π_1 is a subrepresentation of the induced representation from $\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_b$ to $\mathrm{GL}_n(F)$. By the results of Bernstein and Zelevinsky [3], [29], we know that $\sigma_i = \sigma_0 \otimes |\det(\cdot)|^{(b+1)/2-i}$ where σ_0 is a unitary supercuspidal representation of some $\mathrm{GL}_t(F)$ with $n = bt$. Moreover, assume that π_2 is a subrepresentation of the induced representation from $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k \otimes \tau$, where ρ_j is a representation of $\mathrm{GL}_{n_j}(F)$ and τ is one of $\mathrm{GSpin}_{m'}(F)$ with $2(n_1 + n_2 + \cdots + n_k) + m' = m$. Proposition 3.6 then implies the following:

$$(2) \qquad \gamma(s, \pi_2 \times \pi_2) = \prod_{i=1}^{b} \gamma(s, \sigma_i \times \tau) \cdot \prod_{i=1}^{b} \prod_{j=1}^{k} \gamma(s, \sigma_i \times \rho_j)\gamma(s, \sigma_i \times \tilde{\rho}_j).$$

Here $\tilde{\rho}_j$ denotes the contragredient of ρ_j and the two sides of the above equation are rational functions in q^{-s} where q denotes the residual characteristic. The L-function $L(s, \pi_1 \times \pi_2)$ is the inverse of the numerator of the left hand side and our objective is hence to prove that $\gamma(s, \pi_1 \times \pi_2) \neq 0$ for $\Re(s) > 0$.

Write $\rho_j = \rho_{0,j} \otimes |\det(\cdot)|^{\nu_j}$ where $\rho_{0,j}$ is unitary supercuspidal and $\nu_j \in \mathbb{R}$. By Proposition 3.8, the right hand side of (2) can be written as

$$(3) \quad \prod_{i=1}^{b} \gamma\big(s + (b+1)/2 - i, \sigma_0 \times \tau\big) \cdot \prod_{i=1}^{b} \prod_{j=1}^{k} \gamma\big(s + (b+1)/2 - i + \nu_j, \sigma_0 \times \rho_{0,j}\big)\, \gamma\big(s + (b+1)/2 - i - \nu_j, \sigma_0 \times \tilde{\rho_{0,j}}\big).$$

We now show that the above product is non-zero for $\Re(s) > 0$ by carefully studying each of the two terms in it. Since both σ_0 and τ are unitary supercuspidal, Proposition 3.4 and Remark 3.5 imply that the first term of the above product is, up to a monomial in q^{-s}, equal to

$$(4) \qquad \prod_{i=1}^{b} \frac{L\big(s + (b+1)/2 - i - 1, \sigma_0 \times \tau\big)}{L\big(s + (b+1)/2 - i, \sigma_0 \times \tau\big)} = \frac{L\big(s - (b+1)/2, \sigma_0 \times \tau\big)}{L\big(s + (b-1)/2, \sigma_0 \times \tau\big)},$$

which is non-zero since $\Re\big(s + (b-1)/2\big) > 0$ if $\Re(s) > 0$ and $L(s, \sigma_0 \times \tau)$ is holomorphic for $\Re(s) > 0$ by Proposition 5.5. However, the specific term appearing in the numerator is quite important since it may cancel possible zeros coming from the other term in certain cases (*cf.* Example 5.9).

As for the second product in (3), one first observes that one may assume, without loss of generality, that $\rho_{0,j} \simeq \sigma_0$ and σ_0 are self-contragredient (*cf.* [6, p. 573]). The idea of the proof is to show that there are very specific values that ν's can have in order for the representation induced from $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k \otimes \tau$ to have a discrete series subrepresentation and these are exactly what one needs for the zeros of the numerator and the denominator of this product to cancel out. To do this one uses the fact that for σ_0 unitary supercuspidal and π_2 in discrete series, we know that $L(s, \sigma_0 \times \pi_2)$ is holomorphic for $\Re(s) > 0$. To see this note that in this case we have two L-functions ($m = 2$) and the second one (which only depends on σ_0) is of the form in Proposition 5.5 and hence Proposition 5.4 implies that $L(s, \sigma_0 \times \pi_2)$ is holomorphic for $\Re(s) > 0$.

Define a σ_0-chain (or segment) as a sequence of representations of the form $\sigma_0 \otimes |\det(\cdot)|^{\nu_j}$ with $\nu_j \in \mathbb{R}$ and $\nu_j - \nu_{j-1} = 1$. One then uses Theorems 4.4 and 4.5 along with the main tool mentioned above to show that only specific types of chains or pairs of chains ("regular" and "singular") can occur and the values of ν's are always integers or half integers. This will show that all the possible poles of the L-function coming from the second product cancel out with zeros coming from either the second product or possibly the first one. The process of cancellation of zeros in this argument is a very delicate one (*cf.* Example 5.9). We refer to Section 4 of [6] for details. ∎

Following a suggestion by the referee we include the following example in the argument of the proof in order to indicate how delicate the proof could be.

Example 5.9 Consider the following special case of the above theorem. Assume that π_1 is the Steinberg representation, *i.e.*, it is the irreducible subrepresentation of the representation of $\mathrm{GL}_n(F)$ induced from

$$\sigma_0|\det(\cdot)|^{\frac{b-1}{2}} \otimes \sigma_0|\det(\cdot)|^{\frac{b-3}{2}} \otimes \cdots \otimes \sigma_0|\det(\cdot)|^{-\frac{b-1}{2}}$$

and π_2 is an irreducible subrepresentation of the representation of $\mathrm{GSpin}_m(F)$ induced from

$$\sigma_0|\det(\cdot)|^{b} \otimes \sigma_0|\det(\cdot)|^{b-1} \otimes \cdots \otimes \sigma_0|\det(\cdot)| \otimes \tau,$$

where σ_0 is a self-contragredient unitary supercuspidal representation of $\mathrm{GL}_t(F)$. (We have $k = b$ and $\nu_j = b + 1 - j$ for each j.) Here, (σ_0, τ) satisfy $(C1)$, namely, the representation induced from $\sigma_0|\det(\cdot)|^s \otimes \tau$ is reducible at $s = 1$ (*cf.* [6, Definition 4.10]).

As before, the first product is, up to a monomial in q^{-s}, equal to

$$\frac{L\big(s-(b+1)/2, \sigma_0 \times \tau\big)}{L\big(s+(b-1)/2, \sigma_0 \times \tau\big)}. \tag{5}$$

Similarly, the second product is, up to a monomial in q^{-s}, equal to

$$\prod_{i=1}^{b}\prod_{j=1}^{b}\frac{L(s+\frac{b+1}{2}+b-i-j,\sigma_0\times\sigma_0)}{L(s+\frac{b+1}{2}+b+1-i-j,\sigma_0\times\sigma_0)}\cdot\frac{L(s-\frac{b+1}{2}+1-i+j,\sigma_0\times\sigma_0)}{L(s-\frac{b+1}{2}-i+j,\sigma_0\times\sigma_0)}$$

$$=\prod_{i=1}^{b}\frac{L\big(s+(b+1)/2-i,\sigma_0\times\sigma_0\big)}{L\big(s+(b+1)/2+b-i,\sigma_0\times\sigma_0\big)}\cdot\frac{L\big(s-(b+1)/2-i,\sigma_0\times\sigma_0\big)}{L\big(s+(b-1)/2-i,\sigma_0\times\sigma_0\big)}$$

$$=\frac{L\big(s+(b-1)/2,\sigma_0\times\sigma_0\big)}{L\big(s-(b+1)/2,\sigma_0\times\sigma_0\big)}\cdot\prod_{i=1}^{b}\frac{L\big(s-(b+1)/2-i,\sigma_0\times\sigma_0\big)}{L\big(s+(b-1)/2+i,\sigma_0\times\sigma_0\big)}.$$

Observe that the term $L\big(s-(b+1)/2,\sigma_0\times\sigma_0\big)$ could be a potential problem. To get around this problem, one uses the fact that the inverse of this L-function divides the inverse of the numerator of (5) (*cf.* [6, Lemma 4.20]).

The following two results are special cases of Theorem 5.7 and have already been used in [12].

Corollary 5.10 *Let F denote a non-archimedean local field of characteristic zero as before.*

(a) *Let* σ_1 *be an irreducible admissible tempered (hence generic) representation of* $\mathrm{GL}_3(F)$ *and* σ_2 *and* σ_3 *be irreducible admissible tempered (hence generic) representations of* $\mathrm{GL}_2(F)$. *Then the triple product L-function*

$$L(s,\sigma_1\times\sigma_2\times\sigma_3)$$

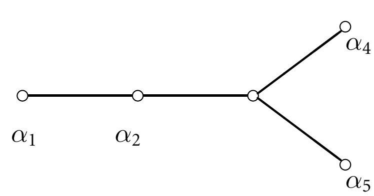
is holomorphic for $\Re(s)>0$.

(b) *Let* σ_1 *be an irreducible admissible tempered (hence generic) representation of* $\mathrm{GL}_3(F)$ *and* σ_2 *be one of* $\mathrm{GL}_4(F)$. *Then the L-function*

$$L(s,\sigma_1\otimes\sigma_2,\rho_3\otimes\wedge^2\rho_4)$$

is holomorphic for $\Re(s)>0$. *Here* ρ_n *denotes the standard representation of* $\mathrm{GL}_n(\mathbb{C})$.

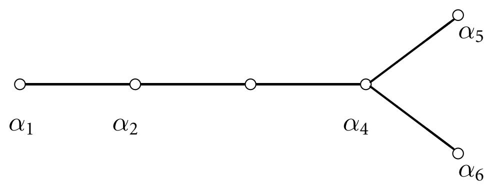
Proof These correspond to the cases D_5-2 and D_6-3 of [18]. For part (a) consider the maximal Levi subgroup $\mathrm{GL}_3\times\mathrm{GSpin}_4$ in GSpin_{10} with the following Dynking diagram where we have omitted the simple root determining this maximal Levi.



https://doi.org/10.4153/CJM-2002-025-8 Published online by Cambridge University Press

In this case there are two L-functions ($m = 2$) and the first one gives the triple product L-function for σ_1, σ_2, and σ_3. Hence we know the holomorphy by our main theorem.

Similarly, for part (b) consider the maximal Levi subgroup $\mathrm{GL}_3 \times \mathrm{GSpin}_6$ in GSpin_{12} with the following Dynkin diagram where we have again omitted the simple root determining the maximal Levi.



Again, there are two L-functions ($m = 2$) in this case and the first one gives the desired L-function in part (b). Although clear from the diagram, we remark that the derived group of GSpin_6 is Spin_6 which is nothing but SL_4. ■

References

[1] M. Asgari, *On the holomorphy of local Langlands L-functions.* PhD thesis, Purdue University, August, 2000.

[2] D. Ban, *Self-duality in the case of* $\mathrm{SO}(2n, F)$. Glas. Mat. Ser. III **34(54)**(1999), 187–196.

[3] I. Bernstein and A. Zelevinsky, *Induced representations of reductive p-adic groups I.* Ann. Sci. École Norm. Sup. **10**(1977), 441–472.

[4] A. Borel, *Automorphic L-functions.* In: Automorphic forms, representations and L-functions, Proc. Sympos. Pure Math. (Oregon State Univ., Corvallis, OR, 1977), Part 2, Amer. Math. Soc., Providence, RI, 1979, 27–61.

[5] W. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups.* Preprint.

[6] W. Casselman and F. Shahidi, *On irreducibility of standard modules for generic representations.* Ann. Sci. École Norm. Sup. **31**(1998), 561–589.

[7] W. Fulton and J. Harris, *Representation theory, A first course.* Graduate Texts in Math. **129**, Springer-Verlag, New York, 1991.

[8] C. Jantzen, *Degenerate principal series for orthogonal groups.* J. Reine Angew. Math. **441**(1993), 61–98.

[9] ———, *Degenerate principal series for symplectic groups.* Mem. Amer. Math. Soc. (488) **102**(1993), xiv + 111 pp.

[10] H. Kim, *On local L-functions and normalized intertwining operators.* Preprint, 2000.

[11] H. Kim and F. Shahidi, *Functorial products for* $\mathrm{GL}_2 \times \mathrm{GL}_3$ *and functorial symmetric cube for* GL_2. C. R. Acad. Sci. Paris. Sér. I Math. **331**(2000), 599–604.

[12] ———, *Functorial products for* $\mathrm{GL}_2 \times \mathrm{GL}_3$ *and functorial symmetric cube for* GL_2. Preprint, 2000.

[13] R. P. Langlands, *Euler products.* A James K. Whittemore Lecture in Mathematics given at Yale University, 1967. Yale Mathematical Monographs **1**, Yale University Press, New Haven, Conn., 1971.

[14] C. Mœglin, *Normalisation des opérateurs d'entrelacement et réductibilité des induites de cuspidales; le cas des groupes classiques p-adiques.* Ann. of Math. (2) **151**(2000), 817–847.

[15] G. Muić, *Some results on square integrable representations; irreducibility of standard representations.* Internat. Math. Res. Notices (14) **1998**(1998), 705–726.

[16] F. Shahidi, *On certain L-functions.* Amer. J. Math. **103**(1981), 297–355.

[17] ———, *Local coefficients and normalization of intertwining operators for* $\mathrm{GL}(n)$. Compositio Math. **48**(1983), 271–295.

https://doi.org/10.4153/CJM-2002-025-8 Published online by Cambridge University Press

[18] ———, *On the Ramanujan conjecture and finiteness of poles for certain L-functions.* Ann. of Math. (2) **127**(1988), 547–584.

[19] ———, *On multiplicativity of local factors.* In: Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989), Weizmann, Jerusalem, 1990, 279–289.

[20] ———, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups.* Ann. of Math. (2) **132**(1990), 273–330.

[21] ———, *Twisted endoscopy and reducibility of induced representations for p-adic groups.* Duke Math. J. **66**(1992), 1–41.

[22] ———, *On non-vanishing of twisted symmetric and exterior square L-functions for* $\mathrm{GL}(n)$. Olga Taussky-Todd: in memoriam, Pacific J. Math. Special Issue (1997), 311–322.

[23] T. A. Springer, *Linear algebraic groups.* 2nd edition, Birkhäuser, Boston, 1998.

[24] M. Tadić, *Notes on representations of non-archimedean* $\mathrm{SL}(n)$. Pacific J. Math. **152**(1992), 375–396.

[25] ———, *Representations of p-adic symplectic groups.* Compositio Math. **90**(1994), 123–181.

[26] ———, *On regular square integrable representations of p-adic groups.* Amer. J. Math. **120**(1998), 159–210.

[27] ———, *Square integrable representations of classical p-adic groups corresponding to segments.* Represent. Theory **3**(1999), 58–59 (electronic).

[28] J. Tate, *Number theoretic background.* In: Automorphic forms, representations and L-functions, Proc. Sympos. Pure Math. (Oregon State Univ., Corvallis, OR, 1977), Part 2, Amer. Math. Soc., Providence, RI, 1979, 3–26.

[29] A. Zelevinsky, *Induced representations of reductive p-adic groups II. On irreducible representations of* $\mathrm{GL}(n)$. Ann. Sci. École Norm. Sup. **13**(1980), 165–210.

Department of Mathematics
The University of Michigan
525 East University Avenue
Ann Arbor, MI 48109-1109
USA
email: asgari@math.lsa.umich.edu

https://doi.org/10.4153/CJM-2002-025-8 Published online by Cambridge University Press