On Errors in Determinants

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§ 1. *Introduction.* Care is needed in dealing with determinants whose elements are subject to experimental error, particularly when a determinant itself is small compared with its first minors. For, as these examples show, a relatively tiny error in one element may be responsible for a large error in the determinant :

$$\begin{vmatrix} -73 & 78 & 24 \\ 92 & 66 & 25 \\ -80 & 37 & 10 \end{vmatrix} = 1; \qquad \begin{vmatrix} -73 & 78 & 24 \\ 92 & 66 & 25 \\ -80 & 37 & 10{\cdot}01 \end{vmatrix} = -118{\cdot}94.$$

Consider a determinant Δ, of order n, with elements a_i $(i = 1, \ldots, n^2)$. Let the actual error in each element a_i be e_i, and let the resultant error in Δ be E. In §§ 2-4, I shall find expressions for (i) E in terms of e_i; (ii) the maximum range of E in terms of the ranges of e_i; (iii) the probability distribution $P(E)$ of E in terms of the probability distributions $p_i(e_i)$ of e_i, assumed independent and (for simplicity) symmetrical. In §§ 5-7, I shall investigate corresponding results for the quotient of two determinants which are identical except for one row or column. Such quotients are of importance in practical work, occurring in the solution of a set of simultaneous linear algebraic equations. In § 8, the method is applied to an arbitrary function whose arguments are subject to error.

To apply the formulae to numerical determinants, it is necessary for first approximations to calculate the complete set of first minors; the second, third, etc., minors are required for closer approximations. The calculation of these minors is very laborious in the case of a determinant of large order. It may be pointed out, however, that all the first minors can be found in the course of evaluating a determinant and its adjugate by extant methods, for example that[1] of T. Smith, *Phil. Mag.* (7), 3 (1927), 1007.

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[1] There is also a method suitable for use with a calculating machine, due to Dr Aitken, and as yet unpublished.

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§ 2. *Actual error in a determinant.* Let A_i, A_{ij}, A_{ijk}, ... denote the first, second, third, ... minors of Δ corresponding to the elements indicated by the suffixes. An error e_i in the term a_i will cause an error $e_i A_i$ in Δ; errors e_i, e_j in the terms a_i, a_j will together cause an error $(e_i A_i + e_j A_j + e_i e_j A_{ij})$ in Δ; and so on. Hence in general:

$$E = \Sigma e_i A_i + \Sigma e_i e_j A_{ij} + \Sigma e_i e_j e_k A_{ijk} + \ldots, \tag{1}$$

the suffixes in each summation running from 1 to n^2. (This result follows at once from Taylor's Theorem, A_i, A_{ij}, A_{ijk}, ... being partial derivatives of Δ.) It is to be observed that none of the errors occur squared or to higher powers in (1).

§ 3. *Range of error.* If $|e_i| \leqslant \epsilon_i$, we deduce from (1):

$$|E| \leqslant \Sigma \epsilon_i |A_i| + \Sigma \epsilon_i \epsilon_j |A_{ij}| + \Sigma \epsilon_i \epsilon_j \epsilon_k |A_{ijk}| + \ldots, \tag{2}$$

vertical bars denoting absolute values. If we put $\epsilon_i = \epsilon$, and neglect ϵ^2, (2) becomes

$$|E| \leqslant \epsilon \Sigma |A_i|. \tag{3}$$

Thus $\Sigma |A_i|$ may be regarded as a measure of the sensitivity of a determinant to small errors whose squares may be neglected.

As an application of these formulae, consider the first determinant in § 1, and suppose that the range of error in each element is $\pm \frac{1}{2}$. We find $\Sigma |A_i| = 33099$, $\Sigma |A_{ij}| = 2 \Sigma |a_i| = 970$, and (for any third order determinant) $\Sigma |A_{ijk}| = 6$. Hence (3) gives ± 16550 as a first approximation to the range of error in Δ. More accurately, (4) gives:

$$|E| \leqslant \tfrac{1}{2} \cdot 33099 + \tfrac{1}{4} \cdot 970 + \tfrac{1}{8} \cdot 6 < 16793.$$

If we suppose that the determinant has occurred in some practical calculation, and that the elements as given are only rough first approximations (with $\epsilon = \frac{1}{2}$), then the important question arises: To what further degree of accuracy η should the elements be evaluated, in order that the error in Δ may not exceed a given limit, say for example 0·2?

We have in general, when the range of error in each element is $\pm \eta$,

$$|E| \leqslant \eta \Sigma |B_i| + \eta^2 \Sigma |B_{ij}| + \ldots,$$

where B_i, B_{ij}, ... are the *correct* values of the minors to which A_i, A_{ij}, ... are approximations.

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Now

$$|A_i - B_i| = \text{error in minor of } a_i$$
$$\leqslant \epsilon \sum_j |A_{ij}| + \epsilon^2 \sum_{jk} |A_{ijk}| + \ldots,$$

and hence

$$|B_i| \leqslant |A_i| + \epsilon \sum_j |A_{ij}| + \epsilon^2 \sum_{jk} |A_{ijk}| + \ldots.$$

Summing for $i = 1, \ldots, n$,

$$\Sigma |B_i| \leqslant \Sigma |A_i| + \epsilon \Sigma |A_{ij}| + \epsilon^2 \Sigma |A_{ijk}| + \ldots.$$

Similarly,

$$\Sigma |B_{ij}| \leqslant \Sigma |A_{ij}| + \epsilon \Sigma |A_{ijk}| + \ldots.$$

Hence

$$|E| \leqslant \eta [\Sigma |A_i| + \epsilon \Sigma |A_{ij}| + \epsilon^2 \Sigma |A_{ijk}| + \ldots]$$
$$+ \eta^2 [\Sigma |A_{ij}| + \epsilon \Sigma |A_{ijk}| + \ldots] + \ldots.$$

This formula gives the range of error in a determinant when the elements are correct to within $\pm \eta$, in terms of the minors in a first approximation in which the elements are correct to within $\pm \epsilon$.

Let us apply this to the determinant already considered, for which

$$\Sigma |A_i| + \epsilon \Sigma |A_{ij}| + \epsilon^2 \Sigma |A_{ijk}| = 33585 \cdot 5.$$

Putting $\eta = \cdot 000005$, we find $|E| < 0 \cdot 17$. Thus, in order that the error in Δ should not exceed $0 \cdot 2$, it is sufficient to evaluate each element to 5 decimal places.

§ 4. *Probability Distribution.* Denote the probability distribution of e_i by $p_i(e_i)$, and the r^{th} moment of this function by

$$m_{ri} \equiv \int_{-\infty}^{\infty} e_i^r p_i(e_i)\, de_i. \tag{4}$$

The functions p_i being given, we can assume that these moments m_{ri} are known; in terms of them we can calculate the r^{th} moment M_r of $P(E)$. We have in fact:

$$M_r \equiv \int_{-\infty}^{\infty} E^r P(E)\, dE = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} E^r . \prod_1^{n^2} [p_i(e_i)\, de_i]. \tag{5}$$

On the right hand side of (5), E^r is to be interpreted as a function of e_i, by means of (1). Consequently, for any value of r, we can expand E^r and integrate term by term, making use of (4), and of the conditions $\int_{-\infty}^{\infty} p_i\, de_i = 1$.

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For example,

$$E^2 = \Sigma\, A_i^2\, e_i^2 + 2\,\Sigma\, A_i\, A_j\, e_i\, e_j + 2\,\Sigma\, A_i\, A_{ij}\, e_i^2\, e_j + 2\,\Sigma\, A_i\, A_{jk}\, e_i\, e_j\, e_k + \ldots .$$

On integration this gives

$$\Sigma A_i^2 m_{2i} + 2\Sigma A_i\, A_j\, m_{1i}\, m_{1j} + 2\Sigma A_i\;\; A_{ij}\, m_{2i}\, m_{1j} + 2\Sigma A_i\, A_{jk}\, m_{1i}\, m_{1j}\, m_{1k} + \ldots$$

as the value of M_2.

The given functions have been assumed to be symmetrical; *i.e.* $p_i(e_i) = p_i(-e_i)$; consequently the odd moments m_{1i}, m_{3i}, ... all vanish. We need, then, only take account of those terms of E^r which contain only *even* powers of the errors e_i; all other terms vanish on integration. We then find

$$M_1 = 0, \tag{6}$$

$$M_2 = \Sigma\, A_i^2\, m_{2i} + \Sigma\, A_{ij}^2\, m_{2i}\, m_{2j} + \Sigma\, A_{ijk}^2\, m_{2i}\, m_{2j}\, m_{2k} + \ldots , \tag{7}$$

$$M_3 = 6\,\Sigma\, A_i\, A_j\, A_{ij}\, m_{2i}\, m_{2j} + 6\,\Sigma\, A_{ij}\, A_{jk}\, A_{ik}\, m_{2i}\, m_{2j}\, m_{2k} + 6\,\Sigma\, A_i A_{jk}\, A_{ijk}\, m_{2i}\, m_{2j}\, m_{2k} + \ldots ,$$

$$M_4 = \Sigma\, A_i^4\, m_{4i} + 6\,\Sigma\, A_i^2\, A_j^2\, m_{2i}\, m_{2j} + \ldots ,$$

and so on.

Each of these expressions has only a finite number of terms, but they rapidly become very complicated. However, if the original errors are known to be sufficiently small, *i.e.* if p_i are practically zero except where e_i are small, then the moments m_{ri} are also small quantities; we can therefore approximate to M_r successfully with only a few terms, and the succeeding moments M_r will rapidly diminish in order of magnitude.

With a change of notation, equation (7) can be rewritten:

$$S^2 = \Sigma\, A_i^2\, \sigma_i^2 + \Sigma\, A_{ij}^2\, \sigma_i^2\, \sigma_j^2 + \Sigma\, A_{ijk}^2\, \sigma_i^2\, \sigma_j^2\, \sigma_k^2 + \ldots \tag{8}$$

giving the standard deviation $S\,(=\sqrt{M_2})$ of the determinant in terms of the standard deviations $\sigma_i\,(=\sqrt{m_{2i}})$ of the elements.

Having thus found M_1, M_2, M_3, ..., we can approximate to $P(E)$ in one of the usual ways. For instance, if we assume that the errors e_i follow the normal law of distribution,

$$p_i(e_i) = \frac{1}{\sigma_i\sqrt{2\pi}} \exp\left(-e_i^2/2\sigma_i^2\right)$$

(or, more generally, that they follow a symmetrical law of Charlier's Type A), then the law for the determinant must necessarily be of Charlier's Type A. We can thus assume

$$P(E) = \left[1 + A\frac{d^3}{dE^3} + B\frac{d^4}{dE^4} + \ldots\right]\frac{1}{S\sqrt{2\pi}}\, e^{-E^2/2S^2}.$$

S is given by (8), and the other constants by:

$$A = -\frac{M_3}{3!}, \quad B = \frac{M_4 - 3S^4}{4!}, \quad etc.,$$

these results being found by comparing the moments of the function with M_2, M_3, ...

To a first approximation,

$$S^2 = \Sigma A_i^2 \sigma_i^2 ; \quad A, B, \ldots . = 0.$$

To a second approximation,

$$S^2 = \Sigma A_i^2 \sigma_i^2 + \Sigma A_i^2 A_j^2 \sigma_i^2 \sigma_j^2 ;$$

$$A = - \Sigma A_i A_j A_{ij} \sigma_i^2 \sigma_j^2 ;$$

$$4! B = \Sigma A_i^4 m_{4i} + 6 \Sigma A_i^2 A_j^2 \sigma_i^2 \sigma_j^2 - 3 (\Sigma A_i^4 \sigma_i^4 + 2 \Sigma A_i^2 A_j^2 \sigma_i^2 \sigma_j^2)$$

$$= 0,$$

since for *normal* distributions $m_{4i} = 3\sigma_i^4$.

For practical purposes a more important case arises when the elements have been calculated correct to x places of decimals, the last figure being forced, *i.e.* the last digit retained is increased by 1 when the first digit not retained is 5, 6, 7, 8 or 9. We may take it that all errors between $\pm \frac{1}{2} 10^{-x}$ are equally likely, and that none exceed these limits. Remembering that $\int_{-\infty}^{\infty} p(e)\, de$ must be unity, we have

$$p(e) = 10^x, \quad \text{if } -\tfrac{1}{2} 10^{-x} \leqslant e < \tfrac{1}{2} 10^{-x},$$
$$= 0, \quad \text{otherwise};$$

the same law of distribution applies to all the elements.

Then we have

$$m_1 = m_3 = \ldots . = 0;$$

$$m_2 = \int_{-\frac{1}{2}10^{-x}}^{\frac{1}{2}10^{-x}} e^2 . 10^x \, de = \frac{1}{12 . 10^{2x}} ;$$

$$m_4 = \int_{-\frac{1}{2}10^{-x}}^{\frac{1}{2}10^{-x}} e^4 . 10^x \, de = \frac{1}{80 . 10^{4x}} ;$$

$$M_1 = 0;$$

$$M_2 = \frac{\Sigma A_i^2}{12 . 10^{2x}} + \frac{\Sigma A_{ij}^2}{144 . 10^{4x}} + \ldots . ;$$

$$M_3 = \frac{\Sigma A_i A_j A_{ij}}{24 . 10^{4x}} + \ldots . ;$$

$$M_4 = \frac{\Sigma A_i^4}{80 . 10^{4x}} + \frac{\Sigma A_i^2 A_j^2}{24 . 10^{4x}} + \ldots .$$

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Hence, neglecting terms involving 10^{-6x}, we find:

$$P(E) = \left[1 - \frac{\Sigma A_i A_j A_{ij}}{144 \cdot 10^{4x}} \frac{d^3}{dE^3} - \frac{\Sigma A_i^4}{288 \cdot 10^{5x}} \frac{d^4}{dE^4}\right] \frac{1}{S\sqrt{2\pi}} e^{-E^2/2S^2}, \tag{9}$$

where

$$S^2 = \frac{\Sigma A_i^2}{12 \cdot 10^{2x}} + \frac{\Sigma A_{ij}^2}{144 \cdot 10^{4x}}. \tag{10}$$

As an application of these formulae, consider the determinant:

$$\begin{vmatrix} \sqrt{7} & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{11} & \sqrt{3} \\ \sqrt{3} & \sqrt{5} & \sqrt{6} \end{vmatrix},$$

the value of which, to five places, is 6·93899. Evaluating each element to two places, we get:

$$\begin{vmatrix} 2{\cdot}65 & 1{\cdot}41 & 1{\cdot}41 \\ 1{\cdot}41 & 3{\cdot}32 & 1{\cdot}73 \\ 1{\cdot}73 & 2{\cdot}24 & 2{\cdot}45 \end{vmatrix},$$

the value of which, to five places, is 6·98983. For the latter determinant we find (correct to the number of places given in each case):

$$\begin{aligned} \Sigma |A_i| &= 26{\cdot}8, \\ \Sigma A_i^2 &= 111{\cdot}9, \\ \Sigma A_{ij}^2 &= 78, \\ \Sigma A_i A_j A_{ij} &= 287, \\ \Sigma A_i^4 &= 1078, \\ \epsilon &= {\cdot}005. \end{aligned}$$

By (3), the maximum range of error is $\pm$ ·135. By (10), the standard deviation (to a first approximation) is ·031. These results may be compared with the actual error ·051. The second approximation to the probability distribution is given by:

$$S = {\cdot}0314, \quad A = -{\cdot}0002, \quad B = -{\cdot}0004.$$

§ 5. *Error in quotient of determinants.* Consider the quotient $\frac{\Delta_1}{\Delta_2}$ of the determinants[1]:

$$\Delta_1 = | \, a_i \, a_{i1} \, a_{i2} \, \ldots . \, a_{i\,n-1} \, |, \quad (i = 1 \ldots . n)$$
$$\Delta_2 = | \, b_i \, a_{i1} \, a_{i2} \, \ldots . \, a_{i\,n-1} \, |.$$

[1] The notation means that the determinants have these as i^{th} rows. The determinants are identical except in their first columns. As pointed out in § 1, such quotients occur in the solution of a set of n simultaneous linear equations.

Suppose that the elements a_i, b_i, a_{ij} are subject to errors e_i, f_i, e_{ij}; and that the resultant errors in Δ_1, Δ_2, $\dfrac{\Delta_1}{\Delta_2}$ are E_1, E_2, E. Form the determinant

$$\Delta_3 = | \; c_i \; a_{i1} \; a_{i2} \; \ldots . \; a_{i\,n-1} \; |,$$

where
$$c_1 = a_i - Xb_i,$$
$$X = \text{the calculated value of } \frac{\Delta_1}{\Delta_2};$$

so that
$$\Delta_3 = \text{the calculated value of } \Delta_1 - X\Delta_2 = 0. \qquad (11)$$

The correct value of $\Delta_1 - X\Delta_2$ is, however, not zero.

Let
$$\begin{aligned} g_i &= \text{error in } c_i = e_i - Xf_i, \\ E_3 &= \text{error in } \Delta_3 = E_1 - XE_2, \\ A_i &= \text{minor of } a_i \text{ in } \Delta_1 \\ &= \text{minor of } b_i \text{ in } \Delta_2 \\ &= \text{minor of } c_i \text{ in } \Delta_3, \\ A_{ij} &= \text{minor of } a_{ij} \text{ in } \Delta_1, \\ B_{ij} &= \text{minor of } a_{ij} \text{ in } \Delta_2, \\ C_{ij} &= \text{minor of } a_{ij} \text{ in } \Delta_3 = A_{ij} - XB_{ij}. \end{aligned}$$

It follows from (11) that the minors of any one row of Δ_3 are proportional to the minors of the first row, *i.e.*

$$C_{ij} = \frac{A_i C_{1j}}{A_1}. \qquad (12)$$

Using (1), we have as first approximations,

$$E_2 = \Sigma f_i A_i + \Sigma e_{ij} B_{ij},$$
$$E_3 = \Sigma g_i A_i + \Sigma e_{ij} C_{ij}.$$

Correct to the second order,

$$\begin{aligned} E &= \frac{\Delta_1 + E_1}{\Delta_2 + E_2} - \frac{\Delta_1}{\Delta_2} \\ &= \frac{\Delta_2 E_1 - \Delta_1 E_2}{\Delta_2(\Delta_2 + E_2)} \\ &= \frac{E_1 - XE_2}{\Delta_2}\left(1 + \frac{E_2}{\Delta_2}\right)^{-1} \\ &= \frac{E_3}{\Delta_2}\left(1 - \frac{E_2}{\Delta_2}\right). \end{aligned}$$

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Thus, to the first order,

$$E = \frac{1}{\Delta_2}[\Sigma\,(e_i - Xf_i)\,A_i + \Sigma\,e_{ij}\,C_{ij}]. \tag{13}$$

To the second order,

$$E = \frac{E_3}{\Delta_2} - \frac{1}{\Delta_2^2}[\Sigma f_i\,A_i + \Sigma\,e_{ij}\,B_{ij}]\,[\Sigma\,(e_i - Xf_i)\,A_i + \Sigma\,e_{ij}\,C_{ij}], \tag{14}$$

$$E^2 = \frac{1}{\Delta_2^2}[\Sigma\,(e_i - Xf_i)\,A_i + \Sigma\,e_{ij}\,C_{ij}]^2. \tag{15}$$

§ 6. *Range of error in quotient of determinants.* Suppose

$$|\,e_i\,| \leqslant \epsilon_i, \quad |\,f_i\,| \leqslant \zeta_i, \quad |\,e_{ij}\,| \leqslant \epsilon_{ij}.$$

From (13), the range of E is given to a first approximation by:

$$|\,E\,| \leqslant \frac{1}{|\,\Delta_2\,|}\,[\Sigma\,(\epsilon_i + |\,X\,|\,\zeta_i)\,|\,A_i\,| + \Sigma\,\epsilon_{ij}\,|\,C_{ij}\,|\,].$$

If the given ranges are all equal, *i.e.*, $\epsilon_i = \zeta_i = \epsilon_{ij} = \epsilon$, we have, using (12)

$$|\,E\,| \leqslant \frac{\epsilon}{|\,\Delta_2\,|}\left[(1 + |\,X\,|)\,\Sigma\,|\,A_i\,| + \Sigma\left|\frac{A_i\,C_{1j}}{A_1}\right|\right]$$

$$\leqslant \frac{\epsilon}{|\,\Delta_2\,A_1\,|}\,\Sigma\,A_i\,.\,[(1 + |\,X\,|)\,|\,A_1\,| + \Sigma\,|\,C_{1j}\,|\,]. \tag{16}$$

This can be expressed in the form

$$|\,E\,| \leqslant \epsilon\,\frac{1 + |\,X\,|}{|\,\Delta_2\,A_1\,|}\,S_1\,S_2\,,$$

where S_1 = sum of absolute values of first minors of first column of D,

S_2 = sum of absolute values of first minors of first row of D,

D being the determinant $\left|\dfrac{a_i - Xb_i}{1 + |\,X\,|} \quad a_{i1}\,a_{i2}\,\ldots\ldots\,a_{i\,n-1}\right|$.

§ 7. *Probability distribution for quotient of determinants.* Let the standard deviations of e_i, f_i, e_{ij} be σ_i, τ_i, σ_{ij}, it being assumed that the first moments are zero. Let M_1, M_2, M_3 .... be the moments of the probability distribution $P(E)$ of E. Using (14) and (15), proceeding as in § 4, and remembering that in accordance with (6) the first moment of E_3 is zero, we get, correct to the second order in the deviations:

$$M_1 = \frac{1}{\Delta_2^2}[X \Sigma \tau_i^2 A_i^2 - \Sigma \sigma_{ij}^2 B_{ij} C_{ij}],$$

$$M_2 = \frac{1}{\Delta_2^2}[\Sigma (\sigma_i^2 + X^2 \tau_i^2) A_i^2 + \Sigma \sigma_{ij}^2 C_{ij}^2],$$

$$M_3 = \ldots\ldots = 0.$$

Thus, assuming that the required function is of Charlier's Type A, we have as a first approximation:

$$P(E) = \frac{1}{S\sqrt{2\pi}} e^{-(E-a)^2/2S^2},$$

where $a = M_1 = [X \Sigma \tau_i^2 A_i^2 - \Sigma \sigma_{ij}^2 B_{ij} C_{ij}] / \Delta_2^2,$

$$S^2 = M_2 - M_1^2 = [\Sigma (\sigma_i^2 + X^2 \tau_i^2) A_i^2 + \Sigma \sigma_{ij}^2 C_{ij}^2] / \Delta_2^2.$$

The second approximation to $P(E)$ involves many complicated summations.

If the given standard deviations are all equal, *i.e.* $\sigma_i = \tau_i = \sigma_{ij} = \sigma$, we have, using (12):

$$a = \frac{\sigma^2}{\Delta_2^2}[X \Sigma A_i^2 - \frac{1}{A_1} \Sigma B_{ij} A_i C_{1j}], \tag{17}$$

$$S^2 = \frac{\sigma^2}{\Delta_2^2 A_1^2} \Sigma A_i^2 . [(1 + X^2) A_1^2 + \Sigma C_{1j}^2]. \tag{18}$$

As in § 6, we note that (18) can be written

$$S^2 = \sigma^2 \frac{1 + X^2}{\Delta_2^2 A_1^2} S_3 S_4,$$

where S_3 = sum of squares of first minors of first column of D',

S_4 = sum of squares of first minors of first row of D',

D' being the determinant $\left| \dfrac{a_i - Xb_i}{(1 + X^2)^{\frac{1}{2}}} a_{i1} a_{i2} \ldots . a_{in-1} \right|.$

As an application of these formulae, consider the value of X found from the equations

$$X\sqrt{7} + Y\sqrt{2} + Z\sqrt{2} = \sqrt{17},$$
$$X\sqrt{3} + Y\sqrt{11} + Z\sqrt{3} = -\sqrt{2},$$
$$X\sqrt{2} + Y\sqrt{5} + Z\sqrt{6} = -\sqrt{3}.$$

To four places, the value is $X = 3{\cdot}1468$. Let us however approximate by evaluating each coefficient to two places. We then have:

$$\Delta_1 = \begin{vmatrix} 4{\cdot}12 & 1{\cdot}41 & 1{\cdot}41 \\ -1{\cdot}41 & 3{\cdot}32 & 1{\cdot}73 \\ -1{\cdot}73 & 2{\cdot}24 & 2{\cdot}45 \end{vmatrix} = 21{\cdot}842244,$$

$$\Delta_2 = \begin{vmatrix} 2{\cdot}65 & 1{\cdot}41 & 1{\cdot}41 \\ 1{\cdot}41 & 3{\cdot}32 & 1{\cdot}73 \\ 1{\cdot}73 & 2{\cdot}24 & 2{\cdot}45 \end{vmatrix} = 6{\cdot}989832.$$

The following results are correct to the number of places given in each case:

$$X = \frac{\Delta_1}{\Delta_2} = 3{\cdot}1249,$$

$$\Delta_3 = \begin{vmatrix} -4{\cdot}16 & 1{\cdot}41 & 1{\cdot}41 \\ -5{\cdot}82 & 3{\cdot}32 & 1{\cdot}73 \\ -7{\cdot}14 & 2{\cdot}24 & 2{\cdot}45 \end{vmatrix} = {\cdot}01,$$

$$A_1 = 4{\cdot}26,$$

$$\Sigma \mid A_i \mid = 6{\cdot}8,$$

$$\Sigma A_i^2 = 23{\cdot}25,$$

$$\Sigma B_{ij} A_i C_{1j} = -264,$$

$$(1 + \mid X \mid) \mid A_1 \mid + \Sigma \mid C_{1j} \mid = 30,$$

$$X \Sigma A_i^2 - \frac{1}{A_1} \Sigma B_{ij} A_i C_{1j} = 134,$$

$$(1 + X^2) A_1^2 + \Sigma C_{1j}^2 = 313,$$

$$\epsilon = {\cdot}005,$$

$$\sigma^2 = \frac{1}{12 \,.\, 10^4}.$$

From (16), (17) and (18) we deduce that the range of error is $\pm {\cdot}035$, and that $a = {\cdot}00002$, $S = {\cdot}008$. The actual error is ${\cdot}022$.

§ 8. *Error in an arbitrary function.* The method of § 4 may also be used to determine the probability distribution of error in an arbitrary function F (instead of the determinant Δ) of the quantities a_i. Equation (1) is then formally the same, but, since $F_{ii} = \partial^2 F / \partial a_i^2 \neq 0$, it includes terms with higher powers of the errors e_i. The following are the corresponding results, suffixes of F indicating partial derivatives.

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If the given laws of distribution are *symmetrical*,

$$M_1 = \tfrac{1}{2}\Sigma F_{ii}\, m_{2i} + \frac{1}{24}\Sigma F_{iiii}\, m_{4i} + \tfrac{1}{4}\Sigma F_{iijj}\, m_{2i}\, m_{2j} + \ldots ;$$

$$M_2 = \Sigma F_i^2 m_{2i} + \tfrac{1}{4}\Sigma F_{ii}^2\, m_{4i} + \Sigma (F_{ii}\, F_{jj} + F_{ij}^2)\, m_{2i}\, m_{2j} + \ldots ;$$

$$M_3 = \tfrac{3}{2}\Sigma F_i^2 F_{ii}\, m_{4i} + (\tfrac{3}{2}\Sigma F_i^2 F_{jj} + 6\,\Sigma F_i\, F_j\, F_{ij})\, m_{2i}\, m_{2j} + \ldots ;$$

$$M_4 = \Sigma F_i^4\, m_{4i} + 6\,\Sigma F_i^2 F_j^2 m_{2i}\, m_{2j} + \ldots \qquad (i \neq j).$$

If the given laws of distribution are *normal*, the required function will be of the form

$$P(E) = \left[1 + A\frac{d^3}{dE^3} + B\frac{d^4}{dE^4} + \ldots\right]\frac{1}{S\sqrt{2\pi}}\, e^{-(E-a)^2/2S^2}.$$

The constants are given by

$$a = M_1,\ S^2 = M_2 - M_1^2,\ A = -\frac{M_3}{6} + \frac{M_1 M_2}{2} - \frac{M_1^3}{3},\ B = \frac{M_4}{24} + \frac{M_1^4}{12} - \frac{M_2^2}{8} + M_1 A.$$

To a first approximation

$$a = \tfrac{1}{2}\Sigma F_{ii}\, \sigma_i^2;\quad S^2 = \Sigma F_i^2\, \sigma_i^2;\quad A, B, \ldots = 0.$$

To a second approximation

$$a = \tfrac{1}{2}\Sigma F_{ii}\, \sigma_i^2 + \tfrac{1}{8}\Sigma F_{iiii}\, \sigma_i^4 + \tfrac{1}{4}\Sigma F_{iijj}\, \sigma_i^2\, \sigma_j^2;$$

$$S^2 = \Sigma F_i^2 \sigma_i^2 + \tfrac{1}{2}\Sigma F_{ii}^2\, \sigma_i^4 + \Sigma F_{ij}^2\, \sigma_i^2 \sigma_j^2;$$

$$A = -\tfrac{1}{2}\Sigma F_{ii}\, F_i^2 \sigma_i^4 - \Sigma F_{ij}\, F_i\, F_j\, \sigma_i^2\, \sigma_j^2;$$

$$B, \ldots = 0.$$

If the arguments of F have been calculated correct to x decimal places, the last figure being forced, then the second approximation is given by:

$$a = \frac{\Sigma F_{ii}}{24\,.\,10^{2x}} + \frac{\Sigma F_{iiii}}{1920\,.\,10^{4x}} + \frac{\Sigma F_{iijj}}{576\,.\,10^{4x}};$$

$$S^2 = \frac{\Sigma F_i^2}{12\,.\,10^{2x}} + \frac{\Sigma F_{ii}^2}{720\,.\,10^{4x}} + \frac{\Sigma F_{ii}\, F_{jj}}{288\,.\,10^{4x}} + \frac{\Sigma F_{ij}^2}{144\,.\,10^{4x}};$$

$$A = -\frac{\Sigma F_{ii}\, F_i^2}{720\,.\,10^{4x}} - \frac{\Sigma F_{ij}\, F_i\, F_j}{144\,.\,10^{4x}};$$

$$B = -\frac{\Sigma F_i^4}{2880\,.\,10^{4x}}.$$