J. Aust. Math. Soc. **90** (2011), 403–430
doi:10.1017/S1446788711001327

IRREDUCIBLE REPRESENTATIONS OF THE HAMILTONIAN ALGEBRA $H(2r; \mathbf{n})$

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(Received 19 December 2009; accepted 12 June 2011)

Communicated by J. Du

Dedicated to the memory of Professor Guang-Yu Shen with deep respect and admiration

Abstract

Let $L = H(2r; \mathbf{n})$ be a graded Lie algebra of Hamiltonian type in the Cartan type series over an algebraically closed field of characteristic $p > 2$. In the generalized restricted Lie algebra setup, any irreducible representation of L corresponds uniquely to a (generalized) p-character χ. When the height of χ is no more than $\min\{p^{n_i} - p^{n_i-1} \mid i = 1, 2, \ldots, 2r\} - 2$, the corresponding irreducible representations are proved to be induced from irreducible representations of the distinguished maximal subalgebra L_0 with the aid of an analogy of Skryabin's category $\mathfrak{C}$ for the generalized Jacobson–Witt algebras and modulo finitely many exceptional cases. Since the exceptional simple modules have been classified, we can then give a full description of the irreducible representations with p-characters of height below this number.

2010 *Mathematics subject classification*: primary 17B10; secondary 17B50, 17B70.

Keywords and phrases: Cartan type Lie algebras, generalized restricted Lie algebras, Hamiltonian algebras, $\mathfrak{C}$-category, exceptional modules.

1. Introduction

In the classification of modular simple Lie algebras there are a variety of Lie algebras of so-called Cartan type as well as classical Lie algebras arising from simple algebraic groups. The simple Lie algebras of Cartan type fall into four classes: types W, S, H and K (see [22]). They are subalgebras of the derivation algebra of the divided power algebra $R = \mathfrak{A}(m; \mathbf{n})$. Here the m-tuple $\mathbf{n}$ of positive integers is an ordered sequence of divided-power exponents $(n_1, \ldots, n_m)$.

The history of the study of representations for Cartan type Lie algebras is a long one. We can trace its beginnings back to the early 1940s when Chang studied representations of the Witt algebra $W(1, 1)$ (see [1]). In the 1980s Shen systematically

This work is partially supported by the NSF of China (No. 10871067), the PCSIRT of China and the Science and Technology Program of Shanghai Maritime University (No. 20110053).

studied graded representations of the Lie algebras of Cartan type (see [13–15]). Shen completely determined the graded simple modules of the so-called exceptional-weight modules and proved that all graded nonexceptional-weight modules are induced modules (see [15]). The results for restricted simple modules were obtained by Nakano [10]. Any simple module of a restricted Cartan type Lie algebra L can be attached to a linear function $\chi \in L^*$ and thereby a height of χ in connection with the filtered structure. Holmes and Zhang completed the work for simple modules of L when the height of χ is not greater than 1. This work follows lines similar to Shen's work on graded modules (see [3, 4, 25]). Furthermore, Zhang and Steffensen studied irreducible modules of L and the rank-two Witt algebra $W(2, \mathbf{1})$ for general χ which are either nonsingular or 'nice', respectively (see [6, 26]).

The second author of this paper found the generalized restricted Lie algebra structure for a Lie algebra of Cartan type L (see [16]). This structure enables one to study the representations of the Lie algebra of Cartan type L by following a program very similar to that for working with restricted Lie algebras. In particular, any simple module of L has a unique generalized p-character χ with a height $\mathrm{ht}(\chi)$ which is an invariant under co-adjoint action of $\mathrm{Aut}(L)$ (see Section 2.3). In such a setting, Shen's simple graded modules are just modules of generalized p-character χ satisfying $\mathrm{ht}(\chi) \leq 1$ and $\chi(L_{[i]}) = 0$ for all $i \neq 0$.

In a generalization of Shen's work, Skryabin studied representations of L more conceptually in [18]. Shen's mixed product combining two modules of R and L is extended to be a so-called (R, L)-module structure in the more general setting of commutative algebras and their differential systems. In his $\mathfrak{C}$-module category, Skryabin proved results parallel to those for simple modules by Shen, Nakano, and Holmes and Zhang with respect to characters with height much greater than 1. A similar argument for (R, L)-modules was given in [11, Section 3.3].

Skryabin's $\mathfrak{C}$-module category has been extended to the case of special Lie algebras of Cartan type by the authors (see [24]). This paper is a continuation of our previous work (see [17, 24]). Recall that Skryabin first introduced the category $\mathfrak{C}$ for the generalized Jacobson–Witt algebra $W(m; \mathbf{n})$ in [18]. Recall that $W(m; \mathbf{n})_0$ consists of 'differential operators' of degree equal to or greater than zero, that is, of the form $\sum_{i=1}^{m} f_i D_i$ with f_i having no constants for $i = 1, \ldots, m$.

In the generalized restricted Lie algebra setup, the 'modified' induced modules for $W(m; \mathbf{n})$ (induced from 'twist' modules of the distinguished maximal subalgebra $W(m; \mathbf{n})_0$) turn out to be objects of the category $\mathfrak{C}$ (see [17]). The category $\mathfrak{C}$ is described based on the understanding that Cartan type Lie algebras are Lie algebras of differential operators on the divided power algebras $\mathfrak{A}(m; \mathbf{n})$. The representations of $W(m; \mathbf{n})$ certainly reflect the connections between the representations of both $W(m; \mathbf{n})$ and $\mathfrak{A}(m; \mathbf{n})$. Furthermore, the induced modules arising from $W(m; \mathbf{n})_0$-modules additionally reflect a close connection between the representations of $W(m; \mathbf{n})_0$ and the representations of the pair $(W(m; \mathbf{n}), \mathfrak{A}(m; \mathbf{n}))$.

Such a connection should exist for all series of simple Lie algebras of Cartan types W, S, H and K. We have successfully worked with the special series $S(m; \mathbf{n})$, by constructing a category with such a 'connection' (see [24]). An idealistic continuation of this work is to find a unified way of defining the 'connection' for all four series of Cartan type Lie algebras. Unfortunately, we have been unable to define such a connection. Indeed, the structure given in this paper does not work for the contact Lie algebra $K(m;\mathbf{n})$ because the canonical graded structure of $K(m;\mathbf{n})$ does not come from the gradation of $\mathfrak{A}(m;\mathbf{n})$. This is a distinguishing feature from the other three cases.

In this paper we construct a counterpart 'connection' in the case of the Hamiltonian algebra $L = H(2r;\mathbf{n})$ in order to study its representations. This algebra consists of differential operators D on the divided power algebra $\mathfrak{A}(2r;\mathbf{n})$ such that $D\omega_H = 0$. Here ω_H is the Hamiltonian differential form (see [9]). Let $L_0 = L \cap W(2r;\mathbf{n})_0$ be the distinguished maximal subalgebra of L and let $R = \mathfrak{A}(2r;\mathbf{n})$. In the generalized restricted Lie algebra setup we can naturally construct induced L-modules from irreducible L_0-modules. Using these constructions, we prove that the induced modules admit an 'admissible' structure involving the representations of L, L_0 and R. The 'admissible' structure enables us to prove that all irreducible L-modules with p-characters of height no more than

$$\min\{p^{n_i} - p^{n_i-1} \mid i = 1, 2, \ldots, 2r\} - 2$$

are induced from irreducible L_0-modules in the so-called nonexceptional cases. The irreducible L_0-modules for the exceptional cases have been described by Shen [15], Holmes [2], and Pu and Jiang [12].

The irreducible modules for the rank-one Hamiltonian algebra $H(2;\mathbf{1})$ were classified by Koreshkov in [8] using a technical computation. Koreshkov's result for the irreducible modules of $H(2;\mathbf{1})$ is more general than the one we give in this paper. However, it seems difficult to generalize his results to general Hamiltonian algebras. In [19] Skryabin extensively studied representations of the restricted Poisson algebra which is a central extension of the restricted Hamiltonian algebra. His work follows a similar approach to that taken in the work of Premet and himself for the Lie algebras of reductive algebraic groups (see [11]). The results of [19] can be applied to estimate dimensions of some irreducible representations of the restricted Lie algebras of Hamiltonian type (see Proposition 4.15).

2. Preliminaries

In this paper we always assume that the ground field F is algebraically closed and of prime characteristic $p > 2$. We let $\mathbb{Z}_{>0}$ (respectively, $\mathbb{Z}_{\geq 0}$) denote the set of all positive (respectively, nonnegative) integers. We fix a positive integer m and an m-tuple $\mathbf{n} = (n_1, n_2, \ldots, n_m) \in \mathbb{Z}^m_{>0}$. All modules (vector spaces) are taken over F and are assumed to be finite-dimensional.

We define

$$A(m;\mathbf{n}) := \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \mid \alpha_i \in \mathbb{Z}_{\geq 0}, \alpha_i < p^{n_i}, \forall i = 1, 2, \ldots, m\}$$

and set

$$\tau = (p^{n_1} - 1, p^{n_2} - 1, \ldots, p^{n_m} - 1).$$

https://doi.org/10.1017/S1446788711001327 Published online by Cambridge University Press

There are natural partial orders '$\preceq$' and '$\prec$' on $A(m; \mathbf{n})$ defined as follows.

(i) We say that $\alpha \preceq \beta$, $\alpha, \beta \in A(m; \mathbf{n})$ if $\alpha_i \leq \beta_i$ for all $i = 1, 2, \ldots, m$.

(ii) We say that $\alpha \prec \beta$ if $\alpha \preceq \beta$ and $\alpha \neq \beta$.

Using this notation, we can rewrite $A(m; \mathbf{n})$ as

$$A(m; \mathbf{n}) = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \mid 0 \preceq \alpha \preceq \tau\}.$$

For brevity we write $\varepsilon_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{im})$.

We use the following componentwise operations in $A(m; \mathbf{n})$. For any elements $\alpha, \beta \in A(m; \mathbf{n})$ we define

$$\alpha \pm \beta := (\alpha_1 \pm \beta_1, \alpha_2 \pm \beta_2, \ldots, \alpha_m \pm \beta_m),$$

$$\alpha! := \prod_{i=1}^{m} \alpha_i!,$$

$$\binom{\alpha}{\beta} := \prod_{i=1}^{m} \binom{\alpha_i}{\beta_i}$$

and

$$|\alpha| := \sum_{i=1}^{m} \alpha_i.$$

2.1. The generalized Jacobson–Witt algebra $W(m; \mathbf{n})$. Let $\mathfrak{A}(m; \mathbf{n})$ denote the divided power algebra which is an F-algebra with an F-basis $\{x^{\alpha} \mid \alpha \in A(m; \mathbf{n})\}$ and multiplication subject to the following rule:

$$x^{\alpha} x^{\beta} = \binom{\alpha + \beta}{\alpha} x^{\alpha+\beta} \quad \forall \alpha, \beta \in A(m; \mathbf{n})$$

with the convention that $x^{(\gamma)} = 0$ if $\gamma \notin A(m; \mathbf{n})$.

For any $i \in \mathbb{Z}_{\geq 0}$ define

$$\mathfrak{A}(m; \mathbf{n})_{[i]} := F\text{-span}\{x^{\alpha} \mid |\alpha| = i\}.$$

Then we have that

$$\mathfrak{A}(m; \mathbf{n}) = \bigoplus_{i=0}^{s} \mathfrak{A}(m; \mathbf{n})_{[i]}$$

which is a natural gradation of $\mathfrak{A}(m; \mathbf{n})$. Here $s = \sum_{i=1}^{m} (p^{n_i} - 1)$. We also write

$$\mathfrak{A}(m; \mathbf{n})_i := \bigoplus_{j \geq i} \mathfrak{A}(m; \mathbf{n})_{[j]}.$$

Then

$$\mathfrak{A}(m; \mathbf{n}) = \mathfrak{A}(m; \mathbf{n})_0 \supseteq \mathfrak{A}(m; \mathbf{n})_1 \supseteq \cdots$$

is the natural filtration associated to the natural gradation given above.

For $1 \leq i \leq m$, let D_i denote the special derivation of $\mathfrak{A}(m; \mathbf{n})$ which satisfies the condition that $D_i(x^{\alpha}) = x^{\alpha - \varepsilon_i}$ for all $\alpha \in A(m; \mathbf{n})$. By definition the generalized

Jacobson–Witt algebra is defined by

$$W(m; \mathbf{n}) = F\text{-span}\{x^\alpha D_i \mid \alpha \in A(m; \mathbf{n}), i = 1, 2, \ldots, m\}$$

and endowed with the Lie bracket satisfying

$$[x^\alpha D_i, x^\beta D_j] = \binom{\alpha + \beta - \varepsilon_i}{\alpha} D_j - \binom{\alpha + \beta - \varepsilon_j}{\beta} D_i$$

for any $\alpha, \beta \in A(m; \mathbf{n})$ and $i, j = 1, 2, \ldots, m$.

Note that all D_i, for $i = 1, \ldots, m$, are mutually commutative. Associated with an element $\alpha \in A(m; \mathbf{n})$ we have a linear operator $D^\alpha := \prod_{i=1}^m D_i^{\alpha_i}$ on $\mathfrak{A}(m; \mathbf{n})$.

For any $i \geq -1$ we define

$$W(m; \mathbf{n})_{[i]} := F\text{-span}\{x^\alpha D_j \mid |\alpha| = i + 1, j = 1, 2, \ldots, m\}.$$

Then

$$W(m; \mathbf{n}) = \bigoplus_{i=-1}^{s-1} W(m; \mathbf{n})_{[i]}$$

is a gradation of $W(m; \mathbf{n})$. Here $s = \sum_{j=1}^m (p^{n_j} - 1)$. Associated with the gradation we have a filtration

$$W(m; \mathbf{n}) = W(m; \mathbf{n})_{-1} \supseteq W(m; \mathbf{n})_0 \supseteq \cdots$$

where $W(m; \mathbf{n})_i := \bigoplus_{j \geq i} W(m; \mathbf{n})_{[j]}$. By [20, Section 4.2], $W(m; \mathbf{n})$ is restricted if and only if $\mathbf{n} = (1, 1, \ldots, 1)$.

2.2. The Hamiltonian algebra $L = H(2r; \mathbf{n})$. Recall that the Hamiltonian algebra $L = H(2r; \mathbf{n})$ is defined to be

$$L = \{D \in W(2r; \mathbf{n}) \mid D\omega_H = 0\}$$

where $\omega_H = \sum_{i=1}^r dx_i \wedge dx_{i+r}$. For the details we refer the interested reader to [9, 20]. This algebra may be described using a linear operator $D_H : \mathfrak{A}(2r; \mathbf{n}) \to W(2r; \mathbf{n})$ which is defined by $x^\alpha \mapsto \sum_{i=1}^{2r} \sigma(i) D_i(x^\alpha) D_{i'}$ with the Lie bracket formula satisfying

$$[D_H(x^\alpha), D_H(x^\beta)] = D_H(D_H(x^\alpha)(x^\beta)) \quad \forall 0 \prec \alpha, \beta \prec \tau.$$

Here we have

$$\sigma(i) := \begin{cases} 1 & \text{if } 1 \leq i \leq r, \\ -1 & \text{if } r + 1 \leq i \leq 2r \end{cases}$$

and

$$i' := \begin{cases} i + r & \text{if } 1 \leq i \leq r, \\ i - r & \text{if } r + 1 \leq i \leq 2r. \end{cases}$$

Thus

$$L = F\text{-span}\{D_H(x^\alpha) \mid 0 \prec \alpha \prec \tau\}$$

(see [20] for the details). Moreover, L is a simple Lie algebra and, furthermore, it is restricted if and only if $\mathbf{n} = (1, 1, \ldots, 1)$. The following facts about $L = H(2r; \mathbf{n})$ are easy to establish.

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(1) There is a natural gradation of L which inherits the gradation of $W(2r; \mathbf{n})$. That is, $L = \bigoplus_{i=-1}^{s-2} L_{[i]}$ where $L_{[i]} = L \cap W(2r; \mathbf{n})_{[i]}$ and $s = \sum_{i=1}^{2r}(p^{n_i} - 1)$.

(2) In the above graded structure of L we have $L_{[0]} \simeq \mathfrak{sp}(2r)$ under the map $\varphi : L_{[0]} \to \mathfrak{sp}(2r)$ with $D_H(x^{2\varepsilon_i}) \mapsto \sigma(i)E_{ii'}$ and

$$D_H(x^{\varepsilon_i+\varepsilon_j}) \mapsto \sigma(j)E_{ij'} + \sigma(i)E_{ji'}$$

for $1 \le i, j \le 2r, i \ne j$.

(3) Associated with this gradation, there is a filtration

$$H(2r; \mathbf{n}) = H(2r; \mathbf{n})_{-1} \supseteq H(2r; \mathbf{n})_0 \supseteq \cdots .$$

Here

$$H(2r; \mathbf{n})_i = H(2r; \mathbf{n}) \cap W(2r; \mathbf{n})_i.$$

According to results of Block and Wilson (see [21]), this filtration is invariant under the action of the automorphism group $\mathrm{Aut}(L)$.

2.3. Generalized restricted Lie algebras and generalized restricted (χ-reduced) representations. It is well known that not all Cartan type Lie algebras are restricted Lie algebras but that these algebras are generalized restricted Lie algebras in the following sense (see [16]).

DEFINITION 2.1. A generalized restricted Lie algebra L over F is a Lie algebra associated with an ordered basis $E = (e_i)|_{i\in I}$ and a mapping $\varphi_{\mathbf{s}} : E \to L$ sending $e_i \mapsto e_i^{\varphi_{\mathbf{s}}}$. Here $\mathbf{s} = (s_i)|_{i\in I}$ where $s_i \in \mathbb{Z}_{>0}$ satisfies the condition that $\mathrm{ad}\, e_i^{\varphi_{\mathbf{s}}} = (\mathrm{ad}\, e_i)^{p^{s_i}}$ for all $i \in I$.

The algebra $H(2r; \mathbf{n})_0$ is restricted under the mapping $D \longmapsto D^{[p]}$. Here $D^{[p]}$ is the usual pth power of the derivation D. So $\mathrm{ad}\, x^{[p]} = (\mathrm{ad}\, x)^p$ for any $x \in H(2r; \mathbf{n})_0$, and this is, in particular, true for any element x taken from a fixed basis E_1 of $H(2r; \mathbf{n})_0$. Set $E = E_1 \cup \{D_1, D_2, \ldots, D_{2r}\}$. Then E is a basis of $H(2r; \mathbf{n})$. After rearrangement, we may assume that $E = (e_i)|_{i=1}^{t}$ is such that $e_i = D_i$, $i = 1, 2, \ldots, 2r$, and $e_j \in E_1$ for $j > 2r$. Here $t = \dim H(2r; \mathbf{n})$ which is equal to $p^{\sum n_i} - 2$. Set $\mathbf{s} = (n_1, n_2, \ldots, n_m, 1, 1, \ldots, 1)$ and define a map $\varphi_{\mathbf{s}} : E \to H(2r; \mathbf{n})$ sending $e_i \mapsto 0$ for $1 \le i \le 2r$ and $e_j \mapsto e_j^{[p]}$ for $j > 2r$. It is then obvious that the condition $\mathrm{ad}\, e_i^{\varphi_{\mathbf{s}}} = (\mathrm{ad}\, e_i)^{p^{s_i}}$ is satisfied for all $i = 1, 2, \ldots, t$. So $H(2r; \mathbf{n})$ is a generalized restricted Lie algebra in the sense of Definition 2.1.

Schur's lemma implies the following fact for a generalized restricted Lie algebra over F.

PROPOSITION 2.2. *Let $(L, \varphi_{\mathbf{s}})$ be a generalized restricted Lie algebra over F associated with a basis $E = (e_i)|_{i\in I}$ and $\varphi_{\mathbf{s}}$ (called the generalized restricted mapping associated with the basis E) where $\mathbf{s} = (s_i)|_{i\in I}$ with $s_i \in \mathbb{Z}_{>0}$ for all $i \in I$. Suppose that (V, ρ) is an irreducible representation of L. Then there exists a unique $\chi \in L^*$ such that*

$$\rho(e_i)^{p^{s_i}} - \rho(e_i^{\varphi_{\mathbf{s}}}) = \chi(e_i)^{p^{s_i}}\mathrm{id}_V \quad \forall e_i \in E. \tag{2.1}$$

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DEFINITION 2.3. The function χ defined above is called a (generalized) p-character of V. A representation (module) of L satisfying (2.1) is called a generalized χ-reduced representation (module). In particular, when $\chi = 0$, such a representation is called a generalized restricted representation (module) of L.

Now suppose that $(L, \varphi_{\mathbf{s}})$ is a generalized restricted Lie algebra associated with a basis $E = (e_i)|_{i\in I}$ and $\varphi_{\mathbf{s}}$ where $\mathbf{s} = (s_i)|_{i\in I}$ satisfies $s_i \in \mathbb{Z}_{>0}$ for all $i \in I$. For any $\chi \in L^*$, define

$$U_{p^{\mathbf{s}}}(L, \chi) := U(L)/(e_i^{p^{s_i}} - e_i^{\varphi_{\mathbf{s}}} - \chi(e_i)^{p^{s_i}} \mid e_i \in E).$$

Here

$$(e_i^{p^{s_i}} - e_i^{\varphi_{\mathbf{s}}} - \chi(e_i)^{p^{s_i}} \mid e_i \in E)$$

denotes the ideal in $U(L)$ generated by the central elements $e_i^{p^{s_i}} - e_i^{\varphi_{\mathbf{s}}} - \chi(e_i)^{p^{s_i}}$ for all $e_i \in E$. The algebra $U_{p^{\mathbf{s}}}(L, \chi)$ is called the generalized χ-reduced enveloping algebra of L. When $\chi = 0$, the algebra $U_{p^{\mathbf{s}}}(L, 0)$ is often called the generalized restricted enveloping algebra of L and is simply denoted by $U_{p^{\mathbf{s}}}(L)$. We have category equivalence between the generalized χ-reduced (respectively, generalized restricted) module category of L and the $U_{p^{\mathbf{s}}}(L, \chi)$ (respectively, $U_{p^{\mathbf{s}}}(L)$)-module category (see [16]).

REMARK 2.4.

(1) A restricted Lie algebra $(g, [p])$ is a generalized restricted Lie algebra associated with an arbitrary given basis E and $\mathbf{s} = \mathbf{1} := (1, 1, \ldots, 1)$. The generalized restricted mapping $\varphi_{\mathbf{s}}$ is the restriction of the usual restricted mapping $[p]$ on E. Furthermore, in this case, a generalized χ-reduced module (enveloping algebra) coincides with the χ-reduced module (enveloping algebra).

(2) The invariance of the filtration for $L = H(2r; \mathbf{n})$ enables us to define the height of a nonzero $\chi \in L^*$ via

$$\mathrm{ht}(\chi) := \max\{i \mid \chi(L_{i-1}) \neq 0\}$$

and $\mathrm{ht}(0) := -1$. Now the height function on L^* is invariant under the action of $\mathrm{Aut}(L)$ defined by $\sigma \cdot \chi = \chi \circ \sigma^{-1}$ for $\sigma \in \mathrm{Aut}(L)$ and $\chi \in L^*$.

2.4. Independent systems of differential operators. Suppose that $\mathfrak{R}$ is an associative commutative F-algebra with unit. Endow the endomorphism algebra $\mathrm{End}_F\,\mathfrak{R}$ with an $\mathfrak{R}$-module structure by putting

$$(f \cdot \varphi)(g) = f\varphi(g), \quad \forall f, g \in \mathfrak{R}, \varphi \in \mathrm{End}_F\mathfrak{R}.$$

DEFINITION 2.5. A system of endomorphisms $\Phi \subseteq \mathrm{End}_F\mathfrak{R}$ is called independent if $\mathrm{Val}\,\Phi' = \mathfrak{R}^n$ for any finite subset $\Phi' = \{\varphi_1, \varphi_2, \ldots, \varphi_n\} \subseteq \Phi$. Here $\mathrm{Val}\,\Phi'$ denotes the submodule of the free $\mathfrak{R}$-module $\mathfrak{R}^n$ generated by all n-tuples $(\varphi_1(g), \varphi_2(g), \ldots, \varphi_n(g))$ with $g \in \mathfrak{R}$.

PROPOSITION 2.6 (See [18, Proposition 3.5]). *Suppose that*

$$\{\partial_i^{p^{r_i}} \mid 1 \leq i \leq 2r, 0 \leq r_i < n_i\}$$

is an independent system of derivations of $\mathfrak{R}$*. For any given subset* $A \subseteq A(2r; \mathbf{n})$ *and n-tuple* $\gamma \in A$*, there exist a finite number of elements* $f_1, f_2, \ldots, f_u, g_1, g_2, \ldots, g_u \in \mathfrak{R}$ *such that the following condition is satisfied:*

$$\sum_{\nu=1}^{u} f_\nu \partial^\alpha g_\nu = \begin{cases} 1 & \text{if } \alpha = \gamma, \\ 0 & \text{if } \alpha \in A \text{ and } \alpha \neq \gamma. \end{cases} \tag{2.2}$$

Remark 2.7. For $\mathfrak{R} = \mathfrak{A}(2r; \mathbf{n})$, one can easily see that

$$\{D_i^{p^{r_i}} \mid 1 \le i \le 2r, 0 \le r_i < n_i\}$$

is independent in the sense of the Definition 2.5.

2.5. Exceptional modules. We turn to the representations of $L_{[0]}$ which can be identified with $\mathfrak{sp}(2r)$ under φ in Section 2.2(2). We define $h_i := E_{ii} - E_{i+r,i+r}$ for $i = 1, 2, \ldots, r$ and

$$\mathfrak{h} = F\text{-span}\{h_i \mid i = 1, 2, \ldots, r\}.$$

Then $\mathfrak{h}$ is a canonical torus of $\mathfrak{sp}(2r)$. The isoclasses of irreducible restricted representations of $\mathfrak{sp}(2r)$ are parameterized by the set of restricted weights

$$\mathfrak{X}(\mathfrak{h}) := \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i)^p = \lambda(h_i), i = 1, \ldots, m\}.$$

A simple module corresponding to λ is denoted by $L_0(\lambda)$ which is a 'highest weight' module with 'highest weight' λ (see [5]). This implies that $L_0(\lambda)$ is generated by a nonzero vector v satisfying the conditions that $h_i.v = \lambda(h_i)v$ for $i = 1, 2, \ldots, r$ and $\mathcal{N} \cdot v = 0$. Here

$$\mathcal{N} = F\text{-span}\{E_{i,j} - E_{j+r,i+r}, E_{i,j+r} + E_{j,i+r}, E_{k,k+r} \mid 1 \le i < j \le r, 1 \le k \le r\}.$$

Let $\varepsilon_i \in \mathfrak{h}^*$ be such that $\varepsilon_i(h_j) = \delta_{ij}$ for $i, j = 1, 2, \ldots, r$. Define $\omega_0 = 0$ and $\omega_i = \sum_{j=1}^{i} \varepsilon_j$ for $i = 1, 2, \ldots, r$. Then $\omega_0, \omega_1, \ldots, \omega_r$ constitute a system of fundamental weights of $\mathfrak{sp}(2r)$. A simple $\mathfrak{sp}(2r)$-module $L_0(\omega_i)$ corresponding to the fundamental weight ω_i $(0 \le i \le r)$ is usually called exceptional. Similarly, a simple module (ρ_0, V) of L_0 is called exceptional if (ρ_0, V) is isomorphic to some $L_0(\omega_i)$ as an $L_{[0]}$-module with a trivial action for $\rho_0(L_1)$.

Proposition 2.8. *Let* $1 \le s_i \le 2r$ *for* $i = 1, 2, 3, 4$*. Suppose that an irreducible representation* ϱ *of the Lie algebra* $\mathfrak{sp}(2r)$ *in a vector space* W *satisfies the following relation:*

$$\sum_{1 \le s < t \le 2r} \sum_{1 \le u < v \le 2r} \delta_{\{s,t,u,v\}\{s_1,s_2,s_3,s_4\}} (\sigma(s)\varrho(E_{ts'}) + \sigma(t)\varrho(E_{st'}))(\sigma(u)\varrho(E_{vu'})$$
$$+ \sigma(v)\varrho(E_{uv'})) + \sum_{s=1}^{2r} \sum_{u=1}^{2r} \delta_{\{s,s,u,u\}\{s_1,s_2,s_3,s_4\}} \sigma(s)\varrho(E_{ss'})\sigma(u)\varrho(E_{uu'})$$

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$$
\begin{aligned}
&+ \sum_{1\le u<v\le 2r} \sum_{s=1}^{2r} \delta_{\{u,v,s,s\}\{s_1,s_2,s_3,s_4\}} \sigma(s)\varrho(E_{ss'})(\sigma(u)\varrho(E_{vu'}) + \sigma(v)\varrho(E_{uv'})) \\
&+ \sum_{1\le s<t\le 2r} \sum_{u=1}^{2r} \delta_{\{s,t,u,u\}\{s_1,s_2,s_3,s_4\}} (\sigma(s)\varrho(E_{ts'}) + \sigma(t)\varrho(E_{st'}))\sigma(u)\varrho(E_{uu'}) \\
&= 0
\end{aligned}
\tag{2.3}
$$

where

$$
\delta_{\{s,t,u,v\}\{s_1,s_2,s_3,s_4\}} = \begin{cases} 1 & \text{if } \{s,t,u,v\} = \{s_1,s_2,s_3,s_4\}, \\ 0 & \text{if } \{s,t,u,v\} \neq \{s_1,s_2,s_3,s_4\}, \end{cases}
$$

with the convention that $\{a_1, a_2, a_3, a_4\} = \{b_1, b_2, b_3, b_4\}$ *if and only if there exists* $\sigma \in \mathfrak{S}_4$ *such that* $a_i = b_{\sigma(i)}$ *for all* $i = 1, \dots, 4$. *Then* W *is exceptional.*

Proof. Let $a \in \{1, 2, \dots, 2r\}$. If we assume that $s_1 = s_2 = s_3 = s_4 = a$ in (2.3), then we obtain that $\varrho(E_{aa'})^2 = 0$. Now we consider

$$
W_1 = \{w \in W \mid \varrho(E_{i,i+r})w = 0 \text{ for all } i = 1, 2, \dots, r\}.
$$

We have $W_1 \neq 0$ since all the $\varrho(E_{i,i+r})$ are mutually commutative and act nilpotently on W.

Fix $b \in \{1, 2, \dots, r\}$ and $a \in \{r+1, r+2, \dots, 2r\}$ such that $b < a'$. Set $s_1 = s_2 = b$ and $s_3 = s_4 = a'$ in (2.3). We obtain that

$$
\varrho(E_{ba} + E_{a'b'})^2 + \varrho(E_{bb'})\varrho(E_{a'a}) + \varrho(E_{a'a})\varrho(E_{bb'}) = 0. \tag{2.4}
$$

Note that $\varrho(E_{ba} + E_{a'b'})$ commutes with $\varrho(E_{i,,i+r})$ for all $i = 1, 2, \dots, r$ and so W_1 is stable under the action of $\varrho(E_{ba} + E_{a'b'})$. Furthermore, by (2.4), $\varrho(E_{ba} + E_{a'b'})$ acts nilpotently on W_1. Now set

$$
\begin{aligned}
W_2 = \{w \in W_1 \mid \varrho(E_{ba} + E_{a'b'})w = 0, \forall b \in \{1, 2, \dots, r\}, \\
a \in \{r+1, r+2, \dots, 2r\} \text{ and } b < a'\}.
\end{aligned}
$$

Then $W_2 \neq 0$ by Jacobson's theorem about weakly nil closed sets (see [20, Theorem 3.1, Ch. I]).

Using a similar argument, one can check that W_2 is stable under the action of $\varrho(E_{ki} - E_{i+r,k+r})$ for all $k, i \in \{1, 2, \dots, r\}$ and $k < i$. Let $1 \le b < a \le r$ and set $s_1 = s_2 = b$ and $s_3 = s_4 = a'$ in (2.3). Then we obtain that

$$
\varrho(E_{ba} - E_{a'b'})^2 - 2\varrho(E_{a'a})\varrho(E_{bb'}) = 0.
$$

Therefore $\varrho(E_{ba} - E_{a'b'})$ acts nilpotently on W_2. Hence Jacobson's theorem about weakly nil closed sets implies that

$$
W_3 = \{w \in W_2 \mid \varrho(E_{ba} - E_{a'b'})w = 0, \text{ for all } 1 \le b < a \le r\} \neq 0.
$$

Let

$$\mathcal{N} = F\text{-span}\{\{E_{ba} - E_{a'b'} \mid 1 \le b < a \le r\} \cup \{E_{i,i+r} \mid 1 \le i \le r\} \cup \{E_{i,j+r} + E_{j,i+r} \mid 1 \le i < j \le r\}\}.$$

Note that

$$W_3 = \{w \in W \mid \varrho(\mathcal{N})w = 0\}.$$

It is obvious that W_3 is stable under the action of

$$\mathfrak{h} = F\text{-span}\{h_i := E_{ii} - E_{i+r,i+r} \mid 1 \le i \le r\}.$$

So there exists a weight vector w in W_3 such that $\varrho(\mathcal{N})w = 0$ and $\varrho(h_i)w = \lambda_i w$ which is a maximal-weight vector.

Next we fix a maximal-weight vector $w \in W_3$. For $i \in \{1, 2, \ldots, r\}$, setting $s_1 = s_2 = i$ and $s_3 = s_4 = i + r$ in (2.3), we obtain that

$$\varrho(E_{ii} - E_{i+r,i+r})^2 - \varrho(E_{i,i+r})\varrho(E_{i+r,i}) - \varrho(E_{i+r,i})\varrho(E_{i,i+r}) = 0. \tag{2.5}$$

Now both sides of (2.5) act on w and so we obtain that $\lambda_i^2 - \lambda_i = 0$. Therefore $\lambda_i = 1$ or 0.

Let $1 \le i < j \le r$. Set $s_1 = i$, $s_2 = j$, $s_3 = i + r$ and $s_4 = j + r$ in (2.3). Then we obtain

$$\begin{aligned}&\varrho(E_{ii} - E_{i+r,i+r})\varrho(E_{jj} - E_{j+r,j+r}) - \varrho(E_{i,j+r} + E_{j,i+r})\varrho(E_{i+r,j} + E_{j+r,i})\\ &\quad + \varrho(E_{ij} - E_{j+r,i+r})\varrho(E_{ji} - E_{i+r,j+r})\\ &\quad = 0.\end{aligned} \tag{2.6}$$

Both sides of (2.6) act on w and so we obtain

$$\lambda_i\lambda_j - 2\lambda_j = 0. \tag{2.7}$$

Now if $\lambda_i = 0$, then by (2.7) we get $\lambda_j = 0$ for all $j > i$. If all $\lambda_i = 0$, then w is an exceptional-weight vector. Otherwise assume that $i_0 = \max\{i \mid \lambda_i \neq 0\}$. Then we have $\lambda_1 = \lambda_2 = \cdots = \lambda_{i_0} = 1$ and $\lambda_{i_0+1} = \lambda_{i_0+2} = \cdots = \lambda_r = 0$. Thus w is also an exceptional-weight vector. In conclusion, W is exceptional and our proof is complete. □

3. The category $\mathfrak{C}$ for the Hamiltonian algebra $H(2r; \mathbf{n})$

From now on we shall always set $L = H(2r; \mathbf{n})$, $L_0 = H(2r; \mathbf{n})_0$ and $R = \mathfrak{A}(2r; \mathbf{n})$.

3.1. The (R, L)-mod and the category $\mathfrak{C}$. In [18] Skryabin introduced the category $\mathfrak{C}$ for the study of representations of the generalized Jacobson–Witt algebra. In this section we shall extend this category to the Hamiltonian algebra $H(2r; \mathbf{n})$.

DEFINITION 3.1. Let (R, L)-**mod** denote the category whose objects are finite-dimensional vector spaces M endowed with an R-module structure (M, ρ_R), an L-module structure (M, ρ_L), an L_0-module structure (M, ϱ) and which satisfy the following 'connection' property:

(R1) $[\rho_L(D), \rho_R(f)] = \rho_R(Df)$.

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Let $\mathfrak{C}$ denote the subcategory of (R, L)-**mod** consisting of those objects which satisfy the additional conditions:

(R2) $[\varrho(D'), \rho_R(f)] = 0$;
(R3) $[\varrho(D'), \rho_L(D_i)] = 0$;
(R4) $\rho_L(D_H(f)) = \sum_{i=1}^{2r} \sigma(i)\rho_R(D_i(f))\rho_L(D_{i'}) + \sum_{|\beta|\geq 2} \rho_R(D^\beta f) \circ \varrho(D_H(x^\beta))$.

Here $f \in R$, $D \in L$ and $D' \in L_0$ for $i = 1, 2, \ldots, 2r$. The morphisms in the categories (R, L)-**mod** and $\mathfrak{C}$ are the mappings which preserve the corresponding module structures.

The objects in $\mathfrak{C}$ (respectively, (R, L)-**mod**) are often called $\mathfrak{C}$-modules (respectively, (R, L)-modules).

For a given R-module (M, ρ_R) and a given set

$$\Phi = \{\varphi_\alpha \in \mathrm{End}_R(M) \mid \alpha \in A(m; \mathbf{n})\},$$

we put

$$\mathrm{Supp}(\Phi) := \{\alpha \in A(m; \mathbf{n}) \mid \varphi_\alpha \neq 0\}$$

and

$$\deg(\Phi) := \max\{|\alpha| \mid \alpha \in \mathrm{Supp}(\Phi)\}.$$

For $f \in R$ we define

$$\Phi(f) = \sum_{\alpha \in A(m;\mathbf{n})} \rho_R(D^\alpha(f))\varphi_\alpha.$$

The following lemma, which is a special case of [18, Lemma 4.5], will be useful in what follows.

Lemma 3.2 [18, Lemma 4.5]. *Let M and Φ be given as above. Suppose that M' is an F-vector subspace of M which does not contain any nonzero R-submodule of M. Then the R-endomorphisms φ_α are nilpotent for all α with $|\alpha| = \deg(\Phi)$ which satisfy the following conditions with respect to Φ:*

(1) *all endomorphisms φ_α with $|\alpha| = \mathrm{Supp}(\Phi)$ are mutually commuting;*
(2) *M' is stable under all endomorphisms $\Phi(f)$ where $f \in R$.*

3.2. Submodules and homomorphisms in the category $\mathfrak{C}$. According to Remark 2.7,

$$\{D_i^{p^{r_i}} \mid 1 \leq i \leq 2r, 0 \leq r_i < n_i\}$$

is independent. For objects $M, N \in \mathfrak{C}$ and a mapping $\varphi : M \to N$, we let $\Gamma(\varphi)$ denote the graph

$$\{(m, \varphi(m)) \mid m \in M\} \subseteq M \oplus N$$

of φ. Then φ respects any of our three module structures if and only if $\Gamma(\varphi)$ is a submodule of $M \oplus N$ with respect to the corresponding module structure. Thus φ is a morphism in $\mathfrak{C}$ if and only if $\Gamma(\varphi)$ is a submodule of $M \oplus N$. We have the following proposition which describes the submodules and homomorphisms in the category $\mathfrak{C}$.

We use the notation

$$A'(2r;\mathbf{n}) := \{\alpha = (\alpha_1, \alpha_2, \ldots \alpha_{2r}) \in A(m;\mathbf{n}) \mid \alpha_i < p^{n_i} - p^{n_i-1}, \forall i = 1, 2, \ldots, 2r\}.$$

Proposition 3.3.

(i) *Let $M \in \mathfrak{C}$ and assume that*

$$\varrho(D_H(x^\alpha)) = 0 \quad \text{for } \alpha \in A(2r;\mathbf{n})\backslash A'(2r;\mathbf{n}). \tag{3.1}$$

Then any (R, L)-submodule M' of M is a $\mathfrak{C}$-submodule.

(ii) *Let $M, N \in \mathfrak{C}$ and assume that both M and N satisfy Equation (3.1). Then any (R, L)-module homomorphism $\varphi : M \to N$ is a morphism in the category $\mathfrak{C}$.*

Proof. (i) We only need to prove that M' is a $\varrho(L_0)$-submodule. Set

$$A := \{\alpha \in A(2r;\mathbf{n}) \mid |\alpha| \geq 2\}$$

and $\varrho(D_H(x^\alpha)) \neq 0\}$. Let

$$A' := A \cup \{\varepsilon_i \mid i = 1, 2, \ldots, 2r\}.$$

Applying Proposition 2.6 to A' and a fixed element $\gamma \in A$, we can find a finite number of elements $f_\nu, g_\nu \in R$ such that

$$\sum_\nu f_\nu D^\alpha g_\nu = \begin{cases} 1 & \text{if } \alpha = \gamma, \\ 0 & \text{if } \alpha \in A'\backslash\gamma. \end{cases} \tag{3.2}$$

Using the above formula, we obtain the equation

$$\begin{aligned} &\sum_\nu \rho_R(f_\nu)\rho_L(D_H(g_\nu)) \\ &\quad = \sum_\nu \rho_R(f_\nu)\Big(\sum_{i=1}^{2r} \sigma(i)\rho_R(D_i(g_\nu))\rho_L(D_{i'}) + \sum_{|\beta|\geq 2} \rho_R(D^\beta(g_\nu))\varrho(D_H(x^\beta))\Big) \\ &\quad = \sum_{i=1}^{2r} \sigma(i)\rho_R(f_\nu D_i(g_\nu))\rho_L(D_{i'}) + \sum_\nu \sum_{|\beta|\geq 2} \rho_R(f_\nu D^\beta(g_\nu))\varrho(D_H(x^\beta)) \\ &\quad = \varrho(D_H(x^\gamma)). \end{aligned}$$

It follows from the above equation and our assumption on M' that M' is stable under the endomorphism $\sum_\nu \rho_R(f_\nu)\rho_L(D_H(g_\nu))$. Hence M' is stable under $\varrho(D_H(x^\gamma))$ for all $\gamma \in A$. Therefore M' is stable under $\varrho(L_0)$ and M' is a $\mathfrak{C}$-submodule.

(ii) The direct sum $M \oplus N$ is an object of the category $\mathfrak{C}$ satisfying Equation (3.1). The graph $\Gamma(\varphi)$ is an (R, L)-submodule of $M \oplus N$. So by (i), $\Gamma(\varphi)$ is a $\varrho(L_0)$-submodule of $M \oplus N$. Thus φ respects the $\varrho(L_0)$-module structure. Therefore φ is a morphism in the category $\mathfrak{C}$. □

Proposition 3.3 enables us to obtain the main result of this section.

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THEOREM 3.4.

(i) *Let $M \in \mathfrak{C}$. Assume that*

$$M \text{ is a completely reducible } \varrho(L_0)\text{-module with no exceptional irreducible direct summands} \tag{MC1}$$

and that

$$\varrho(D_H(x^\alpha)) = 0 \quad \text{for all } \alpha \in A(m; \mathbf{n}) \backslash A'(m; \mathbf{n}). \tag{MC2}$$

Then any L-submodule M' of M is a $\mathfrak{C}$-submodule.

(ii) *Let M, N be two objects of $\mathfrak{C}$ satisfying conditions (MC1) and (MC2). Then any L-module homomorphism $\varphi : M \to N$ is a morphism in $\mathfrak{C}$.*

PROOF. As we showed in the proof of Proposition 3.3, (ii) is a direct consequence of (i). By Proposition 3.3 we only need to prove that M' is a R-submodule of M. We will make use of the strategy that Skryabin proposed for $W(m; \mathbf{n})$ in [18].

Let

$$P = \{m \in M \mid \rho_R(R)m \subseteq M'\}$$

be the largest R-submodule contained in M' and let $Q = \rho_R(R)M'$ be the smallest R-submodule containing M'. By (R1), P and Q are L-submodules. Hence by Proposition 3.3, P and Q are $\mathfrak{C}$-submodules.

We can consider $Q/P \in \mathfrak{C}$ and its L-submodule M'/P. To begin with, we impose the additional assumption that M' contains no nonzero R-submodule of M and that $\rho_R(R)M' = M$. Then it is sufficient to prove that $M = 0$.

We will seek endomorphisms φ of M lying in the associative algebra generated by the endomorphisms $\varrho(D')$. We assume that $D' \in L_0$ has the property that for any $f \in R$ the endomorphism $\rho_R(f)\varphi$ belongs to the associative subalgebra generated by the endomorphisms $\rho_L(D)$ with $D \in L$. This implies that the L-submodule M' is stable under $\rho_R(f)\varphi$ for any $f \in R$. Hence, it contains the R-submodule $\rho_R(R)\varphi(M')$. By the hypothesis we have $\varphi(M') = 0$. By (R2) in Definition 3.1, we know that φ is an R-module endomorphism and so

$$\varphi(M) = \varphi(\rho_R(R)M') = \rho_R(R)\varphi(M') = 0,$$

which implies that $\varphi = 0$. This gives many relations between the endomorphisms $\varrho(D')$ with $D' \in L_0$. These relations will lead us to the conclusion that $M = 0$.

Now assume that $M \neq 0$. By assumption (MC1), we know that M is not a trivial L_0-module. Thus there is some i for which $\varrho(L_i) \neq 0$. Take

$$l = \max\{i \mid \varrho(L_{i-1}) \neq 0\}.$$

First, consider the case where $l \leq 1$. In this case M is a module of the quotient algebra

$$L_0/L_1 \cong L_{[0]} \cong \mathfrak{sp}(2r).$$

For any $s_1, s_2, s_3, s_4 \in \{1, 2, \ldots, 2r\}$ we may apply Proposition 2.6 to

$$A = \{\alpha \in A(m; \mathbf{n}) \mid |\alpha| \leq 4\}$$

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and

$$\gamma = \varepsilon_{s_1} + \varepsilon_{s_2} + \varepsilon_{s_3} + \varepsilon_{s_4}$$

to find $f_\nu, g_\nu \in R = \mathfrak{A}(m; \mathbf{n})$ such that

$$\sum_\nu f_\nu D^\alpha g_\nu = \begin{cases} 1 & \text{if } \alpha = \varepsilon_{s_1} + \varepsilon_{s_2} + \varepsilon_{s_3} + \varepsilon_{s_4}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

The above formula implies that for any $f \in R$ we have

$$\begin{aligned}
&\sum_\nu \rho_L(D_H(ff_\nu))\rho_L(D_H(g_\nu)) \\
&\quad = \sum_\nu \Big(\sum_{i=1}^{2r} \sigma(i)\rho_R(D_i(ff_\nu))\rho_L(D_{i'}) + \sum_{|\beta|\geq 2} \rho_R(D^\beta(ff_\nu))\varrho(D_H(x^\beta)) \Big) \\
&\qquad \times \Big(\sum_{j=1}^{2r} \sigma(j)\rho_R(D_j(g_\nu))\rho_L(D_{j'}) + \sum_{|\gamma|\geq 2} \rho_R(D^\gamma(g_\nu))\varrho(D_H(x^\gamma)) \Big) \\
&\quad = \sum_\nu \Big(\sum_{i=1}^{2r} \sigma(i)\rho_R(D_i(ff_\nu))\rho_L(D_{i'}) \\
&\qquad + \sum_{1\leq s<t\leq 2r} \rho_R(D_sD_t(ff_\nu))(\sigma(s)\varrho(E_{ts'}) + \sigma(t)\varrho(E_{st'})) \\
&\qquad + \sum_{1\leq s\leq 2r} \rho_R(D_sD_s(ff_\nu))\sigma(s)\varrho(E_{ss'}) \Big) \Big(\sum_{j=1}^{2r} \sigma(j)\rho_R(D_j(g_\nu))\rho_L(D_{j'}) \\
&\qquad + \sum_{1\leq u<v\leq 2r} \rho_R(D_uD_v(g_\nu))(\sigma(u)\varrho(E_{vu'}) + \sigma(v)\varrho(E_{uv'})) \\
&\qquad + \sum_{1\leq u\leq 2r} \rho_R(D_uD_u(g_\nu))\sigma(u)\varrho(E_{uu'}) \Big) \\
&\quad = \rho_R(f) \Big(\sum_\nu \sum_{i=1}^{2r} \sum_{1\leq u<v\leq 2r} \sigma(i)\rho_R(f_\nu D_iD_{i'}D_uD_v(g_\nu))(\sigma(u)\varrho(E_{vu'}) + \sigma(v)\varrho(E_{uv'})) \\
&\qquad + \sum_\nu \sum_{i=1}^{2r} \sum_{u=1}^{2r} \sigma(i)\rho_R(f_\nu D_iD_{i'}D_uD_u(g_\nu))\sigma(u)\varrho(E_{uu'}) \\
&\qquad - \sum_\nu \sum_{j=1}^{2r} \sum_{1\leq s<t\leq 2r} \sigma(j)\rho_R(f_\nu D_jD_{j'}D_sD_t(g_\nu))(\sigma(s)\varrho(E_{ts'}) + \sigma(t)\varrho(E_{st'})) \\
&\qquad + \sum_\nu \sum_{1\leq s<t\leq 2r} \sum_{1\leq u<v\leq 2r} \rho_R(f_\nu D_sD_tD_uD_v(g_\nu))(\sigma(s)\varrho(E_{ts'}) \\
&\qquad + \sigma(t)\varrho(E_{st'}))(\sigma(u)\varrho(E_{vu'}) + \sigma(v)\varrho(E_{uv'}))
\end{aligned}$$

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$$
\begin{aligned}
&+\sum_{v}\sum_{1\le s<t\le 2r}\sum_{u=1}^{2r}\rho_R(f_vD_sD_tD_uD_u(g_v))(\sigma(s)\varrho(E_{ts'})\\
&+\sigma(t)\varrho(E_{st'}))\sigma(u)\varrho(E_{uu'})\\
&-\sum_{v}\sum_{j=1}^{2r}\sum_{s=1}^{2r}\sigma(j)\rho_R(f_vD_jD_{j'}D_sD_s(g_v))\sigma(s)\sigma(E_{ss'})\\
&+\sum_{v}\sum_{1\le u<v\le 2r}\sum_{s=1}^{2r}\rho_R(f_vD_uD_vD_sD_s(g_v))\sigma(s)\varrho(E_{ss'})(\sigma(u)\varrho(E_{vu'})\\
&+\sigma(v)\varrho(E_{uv'}))\\
&+\sum_{v}\sum_{s=1}^{2r}\sum_{u=1}^{2r}\rho_R(f_vD_sD_sD_uD_u(g_v))\sigma(s)\varrho(E_{ss'})\sigma(u)\varrho(E_{uu'})\Big)\\
=\,&\rho_R(f)\phi
\end{aligned}
$$

where

$$
\begin{aligned}
\phi=\sum_{v}\Big(&\sum_{1\le s<t\le 2r}\sum_{1\le u<v\le 2r}\rho_R(f_vD_sD_tD_uD_v(g_v))(\sigma(s)\varrho(E_{ts'})\\
&+\sigma(t)\varrho(E_{st'}))(\sigma(u)\varrho(E_{vu'})+\sigma(v)\varrho(E_{uv'}))\\
&+\sum_{1\le s<t\le 2r}\sum_{u=1}^{2r}\rho_R(f_vD_sD_tD_uD_u(g_v))(\sigma(s)\varrho(E_{ts'})+\sigma(t)\varrho(E_{st'}))\sigma(u)\varrho(E_{uu'})\\
&+\sum_{1\le u<v\le 2r}\sum_{s=1}^{2r}\rho_R(f_vD_uD_vD_sD_s(g_v))\sigma(s)\varrho(E_{ss'})(\sigma(u)\varrho(E_{vu'})+\sigma(v)\varrho(E_{uv'}))\\
&+\sum_{s=1}^{2r}\sum_{u=1}^{2r}\rho_R(f_vD_sD_sD_uD_u(g_v))\sigma(s)\varrho(E_{ss'})\sigma(u)\varrho(E_{uu'})\Big).
\end{aligned}
$$

By the previous analysis, we know that $\phi=0$. Keeping the formula (3.3) in mind, we finally arrive at the situation where (2.3) is satisfied for ϱ. By Proposition 2.8 any simple submodule of M is exceptional. This contradicts our assumption on M. Therefore $l>1$. It follows that $\varrho(L_l)=0$ but $\varrho(L_{l-1})$ is a nonzero abelian ideal of $\varrho(L_0)$. For any $f,f_v,g_v\in R$ we have the following computation:

$$
\begin{aligned}
&\sum_{v}\rho_L(D_H(ff_v))\rho_L(D_H(g_v))\\
&\quad=\sum_{v}\Big(\sum_{i=1}^{2r}\sigma(i)\rho_R(D_i(ff_v))\rho_L(D_{i'})+\sum_{|\beta|\ge 2}\rho_R(D^{\beta}(ff_v))\varrho(D_H(x^{\beta}))\Big)\\
&\qquad\times\Big(\sum_{j=1}^{2r}\sigma(j)\rho_R(D_j(g_v))\rho_L(D_{j'})+\sum_{|\gamma|\ge 2}\rho_R(D^{\gamma}(g_v))\varrho(D_H(x^{\gamma}))\Big)
\end{aligned}
$$

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$$
\begin{aligned}
&= \sum_{\nu} \sum_{i=1}^{2r} \sum_{j=1}^{2r} \sigma(i)\sigma(j)\rho_R(D_i(ff_\nu)D_j(g_\nu))\rho_L(D_{i'})\rho_L(D_{j'}) \\
&\quad + \sum_{\nu} \sum_{i=1}^{2r} \sum_{j=1}^{2r} \sigma(i)\sigma(j)\rho_R(D_i(ff_\nu)D_{i'}D_j(g_\nu))\rho_L(D_{j'}) \\
&\quad + \sum_{\nu} \sum_{i=1}^{2r} \sum_{|\gamma|\geq 2} \sigma(i)\rho_R(D_i(ff_\nu)D^{\gamma}(g_\nu))\rho_L(D_{i'})\varrho(D_H(x^{\gamma})) \\
&\quad + \sum_{\nu} \sum_{i=1}^{2r} \sum_{|\gamma|\geq 2} \sigma(i)\rho_R(D_i(ff_\nu)D^{\gamma+\varepsilon_{i'}}(g_\nu))\varrho(D_H(x^{\gamma})) \\
&\quad + \sum_{\nu} \sum_{|\beta|\geq 2} \sum_{j=1}^{2r} \sigma(j)\rho_R(D^{\beta}(ff_\nu)D_j(g_\nu))\rho_L(D_{j'})\varrho(D_H(x^{\beta})) \\
&\quad + \sum_{\nu} \sum_{|\beta|\geq 2} \sum_{|\gamma|\geq 2} \rho_R(D^{\beta}(ff_\nu)D^{\gamma}(g_\nu))\varrho(D_H(x^{\beta}))\varrho(D_H(x^{\gamma})) \\
&= \sum_{\nu} \sum_{i=1}^{2r} \sum_{j=1}^{2r} \sigma(i)\sigma(j)\rho_R(D_i(ff_\nu D_j(g_\nu)))\rho_L(D_{i'})\rho_L(D_{j'}) \\
&\quad - \sum_{\nu} \sum_{i=1}^{2r} \sum_{j=1}^{2r} \sigma(i)\sigma(j)\rho_R(ff_\nu D_iD_j(g_\nu))\rho_L(D_{i'})\rho_L(D_{j'}) \\
&\quad + \sum_{\nu} \sum_{i=1}^{2r} \sum_{j=1}^{2r} \sigma(i)\sigma(j)\rho_R(D_i(ff_\nu D_{i'}D_j(g_\nu)))\rho_L(D_{j'}) \\
&\quad - \sum_{\nu} \sum_{i=1}^{2r} \sum_{j=1}^{2r} \sigma(i)\sigma(j)\rho_R(ff_\nu D_iD_{i'}D_j(g_\nu))\rho_L(D_{j'}) \\
&\quad + \sum_{\nu} \sum_{i=1}^{2r} \sum_{|\gamma|\geq 2} \sigma(i)\rho_R(D_i(ff_\nu D^{\gamma}(g_\nu)))\rho_L(D_{i'})\varrho(D_H(x^{\gamma})) \\
&\quad - \sum_{\nu} \sum_{i=1}^{2r} \sum_{|\gamma|\geq 2} \sigma(i)\rho_R(ff_\nu D^{\gamma+\varepsilon_i}(g_\nu))\rho_L(D_{i'})\varrho(D_H(x^{\gamma})) \\
&\quad + \sum_{\nu} \sum_{i=1}^{2r} \sum_{|\gamma|\geq 2} \sigma(i)\rho_R(D_i(ff_\nu D^{\gamma+\varepsilon_{i'}}(g_\nu)))\varrho(D_H(x^{\gamma})) \\
&\quad - \sum_{\nu} \sum_{i=1}^{2r} \sum_{|\gamma|\geq 2} \sigma(i)\rho_R(ff_\nu D^{\gamma+\varepsilon_i+\varepsilon_{i'}}(g_\nu))\varrho(D_H(x^{\gamma}))
\end{aligned}
$$

$$+\sum_{\nu}\sum_{j=1}^{2r}\sum_{\substack{|\beta|\geq 2\\ \beta=\beta'+\beta''}}\sigma(j)(-1)^{|\beta''|}\binom{\beta}{\beta'}\rho_R(D^{\beta'}(ff_\nu D^{\beta''+\varepsilon_j}(g_\nu)))\rho_L(D_{j'})\varrho(D_H(x^\beta))$$
$$+\sum_{\nu}\sum_{\substack{|\beta|\geq 2\\ \beta=\beta'+\beta''}}\sum_{|\gamma|\geq 2}(-1)^{|\beta''|}\binom{\beta}{\beta'}\rho_R(D^{\beta'}(ff_\nu D^{\beta''+\gamma}(g_\nu)))\varrho(D_H(x^\beta))\varrho(D_H(x^\gamma)).$$

The final equation in the above computation follows from the formulas

$$D_i(f)g=D_i(fg)-fD_i(g)\quad\forall f,g\in R$$

and

$$D^\alpha(f)g=\sum_{\alpha'+\alpha''=\alpha}(-1)^{|\alpha''|}\binom{\alpha}{\alpha'}D^{\alpha'}(fD^{\alpha''}(g))\quad\forall f,g\in R.$$

Let $\gamma\in A(2r;\mathbf{n})$ be such that $|\gamma|=l+1$. Set $t=l+1$. Then for all $\gamma\in A(2r;\mathbf{n})$ which do not satisfy either of the conditions $\gamma=(p-2)\varepsilon_k$ or $n_k=1$ for some k, we can always choose $\gamma'\in A(2r;\mathbf{n})$ such that $\gamma+\gamma'\in A(2r;\mathbf{n})$, $t'=|\gamma'|\geq 2$ and $\binom{\gamma}{\gamma'}\neq 0$. Thus, by Proposition 2.6, there exist $f_\nu, g_\nu\in R$ satisfying

$$\sum_{\nu}f_\nu D^\alpha g_\nu=\begin{cases}0 & \text{if } \alpha\in A(2r;\mathbf{n}),\ |\alpha|\leq 2t \text{ and } \alpha\neq\gamma+\gamma',\\ 1 & \text{if } \alpha=\gamma+\gamma'.\end{cases}\tag{3.4}$$

It follows that

$$\sum_{\nu}\rho_L(D_H(ff_\nu))\rho_L(D_H(g_\nu))$$
$$=\sum_{\substack{|\alpha|\geq 2\\ \alpha=\alpha'+\alpha''}}\sum_{|\beta|\geq 2}\sum_{\nu}(-1)^{|\alpha''|}\binom{\alpha}{\alpha'}\rho_R(D^{\alpha'}(ff_\nu D^{\alpha''+\beta}(g_\nu)))\varrho(D_H(x^\alpha))\varrho(D_H(x^\beta)).\tag{3.5}$$

The right-hand side of the above equation can be written in the form

$$\sum_{\substack{\alpha'\in A(2r;\mathbf{n})\\ |\alpha'|\leq t-t'}}\rho_R(D^{\alpha'}(f))\psi_{\alpha'}$$

which is denoted by $\Psi(f)$. This is a convention that we set previously for a family of R-endomorphisms

$$\Psi=\{\psi_{\alpha'}\in \mathrm{End}_k(M)\mid \alpha'\in A(2r;\mathbf{n}),|\alpha'|\leq t-t'\}$$

satisfying the condition

$$\psi_{\alpha'}=\sum_{\substack{\alpha=\alpha'+\alpha''\\ |\alpha|=t}}(-1)^{t'}\binom{\alpha}{\alpha'}\varrho(D_H(x^\alpha))\varrho(D_H(x^{\gamma+\gamma'-\alpha''}))\quad\text{for } |\alpha'|=t-t'.\tag{3.6}$$

Here the assertion that $\Psi\subset\mathrm{End}_R(M)$ follows from (R2).

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In the case where $\gamma = (p-2)\varepsilon_k$ and $n_k = 1$ for some k, one can choose $\gamma' = \varepsilon_k$ and $\gamma + \gamma' \in A(2r; \mathbf{n})$ such that (3.4) holds. In this case,

$$\begin{aligned}
&\sum_{\nu} \rho_L(D_H(f f_\nu))\rho_L(D_H(g_\nu)) \\
&\quad = \sum_{\substack{|\alpha|\geq 2 \\ \alpha=\alpha'+\alpha''}} \sum_{|\beta|\geq 2} \sum_{\nu} (-1)^{|\alpha''|}\binom{\alpha}{\alpha'} \rho_R(D^{\alpha'}(f f_\nu D^{\alpha''+\beta}(g_\nu)))\varrho(D_H(x^\alpha))\varrho(D_H(x^\beta)) \\
&\qquad + \sigma(k')\rho_R(D_{k'}(f))\varrho(D_H(x^\gamma)) \\
&\quad = \sum_{|\alpha'|\leq t-1} \rho_R(D^{\alpha'}(f))\psi_{\alpha'} + \sigma(k')\rho_R(D_{k'}(f))\varrho(D_H(x^\gamma)) \\
&\quad \triangleq \sum_{|\alpha'|\leq t-1} \rho_R(D^{\alpha'}(f))\widetilde{\psi}_{\alpha'} \\
&\quad \triangleq \widetilde{\Psi}(f)
\end{aligned}$$

where $\widetilde{\Psi}$ denotes the system of R-endomorphisms

$$\{\widetilde{\psi}_{\alpha'} \in \mathrm{End}_R(M) \mid \alpha' \in A(2r; \mathbf{n}), |\alpha'| \leq t-1\}$$

satisfying

$$\widetilde{\psi}_{\alpha'} = \psi_{\alpha'} = \sum_{\substack{\alpha=\alpha'+\alpha'' \\ |\alpha|=t}} -\binom{\alpha}{\alpha'}\varrho(D_H(x^\alpha))\varrho(D_H(x^{\gamma+\gamma'-\alpha''})) \quad \text{for } |\alpha'| = t-1. \tag{3.7}$$

By our assumption M' is stable under $\sum_\nu \rho_L(D_H(f f_\nu))\rho_L(D_H(g_\nu))$. It follows that the above systems Ψ and $\widetilde{\Psi}$ satisfy the two requirements for Lemma 3.2. Lemma 3.2 now implies that those $\psi_{\alpha'}$s in (3.6) and (3.7) are nilpotent. We may use the same inductive arguments found in the proof of [18, Lemma 4.5] to deduce that the constituent $\varrho(D_H(x^\gamma))$s that appear in some $\psi_{\alpha'}$ for $|\alpha'| = l+1$ are also nilpotent. Hence all $\varrho(D_H(x^\alpha))$s with $|\alpha| = l+1$ are nilpotent. It follows that $\varrho(L_{l-1})|_W = 0$ for any irreducible $\varrho(L_0)$-submodule W of M. The complete reducibility of M as a $\varrho(L_0)$-module implies that $\varrho(L_{l-1}) = 0$. This contradicts our choice of l.

The proof is now complete. □

4. Irreducible representations of the Hamiltonian algebra

4.1. Nonexceptional modules. We use the same notation as we used earlier. In particular, we set

$$R = \mathfrak{A}(m; \mathbf{n}), \qquad L = H(2r; \mathbf{n}).$$

Recall that the height of $\chi \in L^*$ is defined as

$$\mathrm{ht}(\chi) := \max\{i \mid \chi(L_{i-1}) \neq 0\}.$$

This definition is given in Remark 2.4(2) with the convention that $\mathrm{ht}(0) = -1$. Since L_0 is a restricted subalgebra, the Schur lemma implies that any irreducible L_0-module is associated to a unique $\zeta \in L_0^*$. Let (V, ρ_0) be a $\chi|_{L_0}$-reduced representation of L_0 for

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some $\chi \in L^*$. Then we have an induced module

$$\mathcal{V} := \mathbf{Ind}_{U(L_0,\chi)}^{U_{p^{\mathbf{s}}}(L,\chi)} V = U_{p^{\mathbf{s}}}(L,\chi) \otimes_{U(L_0,\chi)} V.$$

Here $\mathbf{s} = (n_1, n_2, \ldots, n_m, 1, 1, \ldots, 1)$ and $U_{p^{\mathbf{s}}}(L,\chi)$ is the generalized χ-reduced enveloping algebra of L (see Section 2.3). In addition, $U(L_0,\chi)$ is the $\chi|_{L_0}$-reduced enveloping algebra of L_0. By the Poincaré–Birkhoff–Witt theorem we have $\mathcal{V} = \sum_\beta F E^\beta \otimes V$ as a vector space. Here $E^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_{2r}^{\alpha_{2r}}$ where $0 \le \alpha_i \le p^{n_i} - 1$ for $1 \le i \le 2r$.

Next we show that $\mathcal{V}$ becomes an object of the category $\mathfrak{C}$ and then apply the results on the category $\mathfrak{C}$ to $\mathcal{V}$. The argument will proceed in steps.

Step 1. The R-module structure ρ_R is defined via

$$\rho_R(x^\alpha) E^\beta \otimes v = (-1)^{|\alpha|} \binom{\beta}{\alpha} E^{\beta-\alpha} \otimes v. \tag{4.1}$$

It is routine to verify that $\mathcal{V}$ is an R-module with the corresponding module structure defined by (4.1).

Step 2. The L-module structure on ρ_L is defined via

$$\begin{aligned} &\rho_L(D_H(x^\alpha)) E^\beta \otimes v \\ &\quad = \sum_{i=1}^{r} (-1)^{|\alpha|-1} \left(\binom{\beta+\varepsilon_{i'}}{\alpha-\varepsilon_i} - \binom{\beta+\varepsilon_i}{\alpha-\varepsilon_{i'}} \right) E^{\beta+\varepsilon_i+\varepsilon_{i'}-\alpha} \otimes v \\ &\qquad + \sum_{\substack{0<\gamma\le\alpha \\ |\gamma|\ge 2}} (-1)^{|\alpha|-|\gamma|} \binom{\beta}{\alpha-\gamma} E^{\beta+\gamma-\alpha} \otimes \rho_0(D_H(x^\gamma)) v. \end{aligned} \tag{4.2}$$

Let **ind** denote the induced representation of L on $\mathcal{V} = \mathbf{Ind}_{U(L_0,\chi)}^{U_{p^{\mathbf{s}}}(L,\chi)} V$. Note that for any $x^\alpha \in \mathfrak{A}(m; \mathbf{n})$ we have $D_H(x^\alpha) = \sum_{i=1}^{r} D_{i'i}(x^\alpha)$. Here, and later on, the divergence map D_{ij} for $1 \le i, j \le 2r$ is defined to be a linear map from the divided power algebra $\mathfrak{A}(2r; \mathbf{n})$ to the generalized Jacobson–Witt algebra $W(2r; \mathbf{n})$ via

$$D_{ij}(x^\alpha) = x^{\alpha-\varepsilon_j} D_i - x^{\alpha-\varepsilon_i} D_j$$

for $\alpha \in A(2r; \mathbf{n})$ (see [20, Section 4.3]).

Remark 4.1. Using the same arguments as in [24, Proposition 5.1], it is easy to see that the action of L on $\mathcal{V}$ defined by (4.2) coincides with **ind**. So $\mathcal{V}$ becomes a generalized χ-reduced L-module with the corresponding L-module structure defined by (4.2).

Step 3. The L_0-module structure on ϱ is defined via

$$\varrho(D') E^\beta \otimes v = E^\beta \otimes \rho_0(D') v. \tag{4.3}$$

It is obvious that $\mathcal{V}$ becomes a $\chi|_{L_0}$-reduced L_0-module with the corresponding module structure defined via (4.3) since (V, ρ_0) is a $\chi|_{L_0}$-reduced representation of L_0.

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In the following theorem we prove that $\mathcal{V}$ is an object of the category $\mathfrak{C}$.

THEOREM 4.2. *$\mathcal{V}$ belongs to the category $\mathfrak{C}$.*

PROOF. We need to check that (R1)–(R4) of Definition 3.1 hold.

(1) For any $\alpha, \beta, \gamma \in A(m; \mathbf{n})$ and $v \in V$,

$$\begin{aligned}
&[\rho_L(D_H(x^\alpha)), \rho_R(x^\beta)](E^\gamma \otimes v)\\
&\quad = \rho_L(D_H(x^\alpha)) \circ \rho_R(x^\beta)(E^\gamma \otimes v) - \rho_R(x^\beta) \circ \rho_L(D_H(x^\alpha))(E^\gamma \otimes v)\\
&\quad = (-1)^{|\beta|}\binom{\gamma}{\beta} D_H(x^\alpha)E^{\gamma-\beta} \otimes v - \rho_R(x^\beta)D_H(x^\alpha)E^\gamma \otimes v\\
&\quad = \sum_{i=1}^{r}(-1)^{|\beta|}\binom{\gamma}{\beta} D_{i'i}(x^\alpha)E^{\gamma-\beta} \otimes v - \sum_{i=1}^{r}\rho_R(x^\beta)D_{i'i}(x^\alpha)E^\gamma \otimes v\\
&\quad = \sum_{i=1}^{r}(-1)^{|\beta|}\binom{\gamma}{\beta} D_{i'i}(x^\alpha)E^{\gamma-\beta} \otimes v - \rho_R(x^\beta)D_{i'i}(x^\alpha)E^\gamma \otimes v\\
&\quad = \sum_{i=1}^{r}\rho_R(D_{i'i}(x^\alpha)(x^\beta))E^\gamma \otimes v\\
&\quad = \rho_R(D_H(x^\alpha)(x^\beta))E^\gamma \otimes v,
\end{aligned}$$

where the fifth identity follows from (1) in the proof of [24, Theorem 5.3]. Therefore

$$[\rho_L(D_H(x^\alpha)), \rho_R(x^\beta)] = \rho_R(D_H(x^\alpha)(x^\beta)).$$

Hence (R1) holds.

(2) For any $\alpha, \beta, \gamma \in A(m; \mathbf{n})$ and $v \in V$,

$$\begin{aligned}
&[\varrho(D_H(x^\alpha)), \rho_R(x^\beta)](E^\beta \otimes v)\\
&\quad = \varrho(D_H(x^\alpha)) \circ \rho_R(x^\beta)(E^\beta \otimes v) - \rho_R(x^\beta) \circ \varrho(D_H(x^\alpha))(E^\beta \otimes v)\\
&\quad = (-1)^{|\beta|}\binom{\gamma}{\beta} E^{\gamma-\beta} \otimes \rho_0(D_H(x^\alpha))v - (-1)^{|\beta|}\binom{\gamma}{\beta} E^{\gamma-\beta} \otimes \rho_0(D_H(x^\alpha))v\\
&\quad = 0.
\end{aligned}$$

Therefore

$$[\varrho(D_H(x^\alpha)), \rho_R(x^\beta)] = 0.$$

Hence (R2) holds.

(3) For any $\alpha, \beta \in A(m; \mathbf{n})$ and $v \in V$ and $D_i \in L_{[-1]}$, $i = 1, 2, \ldots, 2r$,

$$\begin{aligned}
&[\rho_L(D_i), \varrho(D_H(x^\alpha))](E^\beta \otimes v)\\
&\quad = \rho_L(D_i) \circ \varrho(D_H(x^\alpha))(E^\beta \otimes v) - \varrho(D_H(x^\alpha)) \circ \rho_L(D_i)(E^\beta \otimes v)\\
&\quad = E^{\beta+\varepsilon_i} \otimes \rho_0(D_H(x^\alpha))v - E^{\beta+\varepsilon_i} \otimes \rho_0(D_H(x^\alpha))v\\
&\quad = 0.
\end{aligned}$$

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Therefore
$$[\rho_L(D_i), \varrho(D_H(x^\alpha))] = 0.$$
Hence (R3) holds.

(4) For any $\alpha, \beta \in A(m; \mathbf{n})$ and $v \in V$,
$$\begin{aligned}\rho_L(D_H(x^\alpha))(E^\beta \otimes v) \\ &= \sum_{i=1}^{r}(-1)^{|\alpha|-1}\Big[\binom{\beta+\varepsilon_{i'}}{\alpha-\varepsilon_i} - \binom{\beta+\varepsilon_i}{\alpha-\varepsilon_{i'}}\Big]E^{\beta+\varepsilon_i+\varepsilon_{i'}-\alpha} \otimes v \\ &\quad + \sum_{\substack{0<\gamma\le\alpha \\ |\gamma|\ge 2}}(-1)^{|\alpha|-|\gamma|}\binom{\beta}{\alpha-\gamma}E^{\beta+\gamma-\alpha} \otimes \rho_0(D_H(x^\gamma))v,\end{aligned}$$
while
$$\begin{aligned}\Big(\sum_{i=1}^{2r}\sigma(i)\rho_R(D_i(x^\alpha))\rho_L(D_{i'}) + \sum_{|\gamma|\ge 2}\rho_R(x^{\alpha-\gamma})\varrho(D_H(x^\gamma))\Big)(E^\beta \otimes v) \\ &= \sum_{i=1}^{r}(-1)^{|\alpha|-1}\Big[\binom{\beta+\varepsilon_{i'}}{\alpha-\varepsilon_i} - \binom{\beta+\varepsilon_i}{\alpha-\varepsilon_{i'}}\Big]E^{\beta+\varepsilon_i+\varepsilon_{i'}-\alpha} \otimes v \\ &\quad + \sum_{\substack{0<\gamma\le\alpha \\ |\gamma|\ge 2}}(-1)^{|\alpha|-|\gamma|}\binom{\beta}{\alpha-\gamma}E^{\beta+\gamma-\alpha} \otimes \rho_0(D_H(x^\gamma))v.\end{aligned}$$
Therefore
$$\rho_L(D_H(x^\alpha)) = \sum_{i=1}^{2r}\sigma(i)\rho_R(D_i(x^\alpha))\rho_L(D_{i'}) + \sum_{|\gamma|\ge 2}\rho_R(x^{\alpha-\gamma})\varrho(D_H(x^\gamma)).$$
Hence (R4) holds.

Since $\mathcal{V}$ satisfies (1)–(4), it belongs to the category $\mathfrak{C}$. □

As we pointed out previously, we have $L_{[0]} \cong \mathfrak{sp}(2r)$. For $i = 1, 2, \ldots, 2r$ set
$$h_i := -D_H(x^{\varepsilon_i+\varepsilon_{i'}}) = \sigma(i')x^{\varepsilon_{i'}}D_{i'} + \sigma(i)x^{\varepsilon_i}D_i.$$
Then $h_i = h_{i'}$ for all $i = 1, 2, \ldots, 2r$. We continue to use $\mathfrak{h}$ to denote the canonical torus of $L_{[0]}$. We have
$$\mathfrak{h} = F\text{-span}\{h_i \mid i = 1, 2, \ldots, r\}.$$
Let (V, ρ_0) be a representation of L_0 and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2r}) \in F^{2r}$. If $0 \ne v \in V$ satisfies $\rho_0(h_i)v = \lambda_i v$ for $i = 1, 2, \ldots r$, then v is called a weight vector of weight λ. If, in addition, $\rho_0(\mathcal{N} + L_1)v = 0$ where
$$\mathcal{N} = F\text{-span}\{D_H(x^{\varepsilon_i+\varepsilon_{j'}}), D_H(x^{\varepsilon_i+\varepsilon_j}), D_H(2x^{\varepsilon_k}) \mid 1 \le i < j \le r, 1 \le k \le r\},$$
then v is called a maximal-weight vector of weight λ.

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We choose $\varepsilon_i \in \mathfrak{h}^*$ such that $\varepsilon_i(h_j) = \delta_{ij}$ for $i, j = 1, 2, \ldots, r$. We let $\omega_0 = 0$ and $\omega_i = \sum_{j=1}^{i} \varepsilon_j$ for $i = 1, 2, \ldots, r$. We have the following result, which is a corollary to Theorems 3.4 and 4.2.

Theorem 4.3. *Let $\chi \in L^*$ satisfy the condition that*

$$\mathrm{ht}(\chi) \le \min\{p^{n_i} - p^{n_i-1} \mid 1 \le i \le 2r\} - 2.$$

If V is an irreducible L_0-module with character χ and V is not exceptional, then $(\mathcal{V}, \rho_L)$ is an irreducible L-module.

Proof. Set $R = \mathfrak{A}(2r; \mathbf{n})$ and $L = H(2r; \mathbf{n})$. By Theorem 4.2, $\mathcal{V}$ belongs to the category $\mathfrak{C}$. Set

$$\mathcal{V}_\theta = F\text{-span}\{E^\theta \otimes v \mid v \in V\}$$

for some $\theta \in A(m; \mathbf{n})$. Then

$$\mathcal{V} = \bigoplus_{\theta \in A(m;\mathbf{n})} \mathcal{V}_\theta$$

and $\mathcal{V}_\theta \cong V$ as $\varrho(L_0)$-modules. Therefore $\mathcal{V}$ is completely reducible as a $\varrho(L_0)$-module and none of its irreducible direct summands are exceptional. This implies that the first condition of Theorem 3.4 is satisfied.

The assumption that

$$\mathrm{ht}(\chi) \le \min\{p^{n_i} - p^{n_i-1} \mid 1 \le i \le 2r\} - 2$$

ensures that the second condition of Theorem 3.4 is satisfied. Therefore, by Theorem 3.4, any L-submodule $\mathcal{V}'$ of $\mathcal{V}$ is also an R-submodule of $\mathcal{V}$.

Suppose now that $\mathcal{V}'$ is an arbitrary nonzero L-submodule of $\mathcal{V}$. Next we shall prove that $\mathcal{V}' = \mathcal{V}$. Suppose that

$$0 \ne v = \sum_{i=1}^{t} E^{\theta_i} \otimes v_i \in \mathcal{V}'$$

where $\theta_i \in A(m; \mathbf{n})$ and $0 \ne v_i \in V$. Define a total order '$\rhd$' on $A(m; \mathbf{n})$ by the lexicographic order, that is,

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \rhd \beta = (\beta_1, \beta_2, \ldots, \beta_m)$$

if and only if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and there exists some $i \in \{1, 2, \ldots, 2r\}$ such that $\alpha_j = \beta_j$ for $j < i$ and $\alpha_i < \beta_i$. Without loss of generality, we may assume that $\theta_1 = \max\{\theta_i \mid i = 1, 2, \ldots, t\}$. Then $\theta_j \rhd \theta_1$ for all $j > 1$. We now have

$$\rho_R(x^{\theta_1})v = (-1)^{|\theta_1|} 1 \otimes v_1 \in \mathcal{V}'.$$

Therefore $\mathcal{V}' = \mathcal{V}$ by the simplicity of V as an L_0-module, and our result is established. □

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Remark 4.4. For $\mathbf{n} = \mathbf{1}$, that is, the restricted case, the result of Theorem 4.3 can be deduced by combining [26, Theorem 2.5, Proposition 2.6]. In this case, the result also coincides with a recent theorem of Wu, Jiang and Pu (see [23, Theorem 1]). In the case of the rank-one Hamiltonian algebra $H(2; \mathbf{1})$, the result of Theorem 4.3 can be obtained from [8] where the author gives a complete determination of the simple modules of $H(2; \mathbf{1})$.

Definition 4.5. An irreducible L-module M is called exceptional if M contains an irreducible exceptional L_0-submodule.

Finally, we may deduce the following theorem from Theorem 4.3.

Theorem 4.6. *Let $\chi \in L^*$ satisfy the condition that*

$$\mathrm{ht}(\chi) \leq \min\{p^{n_i} - p^{n_i - 1} \mid 1 \leq i \leq 2r\} - 2.$$

Suppose that M is an irreducible generalized χ-reduced L-module which is not exceptional. Then all irreducible L_0-submodules of M are isomorphic and M is isomorphic to the induced module from any one of its irreducible L_0-submodules. Furthermore, if N is another nonexceptional irreducible generalized χ-reduced L-module, then $M \cong N$ if and only if all irreducible L_0-submodules of M and N are isomorphic.

4.2. Exceptional modules. In the exceptional case the irreducible modules were described by Shen in [15] and Holmes in [2] for $\chi = 0$ (the height of 0 is defined to be -1). For $\chi \neq 0$ with height 0, they were described by Pu and Jiang in [12].

In this subsection we list some results about the descriptions of exceptional modules for completeness. The detailed arguments are found in [2, 12, 15]. Moreover, we can obtain some more precise descriptions of irreducible representations with character height not larger than 1.

Theorem 4.7 [2, 12, 15]. *Let $L = H(2r; \mathbf{n})$ and let $\chi \in L^*$ be such that $\mathrm{ht}(\chi) \in \{-1, 0\}$. Assume that $p > r$ and let $L^\chi(\omega_i)$ denote an exceptional irreducible L-module with exceptional weight ω_i for $i = 0, 1, \ldots, r$.*

(1) *If $\mathrm{ht}(\chi) = -1$, then*

$$L^\chi(\omega_i) \ncong L^\chi(\omega_j) \quad \text{if } i \neq j$$

and

$$\dim_F L^\chi(\omega_i) = \begin{cases} 1 & \text{if } i = 0, \\ p^{\sum n_i}\left[\binom{2r-2}{i-1} - \binom{2r-2}{i-3}\right] - 2\binom{2r-1}{i-1} & \text{if } 1 \leq i \leq r. \end{cases}$$

(2) *If $\mathrm{ht}(\chi) = 0$, then*

$$L^\chi(\omega_i) \ncong L^\chi(\omega_j), \quad \text{if } i \neq j \text{ and } \{i, j\} \neq \{0, 1\},$$

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while $L^\chi(\omega_0) \cong L^\chi(\omega_1)$ *and*

$$\dim_F L^\chi(\omega_i) = p^{\sum n_i}\left[\binom{2r-1}{i-1} - \binom{2r-1}{i-2}\right], \quad i = 1, \ldots, r.$$

Thus we have the following theorem.

THEOREM 4.8. *Let* $L = H(2r; \mathbf{n})$ *and let* $\chi \in L^*$ *be such that*

$$\mathrm{ht}(\chi) \le \min\{p^{n_i} - p^{n_i - 1} \mid 1 \le i \le 2r\} - 2.$$

(I) *In the case of nonexceptional irreducible L-modules:*

  (1) *all nonexceptional irreducible* $U_{p^s}(L,\chi)$*-modules are induced from any irreducible* $U(L_0,\chi)$*-submodule. Moreover, all irreducible* $U(L_0,\chi)$*-submodules of a nonexceptional irreducible* $U_{p^s}(L,\chi)$*-module are isomorphic.*

  (2) *Let* V, W *be two nonexceptional irreducible* $U_{p^s}(L,\chi)$*-modules and* V_0, W_0 *be any irreducible* $U(L_0,\chi)$*-submodules of* V *and* W*, respectively. Then* $V \cong W$ *if and only if* $V_0 \cong W_0$.

(II) *In the case of exceptional irreducible L-modules we shall assume, further, that* $p > r$.

  (1) *If* $\mathrm{ht}(\chi) = -1$*, then* $L^\chi(\omega_i) \ncong L^\chi(\omega_j)$ *if* $i \ne j$ *and*

$$\dim_F L^\chi(\omega_i) = \begin{cases} 1 & \text{if } i = 0, \\ p^{\sum n_i}\left[\binom{2r-2}{i-1} - \binom{2r-2}{i-3}\right] - 2\binom{2r-1}{i-1} & \text{if } 1 \le i \le r. \end{cases}$$

  (2) *If* $\mathrm{ht}(\chi) = 0$*, then* $L^\chi(\omega_i) \ncong L^\chi(\omega_j)$ *if* $i \ne j$ *and* $\{i, j\} \ne \{0, 1\}$*. However,* $L^\chi(\omega_0) \cong L^\chi(\omega_1)$ *and*

$$\dim_F L^\chi(\omega_i) = p^{\sum n_i}\left[\binom{2r-1}{i-1} - \binom{2r-1}{i-2}\right], \quad i = 1, \ldots, r.$$

Combining Theorems 4.3, 4.6, 4.8 and classical results on restricted irreducible representations of the classical Lie algebra $\mathfrak{sp}(2r)$ (see [7]) gives us the following theorem which describes the isomorphism classes and dimensions of irreducible generalized χ-reduced representations of $L = H(2r; \mathbf{n})$ with $\mathrm{ht}(\chi) = 0$.

THEOREM 4.9. *Let* $L = H(2r; \mathbf{n})$ *and* $\chi \in L^*$ *satisfy* $\mathrm{ht}(\chi) = 0$*. Assume that* $p > r$*. Then the following statements hold.*

(i) *Irreducible* $U_{p^s}(L,\chi)$*-modules are parameterized by 'highest weights'. Up to isomorphism, there are* $p^r - 1$ *distinct irreducible* $U_{p^s}(L,\chi)$*-modules. These modules are represented by* $\{L^\chi(\lambda) \mid \lambda \in \mathbb{F}_p^r \setminus 0\}$.

(ii) *We have* $L^\chi(\lambda) \cong \mathbf{Ind}(L_0(\lambda))$ *if and only if* $\lambda \notin \{\omega_1, \ldots, \omega_r\}$ *and* $L^\chi(\omega_0) \cong L^\chi(\omega_1)$*. Here* $L_0(\lambda)$ *denotes the irreducible restricted* $\mathfrak{sp}(2r)$*-module with 'highest weight'* λ *which can be considered as a restricted irreducible* L_0*-module with trivial* L_1*-actions.*

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(iii) *If λ is not exceptional, then*

$$\dim_F L^{\chi}(\lambda) = p^{\sum n_i} \dim_F L_0(\lambda).$$

In addition,

$$\dim_F L^{\chi}(\omega_i) = p^{\sum n_i}\left[\binom{2r-1}{i-1} - \binom{2r-1}{i-2}\right], \quad i = 1, \ldots, r.$$

We can also give some descriptions of the irreducible representations with character height equal to 1. For this we first note that if $\mathrm{ht}(\chi) = 1$, then $\chi(L_1) = 0$. As L_1 is a p-nilpotent ideal of L_0, L_1 acts trivially on any irreducible $U(L_0, \chi)$-module (see [20, Corollary 3.8, Ch. I]). Therefore the collection of irreducible $U(L_0, \chi)$-modules coincides with the collection of irreducible $U(L_{[0]}, \chi|_{L_{[0]}})(\cong U(\mathfrak{sp}(2r), \chi|_{L_{[0]}}))$-modules. If we combine this observation and Theorem 4.6, then it is easy to obtain the following descriptions of the isomorphism classes and dimensions of irreducible L-modules with character height 1.

THEOREM 4.10. *Let $L = H(2r; \mathbf{n})$ and let $\chi \in L^*$ satisfy $\mathrm{ht}(\chi) = 1$. Suppose that $\{S \mid S \in \mho\}$ is a set of representatives for the isomorphism classes of irreducible $U(L_{[0]}, \chi|_{L_{[0]}}) \cong U(\mathfrak{sp}(2r), \chi|_{L_{[0]}})$-modules. Then the following statements hold.*

(1) *Up to isomorphism there are $|\mho|$ distinct irreducible $U_{p^s}(L, \chi)$-modules. They are represented by $\{L^{\chi}(S) \mid S \in \mho\}$.*
(2) *We have $L^{\chi}(S) \cong \mathbf{Ind}(S)$ for any $S \in \mho$.*
(3) *We have $\dim_F L^{\chi}(S) = p^{\sum n_i} \dim_F S$ for any $S \in \mho$.*

REMARK 4.11. In the case where $\mathbf{n} = \mathbf{1}$, that is, L is restricted, the results of Theorems 4.9 and 4.10 have been obtained in [4, Theorem 4.4] and [25, Lemma 2.2.3, Theorem 2.3.4].

In the final part of this paper we combine the observation that the Poisson algebra is a central extension of the Hamiltonian algebra with a result (see [19, Corollary 5.4]) of Skryabin on representations of the restricted Poisson algebra to estimate the dimensions of some simple modules of the Hamiltonian algebras. In order to do this, we define a truncated polynomial algebra

$$B_{2r} = F[x_1, x_2, \ldots, x_{2r}]/(x_1^p, x_2^p, \ldots, x_{2r}^p)$$

over F. One can define a Poisson bracket on B_{2r} as follows:

$$[f, g] = \sum_{i=1}^{2r} \sigma(i) D_i(f) D_{i'}(g) \quad \forall f, g \in B_{2r}.$$

It is well known that B_{2r} is a restricted Lie algebra with the p-mapping $[p]$ satisfying the condition that

$$(x^{\alpha})^{[p]} = \begin{cases} x^{\alpha} & \text{if } \alpha = \varepsilon_i + \varepsilon_{i+r}, i = 1, 2, \ldots, r, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly B_{2r} has a one-dimensional center generated by 1 which we denote by $\mathbf{F}$. Let $\overline{B}_{2r} = B_{2r}/\mathbf{F}$. For any $x \in B_{2r}$ we also use x to denote the coset of x in $\overline{B}_{2r}$ for brevity. Note that $\overline{B}_{2r} = H \oplus Fx^{\tau}$ as vector spaces, where $\tau = (p-1, p-1, \ldots, p-1)$ and $H = F\text{-span}\{x^{\alpha} \mid \alpha \prec \tau\}$ with $H \cong H(2r; \mathbf{1})$. Furthermore, H is a restricted ideal of $\overline{B}_{2r}$. The following lemma is due to Skryabin.

LEMMA 4.12 [19, Corollary 5.4]. *There exists an open dense subset $U \subset B_{2r}^*$ such that for any $\xi \in U$ all irreducible $U_\xi(B_{2r})$-modules have the same dimension $p^{\frac{1}{2}(p^{2r}-p^r)}$. Moreover, for any $\xi \in U$ with $\xi(1) = 0$, $\mathbf{F}$ acts trivially on any irreducible $U_\xi(B_{2r})$-module. So there is a one-to-one correspondence between the set of irreducible $U_\xi(B_{2r})$-modules and the set of irreducible $U_\xi(\overline{B}_{2r})$-modules.*

REMARK 4.13. The open dense subset U in Lemma 4.12 consists of the so-called 'good' elements of B_{2r}^* in the sense of [19].

For any irreducible H-module V with character χ, one can consider a $\overline{B}_{2r}$-module $U_{\bar{\chi}}(\overline{B}_{2r}) \otimes_{U_\chi(H)} V$ which is a $U_{\bar{\chi}}(\overline{B}_{2r})$-module. Here $\bar{\chi}$ is a trivial extension of χ to $\overline{B}_{2r}^*$, that is, $\bar{\chi}|_H = \chi$ and $\bar{\chi}(x^{\tau}) = 0$.

Consider the restricted Hamiltonian algebra $H(2r; \mathbf{1})$ canonically as a subalgebra of $\overline{B}_{2r}$. Then for any $\chi \in H(2r; \mathbf{1})^*$, one can also consider χ as a linear function on $\overline{B}_{2r}$ with the trivial action on Fx^{τ}, and furthermore as a linear function on B_{2r} with the trivial action on $\mathbf{F}$. When we refer to $\chi \in H(2r; \mathbf{1})^*$ as an element of $\overline{B}_{2r}^*$ or B_{2r}^*, we always obey this convention.

By Lemma 4.12 we immediately have the following proposition for estimating dimensions of irreducible representations of $H(2r; \mathbf{1})$ with 'good' character χ in the sense of the following definition.

DEFINITION 4.14. A character $\chi \in H(2r; \mathbf{1})^*$ is called a 'good' character if we have $\chi \in U$ when χ is referred to as an element of B_{2r}^* in the way stated above.

PROPOSITION 4.15. *Let $\chi \in H(2r; \mathbf{1})^*$ be a 'good' character. Then for any irreducible $U_\chi(H(2r; \mathbf{1}))$-module V we have $\dim_F V \geq p^{\frac{1}{2}(p^{2r}-p^r)-1}$.*

PROOF. Consider the $\overline{B}_{2r}$-module

$$\mathfrak{V} = \mathbf{Ind}_H^{\overline{B}_{2r}} V := U_\chi(\overline{B}_{2r}) \otimes_{U_\chi(H)} V.$$

By Lemma 4.12 we have $\dim_F \mathfrak{V} \geq p^{\frac{1}{2}(p^{2r}-p^r)}$ and our result follows immediately. □

The following example shows that 'good' characters may have very large heights.

EXAMPLE 4.16. Let $r = 1$. Define $\chi \in H(2; \mathbf{1})^*$ such that $\chi(D_H(x^{\alpha})) = \varphi(x_1 x^{\alpha})$. Here

$$\begin{aligned} \varphi : B_2 &\longrightarrow F \\ \textstyle\sum k_\alpha x^{\alpha} &\longmapsto k_\tau \end{aligned} \tag{4.4}$$

Then χ is 'good' in the sense of [19]. So $\chi \in U$. One can easily check that $\mathrm{ht}(\chi) = 2p - 4$ which is the highest possible character height. By Proposition 4.15, we have $\dim_F V \geq p^{\frac{1}{2}(p^2-p)-1}$ for any irreducible $H(2; \mathbf{1})$-module V with character χ. This can also be deduced from [8].

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https://doi.org/10.1017/S1446788711001327 Published online by Cambridge University Press

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