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SOBOLEV SPACES ON LOCALLY COMPACT ABELIAN GROUPS AND THE BOSONIC STRING EQUATION

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Abstract

Motivated by a class of nonlinear nonlocal equations of interest for string theory, we introduce Sobolev spaces on arbitrary locally compact abelian groups and we examine some of their properties. Specifically, we focus on analogs of the Sobolev embedding and Rellich–Kondrachov compactness theorems. As an application, we prove the existence of continuous solutions to a generalized bosonic string equation posed on an arbitrary compact abelian group, and we also remark that our approach allows us to solve very general linear equations in a p-adic context.

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1. Introduction

In his seminal paper [41], Witten introduced the bosonic string action which contains an infinite number of fields and yields—via a formal application of the variational principle—an infinite number of equations in infinitely many variables; see for instance [8, 9] and references therein. From this principle, physicists [8, 9] have extracted nonlocal actions such as

$$\mathcal{L}(\phi) = \phi\Delta e^{-c\Delta}\phi - \mathcal{U}(x,\phi), \quad c > 0. \tag{1.1}$$

The Euler–Lagrange equation arising from (1.1) is

$$\Delta e^{-c\Delta}\phi - U(x,\phi) = 0, \quad c > 0, \tag{1.2}$$

where $U = \partial\mathcal{U}/\partial\phi$. Equation (1.2) is a general type of equation encompassing (in Lorentzian signature) the bosonic string [27] and a simplified case of the supersymmetric string [9].

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Nonlocal equations (or equations in infinitely many derivatives) appear not only in string theory [11, 38, 39, 41], but also as field equations of motion in particle physics [29] and in gravity and cosmology [2, 4, 5, 28]. Thus, it is clearly necessary to understand them in detail. In our recent papers [17–19], we have undertaken the study of nonlinear nonlocal equations such as (1.2) on Euclidean space and also on compact Riemannian manifolds. For example, we have proven that under some appropriate growth and symmetry assumptions on the nonlinearity U appearing in (1.2), this equation admits *real-analytic* solutions on Euclidean space.

On the other hand, if (1.2) is posed on spaces on which the Laplace operator does not have a positive-definite symbol, the studies carried out in [17–19] do not apply. One of the problems which appears is that the standard Sobolev norm, see for example [36], is no longer well defined. Thus, motivated by the papers [7, 41], we wonder if it is possible to define a 'Laplace operator' and Sobolev spaces in settings other than Riemannian manifolds. If so, then in such a setting we can hope to be able to consider nonlinear nonlocal equations rigorously. Topological groups appear as a natural testing ground due to the existence of the Fourier transform. In this context, if G is a locally compact abelian group, we simply set $\Delta(\phi) = \mathcal{F}^{-1}(\gamma(\cdot)^2\mathcal{F}(\phi))$, in which $\mathcal{F}$ indicates the Fourier transform and γ is a weakly subadditive $[0, \infty)$-valued function on the dual group of G (functions of this type appear for example in [15]). This definition allows us to consider Δ independently of any (pseudo)metric the space G may be equipped with.

Developing this idea, in this paper we define Sobolev spaces on arbitrary locally compact abelian groups and we examine analogs to the Sobolev embedding and Rellich–Kondrachov compactness theorems. These results, in turn, allow us to construct a suitable domain for the operator $\Delta e^{-c\Delta}$ and to prove rigorously existence of regular solutions to (1.2) in compact abelian groups, a class of spaces which is certainly different to the class of compact Riemannian manifolds considered in [17–19]. For instance, we can consider Equation (1.2) for functions ϕ 'depending on an infinite number of variables' if we pose it on an infinite product of tori. In fact, motivated by [19], we are confident that our techniques will allow us to handle very general nonlocal equations with a finite number of dependent variables and infinitely many *independent* variables. We mention that classes of differential equations with infinitely many independent variables appear in areas of mathematical physics such as statistical mechanics, fluid dynamics and classical field theory; see [6, 33, 37]. Thus, we expect that our methods will also apply in these contexts.

Added in proof. A preliminary version of this paper appeared as arXiv.org e-Print 1208.3053 in 2012.

2. Sobolev spaces

Sobolev spaces are well understood on (domains of) $\mathbb{R}^n$, see [1], compact and complete Riemannian manifolds [24] and metric measure spaces (the so-called Hajlasz–Sobolev spaces, see [22, 23], and Newtonian spaces [32]). There are also some works on Sobolev spaces in the p-adic context, see [30] and references therein, and in special cases of locally compact groups such as the Heisenberg group [3]. We also remark that pseudodifferential operators defined on locally compact abelian groups and compact Lie groups have been studied in [21, 31], respectively.

We start with some standard notation from harmonic analysis [25]. Let us fix a locally compact abelian group G. We denote by μ_G the unique Haar measure of G. We also consider the *dual group* of the group G (that is, the locally compact abelian group of all continuous group homomorphisms from G to the circle group T), and we denote it by $G^\wedge$. L^p spaces over G are defined as usual:

$$L^p(G) = \left\{ f : G \to \mathbb{C} : f \text{ is measurable and } \int_G |f(x)|^p \, d\mu_G(x) < \infty \right\},$$

and the Fourier transform on G is defined as follows: if $f \in L^1(G)$, then its Fourier transform is the function $\hat{f} : G^\wedge \to \mathbb{C}$ given by

$$\hat{f}(\xi) = \int_G \overline{\xi(x)} f(x) \, d\mu_G(x).$$

Next, we denote by Γ the following set:

$$\Gamma = \{\gamma : G^\wedge \to [0, \infty) : \gamma \text{ is measurable and } \exists_{c_\gamma} \forall_{\alpha, \beta \in G^\wedge} \gamma(\alpha\beta) \leq c_\gamma[\gamma(\alpha) + \gamma(\beta)]\},$$

and we are in position to introduce Sobolev spaces.

Definition 2.1. Let us fix a map $\gamma \in \Gamma$ and a nonnegative real number s. We shall say that $f \in L^2(G)$ belongs to $H^s_\gamma(G)$ if the following integral is finite:

$$\int_{G^\wedge} (1 + \gamma(\xi)^2)^s |\hat{f}(\xi)|^2 \, d\mu_{G^\wedge}(\xi). \tag{2.3}$$

Moreover, for $f \in H^s_\gamma(G)$, its norm $\|f\|_{H^s_\gamma(G)}$ is defined as follows:

$$\|f\|_{H^s_\gamma(G)} = \left(\int_{G^\wedge} (1 + \gamma(\xi)^2)^s |\hat{f}(\xi)|^2 \, d\mu_{G^\wedge}(\xi) \right)^{1/2}. \tag{2.4}$$

Remark. We note that by taking appropriate functions γ, we obtain the classical Sobolev spaces on $\mathbb{T}^n$ and $\mathbb{R}^n$; see [15] and [36, Ch. 4]. The use of the Fourier transform and the duality theory of locally compact abelian groups is crucial in the present general context, since we do not have differential calculus at our disposal. A particular instance of Definition 2.1 appears in the paper [15] by Feichtinger and Werther. The function γ used in that paper is called by the authors a *weakly subadditive weight*. We also note that in p-adic analysis Sobolev spaces are defined in a way analogous to our Definition 2.1: if we take $\gamma(\xi) = \|\xi\|_p$, where $\|\cdot\|_p$ is a p-adic norm on $\mathbb{Q}_p^n \simeq \mathbb{Q}_p^{n\wedge}$, then (2.3) and (2.4) allow us to recover the p-adic Sobolev spaces considered in [30].

Remark. In the paper [14], the authors introduce harmonic Hilbert spaces in a way strongly resembling our definition of Sobolev spaces. What separates the two notions are the properties of the 'weights' $(1 + \gamma(\xi)^2)^{s/2}$ for $\gamma \in \Gamma$. In particular, it is proven in

[14] that their spaces are embedded in a space of continuous functions if their weights ω (which are submultiplicative and not subadditive as in our case) satisfy the condition $\sum n^{-2} \log \omega(n\xi) < \infty$ for all $\xi \in \widehat{G}$. In this paper, we prove a similar embedding result, see Theorem 3.3 below, but we do not need such an assumption. We also point out that it is important to take $\gamma \in \Gamma$ in the definition of our weights because it allows us to prove that—as happens in standard contexts, see [15, 36]—our spaces $H^s_\gamma(G)$ are algebras, see Theorem 3.4 below. The other theorems of Sections 2 and 3 of this paper hold true without assuming that $\gamma \in \Gamma$.

Remark. Our spaces $H^s_\gamma(G)$ are contained in the $A^p_{\mathrm{w},\omega}(G)$ spaces introduced by Feichtinger and Gürkanli in [13] in the following sense: if (notation as in [13]) $\mathrm{w} \in L^2(G)$ and we take $\omega = (1+\gamma^2)^{s/2}$, then $H^s_\gamma(G) \hookrightarrow A^2_{\mathrm{w},\omega}(G)$.

3. Continuous embedding theorems

Embedding properties of Sobolev spaces are essential for proving existence and regularity of solutions to partial differential equations [36] and for the analysis of pseudodifferential operators; see for instance [30]. Thus, we begin by proving a Sobolev embedding type theorem in our general setting.

Let us start with two elementary observations. First, we show in Proposition 3.1 that our spaces $H^s_\gamma(G)$ are included in $L^2(G)$. Then, we prove in Proposition 3.2 that in fact we have a 'scale' of spaces with respect to continuous inclusion.

Proposition 3.1. *If G is a locally compact abelian group, then*

$$H^s_\gamma(G) \hookrightarrow L^2(G).$$

Moreover, for each $f \in H^s_\gamma(G)$, the following inequality holds:

$$\|f\|_{L^2(G)} \leq \|f\|_{H^s_\gamma(G)}.$$

Proof. By Pontryagin duality and a basic inequality,

$$\begin{aligned} \|f\|_{L^2(G)} = \|\hat{f}\|_{L^2(G^\wedge)} &= \bigg(\int_{G^\wedge} |\hat{f}(\xi)|^2 \, d\mu_{G^\wedge}(\xi)\bigg)^{1/2} \\ &\leq \bigg(\int_{G^\wedge} (1+\gamma(\xi)^2)^s |\hat{f}(\xi)|^2 \, d\mu_{G^\wedge}(\xi)\bigg)^{1/2} = \|f\|_{H^s_\gamma(G)}. \end{aligned}$$ □

Proposition 3.2. *If $s > \sigma$, then $H^s_\gamma(G) \hookrightarrow H^\sigma_\gamma(G)$. Moreover, the inequality*

$$\|f\|_{H^\sigma_\gamma(G)} \leq \|f\|_{H^s_\gamma(G)}$$

holds.

Proof. The proof follows from an elementary inequality. □

The classical Sobolev embedding theorem, see for instance [1], reads in our context as follows.

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THEOREM 3.3. *If* $1/(1+\gamma(\cdot)^2)^s \in L^1(G^\wedge)$, *then*

$$H^s_\gamma(G) \hookrightarrow C(G),$$

in which $C(G)$ *denotes the space of bounded continuous complex-valued functions on* G. *Moreover, there exists a constant* $C(\gamma, s)$ *such that for each* $f \in H^s_\gamma(G)$, *the following inequality holds:*

$$\|f\|_{C(G)} \le C(\gamma, s)\|f\|_{H^s_\gamma(G)}.$$

PROOF. First of all, we note that

$$\int_{G^\wedge} |\hat{f}(\xi)|\, d\mu_{G^\wedge}(\xi) \le \Big(\int_{G^\wedge} (1+\gamma(\xi)^2)^s |\hat{f}(\xi)|^2\, d\mu_{G^\wedge}(\xi)\Big)^{1/2} \Big\|\frac{1}{(1+\gamma(\cdot)^2)^s}\Big\|^{1/2}_{L^1(G^\wedge)}$$

and therefore $\widehat{f} \in L^1(G^\wedge)$. Thus, we can use the inverse Fourier transform:

$$\begin{aligned}|f(x)| &= \Big|\int_{G^\wedge} \hat{f}(\xi)\xi(x)\, d\mu_{G^\wedge}(\xi)\Big| \le \int_{G^\wedge} |\hat{f}(\xi)|\, d\mu_{G^\wedge}(\xi)\\ &\le \|f\|_{H^s_\gamma(G)} \Big\|\frac{1}{(1+\gamma(\cdot)^2)^s}\Big\|^{1/2}_{L^1(G^\wedge)}.\end{aligned}$$

From [25, Vol. 2, Theorem 31.5], we have $f \in C(G)$. □

The following theorem tells us that, under a technical assumption involving our function γ, the space $H^s_\gamma(G)$ is an algebra. It is well known that such a property is important, for instance, for the study of existence of solutions to partial differential equations. A recent example appears in our paper [20].

THEOREM 3.4. *If* $1/(1+\gamma(\cdot)^2)^s \in L^1(G^\wedge)$, *then* $H^s_\gamma(G)$ *is an algebra. There exists a constant* $D(\gamma, s)$ *such that for each* $f, g \in H^s_\gamma(G)$, *the following inequality holds:*

$$\|fg\|_{H^s_\gamma(G)} \le D(\gamma, s)\|f\|_{H^s_\gamma(G)}\|g\|_{H^s_\gamma(G)}.$$

PROOF. First of all, let us notice that for each $\xi, \eta \in G^\wedge$, the following inequality holds:

$$(1+\gamma(\xi)^2) \le (2+2c_\gamma^2)(2+\gamma(\xi\eta^{-1})^2+\gamma(\eta)^2).$$

Hence,

$$\begin{aligned}&|(1+\gamma(\xi)^2)^{s/2}\widehat{fg}(\xi)|\\ &\quad= \Big|\int_{G^\wedge} (1+\gamma(\xi)^2)^{s/2}\widehat{f}(\xi\eta^{-1})\widehat{g}(\eta)\, d\mu_{G^\wedge}(\eta)\Big|\\ &\quad\le 2^{s/2}(1+c_\gamma^2)^{s/2}\int_{G^\wedge} (1+\gamma(\xi\eta^{-1})^2+1+\gamma(\eta)^2)^{s/2}|\widehat{f}(\xi\eta^{-1})\widehat{g}(\eta)|\, d\mu_{G^\wedge}(\eta)\\ &\quad\le 2^{s}(1+c_\gamma^2)^{s/2}\int_{G^\wedge} (1+\gamma(\xi\eta^{-1})^2)^{s/2}|\widehat{f}(\xi\eta^{-1})\widehat{g}(\eta)|\, d\mu_{G^\wedge}(\eta)\\ &\qquad+ 2^{s}(1+c_\gamma^2)^{s/2}\int_{G^\wedge} (1+\gamma(\eta)^2)^{s/2}|\widehat{f}(\xi\eta^{-1})\widehat{g}(\eta)|\, d\mu_{G^\wedge}(\eta)\\ &\quad= 2^{s}(1+c_\gamma^2)^{s/2}(((1+\gamma(\cdot)^2)^{s/2}|\widehat{f}|) * |\widehat{g}| + |\widehat{f}| * (|\widehat{g}|(1+\gamma(\cdot)^2)^{s/2})).\end{aligned}$$

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Thus,

$$\begin{aligned}\|fg\|^2_{H^s_\gamma(G)} &= \|(1+\gamma(\xi)^2)^{s/2}\widehat{fg}(\xi)\|^2_{L^2(G^\wedge)}\\ &\le 2^{2s+1}(1+c_\gamma^2)^s(\|((1+\gamma(\cdot)^2)^{s/2}|\widehat{f}|)*|\widehat{g}|\,\|^2_{L^2(G^\wedge)}\\ &\quad+\|\,|\widehat{f}|*(|\widehat{g}|(1+\gamma(\cdot)^2)^{s/2})\|^2_{L^2(G^\wedge)}).\end{aligned}$$

Next, by the Young inequality, $\|u*v\|_{L^2(G^\wedge)} \le c_y\|u\|_{L^2(G^\wedge)}\|v\|_{L^1(G^\wedge)}$,

$$\|fg\|^2_{H^s_\gamma(G)} \le 2^{2s+1}(1+c_\gamma^2)^s c_y^2(\|f\|^2_{H^s_\gamma(G)}\|\widehat{g}\|^2_{L^1(G^\wedge)}+\|\widehat{f}\|^2_{L^1(G^\wedge)}\|g\|^2_{H^s_\gamma(G)}).$$

The result now follows from the proof of the previous theorem. □

Now we prove a second embedding result. While Theorem 3.3 tells us that functions belonging to $H^s_\gamma(G)$ are continuous, Theorem 3.5 below tells us that they possess 'higher integrability properties'.

THEOREM 3.5. *If $\alpha > s$ and $1/(1+\gamma(\cdot)^2) \in L^\alpha(G^\wedge)$, then*

$$H^s_\gamma(G) \hookrightarrow L^{\alpha^*}(G),$$

where $\alpha^* = 2\alpha/(\alpha - s)$. Moreover, there exists a constant $D(\gamma, s)$ such that for each $f \in H^s_\gamma(G)$, the following inequality holds:

$$\|f\|_{L^{\alpha^*}(G)} \le D(\gamma, s)\|f\|_{H^s_\gamma(G)}.$$

PROOF. By a standard corollary of the Hausdorff–Young inequality (see [25, Vol. 2]),

$$\|f\|_{L^{\alpha^*}(G)} \le \|\hat{f}\|_{L^p(G^\wedge)},$$

where p is the conjugate of α^*, that is, $p = 2\alpha/(\alpha + s)$. Next, using the Hölder inequality with exponents $2/p$ and $2/(2-p)$,

$$\begin{aligned}\|\hat{f}\|_{L^p(G^\wedge)} &= \Big(\int_{G^\wedge}|\hat{f}(\xi)|^p\frac{(1+\gamma(\xi)^2)^{sp/2}}{(1+\gamma(\xi)^2)^{sp/2}}\,d\mu_{G^\wedge}(\xi)\Big)^{1/p}\\ &\le \|f\|_{H^s_\gamma(G)}\Big(\int_{G^\wedge}\frac{1}{(1+\gamma(\xi)^2)^{sp/(2-p)}}\,d\mu_{G^\wedge}(\xi)\Big)^{(2-p)/2p}.\end{aligned}$$

Since $sp/(2-p) = \alpha$,

$$\|f\|_{L^{\alpha^*}(G)} \le \Big\|\frac{1}{(1+\gamma(\cdot)^2)}\Big\|^{s/2}_{L^\alpha(G^\wedge)}\|f\|_{H^s_\gamma(G)}.$$ □

4. Compact embedding theorems

In this section, we prove a Rellich–Kondrachov type theorem. As is well known, this theorem plays a crucial role in proving compactness of operators and in fixed point arguments. Now, the standard Rellich–Kondrachov theorem [1, 24] is valid only on spaces with finite measure such as compact Riemannian manifolds. It is then natural to

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assume that in our case the condition $\mu_G(G) < \infty$ holds or, equivalently (see [25]), that the locally compact abelian group G is actually compact. We stress that even with this restriction our results go beyond the standard case: besides infinite products of basic examples of compact abelian groups, other interesting instances of compact groups are the dyadic group (see for instance [34]) and the compact group appearing in the recent paper [35].

In the theorem below, we use the following convention: $g(h) \to 0$ as $h \to e$ means that for all $\epsilon > 0$, there exists an open set U_ϵ with $e \in U_\epsilon$ such that for all $h \in U_\epsilon$, we have $|g(h)| \leq \epsilon$. Also, the notation $A \hookrightarrow\hookrightarrow B$ means that the space A is compactly embedded into B.

Remark. It is known that in the case of $\mathbb{R}^n$, the Kolmogorov–Riesz–Weil theorem (see [26, 40]) can be used to prove the Rellich–Kondrachov theorem. Similar compactness results exist for locally compact abelian groups, see [12], which presumably would yield another approach to the problem of compact embeddings. We present a direct proof.

Theorem 4.1. *Let $1/(1 + \gamma(\cdot)^2) \in L^\alpha(G^\wedge)$ for some $\alpha > s$ and assume that*

$$\frac{|\xi(h) - 1|}{(1 + \gamma(\xi)^2)^s} \underset{h\to e}{\longrightarrow} 0 \quad \textit{uniformly with respect to } \xi \in G^\wedge. \tag{4.5}$$

If G is compact, then, for all $p < \alpha^*$,

$$H^s_\gamma(G) \hookrightarrow\hookrightarrow L^p(G).$$

Before proving Theorem 4.1, we note that if G is $\mathbb{R}^n$ or $\mathbb{T}^n$, Condition (4.5) is satisfied. Indeed, if $G = \mathbb{R}^n$, then $G^\wedge = \mathbb{R}^n$ and a straightforward calculation yields

$$\frac{|\xi(h) - 1|}{(1 + \gamma(\xi)^2)^s} = \frac{|e^{i\xi h} - 1|}{(1 + |\xi|^2)^s} \leq \frac{\sqrt{2}|\sin(\xi h)|}{(1 + |\xi|^2)^s}$$

and, if $G = \mathbb{T}^n$, then $G^\wedge = \mathbb{Z}^n$; we can show that

$$\frac{|m(h) - 1|}{(1 + \gamma(m)^2)^s} = \frac{|e^{imh} - 1|}{(1 + |m|^2)^s} \leq \frac{\sqrt{2}|\sin(mh)|}{(1 + |m|^2)^s}.$$

We also note that a condition similar to (4.5) appears in the characterization of (pre)compact sets in $L^2(G)$ via the Fourier transform (Pego's theorem, see [16]).

Proof. Let us start with the following lemma.

Lemma 4.2. *Let $f \in H^s_\gamma(G)$ and assume that $|\xi(h) - 1|/(1 + \gamma(\xi)^2)^s \underset{h\to e}{\longrightarrow} 0$ uniformly with respect to $\xi \in G^\wedge$. Then, for each $h \in G$,*

$$\int_G |f(xh) - f(x)|^2 \, d\mu_G(x) \leq C(h)\|f\|^2_{H^s_\gamma(G)},$$

where $C(h) \underset{h\to e}{\longrightarrow} 0$.

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PROOF. By Pontryagin duality,

$$\int_G |f(xh) - f(x)|^2 \, d\mu_G(x) = \int_{G^\wedge} |\widehat{f(.h)}(\xi) - \hat{f}(\xi)|^2 \, d\mu_{G^\wedge}(\xi).$$

Since the measure μ_G is invariant,

$$\begin{aligned}\widehat{f(.h)}(\xi) &= \int_G \overline{\xi(x)} f(xh) \, d\mu_G(x) = \int_G \overline{\xi(yh^{-1})} f(y) \, d\mu_G(y) \\ &= \int_G \overline{\xi(y)\xi(h^{-1})} f(y) \, d\mu_G(y) = \xi(h)\hat{f}(\xi).\end{aligned}$$

Hence,

$$\begin{aligned}&\int_G |f(xh) - f(x)|^2 \, d\mu_G(x) \\ &\quad = \int_{G^\wedge} |\hat{f}(\xi)|^2 |\xi(h) - 1|^2 \, d\mu_{G^\wedge}(\xi) \\ &\quad = \int_{G^\wedge} |\hat{f}(\xi)|^2 (1 + \gamma(\xi)^2)^s \frac{|\xi(h) - 1|^2}{(1 + \gamma(\xi)^2)^s} \, d\mu_{G^\wedge}(\xi) \le C(h) \|f\|^2_{H^s_\gamma(G)},\end{aligned}$$

where $C(h) = \| \, |\xi(h) - 1|^2/(1 + \gamma(\xi)^2)^s \|_{L^\infty(G^\wedge)} \underset{h \to e}{\longrightarrow} 0.$ □

We continue the proof of Theorem 4.1. Let $\mathcal{I}$ be the set of all symmetric unit neighborhoods, partially ordered by the inverse inclusion. Then, using the Urysohn lemma, we can construct the so-called Dirac net $(\phi_U)_{U \in I}$ in $C_c(G)$ (see [10]). Each function ϕ_U is nonnegative, satisfies $\int_G \phi_U(x) \, d\mu_G(x) = 1$ and the support of ϕ_U shrinks. We are in position to formulate the next lemma.

LEMMA 4.3. *Let $(\phi_U)_{U \in \mathcal{I}}$ be a Dirac net and $f \in H^s_\gamma(G)$. Then*

$$\int_G |f * \phi_U(x) - f(x)|^2 \, d\mu_G(x) \le \|f\|^2_{H^s_\gamma(G)} \sup_{y \in U} C(y).$$

PROOF.

$$\begin{aligned}&|f * \phi_U(x) - f(x)|^2 \\ &\quad = \left| \int_G \phi_U(y) f(y^{-1}x) \, d\mu_G(y) - f(x) \right|^2 \\ &\quad = \left| \int_G \phi_U(y)(f(y^{-1}x) - f(x)) \, d\mu_G(y) \right|^2 \le \int_G \phi_U(y) |f(y^{-1}x) - f(x)|^2 \, d\mu_G(y) \\ &\quad = \int_U \phi_U(y) |f(y^{-1}x) - f(x)|^2 \, d\mu_G(y).\end{aligned}$$

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Hence, by the Fubini theorem and the invariance of the measure,

$$\begin{aligned}\int_G |f * \phi_U(x) - f(x)|^2 \, d\mu_G(x) &\le \int_G \int_U \phi_U(y) |f(y^{-1}x) - f(x)|^2 \, d\mu_G(y) \, d\mu_G(x) \\ &= \int_U \int_G \phi_U(y) |f(y^{-1}x) - f(x)|^2 \, d\mu_G(x) \, d\mu_G(y) \\ &= \int_U \phi_U(y) \int_G |f(z) - f(yz)|^2 \, d\mu_G(z) \, d\mu_G(y).\end{aligned}$$

By the previous lemma,

$$\begin{aligned}\int_G |f * \phi_U(x) - f(x)|^2 \, d\mu_G(x) &\le \int_U \phi_U(y) \|f\|^2_{H^s_\gamma(G)} C(y) \, d\mu_G(y) \\ &\le \int_U \phi_U(y) \, d\mu_G(y) \|f\|^2_{H^s_\gamma(G)} \sup_{y \in U} C(y) = \|f\|^2_{H^s_\gamma(G)} \sup_{y \in U} C(y).\end{aligned}$$

This finishes the proof of the lemma. □

Now we can finish the proof of the theorem. Let us take any sequence f_n bounded in the space $H^s_\gamma(G)$; then, by Theorem 3.5, the sequence is bounded in $L^{\alpha^*}(G)$. Hence, there exists a subsequence f_{n_k} of f_n such that

$$f_{n_k} \rightharpoonup f \quad \text{in } L^{\alpha^*}(G).$$

We claim that $f_{n_k} \to f$ in $L^q(G)$, where $q < \alpha^*$: for every $f \in L^2(G)$, we denote by $f_{(U)}$ the function $f_{(U)} = f * \phi_U$. Also, for simplicity, we write f_n instead of f_{n_k}. By Lemma 4.3,

$$\sup_n \int_G |f_{n_{(U)}}(x) - f_n(x)|^2 \, d\mu_G(x) \le \sup_n \|f_n\|^2_{H^s_\gamma(G)} \sup_{y \in U} C(y) \le C \sup_{y \in U} C(y).$$

Moreover, we can show that $\|f_{(U)} - f\|_{L^2(G)} \to 0$ in the sense that for each $\epsilon > 0$, there exists U_ϵ such that for each $U \in \mathcal{I}$, $U \subset U_\epsilon$, the inequality $\|f_{(U)} - f\|_{L^2(G)} \le \epsilon$ holds. Next, by the Minkowski inequality,

$$\|f_n - f\|_{L^2(G)} \le \|f_n - f_{n_{(U)}}\|_{L^2(G)} + \|f_{n_{(U)}} - f_{(U)}\|_{L^2(G)} + \|f_{(U)} - f\|_{L^2(G)}.$$

Now we fix $\epsilon > 0$. There exists $U_\epsilon \in I$ such that for each $U \in \mathcal{I}$, $U \subset U_\epsilon$, the following inequality holds:

$$\|f_n - f\|_{L^2(G)} \le \tfrac{2}{3}\epsilon + \|f_{n_{(U)}} - f_{(U)}\|_{L^2(G)}.$$

Thus, in order to show that $\|f_n - f\|_{L^2(G)} \to 0$, it is enough to check the limit

$$\|f_{n_{(U_\epsilon)}} - f_{(U_\epsilon)}\|_{L^2(G)} \underset{n\to\infty}{\longrightarrow} 0.$$

In fact, since $f_n \rightharpoonup f$ in $L^{\alpha^*}(G)$,

$$f_{n_{(U_\epsilon)}}(x) = \int_G \phi_{U_\epsilon}(xy^{-1}) f_n(y) \, d\mu_G(y) \to \int_G \phi_{U_\epsilon}(xy^{-1}) f(y) \, d\mu_G(y) = f_{(U_\epsilon)}(x).$$

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Moreover, since G is compact,

$$\begin{aligned}
&|f_{n_{(U_\epsilon)}}(x) - f_{(U_\epsilon)}(x)|^2 \\
&\quad = \left|\int_G (f_n(y) - f(y))\phi_{U_\epsilon}(y^{-1}x)\, d\mu_G(y)\right|^2 \\
&\quad \leq \int_G |f_n(y) - f(y)|^2 \phi_{U_\epsilon}(y^{-1}x)\, d\mu_G(y) \leq \sup_{z\in U_\epsilon} \phi_{U_\epsilon}(z) \int_G |f_n(y) - f(y)|^2\, d\mu_G(y)
\end{aligned}$$

and, finally, since we are assuming that G is of finite measure, we can apply the Lebesgue theorem and obtain

$$\|f_{n_{(U_\epsilon)}} - f_{(U_\epsilon)}\|_{L^2(G)} \underset{n\to\infty}{\longrightarrow} 0.$$

So, we have obtained that $f_{n_k} \to f$ in $L^2(G)$. Finally, since the sequence is bounded in $L^{\alpha^*}(G)$, we can apply the Vitali convergence theorem and we obtain that $f_{n_k} \to f$ in $L^p(G)$, where $p < \alpha^*$. □

5. An application: the generalized bosonic string

We recall from Section 1 that the generalized bosonic string equation, [9], is

$$\Delta e^{-c\Delta}\phi = U(x, \phi), \quad c > 0. \tag{5.6}$$

Suppose for a moment that we are working on Euclidean space and that f is the real function $f(s) = s\exp(-cs)$ so that, formally, the left-hand side of (5.6) is $f(\Delta)$. We expand f as a power series, $f(s) = \sum_{n=0}^{\infty} (f^{(n)}(0)/n!)s^n$. Then, formally,

$$f(\Delta)u = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \Delta^n u.$$

Applying the Fourier transform, we obtain (we set $\widehat{f} = \mathcal{F}(f)$ for clarity)

$$\begin{aligned}
\mathcal{F}(f(\Delta)u)(\xi) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \mathcal{F}(\Delta^n u)(\xi) \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (-|\xi|^2)^n \mathcal{F}(u)(\xi) = f(-|\xi|^2)\mathcal{F}(u)(\xi)
\end{aligned}$$

so that, naturally, we may interpret $f(\Delta)u$ in a way that reminds us of the classical definitions of pseudodifferential operators (see for instance [31]):

$$f(\Delta)u = \mathcal{F}^{-1}(f(-|\xi|^2)\mathcal{F}(u)(\xi)) = -\mathcal{F}^{-1}(|\xi|^2 e^{c|\xi|^2}\mathcal{F}(u)(\xi)).$$

Motivated by these remarks, we make two general definitions. First, we write down the correct domain for the operator $L_c = \Delta e^{-c\Delta} - \mathrm{Id}$ on locally compact abelian groups. Then, we define the action of L_c.

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Definition 5.1. The space $\mathcal{H}^{c,\infty}(G)$, $c > 0$, is given by

$$\mathcal{H}^{c,\infty}(G) = \left\{ f \in L^2(G) : \int_{G^\wedge} (1 + \gamma(\xi)^2 e^{c\gamma(\xi)^2})^2 |\widehat{f}(\xi)|^2 \, d\mu_{G^\wedge}(\xi) < \infty \right\}.$$

Definition 5.2. The operator $L_c = \Delta e^{-c\Delta} - \mathrm{Id}$ is defined as

$$L_c u = -\mathcal{F}^{-1}(\mathcal{F}(u) + \gamma(\xi)^2 e^{c\gamma(\xi)^2} \mathcal{F}(u))$$

for any $u \in \mathcal{H}^{c,\infty}(G)$.

We note that L_c is an isometry from $\mathcal{H}^{c,\infty}(G)$ into $L^2(G)$; we also remark that related definitions of pseudodifferential operators in p-adic analysis appear for instance in [30].

We state two important technical observations on the structure of the space $\mathcal{H}^{c,\infty}(G)$.

Lemma 5.3.

(1) *For each nonnegative $s \in \mathbb{R}$, the embedding $\mathcal{H}^{c,\infty}(G) \hookrightarrow H^s_\gamma(G)$ holds. In other words, $\|f\|_{H^s_\gamma(G)} \le C(s)\|f\|_{\mathcal{H}^{c,\infty}(G)}$ for some constant $C(s) > 0$.*

(2) *Assume that $1/(1 + \gamma(\cdot)^2)^2 \in L^1(G^\wedge)$. Then the embedding $\mathcal{H}^{c,\infty}(G) \hookrightarrow C(G)$ holds.*

Proof. The first claim follows immediately from the elementary properties of the map $x \mapsto e^x$. The second claim is a consequence of the first one combined with our Sobolev embedding result, Theorem 3.3. □

Now we show that the linear problem $L_c u = g$, $g \in L^2(G)$, can be solved completely using our set-up.

Theorem 5.4. *For each $c > 0$ and $g \in L^2(G)$, there exists a unique solution $u_g \in \mathcal{H}^{c,\infty}(G)$ to the linear problem*

$$L_c u = g. \tag{5.7}$$

Moreover, the equation

$$\|u_g\|_{\mathcal{H}^{c,\infty}(G)} = \|g\|_{L^2(G)} \tag{5.8}$$

holds.

Proof. It is easy to see that u_g, given by

$$u_g = -\mathcal{F}^{-1}\left(\frac{\mathcal{F}(g)}{1 + \gamma(\xi)^2 e^{c\gamma(\xi)^2}} \right),$$

is an element of $\mathcal{H}^{c,\infty}(G)$ which solves Equation (5.7). Now, applying the Fourier transform,

$$(1 + \gamma(\xi)^2 e^{c\gamma(\xi)^2}) \mathcal{F}(u_g)(\xi) = -\mathcal{F}(g)(\xi)$$

and so the Plancherel theorem implies that $\|u_g\|_{\mathcal{H}^{c,\infty}(G)} = \|g\|_{L^2(G)}$. Equation (5.8) tells us that the operator L_c has trivial kernel. Uniqueness then follows immediately. □

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Remark. Motivated by [7], we remark that it follows from Theorem 5.4 that we are able to solve completely the linear equation $L_c u = g$ in p-adic analysis. In fact, this observation can be generalized to a very large class of linear equations over p-adic fields, reasoning as in [19].

We are ready to show that the generalized bosonic string Equation (5.6) admits continuous solutions.

Theorem 5.5. *Assume that G is a compact abelian group and that $1/(1+\gamma(\cdot)^2) \in L^\delta(G^\wedge)$, where $\delta > 1$. Let $U : G \times \mathbb{R} \to \mathbb{R}$ be a function which is differentiable with respect to its second argument and suppose that there exist constants $\alpha > 1$, $\beta \in [0, \alpha - 1]$ and $C > 0$ and functions $h \in L^2(G)$ and $f \in L^{2\alpha/(\alpha-1)}(G)$ such that the following two inequalities hold:*

$$|U(x,y) - y| \le C(|h(x)| + |y|^\alpha), \quad \left|\frac{\partial}{\partial y}(U(x,y) - y)\right| \le C(|f(x)| + |y|^\beta). \tag{5.9}$$

If $\|h\|_{L^2(G)}$ is suitably small and $|\xi(h) - 1|/(1+\gamma(\xi)^2)^{\delta-(\delta/\alpha)} \underset{h\to e}{\longrightarrow} 0$ uniformly with respect to $\xi \in G^\wedge$, then there exists a solution $\phi \in \mathcal{H}^{c,\infty}(G) \cap C(G)$ to the nonlinear problem

$$\Delta e^{-c\Delta}\phi - U(x,\phi) = 0. \tag{5.10}$$

Proof. Let us set $V(\cdot, u(\cdot)) = U(\cdot, u(\cdot)) - u(\cdot)$. Then the nonlinear Equation (5.10) is formally equivalent to $L_c u = V(\cdot, u)$. We can easily see that the function V belongs to $L^2(G)$ using (5.9); see (5.11) below. We define the set

$$Y_\epsilon = \{u \in L^{2\alpha}(G) : \|u\|_{L^{2\alpha}(G)} \le \epsilon\}$$

for $\epsilon > 0$. It is easy to see that Y_ϵ is a bounded, closed, convex and nonempty subset of the Banach space $L^{2\alpha}(G)$. We define a map $\mathcal{G}$ as follows:

$$\mathcal{G} : Y_\epsilon \to L^{2\alpha}(G), \quad \mathcal{G}(u) = \tilde{u},$$

where $\tilde{u}$ is the unique solution to the nonhomogeneous linear problem

$$L_c\tilde{u} = V(\cdot, u).$$

Theorem 5.4 implies that $\mathcal{G}$ is well defined. We show that there exists $\epsilon > 0$ such that $\mathcal{G} : Y_\epsilon \to Y_\epsilon$. Indeed, let us take $u \in Y_\epsilon$; then we get, using (5.9),

$$\begin{aligned}\|\mathcal{G}(u)\|^2_{H^{c,\infty}(G)} &= \|V(\cdot,u)\|^2_{L^2(G)} \\ &\le C^2 \int_G ||h(x)| + |u(x)|^\alpha|^2 \, d\mu_G(x) \\ &\le 2C^2 \int_G ||h(x)|^2 + |u(x)|^{2\alpha}| \, d\mu_G(x) \\ &\le 2C^2(\|h\|^2_{L^2(G)} + \|u\|^{2\alpha}_{L^{2\alpha}(G)}).\end{aligned} \tag{5.11}$$

Now, let us fix $s \in (\delta - (\delta/\alpha), \delta)$. Using again Lemma 5.3 and Theorem 3.5,

$$H^{c,\infty}(G) \hookrightarrow H^s_\gamma(G) \hookrightarrow L^{2\alpha}(G).$$

Hence, since $u \in Y_\epsilon$, inequality (5.11) implies that there exists a constant D such that

$$\|\mathcal{G}(u)\|^2_{L^{2\alpha}(G)} \leq D(\|h\|^2_{L^2(G)} + \epsilon^{2\alpha}).$$

Since we are assuming that $\|h\|_{L^2(G)}$ is suitably small and $\alpha > 1$, we can find ϵ such that $\|\mathcal{G}(u)\|_{L^{2\alpha}(G)} \leq \epsilon$. This implies that $\mathcal{G} : Y_\epsilon \to Y_\epsilon$.

Now we apply a fixed point argument. We skip the details, as similar proofs appear in [18]. First, we note that Theorem 4.1 implies that $H^s_\gamma(G) \hookrightarrow\hookrightarrow L^{2\alpha}(G)$ and therefore the map $\mathcal{G}$ is compact. Second, standard reasoning using the mean value theorem and our assumptions on the derivative of V implies that the map $\mathcal{G}$ is continuous. Application of Schauder's fixed point theorem finishes the proof. □

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References

[1] R. A. Adams, *Sobolev Spaces* (Academic Press, New York, 1975).
[2] I. Ya. Aref'eva and I. V. Volovich, 'Cosmological daemon', *J. High Energy Phys.* **102**(8) (2011), 32.
[3] H. Bahouri, C. Fermanian-Kammerer and I. Gallagher, 'Analyse de l'espace des phases et calcul pseudo-differential sur le groupe de Heisenberg', *C. R. Math. Acad. Sci. Paris* **347**(17–18) (2009), 1021–1024.
[4] N. Barnaby, 'A new formulation of the initial value problem for nonlocal theories', *Nuclear Phys.* B **845** (2011), 1–29.
[5] N. Barnaby, T. Biswas and J. M. Cline, 'p-adic inflation', *J. High Energy Phys.* **0704** (2007), 056 35 pp.
[6] Y. M. Berezansky and Y. G. Kondratiev, *Spectral Methods in Infinite-Dimensional Analysis,* Vols. 1–2 (Kluwer, Dordrecht, 1995).
[7] L. Brekke and P. G. O. Freund, 'p-adic numbers in physics', *Phys. Rep.* **233** (1993), 1–66.
[8] G. Calcagni, M. Montobbio and G. Nardelli, 'Route to nonlocal cosmology', *Phys. Rev.* D **76** (2007), 126001, 20 pp.
[9] G. Calcagni, M. Montobbio and G. Nardelli, 'Localization of nonlocal theories', *Phys. Lett.* B **662** (2008), 285–289.
[10] A. Deitmer and S. Echterhoff, *Principles of Harmonic Analysis* (Springer, New York, 2009).
[11] D. A. Eliezer and R. P. Woodard, 'The problem of nonlocality in string theory', *Nuclear Phys.* B **325** (1989), 389–469.
[12] H. G. Feichtinger, 'Compactness in translation invariant Banach spaces of distributions and compact multipliers', *J. Math. Anal. Appl.* **102** (1984), 289–327.
[13] H. G. Feichtinger and A. T. Gürkanli, 'On a family of weighted convolution algebras', *Int. J. Math. Math. Sci.* **13** (1990), 517–526.
[14] H. G. Feichtinger, S. S. Pandey and T. Werther, 'Minimal norm interpolation in harmonic Hilbert spaces and Wiener amalgam spaces on locally compact abelian groups', *J. Math. Kyoto Univ.* **47**(1) (2007), 65–78.

https://doi.org/10.1017/S1446788714000433 Published online by Cambridge University Press

[15] H. G. Feichtinger and T. Werther, 'Robustness of regular sampling in Sobolev algebras', in: *Sampling, Wavelets and Tomography* (eds. J. Benedetto and A. I. Zayed) (Birkhäuser, Boston, MA, 2004), 83–113.
[16] P. Górka, 'Pego theorem on locally compact abelian groups', *J. Algebra Appl.* **13** 1350143 (2014).
[17] P. Górka, H. Prado and E. G. Reyes, 'Nonlinear equations with infinitely many derivatives', *Complex Anal. Oper. Theory* **5** (2011), 313–323.
[18] P. Górka, H. Prado and E. G. Reyes, 'Generalized euclidean bosonic string equations', in: *Operator Theory: Advances and Applications,* Vol. 224 (Springer, Basel, 2012), 147–169.
[19] P. Górka, H. Prado and E. G. Reyes, 'On a general class of nonlocal equations', *Ann. Henri Poincaré* **14** (2013), 947–966.
[20] P. Górka and E. G. Reyes, 'The modified Camassa–Holm equation', *Int. Math. Res. Not. IMRN* **2011**(12) (2011), 2617–2649.
[21] K. Gröchenig and T. Strohmer, 'Pseudodifferential operators on locally compact abelian groups and Sjöstrand's symbol class', *J. reine angew. Math.* **613** (2007), 121–146.
[22] P. Hajłasz, 'Sobolev spaces on an arbitrary metric space', *Potential Anal.* **5**(4) (1996), 403–415.
[23] P. Hajłasz and P. Koskela, 'Sobolev met Poincaré', *Mem. Amer. Math. Soc.* **145**(688) (2000).
[24] E. Hebey, *Sobolev Spaces on Riemannian Manifolds*, Lecture Notes in Mathematics, 1635 (Springer, Berlin, 1996).
[25] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis, Vol. I: Structure of Topological Groups. Integration Theory. Group Representations; Vol. II: Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups* (Springer, New York–Berlin, 1963, 1970).
[26] A. N. Kolmogorov, 'Über Kompaktheit der Funktionenmengen bei der Konvergenz im Mittel', *Nachr. Ges. Wiss. Göttingen* **9** (1931), 60–63.
[27] V. A. Kostelecký and S. Samuel, 'On a nonperturbative vacuum for the open bosonic string', *Nuclear Phys.* B **336** (1990), 263–296.
[28] N. Moeller and B. Zwiebach, 'Dynamics with infinitely many time derivatives and rolling tachyons', *J. High Energy Phys.* **34**(10) (2002), 38.
[29] J. W. Moffat, 'Ultraviolet complete electroweak model without a Higgs particle', *Eur. Phys. J. Plus* **126** (2011), 53.
[30] J. J. Rodríguez-Vega and W. A. Zúñiga-Galindo, 'Elliptic pseudodifferential equations and Sobolev spaces over p-adic fields', *Pacific J. Math.* **246**(2) (2010), 407–420.
[31] M. Ruzhansky and V. Turunen, *Pseudo-Differential Operators and Symmetries. Background Analysis and Advanced Topics* (Birkhäuser, Basel–Boston–Berlin, 2010).
[32] N. Shanmugalingam, 'Newtonian spaces: an extension of Sobolev spaces to metric measure spaces', *Rev. Mat. Iberoam.* **16**(2) (2000), 243–279.
[33] D. W. Stroock, 'Logarithmic Sobolev inequalities for Gibbs states', in: *Corso CIME* (Springer, Berlin–Heidelberg, 1993).
[34] J. Tateoka, 'On the characterization of Hardy–Besov spaces on the dyadic group and its applications', *Studia Math.* **110** (1994), 127–148.
[35] D. V. Tausk, 'A locally compact non divisible abelian group whose character group is torsion free and divisible', *Canad. Math. Bull.* **56** (2013), 213–217.
[36] M. E. Taylor, *Partial Differential Equations,* Vol. I (Springer, New York, 1996).
[37] M. I. Vishik and A. V. Fursikov, *Mathematical Problems of Statistical Hydromechanics* (Kluwer Academic, Dordrecht–Boston–London, 1980).
[38] V. S. Vladimirov, 'The equation of the p-adic open string for the scalar tachyon field', *Izv. Math.* **69** (2005), 487–512.
[39] V. S. Vladimirov and Ya. I. Volovich, 'Nonlinear dynamics equation in p-adic string theory', *Teoret. Mat. Fiz.* **138** (2004), 355–368; translation in *Theoret. Math. Phys.* **138** (2004), 297–309.
[40] A. Weil, *L'intégration dans les groupes topologiques et ses applications* (Hermann, Paris, 1940).
[41] E. Witten, 'Noncommutative geometry and string field theory', *Nucl. Phys.* B **268** (1986), 253–294.

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