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UNITARIZATION OF LOOP GROUP REPRESENTATIONS OF FUNDAMENTAL GROUPS

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Abstract. In this paper, we give a characterization of the simultaneous unitarizability of any finite set of $\mathrm{SL}(2,\mathbb{C})$-valued functions on $\mathbb{S}^1$ and determine all possible ways of the unitarization. Such matrix functions can be regarded as images of the generators for the fundamental group of a surface in an $\mathbb{S}^1$-family, and the results of this paper have applications in the construction of constant mean curvature surfaces in space.

In recent years, a Weierstrass type representation [4] has been used to construct surfaces of non-zero constant mean curvature in space (CMC surfaces). This construction starts with an $\mathrm{sl}(2,\mathbb{C})$-valued holomorphic or meromorphic 1-form on some (compact or non-compact) Riemann surface $\mathcal{S}$ other than $\mathbb{S}^2$. Pulling back to the simply connected cover $\widetilde{\mathcal{S}}$ of $\mathcal{S}$, one obtains a 1-form η on $\widetilde{\mathcal{S}}$ which is invariant under the fundamental group $\pi_1(\mathcal{S})$ of $\mathcal{S}$. In the next step of the construction, one solves the ordinary differential equation $\mathrm{d}H = H\eta$ on $\widetilde{\mathcal{S}}$ with the initial condition I at some base point z_0. Then, every $\gamma \in \pi_1(\mathcal{S})$ induces a monodromy matrix ρ_γ via H. As a special feature of this construction, everything depends also on a parameter $\lambda \in \mathbb{S}^1$. In particular, for $\gamma \in \pi_1(\mathcal{S})$, the matrix $\rho_\gamma = \rho_\gamma(\lambda)$ depends holomorphically on λ on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. In this way, one obtains a map ρ from $\pi_1(\mathcal{S})$ into the loops in $\mathrm{SL}(2,\mathbb{C})$ satisfying $\rho_{\gamma\delta} = \rho_\gamma \rho_\delta$, i.e., ρ is a representation of $\pi_1(\mathcal{S})$ in the $\mathrm{SL}(2,\mathbb{C})$-loop group. When the CMC surface to be constructed has umbilic points, in order to obtain eventually an immersion from $\mathcal{S}$ into $\mathbb{R}^3$, not just from $\widetilde{\mathcal{S}}$ into $\mathbb{R}^3$, it is necessary that $\rho_\gamma(\lambda) \in \mathrm{SU}(2)$ for all $\gamma \in \pi_1(\mathcal{S})$ and $\lambda \in \mathbb{S}^1$. At this point, no criterion on η is known which would characterize the 1-forms on $\widetilde{\mathcal{S}}$ corresponding to unitary representations ρ. Therefore, one tries to find conditions under which one can conjugate ρ such that the new representation $\hat{\rho}$ is unitary. The new representation $\hat{\rho}$ can always be derived from some 1-form $\hat{\eta}$, which can be obtained from η by a gauge transformation. This actually corresponds to

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using an initial condition at z_0 different from I when solving the differential equation $\mathrm{d}H = H\eta$.

In this paper, we present an algorithm testing and characterizing the unitarizability of a monodromy representation ρ. It suffices to check whether a set of generators $L_1, L_2, \ldots, L_m$ of the image of $\pi_1(\mathcal{S})$ under ρ can be simultaneously unitarized. It turns out that this is possible only if each L_i has an eigenvalue function $v_i(\lambda)$ which is holomorphic on a neighborhood of $\mathbb{S}^1$. We will also assume that no L_i has the λ-independent eigenvalues ± 1, which is satisfied in the geometric context mentioned above. Then, the unitarizability of one matrix function L can be characterized by the obvious conditions on the trace of L, see Proposition 2.2. Also, a specific matrix function T can be found conjugating L into some canonical unitary matrix function. Given two matrix functions L_1 and L_2 satisfying these conditions and thus being unitarizable separately by some T_1 and T_2, respectively, we first consider the case where L_1 and L_2 commute. This can be characterized by $T_1 T_2^{-1}$ being diagonal or off-diagonal, see Proposition 3.1. Since commuting with each other is an equivalence relation, we can assume that no pair of the generators $L_1, L_2, \ldots, L_m$ commute. In this case, we characterize the simultaneous unitarizability of L_1 and L_2 by simple conditions on the entry functions of $T_1 T_2^{-1}$, see Theorem 3.5. Similarly, the simultaneous unitarizability of L_1, L_2 and L_3 is characterized by simple conditions on the entry functions of $T_1 T_2^{-1}$ and $T_2 T_3^{-1}$, with each T_i having the same meaning as above, see Theorem 4.1. Finally, for the case of four or more generators, it turns out that it suffices to verify that every three consecutive generators are simultaneously unitarizable, see Theorem 4.14.

In the geometric applications to CMC surfaces mentioned above, one actually uses the twisted loop group consisting of the loops in $\mathrm{SL}(2,\mathbb{C})$ which reflect some invariance under the Cartan involution of $\mathrm{SL}(2,\mathbb{C})$ corresponding to the symmetric space $\mathbb{S}^2 = \mathrm{SU}(2)/\mathrm{U}(1)$. Therefore, it is natural that in view of these applications, we use twisted loop groups throughout this paper. However, there is also some technical advantage: while the twisted and untwisted loop groups are isomorphic in the case of $\mathrm{SL}(2,\mathbb{C})$, this isomorphism does not preserve the so-called "positive loops", see (1.10). Thus, writing this paper in the untwisted setting would require replacing frequently used standard loop group splittings (Birkhoff and Iwasawa splittings) by closely related ones, for which at least some of the properties would need to be proven. Finally, from the point of view of Kac-Moody groups, twisted and untwisted groups are equally natural, and the untwisted version should not

be preferred.

Another issue is how to deal with cases where some eigenvalue function vanishes at some points of $\mathbb{S}^1$. Since these eigenvalue functions do not vanish identically on $\mathbb{S}^1$ and are holomorphic in some neighborhood of $\mathbb{S}^1$, it is natural to consider r-circles which are sufficiently close to $\mathbb{S}^1$. Fortunately, for these r-loops, there is already a well-developed theory for us to use.

It is worth pointing out that up to conjugations by unitary matrix functions, there is at most one way to simultaneously unitarize two or more matrix functions, unless they commute for all values of $\lambda \in \mathbb{S}^1$, see Theorems 3.5, 3.24, 4.1 and 4.14. This result will be essential in proving the so-called "closing conditions" when r-dressing with $r < 1$ is needed. In the geometric applications discussed above, these closing conditions ensure that the CMC immersion from $\widetilde{\mathcal{S}}$ into $\mathbb{R}^3$ constructed from η actually descends down to $\mathcal{S}$ and yields a CMC immersion from $\mathcal{S}$ into $\mathbb{R}^3$.

We mention that it seems more natural to consider the unitarization problem in the algebra of matrix functions meromorphic in a neighborhood of $\mathbb{S}^1$. However, in general, the problem does not have a solution in that algebra, see the proof of Theorem 2.6 together with Remark 2.14. Moreover, the topology on the Wiener algebra needed for the geometric applications that we are primarily interested in adds severe difficulties to the problem.

This paper is organized as follows. In Section 1, we introduce the basic definitions concerning loop groups and recall some fundamental results. In Section 2, we characterize the unitarizability of one matrix function L and determine all the matrix functions unitarizing L. Section 3 is devoted to the characterization of the simultaneous unitarizability of two matrix functions, while Section 4 considers three or more matrix functions. Finally, in Section 5, we discuss the relation between the simultaneous unitarizability of matrix functions by dressing and their pointwise simultaneous unitarizability, and deal with the closing conditions.

§1. Notation and basic results on loop groups

In this paper, for any $r > 0$ and each $c \in \mathbb{C}$, we will use $\mathbb{S}^1_r(c)$, $\mathcal{D}_r(c)$ and $\mathcal{D}^*_r(c)$ to denote the circle $\{z \in \mathbb{C}\,;\, |z-c| = r\}$, the disk $\{z \in \mathbb{C}\,;\, |z-c| < r\}$ and the punctured disk $\{z \in \mathbb{C}\,;\, 0 < |z-c| < r\}$, respectively. When any of $c = 0$ and $r = 1$ happens, we will omit the corresponding part from the notation: $\mathbb{S}^1 = \mathbb{S}^1_1 = \mathbb{S}^1_1(0)$, $\mathcal{D} = \mathcal{D}_1 = \mathcal{D}_1(0)$, etc.

First, we state a result about holomorphic functions. This result will be needed in Sections 2 and 3.

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LEMMA 1.1. *If a function f is even, holomorphic on a neighborhood of $\mathbb{S}^1_r$ and real-valued on $\mathbb{S}^1_r$, and any zero of f on $\mathbb{S}^1_r$ has an even order, then there is a function g such that $g^2 = f$ and g is also holomorphic on a neighborhood of $\mathbb{S}^1_r$, and each such g is either even in λ or odd in λ.*

Proof. Note that either $f \geq 0$ on $\mathbb{S}^1_r$ or $f \leq 0$ on $\mathbb{S}^1_r$. Without loss of generality, we assume further that $f \geq 0$ on $\mathbb{S}^1_r$. Locally, f has two holomorphic square roots which are real at points of $\mathbb{S}^1_r$. If they change sign at a zero $\lambda_0 \in \mathbb{S}^1_r$ of f, then the corresponding local holomorphic square roots near $-\lambda_0$ also change sign at $-\lambda_0$, since f is even in λ. Thus, local holomorphic square roots can be extended to become global holomorphic square roots on a neighborhood of $\mathbb{S}^1_r$: consider a $\lambda_* \in \mathbb{S}^1_r$ which is not a zero of f, and along $\mathbb{S}^1_r$ extend holomorphically the positive local holomorphic square root near λ_*, then the extension is still positive at λ_* (and hence agrees with the original square root at λ_*) when $\mathbb{S}^1_r$ is exhausted. Note that two local holomorphic square roots of f only differ by a sign. Since again f is even in λ, g must be either even in λ or is odd in λ. □

We denote by $\mathcal{W}_r$ the Wiener algebra [8] on $\mathbb{S}^1_r$, i.e., the Banach algebra of continuous functions f on $\mathbb{S}^1_r$ whose Fourier expansion $f(\lambda) = \sum_{i\in\mathbb{Z}} f_i\lambda^i$ exists and satisfies

$$\|f\|_r := \sum_{i\in\mathbb{Z}} |f_i| r^i < +\infty. \tag{1.2}$$

Moreover, $\mathcal{W}_r^*$ stands for the subalgebra of $\mathcal{W}_r$ consisting of elements which never vanish on $\mathbb{S}^1_r$, $\mathcal{W}_r^-$ the subalgebra consisting of elements whose Fourier expansion has only negative powers of λ, and $\mathcal{W}_r^+$ the subalgebra consisting of elements whose Fourier expansion does not involve any negative power of λ. We will need the following results later.

LEMMA 1.3. *Let $n \in \mathbb{N}$. If $h \in \mathcal{W}_r^*$ is always positive on $\mathbb{S}^1_r$, then there is a $k_+ \in \mathcal{W}_r^+$ never vanishing on $\mathcal{D}_r$ and satisfying*

$$h(\lambda) = k_+(\lambda)^n \overline{k_+\big(r^2/\overline{\lambda}\big)^n} \quad \text{on } \mathbb{S}^1_r. \tag{1.4}$$

Moreover, if h is even in λ, then so is k_+; if $h(-\lambda) = 1/h(\lambda)$ for all $\lambda \in \mathbb{S}^1_r$, then $k_+ = \mathrm{e}^{\alpha_+}$ for some $\alpha_+ \in \mathcal{W}_r^+$ which is odd in λ.

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Proof. By the Birkhoff decomposition on $\mathbb{S}^1_r$ [6],

$$h(\lambda) = c\lambda^m h_-(\lambda)h_+(\lambda) \quad \text{on } \mathbb{S}^1_r, \tag{1.5}$$

where c is a non-zero constant, $m \in \mathbb{Z}$, $h_- \in \mathcal{W}_r^-$ never vanishes on $\overline{\mathcal{E}_r} := \{|\lambda| \geq r\} \cup \{\infty\}$ and satisfies $h_-(\infty) = 1$, while $h_+ \in \mathcal{W}_r^+$ never vanishes on the closure $\overline{\mathcal{D}_r}$ of $\mathcal{D}_r$ and satisfies $h_+(0) = 1$. By the uniqueness of the Birkhoff decomposition, the reality of h on $\mathbb{S}^1_r$ is equivalent to that

$$c \in \mathbb{R}, \quad m = 0, \quad h_-(\lambda) = \overline{h_+(\lambda)} = \overline{h_+\big(r^2/\overline{\lambda}\big)} \quad \text{on } \mathbb{S}^1_r. \tag{1.6}$$

So, $c > 0$ by the positivity of h on $\mathbb{S}^1_r$. Since $h_+ \in \mathcal{W}_r^+$ never vanishes on $\overline{\mathcal{D}_r}$, it has an n-th root in $\mathcal{W}_r^+$, say l_+. Therefore, we can set $k_+ = c' l_+$, where $c' > 0$ is a $2n$-th root of c.

If h is even in λ, then (1.4) and the uniqueness of the Birkhoff decomposition yield the evenness of k_+. If $h(-\lambda) = 1/h(\lambda)$ for all $\lambda \in \mathbb{S}^1_r$, then $k_+(\lambda)k_+(-\lambda) = 1$, and hence $k_+ = \mathrm{e}^{\alpha_+}$ for some $\alpha_+ \in \mathcal{W}_r^+$ which is odd in λ. ∎

We will frequently use the following relation between the Wiener algebra and the set of analytic functions: let $0 \leq r < R \leq +\infty$, then a function on $\{\lambda \in \mathbb{C} \,;\, r < |\lambda| < R\}$ is in $\mathcal{W}_s$ for all $s \in (r, R)$ if and only if it is analytic on the annulus.

Set $\boldsymbol{G} = \mathrm{SU}(2)$. Then, the complexification of $\boldsymbol{G}$ is $\boldsymbol{G}^{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$. Let $r \in (0, +\infty)$. We introduce the following twisted r-loop groups (see [4], [2] and [3] for details), in which $\lambda \in \mathbb{S}^1_r$ serves as the loop parameter:

$$\Lambda_r \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} = \left\{ G : \mathbb{S}^1_r \to \boldsymbol{G}^{\mathbb{C}} \,;\, \begin{array}{c} g_{ij} \in \mathcal{W}_r \text{ for each } i,\, j \\ g_{11},\, g_{22} \text{ are even in } \lambda \\ g_{12},\, g_{21} \text{ are odd in } \lambda \end{array} \right\}, \tag{1.7}$$

$$\Lambda_r \boldsymbol{G}_{\mathrm{t}} = \left\{ G \in \Lambda_r \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} \,;\, \begin{array}{l} G \text{ has a continuous extension} \\ \text{up to } \mathbb{S}^1 \text{ and is unitary on } \mathbb{S}^1 \end{array} \right\}, \tag{1.8}$$

$$\Lambda^-_{r,*} \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} = \left\{ G \in \Lambda_r \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} \,;\, \begin{array}{l} G \text{ extends holomorphically} \\ \text{to } \mathcal{E}_r, \text{ and } G(\infty) = I \end{array} \right\}, \tag{1.9}$$

$$\Lambda^+_r \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} = \big\{ G \in \Lambda_r \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} \,;\, G \text{ extends holomorphically to } \mathcal{D}_r \big\}, \tag{1.10}$$

$$\Lambda^+_{r,\boldsymbol{B}} \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} = \big\{ G \in \Lambda^+_r \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} \,;\, G(0) \in \boldsymbol{B} \big\}, \tag{1.11}$$

where the g_{jk}'s are the entries of G, $\mathcal{E}_r = \{z \in \mathbb{C} \,;\, |z| > r\} \cup \{\infty\}$ and $\boldsymbol{B}$ is the solvable subgroup

$$\left\{ \begin{pmatrix} \mathrm{e}^s & 0 \\ 0 & \mathrm{e}^{-s} \end{pmatrix} \,;\, s \in \mathbb{R} \right\} \tag{1.12}$$

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of $\boldsymbol{G}^{\mathbb{C}}$. Then $\Lambda_r\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ is a complex Banach Lie group. Note that the map

$$G(\lambda) \longmapsto \overline{G(1/\overline{\lambda})} \tag{1.13}$$

is a Lie group isomorphism from $\Lambda_r\boldsymbol{G}_{\mathrm{t}}$ onto $\Lambda_{1/r}\boldsymbol{G}$, and each element of them is holomorphic in λ on the open annulus between $\mathbb{S}^1_r$ and $\mathbb{S}^1_{1/r}$. The Lie algebra of $\Lambda_r\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ is denoted by $\Lambda_r\mathcal{G}^{\mathbb{C}}$, and one has similar notation for the Lie algebras of the other Lie groups defined above. When $r = 1$, we will always omit the corresponding subindex. The following results are from Theorem 2.3 of [4].

THEOREM 1.14. *The multiplication* $\Lambda_r\boldsymbol{G}_{\mathrm{t}} \times \Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} \to \Lambda_r\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ *is an onto diffeomorphism. The multiplication* $\Lambda^-_{r,*}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} \times \Lambda^+_r\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} \to \Lambda_r\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ *is a diffeomorphism onto the open and dense subset* $\Lambda^-_{r,*}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} \cdot \Lambda^+_r\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$.

The splitting $\Lambda_r\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} = \Lambda_r\boldsymbol{G}_{\mathrm{t}} \times \Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ will be called the *r-Iwasawa splitting*, while the splitting $\Lambda^-_{r,*}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} \cdot \Lambda^+_r\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} = \Lambda^-_{r,*}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} \times \Lambda^+_r\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} \subseteq \Lambda_r\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ is usually called the *r-Birkhoff splitting*. Both of these splittings are analytic. The elements of $\Lambda_r\boldsymbol{G}_{\mathrm{t}}$ will sometimes be called *unitary r-loops* or said to be *r-unitary*. When $r = 1$, we will also always omit the corresponding prefix.

Remark 1.15. If $r \in (0,1)$, $G \in \Lambda_r\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ is holomorphic on $\mathcal{D}_r \setminus \overline{\mathcal{D}_\rho}$ for some $\rho \in (0,r)$ and $G = G_{\mathrm{u}}G_+$ is its r-Iwasawa splitting, then G_{u} is holomorphic on $\mathcal{D}_{1/\rho} \setminus \overline{\mathcal{D}_\rho}$.

The untwisted loop group $\Lambda_r\boldsymbol{G}^{\mathbb{C}}$ can be defined by removing the even-odd conditions in (1.7), and its corresponding subgroups can be introduced analogously. We omit the details.

§2. Unitarization of one loop matrix

In this section, we give conditions for the unitarizability of one loop matrix and determine all the possible ways to do the unitarization. As pointed out in the introduction, the results on all the possible ways for unitarizing one loop matrix will be used in Sections 3 and 4 in the unitarization of two or more loop matrices. We also give examples showing that in general, it is necessary to use r-dressing with $r < 1$ for unitarization.

Let $r \in (0,1]$. We will use the notation $\Lambda\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} \cap \Lambda_r\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ to denote the elements of $\Lambda\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ whose entries also converge under the $\mathcal{W}_r$-topology, i.e.,

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the "common part" of $\Lambda\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ and $\Lambda_r\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$. An element $L \in \Lambda\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}} \cap \Lambda_r\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ is said to be *unitarizable via r-dressing* if there is a $T_+ \in \Lambda_r^+\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ such that the element $T_+LT_+^{-1}$ of $\Lambda_r\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ is r-unitary, i.e., has a continuous extension up to $\mathbb{S}^1$ and is unitary on $\mathbb{S}^1$. In this case, if $0 < r < 1$, then $T_+LT_+^{-1}$ is actually holomorphic on $\{\lambda \in \mathbb{C}\,;\, r < |\lambda| < 1/r\}$.

Sometimes, we will abbreviate unitarization via 1-dressing as unitarization via dressing.

The following proposition shows that all the elements of $\Lambda_r\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ (not only just the elements of $\Lambda_r^+\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$) can be used for unitarization.

PROPOSITION 2.1. *Let $r \in (0,1]$. An element $L \in \Lambda\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}} \cap \Lambda_r\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ is unitarizable via r-dressing if and only if there is a $T \in \Lambda_r\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ such that the element TLT^{-1} of $\Lambda_r\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ is r-unitary.*

Proof. The necessity is clear. To show the sufficiency, let $T = T_{\mathrm{u}}T_+$ be the r-Iwasawa splitting of T with $T_{\mathrm{u}} \in \Lambda_r\boldsymbol{G}_{\mathrm{t}}$ and $T_+ \in \Lambda_{r,\boldsymbol{B}}^+\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$, then $T_+LT_+^{-1} = T_{\mathrm{u}}^{-1} \cdot TLT^{-1} \cdot T_{\mathrm{u}}$ is r-unitary, and hence L is unitarizable via r-dressing. ∎

Conditions for unitarizability

The following proposition (see Theorem 3.5 in [3]) gives a characterization of an important class of unitarizable loop matrices. Recall that an element of $\mathrm{SL}(2,\mathbb{C})$ is semi-simple if and only if it is diagonalizable.

PROPOSITION 2.2. i) *Let $r \in (0,1]$. If $L \in \Lambda\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}} \cap \Lambda_r\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ is unitarizable via r-dressing, then*

$$\operatorname{tr} L(\lambda) \in \mathbb{R} \quad \text{and} \quad \big|\operatorname{tr} L(\lambda)\big| \le 2 \quad \text{for all } \lambda \in \mathbb{S}^1. \tag{2.3}$$

ii) *Assume that $L \in \Lambda\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ is holomorphic on a neighborhood of $\mathbb{S}^1$ and semi-simple on $\mathbb{S}^1$. If L satisfies* (2.3)*, then it is unitarizable via r-dressing for some $r \in (0,1]$.*

Remark 2.4. Note that $L \in \Lambda\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ satisfies (2.3) if and only if the eigenvalues of $L(\lambda)$ are unimodular for all $\lambda \in \mathbb{S}^1$.

In this paper, we will use $v(\lambda)$ to denote an eigenvalues of $L(\lambda)$, opposite to [3], in which $v(\lambda)$ is used to stand for the imaginary part of an eigenvalues. So, when $L(\lambda)$ satisfies (2.3), its eigenvalues are $v(\lambda)$ and $1/v(\lambda)$.

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PROPOSITION 2.5. *Assume that $L \in \Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ is holomorphic on a neighborhood of $\mathbb{S}^1$ and satisfies* (2.3). *Then, the eigenvalues $v(\lambda)$ and $1/v(\lambda)$ of $L(\lambda)$ can be chosen to depend holomorphically on λ on a neighborhood of $\mathbb{S}^1$. The real parts of v and $1/v$ are both even in λ, and the imaginary parts of v and $1/v$ are either both even in λ or both odd in λ.*

Proof. Note that $4 - (\operatorname{tr} L)^2$ is even in λ. From (2.3) we deduce that $4 - (\operatorname{tr} L)^2$ is real on $\mathbb{S}^1$ and only has zeros of even order on $\mathbb{S}^1$. Thus, by Lemma 1.1, there is a function g which is holomorphic on a neighborhood of $\mathbb{S}^1$ and satisfies $g^2 = 4 - (\operatorname{tr} L)^2$. So, we can take $v = (\operatorname{tr} L + \mathrm{i}g)/2$ and $1/v = (\operatorname{tr} L - \mathrm{i}g)/2$. The real parts both equal $\operatorname{tr} L/2$ and are even in λ, while g must be either even in λ or odd in λ. □

The eigenvalues $v(\lambda)$ and $1/v(\lambda)$ of $L(\lambda)$ that depend holomorphically on λ on a neighborhood of $\mathbb{S}^1$, as produced in Proposition 2.5, will be called *holomorphic eigenvalues*. We will always use such eigenvalues. Moreover, when the imaginary parts of v and $1/v$ are even in λ, we will say that we have the *even case*; while when the imaginary parts of v and $1/v$ are odd in λ, the situation will called the *odd case*.

We note that the holomorphic eigenvalues of the monodromy of a non-cylindrical Delaunay end are even in λ, while those of a cylindrical end are not. These examples motivate our distinction between the even case and odd case.

Using the concept of holomorphic eigenvalues, we have the following refinement of Part ii) of Proposition 2.2.

THEOREM 2.6. *Let $L = (l_{ij}) \in \Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ be holomorphic on a neighborhood of $\mathbb{S}^1$ and satisfy* (2.3). *Assume that the holomorphic eigenvalues v and $1/v$ of L are not always equal on $\mathbb{S}^1$, i.e., $\operatorname{tr} L$ is not always equal to ± 2. Then, for each $r \in (0,1)$ sufficiently close to 1, L is unitarizable via r-dressing.*

Proof. Our proof is basically the one of Theorem 3.5 in [3]. It is presented here because some details will be needed later.

In the even case, if we set

$$T = \begin{pmatrix} 1 & l_{12}/(v - l_{22}) \\ -l_{21}/(v - 1/v) & (l_{11} - 1/v)/(v - 1/v) \end{pmatrix} \tag{2.7}$$

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when L is not triangular (i.e., $l_{12} \not\equiv 0 \not\equiv l_{21}$, which imply that neither of l_{11} and l_{22} is always equal to any of v and $1/v$ on $\mathbb{S}^1$),

$$T = \begin{pmatrix} 1 & 0 \\ l_{21}/(1/v - v) & 1 \end{pmatrix} \tag{2.8}$$

when L is lower triangular (i.e., $l_{12} \equiv 0$), and

$$T = \begin{pmatrix} 1 & l_{12}/(v - 1/v) \\ 0 & 1 \end{pmatrix} \tag{2.9}$$

when L is upper triangular (i.e., $l_{21} \equiv 0$), then for each $r \in (0, 1)$ sufficiently close to 1, $T \in \Lambda_r \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ and it diagonalizes L into $\Lambda_r \boldsymbol{G}_{\mathrm{t}}$, i.e.,

$$TLT^{-1} = \begin{pmatrix} v & 0 \\ 0 & 1/v \end{pmatrix} \in \Lambda_r \boldsymbol{G}_{\mathrm{t}}. \tag{2.10}$$

In the odd case, $l_{12}v_{\mathrm{I}} \not\equiv 0$ is even in λ, where $v = v_{\mathrm{R}} + \mathrm{i}v_{\mathrm{I}}$ with v_{R} and v_{I} being real on $\mathbb{S}^1$. Moreover, for each $r \in (0, 1)$ sufficiently close to 1, we have that $T \in \Lambda_r \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ and

$$TLT^{-1} = \begin{pmatrix} v_{\mathrm{R}} & \mathrm{i}v_{\mathrm{I}} \\ \mathrm{i}v_{\mathrm{I}} & v_{\mathrm{R}} \end{pmatrix} \in \Lambda_r \boldsymbol{G}_{\mathrm{t}}, \tag{2.11}$$

where

$$T = \frac{\mathrm{i}}{\sqrt{\mathrm{i}\, l_{12} v_{\mathrm{I}}}} \begin{pmatrix} v_{\mathrm{R}} - l_{11} & -l_{12} \\ -\mathrm{i}v_{\mathrm{I}} & 0 \end{pmatrix} \tag{2.12}$$

if $\mathrm{i}\, l_{12} v_{\mathrm{I}}$ has an even root $\sqrt{\mathrm{i}\, l_{12} v_{\mathrm{I}}}$ which is holomorphic along $\mathbb{S}^1_r$, and

$$T = \frac{1}{\sqrt{\mathrm{i}\, l_{12} v_{\mathrm{I}}}} \begin{pmatrix} \mathrm{i}v_{\mathrm{I}} & 0 \\ l_{11} - v_{\mathrm{R}} & l_{12} \end{pmatrix} \tag{2.13}$$

if $\mathrm{i}\, l_{12} v_{\mathrm{I}}$ has an odd root $\sqrt{\mathrm{i}\, l_{12} v_{\mathrm{I}}}$ which is holomorphic along $\mathbb{S}^1_r$. □

Remark 2.14. We emphasize that the matrices in (2.7)–(2.9) are meromorphic on a neighborhood of $\mathbb{S}^1$, while the matrices in (2.12) and (2.13) are holomorphic on $\mathcal{D} \setminus \overline{\mathcal{D}_s}$ for some $s \in (0, 1)$ and the product of any two of their entries has a meromorphic extension to a neighborhood of $\mathbb{S}^1$.

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The form given in (2.11) for elements of $\Lambda_r \boldsymbol{G}_{\mathrm{t}}$ is called the *anti-diagonal form*, while the one in (2.10) the *diagonal form*. All the elements of $\Lambda_r \boldsymbol{G}_{\mathrm{t}}$ in the anti-diagonal form make up a subgroup of $\Lambda_r \boldsymbol{G}_{\mathrm{t}}$, to be denoted by $\Lambda_r^{\mathrm{a}} \boldsymbol{G}_{\mathrm{t}}$ (and sometimes abbreviated as $\Lambda^{\mathrm{a}} \boldsymbol{G}_{\mathrm{t}}$ when $r = 1$). Similarly, we will use $\Lambda_r^{\mathrm{d}} \boldsymbol{G}_{\mathrm{t}}$ to denote the subgroup of $\Lambda_r \boldsymbol{G}_{\mathrm{t}}$ consisting of the diagonal elements of $\Lambda_r \boldsymbol{G}_{\mathrm{t}}$ (and frequently abbreviate $\Lambda_1^{\mathrm{d}} \boldsymbol{G}_{\mathrm{t}}$ as $\Lambda^{\mathrm{d}} \boldsymbol{G}_{\mathrm{t}}$).

Remark 2.15. Let $r \in (0, 1]$.

i) If $L \in \Lambda_r \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ is diagonalizable in $\Lambda_r \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$, then its eigenvalues are equal for all $\lambda \in \mathbb{S}_r^1$ if and only if $L \equiv I$ or $-I$.

ii) Assume that $L \in \Lambda_r \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}} \setminus \{\pm I\}$ is diagonalized by $T \in \Lambda_r \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$, and the eigenvalues of L are holomorphic on a neighborhood of $\mathbb{S}_r^1$. Then, the general $S \in \Lambda_r \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ diagonalizing L differs from T by a diagonal or off-diagonal factor on the left. Note that a diagonal factor on the left does not change TLT^{-1}, while an off-diagonal one only interchanges the eigenvalues in TLT^{-1}.

General form of all unitarizing matrices

If the eigenvalues of a matrix $L \in \mathrm{SL}(2, \mathbb{C})$ are distinct and have unit modulus, then there is a matrix $T \in \mathrm{SL}(2, \mathbb{C})$ such that TLT^{-1} is diagonal (and hence unitary), and the general $S \in \mathrm{SL}(2, \mathbb{C})$ unitarizing L has the form $S = UDT$ with unitary U and diagonal D. We generalize this matrix result to the loop group setting.

THEOREM 2.16. *Let $L \in \Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ be holomorphic on a neighborhood of $\mathbb{S}^1$.*

i) *If $T \in \Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ diagonalizes L into $\Lambda \boldsymbol{G}_{\mathrm{t}}$ and $L \notin \{\pm I\}$, then the general $S \in \Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ unitarizing L has the form $S = UG_+T$ with $U \in \Lambda \boldsymbol{G}_{\mathrm{t}}$ and* $\operatorname{diagonal} G_+ \in \Lambda^+ \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$.

ii) *If $T \in \Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ conjugates L into $\Lambda^{\mathrm{a}} \boldsymbol{G}_{\mathrm{t}}$ and $L \notin \{\pm I\}$, then the general $S \in \Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ unitarizing L has the form $S = UG_+T$ with $U \in \Lambda \boldsymbol{G}_{\mathrm{t}}$ and*

$$G_+ = \begin{pmatrix} \cosh \alpha_+ & \sinh \alpha_+ \\ \sinh \alpha_+ & \cosh \alpha_+ \end{pmatrix} \tag{2.17}$$

for an odd $\alpha_+ \in \mathcal{W}^+$.

iii) *Assume that L satisfies* (2.3)*, and the two holomorphic eigenvalues of L are even in λ and not always equal to each other. Let T be the matrix given by* (2.7) *or* (2.8) *or* (2.9) *for L. Then, for any $r \in (0, 1)$ sufficiently*

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close to 1, *the general* $S \in \Lambda_r \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ *unitarizing* L *has the form* $S = UG_+T$ *with* $U \in \Lambda_r \boldsymbol{G}_{\mathrm{t}}$ *and* $\operatorname{diagonal} G_+ \in \Lambda_r^+ \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$.

iv) *Assume that* L *satisfies* (2.3), *and the two holomorphic eigenvalues of* L *are not even in* λ. *Let* T *be the matrix given by* (2.12) *or* (2.13) *for* L. *Then, for any* $r \in (0,1)$ *sufficiently close to* 1, *the general* $S \in \Lambda_r \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ *unitarizing* L *has the form* $S = UG_+T$ *with* $U \in \Lambda_r \boldsymbol{G}_{\mathrm{t}}$ *and* $G_+ \in \Lambda_r^+ \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ *given by* (2.17) *for an odd* $\alpha_+ \in \mathcal{W}_r^+$.

Proof. First, we show Parts i) and iii). We only need to prove the necessity of the form of S, since the sufficiency is trivial. Set

$$TLT^{-1} = \begin{pmatrix} v & 0 \\ 0 & 1/v \end{pmatrix}. \tag{2.18}$$

By our assumptions, $v - 1/v$ has at most finitely many zeros on $\mathbb{S}^1$. Note also that $1/v = \overline{v}$ on $\mathbb{S}^1$.

i) Now, assume that $S \in \Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ unitarizes L. Let $ST^{-1} = UG_+$ be the Iwasawa splitting of ST^{-1} and set

$$G_+ = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.19}$$

then

$$\begin{aligned} &\begin{pmatrix} v + bc(v - 1/v) & ab(1/v - v) \\ cd(v - 1/v) & 1/v + bc(1/v - v) \end{pmatrix} \\ &\quad = G_+ \begin{pmatrix} v & 0 \\ 0 & 1/v \end{pmatrix} G_+^{-1} = U^{-1} \cdot SLS^{-1} \cdot U \end{aligned} \tag{2.20}$$

is unitary, i.e.,

$$bc(1/v - v) = \overline{bc(v - 1/v)}, \quad cd(v - 1/v) = -\overline{ab(1/v - v)} \quad \text{on } \mathbb{S}^1. \tag{2.21}$$

The first identity of (2.21) implies that bc is real on $\mathbb{S}^1$. Hence, bc has a holomorphic extension to the whole Riemann sphere $\mathbb{S}^2$, since it is holomorphic inside $\mathcal{D}$ (because $b, c \in \mathcal{W}^+$). So, bc is constant and actually equals 0, since $b(0)c(0) = 0$. Thus, either $b = 0$ or $c = 0$. Moreover, $ad = ad - bc = 1$. Therefore, the second identity of (2.21) yields that $b = c = 0$, i.e., G_+ is diagonal.

iii) By our assumptions, there is an $s \in (0,1)$ such that L and its eigenvalues v and $1/v$ are holomorphic on $\mathcal{N} := \mathcal{D}_{1/s} \setminus \overline{\mathcal{D}_s}$, $v - 1/v$ does

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not have any zero on $\mathcal{N} \setminus \mathbb{S}^1$, and T is holomorphic on $\mathcal{N} \setminus \mathbb{S}^1$. Fix an $r \in (s, 1)$. Note that this r can be as close to 1 as one wants. Now, assume that $S \in \Lambda_r \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ unitarizes L. Let $ST^{-1} = UG_+$ be the r-Iwasawa splitting of ST^{-1} and define a, b, c and d by (2.19). Then, $U^{-1} \cdot SLS^{-1} \cdot U$ is given by (2.20) and is r-unitary. Thus, ab, bc and cd have meromorphic extensions to $\mathcal{D}_{1/r}$, their poles in $\mathcal{D}_{1/r}$ are on $\mathbb{S}^1$, and

$$bc = \overline{bc}, \quad ab = -\overline{cd} \quad \text{on } \mathbb{S}^1. \tag{2.22}$$

So, ab, bc, cd and $ad = 1 + bc$ are meromorphic functions on the whole Riemann sphere $\mathbb{S}^2$ with poles only on $\mathbb{S}^1$, and bc is real on $\mathbb{S}^1$. Since $ad \cdot bc = ab \cdot cd$ on $\mathcal{D}_r$ and hence on $\mathbb{S}^2$, from the second identity of (2.22) we obtain that $ad \cdot bc = -ab \cdot \overline{ab}$ on $\mathbb{S}^1$, which together with $ad = 1 + bc$ yield that

$$(1 + bc)bc = -ab \cdot \overline{ab} \leq 0 \quad \text{on } \mathbb{S}^1. \tag{2.23}$$

The reality of bc on $\mathbb{S}^1$ now implies that $-1 \leq bc \leq 0$ on $\mathbb{S}^1$. Thus, bc does not have any pole on $\mathbb{S}^1$ and hence is actually holomorphic on the whole Riemann sphere $\mathbb{S}^2$. The rest of the arguments for this part are the same as the corresponding ones for Part i).

Next, we show Parts ii) and iv). We note that Parts i) and iii) are also true for the untwisted loop group (in this setting, $\boldsymbol{B}$ is replaced by the subgroup of $\mathrm{SL}(2, \mathbb{C})$ consisting of the upper triangular matrices with positive diagonal), since for any G_+ given by (2.19) in the untwisted version of $\Lambda^+_{\boldsymbol{B}} \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ or $\Lambda^+_{r,\boldsymbol{B}} \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ we still have that $c(0) = 0$.

ii) Let S, U, G_+, a, b, c and d be as in the proof of Part i). Set

$$T_* = \frac{\mathrm{i}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{2.24}$$

then $T_* \in \mathrm{SU}(2)$, and

$$T_* \begin{pmatrix} v_{\mathrm{R}} & \mathrm{i}v_{\mathrm{I}} \\ \mathrm{i}v_{\mathrm{I}} & v_{\mathrm{R}} \end{pmatrix} T_*^{-1} = \begin{pmatrix} v & 0 \\ 0 & 1/v \end{pmatrix}, \tag{2.25}$$

which is only in the untwisted loop group in general. Note that T_*T diagonalizes L. Split G_+ into its constant term times its "strictly positive

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part":

$$G_+(\lambda) = \begin{pmatrix} \mathrm{e}^{\alpha_0} & 0 \\ 0 & \mathrm{e}^{-\alpha_0} \end{pmatrix} G_{++}(\lambda) \tag{2.26}$$
$$= \begin{pmatrix} \mathrm{e}^{\alpha_0} & 0 \\ 0 & \mathrm{e}^{-\alpha_0} \end{pmatrix} (I + G_1\lambda + G_2\lambda^2 + \cdots)$$

with $\alpha_0 \in \mathbb{R}$. Then, the Iwasawa splitting of $S(T_*T)^{-1} = UG_+T_*^{-1}$ in the untwisted loop group is

$$UT_*^{-1}\frac{1}{\beta}\begin{pmatrix} \cosh\alpha_0 & -\sinh\alpha_0 \\ \sinh\alpha_0 & \cosh\alpha_0 \end{pmatrix}\cdot\begin{pmatrix} \beta & (\beta+\frac{1}{\beta})\tanh\alpha_0 \\ 0 & \frac{1}{\beta} \end{pmatrix} T_*G_{++}T_*^{-1}, \tag{2.27}$$

where $\beta = \sqrt{\cosh^2\alpha_0 + \sinh^2\alpha_0}$. Thus,

$$\begin{pmatrix} \beta & (\beta+\frac{1}{\beta})\tanh\alpha_0 \\ 0 & \frac{1}{\beta} \end{pmatrix} T_*G_{++}T_*^{-1} \tag{2.28}$$

is diagonal. So, $\alpha_0 = 0$, and $T_*G_{++}T_*^{-1} = \mathrm{diag}\{d_{++}, 1/d_{++}\}$ for some invertible $d_{++} \in \mathcal{W}^+$ satisfying $d_{++}(0) = 1$. Hence, G_+ is given by (2.17) with $\alpha_+ \in \mathcal{W}^+$ such that $\mathrm{e}^{\alpha_+} = d_{++}$. From $G_+ \in \Lambda^+\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ and the identity

$$\begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix}\begin{pmatrix} \cosh\delta & \sinh\delta \\ \sinh\delta & \cosh\delta \end{pmatrix} = \begin{pmatrix} \cosh(\gamma+\delta) & \sinh(\gamma+\delta) \\ \sinh(\gamma+\delta) & \cosh(\gamma+\delta) \end{pmatrix} \tag{2.29}$$

we see that α_+ is odd.

iv) This part can be shown similar to Part ii). □

Remark 2.30. In the odd case, i.e., in Parts ii) and iv), we just showed that

$$S = UT_*^{-1}\cdot\begin{pmatrix} \mathrm{e}^{\alpha_+} & 0 \\ 0 & \mathrm{e}^{-\alpha_+} \end{pmatrix}\cdot\widehat{T} \tag{2.31}$$

with $U \in \Lambda_r\boldsymbol{G}_{\mathrm{t}}$, odd $\alpha_+ \in \mathcal{W}_r^+$ and $\widehat{T} = T_*T$, where $r = 1$ for Part ii).

We note that in Theorem 2.16, whence T is chosen, the factorization $S = UG_+T$ of S with U in $\Lambda\boldsymbol{G}_{\mathrm{t}}$ or $\Lambda_r\boldsymbol{G}_{\mathrm{t}}$ and G_+ in $\Lambda_{\boldsymbol{B}}^+\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ or $\Lambda_{r,\boldsymbol{B}}^+\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ comes from the Iwasawa decomposition of ST^{-1}, and hence is unique. Moreover, Theorem 2.16 can be restated as the following: G_+ either is diagonal or has the form given by (2.17), and this is the only condition on S.

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Lower and upper triangular loop matrices

The following result gives a necessary and sufficient condition for unitarization via dressing for the lower triangular loop matrices. Note that now we are in the even case.

PROPOSITION 2.32. *Let $L \in \Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ be holomorphic on a neighborhood of $\mathbb{S}^1$ and satisfy* (2.3)*. Assume that $l_{12} = 0$.*

i) *If $l_{11} \equiv l_{22}$, then L is unitarizable via dressing if and only if $L = \pm I$.*

ii) *If $l_{11} \not\equiv l_{22}$, then L is unitarizable via dressing if and only if the meromorphic function $l_{21}/(l_{11} - l_{22})$ does not have any pole on $\mathbb{S}^1$, i.e., if and only if the vanishing order of l_{21} at any $\lambda_0 \in \mathbb{S}^1$ is at least that of $l_{11} - l_{22}$ at λ_0.*

Proof. Since $l_{12} = 0$, the holomorphic eigenvalues of L are l_{11} and l_{22}. The trace conditions imply that on $\mathbb{S}^1$, l_{11} and l_{22} are unimodular, and $l_{22} = 1/l_{11} = \overline{l_{11}}$.

i) From $l_{11} \equiv l_{22}$ we obtain that $l_{11} \equiv l_{22} \equiv \pm 1$ and hence the claim.

ii) Define T by (2.8) with $v = l_{11}$, then $T \in \Lambda_r \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ for every real $r \neq 1$ sufficiently close to 1, and

$$TLT^{-1} = \begin{pmatrix} l_{11} & 0 \\ 0 & l_{22} \end{pmatrix} \in \Lambda_r \boldsymbol{G}_{\mathrm{t}} \tag{2.33}$$

for each real r in a neighborhood of 1.

If $l_{21}/(l_{11} - l_{22})$ does not have any pole on $\mathbb{S}^1$, then it is holomorphic on a neighborhood of $\mathbb{S}^1$, and hence $T \in \Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ unitarizes L.

Conversely, assume that L is unitarizable via dressing, then there is a $G_+ \in \Lambda^+ \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ such that $G_+ L G_+^{-1} \in \Lambda \boldsymbol{G}_{\mathrm{t}}$. Let $r \in (0,1)$ be sufficiently close to 1. Then, $G_+ L G_+^{-1} \in \Lambda_r \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$, $G_+ L G_+^{-1}$ is hence r-unitary, and $G_+ = UD_+T$ for some $U \in \Lambda_r \boldsymbol{G}_{\mathrm{t}}$ and diagonal $D_+ \in \Lambda_r^+ \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ by Theorem 2.16 Part iii). From $D_+ = U^{-1} G_+ T^{-1}$ we see that D_+ has a continuous extension to $\overline{\mathcal{D}}$ with a finite number of points taken away that is holomorphic on $\mathcal{D}$. Set

$$D_+ = \begin{pmatrix} d_+ & 0 \\ 0 & 1/d_+ \end{pmatrix}, \tag{2.34}$$

then

$$U^{-1} G_+ = D_+ T = \begin{pmatrix} d_+ & 0 \\ l_{21}/[(l_{22} - l_{11}) d_+] & 1/d_+ \end{pmatrix} \in \Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}. \tag{2.35}$$

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This implies that

$$d_+ \in \mathcal{W}^*, \quad l_{21}/[(l_{11}-l_{22})d_+] \in \mathcal{W}. \tag{2.36}$$

As a consequence, $l_{21}/(l_{11}-l_{22})$ is in $\mathcal{W}$ and hence does not have any pole on $\mathbb{S}^1$. □

Note that the above proof also shows that in the case of Part ii) of the proposition, the general element of $\Lambda\boldsymbol{G}_{\mathrm{t}}$ unitarizing L is of the form UD_+T, where $U \in \Lambda\boldsymbol{G}_{\mathrm{t}}$, $D_+ \in \Lambda^+\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ is diagonal, and T is given by (2.8) with $v = l_{11}$.

Taking transpose inverse in the above proposition, one immediately obtains the corresponding result for upper triangular loop matrices.

Necessity of using r-dressing

Now, we present some loop matrices that can be unitarized via r-dressing with suitable $r < 1$, but not via dressing. So, *in general, r-dressing can not be replaced by dressing.*

EXAMPLE 2.37. For $s \in (0,+\infty) \setminus \{1/4, 1/2\}$, let $t = 1/2 - s$ and

$$f(\lambda) = 2\pi\sqrt{s^2 + st(\lambda^{-2}+\lambda^2) + t^2}. \tag{2.38}$$

Since $s^2+st(\lambda^{-2}+\lambda^2)+t^2$ is positive on $\mathbb{S}^1$ and even in λ, Lemma 1.1 implies that f is holomorphic on a neighborhood of $\mathbb{S}^1$. Thus, by Proposition 2.32, for any odd function g which is also holomorphic on a neighborhood of $\mathbb{S}^1$,

$$L(\lambda) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}f(\lambda)} & 0 \\ g(\lambda) & \mathrm{e}^{-\mathrm{i}f(\lambda)} \end{pmatrix} \in \Lambda\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}} \tag{2.39}$$

is unitarizable via dressing if and only if ± 1 are zeros of g of order ≥ 2. This is because ± 1 are the only zeros of $2\mathrm{i}\sin f = \mathrm{e}^{\mathrm{i}f} - \mathrm{e}^{-\mathrm{i}f}$, and they are double zeros. Note that L is always unitarizable via r-dressing for each $r \in (0,1)$ sufficiently close to 1.

In particular, the case with $g(\lambda) = \lambda$ yields

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}f(\lambda)} & 0 \\ \lambda & \mathrm{e}^{-\mathrm{i}f(\lambda)} \end{pmatrix} \in \Lambda\boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}, \tag{2.40}$$

which is not unitarizable via dressing, but always unitarizable via r-dressing for each $r \in (0,1)$ sufficiently close to 1.

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§3. Simultaneous unitarization of two loop matrices

In this section, we give conditions for the simultaneous unitarization of two loop matrices and show that under these conditions, there is a unique element of $\Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ unitarizing them, unless the two loop matrices commute. As pointed out in the introduction, this uniqueness result will be used in Section 5 to deal with the closing conditions.

If one of the two loop matrices to be simultaneously unitarized is equal to I or $-I$, then we only need to unitarize the other one. So, in the rest of this section, we will frequently exclude $\pm I$ from the two loop matrices to be simultaneously unitarized.

Commutativity of two loop matrices

First, we characterize the commutativity of two given loop matrices in terms of conditions on the loop matrices diagonalizing or anti-diagonalizing them individually.

PROPOSITION 3.1. *Let $0 < r \leq 1$, and $L_1, L_2 \in \Lambda_r\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} \setminus \{\pm I\}$. Assume that for $i = 1$ and 2, there exists a $T_i \in \Lambda_r\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ such that $T_iL_iT_i^{-1}$ is in $\Lambda^{\mathrm{d}}_r\boldsymbol{G}_{\mathrm{t}} \cup \Lambda^{\mathrm{a}}_r\boldsymbol{G}_{\mathrm{t}}$ and holomorphic on a neighborhood of $\mathbb{S}^1_r$. Then, the following statements are equivalent*:

i) *L_1 and L_2 commute*;
ii) *T_1 conjugates L_1 and L_2 both into $\Lambda^{\mathrm{d}}_r\boldsymbol{G}_{\mathrm{t}}$ or both into $\Lambda^{\mathrm{a}}_r\boldsymbol{G}_{\mathrm{t}}$*;
iii) *L_1 and L_2 are both in the even case or both in the odd case, and $T_*T_1T_2^{-1}T_*^{-1}$ is either diagonal or off-diagonal, where $T_* = I$ in the even case and is given by* (2.24) *in the odd case.*

Proof. Let $i = 1$ or 2. Set $\widehat{T}_i = T_*T_i$, where $T_* = I$ if L_i is in the even case and T_* is given by (2.24) if L_i is in the odd case. Then, $D_i := \widehat{T}_iL_i\widehat{T}_i^{-1}$ is diagonal, and its diagonal entries are holomorphic on a neighborhood of $\mathbb{S}^1_r$ and not always equal. Thus, an S in the untwisted r-loop group $\Lambda_r\boldsymbol{G}^{\mathbb{C}}$ diagonalizes L_i if and only if S differs from $\widehat{T}_i$ by a diagonal or off-diagonal factor on the left, and an off-diagonal factor on the left just interchanges the diagonal entries of $\widehat{T}_iL_i\widehat{T}_i^{-1}$.

i) $\Longrightarrow$ ii). From $\widehat{T}_1^{-1}D_1\widehat{T}_1L_2 = L_1L_2 = L_2L_1 = L_2\widehat{T}_1^{-1}D_1\widehat{T}_1$ we obtain

$$D_1 \cdot \widehat{T}_1L_2\widehat{T}_1^{-1} = \widehat{T}_1L_2\widehat{T}_1^{-1} \cdot D_1, \tag{3.2}$$

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which implies that $\widehat{T}_1 L_2 \widehat{T}_1^{-1}$ is diagonal. Thus, if L_1 is in the even case, then T_1 conjugates L_1 and L_2 both into $\Lambda_r^{\mathrm{d}} \boldsymbol{G}_{\mathrm{t}}$; if L_1 is in the odd case, then T_1 conjugates L_1 and L_2 both into $\Lambda_r^{\mathrm{a}} \boldsymbol{G}_{\mathrm{t}}$.

ii) $\Longrightarrow$ i). This is trivial.

i) $\Longrightarrow$ iii). By Part ii), L_1 and L_2 are both in the even case or both in the odd case. Substituting $L_2 = \widehat{T}_2^{-1} D_2 \widehat{T}_2$ into (3.2) yields that

$$D_1 \cdot \widehat{T}_1 \widehat{T}_2^{-1} D_2 \widehat{T}_2 \widehat{T}_1^{-1} = \widehat{T}_1 \widehat{T}_2^{-1} D_2 \widehat{T}_2 \widehat{T}_1^{-1} \cdot D_1, \tag{3.3}$$

from which we see that $\widehat{T}_1 \widehat{T}_2^{-1} D_2 \widehat{T}_2 \widehat{T}_1^{-1}$ is diagonal, and its entries are the same as those of D_2, but perhaps in different order. Thus, $T_* T_1 T_2^{-1} T_*^{-1} = \widehat{T}_1 \widehat{T}_2^{-1}$ is either diagonal or off-diagonal.

iii) $\Longrightarrow$ i). Since now $\widehat{T}_1 \widehat{T}_2^{-1} D_2 \widehat{T}_2 \widehat{T}_1^{-1}$ is diagonal by assumption, (3.3) follows, and hence

$$\widehat{T}_1^{-1} D_1 \widehat{T}_1 \cdot \widehat{T}_2^{-1} D_2 \widehat{T}_2 = \widehat{T}_2^{-1} D_2 \widehat{T}_2 \cdot \widehat{T}_1^{-1} D_1 \widehat{T}_1, \tag{3.4}$$

i.e., $L_1 L_2 = L_2 L_1$. □

CONVENTION. Suggested by the above proof, whenever TLT^{-1} is in $\Lambda_r^{\mathrm{d}} \boldsymbol{G}_{\mathrm{t}} \cup \Lambda_r^{\mathrm{a}} \boldsymbol{G}_{\mathrm{t}}$ and holomorphic on a neighborhood of $\mathbb{S}_r^1$ for some $r \in (0, 1]$, we set $\widehat{T} = T_* T$, where $T_* = I$ if L is in the even case and T_* is given by (2.24) if L is in the odd case.

Simultaneous unitarization of two loop matrices via dressing

Now, we study the simultaneous unitarization of two loop matrices via dressing. It turns out that when the two loop matrices do not commute, there is at most one element of $\Lambda_{\boldsymbol{B}}^{+} \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ simultaneously unitarizing them.

THEOREM 3.5. *Let $L_1, L_2 \in \Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}} \setminus \{\pm I\}$ be holomorphic on a neighborhood of $\mathbb{S}^1$. Assume that for $i = 1$ and 2, there exists a $T_i \in \Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ such that $T_i L_i T_i^{-1}$ is in $\Lambda^{\mathrm{d}} \boldsymbol{G}_{\mathrm{t}} \cup \Lambda^{\mathrm{a}} \boldsymbol{G}_{\mathrm{t}}$, and let $\widehat{T}_i$ be as in the above convention. Set*

$$\widehat{T}_1 \widehat{T}_2^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.6}$$

Then, L_1 and L_2 can be simultaneously unitarized via dressing if and only if either $b = c = 0$, or $a = d = 0$, or

$$abcd \not\equiv 0, \quad \text{and } \frac{d}{\overline{a}}, \ -\frac{c}{\overline{b}} \text{ are always finite, real and positive on } \mathbb{S}^1. \tag{3.7}$$

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In the first two cases, L_1 and L_2 commute; while in the last case, there is a unique element of $\Lambda^+_{\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ simultaneously unitarizing L_1 and L_2.

Proof. By Proposition 3.1, L_1 and L_2 can be simultaneously conjugated into $\Lambda^{\mathrm{d}}\boldsymbol{G}_{\mathrm{t}}$ or into $\Lambda^{\mathrm{a}}\boldsymbol{G}_{\mathrm{t}}$ if and only if $\widehat{T}_1\widehat{T}_2^{-1}$ is either diagonal or off-diagonal, i.e., if and only if L_1 and L_2 commute. So, for the rest of this proof, we can assume that $\widehat{T}_1\widehat{T}_2^{-1}$ is neither diagonal nor off-diagonal.

For $i = 1$ and 2, let $T_{*,i} = I$ if L_i is in the even case and $T_{*,i}$ equals the matrix in (2.24) if L_i is in the odd case.

From Parts i) and ii) of Theorem 2.16 we know that L_1 and L_2 can be simultaneously unitarized via dressing using an $S \in \Lambda\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ if and only if there are $U_1, U_2 \in \Lambda\boldsymbol{G}$ and $G_1, G_2 \in \Lambda^+_{\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ such that each of G_1 and G_2 is diagonal or has the form given in (2.17), and $S = U_1G_1T_1 = U_2G_2T_2$. The latter is equivalent to $S = V_1K_1\widehat{T}_1 = V_2K_2\widehat{T}_2$, where $V_i = U_iT_{*,i}^{-1}$ and $K_i = T_{*,i}G_iT_{*,i}^{-1}$ for $i = 1$ and 2. Note that K_1 and K_2 are diagonal by Remark 2.30, and that V_1, V_2, K_1, K_2, $\widehat{T}_1$ and $\widehat{T}_2$ are only in the untwisted loop group $\Lambda\boldsymbol{G}^{\mathbb{C}}$ in general, and V_1 and V_2 are in $\Lambda\boldsymbol{G}$.

Now, assume that L_1 and L_2 can be simultaneously unitarized via dressing. Then, using the notation of the last paragraph, $K_1\widehat{T}_1\widehat{T}_2^{-1}K_2^{-1} = V_1^{-1}V_2 \in \Lambda\boldsymbol{G}^{\mathbb{C}}$ is unitary. Set

$$K_i = \begin{pmatrix} k_i & 0 \\ 0 & 1/k_i \end{pmatrix} \quad \text{for } i = 1 \text{ and } 2. \tag{3.8}$$

Then,

$$K_1\widehat{T}_1\widehat{T}_2^{-1}K_2^{-1} = \begin{pmatrix} ak_1/k_2 & bk_1k_2 \\ c/(k_1k_2) & dk_2/k_1 \end{pmatrix}, \tag{3.9}$$

and hence

$$dk_2/k_1 = \overline{ak_1/k_2},\ \ c/(k_1k_2) = -\overline{bk_1k_2} \quad \text{on } \mathbb{S}^1. \tag{3.10}$$

So, none of a, b, c and d is identically zero on $\mathbb{S}^1$ (since $\widehat{T}_1\widehat{T}_2^{-1}$ is neither diagonal nor off-diagonal by our assumption), and $d/\overline{a}$ and $-c/\overline{b}$ are always finite, real and actually positive on $\mathbb{S}^1$. Thus, (3.7) is necessary.

Conversely, to show sufficiency, assume that (3.7) is true. By the Birkhoff decomposition on $\mathbb{S}^1$,

$$\frac{d(\lambda)}{\overline{a(\lambda)}} = \frac{1}{m_1}\lambda^m f_-(\lambda)f_+(\lambda) \quad \text{on } \mathbb{S}^1, \tag{3.11}$$

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where m_1 is a non-zero constant, $m \in \mathbb{Z}$, $f_- \in \mathcal{W}^-$ never vanishes on $\overline{\mathcal{E}} := \{|\lambda| \geq 1\} \cup \{\infty\}$ and satisfies $f_-(\infty) = 1$, while $f_+ \in \mathcal{W}^+$ never vanishes on $\overline{\mathcal{D}}$ and satisfies $f_+(0) = 1$. By the uniqueness of the Birkhoff decomposition, the reality of $d/\overline{a}$ on $\mathbb{S}^1$ is equivalent to that

$$m_1 \in \mathbb{R}, \quad m = 0, \quad \overline{f_+(\lambda)} = f_-(\lambda) \quad \text{on } \mathbb{S}^1. \tag{3.12}$$

Thus,

$$\frac{d(\lambda)}{\overline{a(\lambda)}} = \frac{1}{m_1}\overline{f_+(\lambda)} f_+(\lambda) \quad \text{on } \mathbb{S}^1. \tag{3.13}$$

The positivity of $d/\overline{a}$ on $\mathbb{S}^1$ then implies that $m_1 > 0$. Similarly,

$$-\frac{c(\lambda)}{\overline{b(\lambda)}} = n_1\overline{g_+(\lambda)} g_+(\lambda) \quad \text{on } \mathbb{S}^1, \tag{3.14}$$

where $n_1 > 0$, and $g_+ \in \mathcal{W}^+$ never vanishes on $\overline{\mathcal{D}}$ and satisfies $g_+(0) = 1$. Set

$$k_1 = (n_1/m_1)^{1/4}(f_+ g_+)^{1/2}, \;\; k_2 = (m_1 n_1)^{1/4}(g_+/f_+)^{1/2} \quad \text{on } \overline{\mathcal{D}}. \tag{3.15}$$

The roots here are possible since f_+ and g_+ can be represented in the form $\exp h_+$ with $h_+ \in \mathcal{W}_+$. Then, k_1 and k_2 belong to $\mathcal{W}^+$, never vanish on $\overline{\mathcal{D}}$, and satisfy (3.10). Note that $k_1(0) > 0$ and $k_2(0) > 0$. Define K_1 and K_2 by (3.8), then we have (3.9), and hence $V := K_1\widehat{T}_1\widehat{T}_2^{-1}K_2^{-1} \in \Lambda\boldsymbol{G}^{\mathbb{C}}$ is unitary. Moreover, from (3.10) or direct calculations we deduce that

$$k_1^2\overline{k_1}^2 = -\frac{cd}{\overline{ab}}, \;\; k_2^2\overline{k_2}^2 = -\frac{\overline{a}c}{\overline{b}d} = -\frac{\overline{ac}}{bd} \quad \text{on } \mathbb{S}^1. \tag{3.16}$$

Next, we proceed in three non-overlapping situations separately.

If L_1 and L_2 are both in the even case, then $\widehat{T}_1\widehat{T}_2^{-1} = T_1T_2^{-1}$, which implies that a and d are even in λ, while b and c are odd in λ. Thus, $cd/\overline{ab}$ and $\overline{ac}/(bd)$ are even in λ. By (3.16) and Lemma 1.3, k_1 and k_2 are also even in λ. Hence, $G_i := K_i \in \Lambda^+_{\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ for $i = 1$ and 2, and $U := V = K_1T_1T_2^{-1}K_2^{-1} \in \Lambda\boldsymbol{G}_{\mathrm{t}}$. So, L_1 and L_2 can be simultaneously unitarized via dressing using $G_1T_1 = UG_2T_2$.

If one of L_1 and L_2 is in the even case and the other is in the odd case, without loss of generality, we can assume that L_1 is in the even case and L_2 is in the odd case. Set

$$T_1T_2^{-1} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}. \tag{3.17}$$

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Then, $\tilde{a}$ and $\tilde{d}$ are even in λ, $\tilde{b}$ and $\tilde{c}$ are odd in λ, and

$$\frac{cd}{\overline{ab}} = \frac{\tilde{c}^2 - \tilde{d}^2}{\overline{\tilde{a}^2 - \tilde{b}^2}}, \quad \frac{\overline{ac}}{bd} = \frac{ac}{\overline{bd}} = \frac{\overline{(\tilde{a}+\tilde{b})(\tilde{c}+\tilde{d})}}{(\tilde{a}-\tilde{b})(\tilde{c}-\tilde{d})} = \frac{(\tilde{a}+\tilde{b})(\tilde{c}+\tilde{d})}{\overline{(\tilde{a}-\tilde{b})(\tilde{c}-\tilde{d})}}. \tag{3.18}$$

So, $cd/\overline{ab}$ is even in λ, and

$$\Big(\frac{\overline{ac}}{bd}\Big)(-\lambda) = \Big(\frac{bd}{\overline{ac}}\Big)(\lambda). \tag{3.19}$$

Thus, by (3.16) and Lemma 1.3, k_1 is even in λ, and $k_2 = \mathrm{e}^{\alpha_2}$ for some odd $\alpha_2 \in \mathcal{W}^+$. Then, we deduce that $G_1 := K_1 \in \Lambda^+_{\boldsymbol{B}} \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$, $G_2 := (T_{*,2})^{-1} K_2 T_{*,2} \in \Lambda^+_{\boldsymbol{B}} \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$, and

$$U := VT_{*,2} = K_1 \cdot T_1 T_2^{-1} \cdot T_{*,2}^{-1} K_2^{-1} T_{*,2} = G_1 \cdot T_1 T_2^{-1} \cdot G_2^{-1} \in \Lambda \boldsymbol{G}_{\mathrm{t}}. \tag{3.20}$$

Thus, $G_1 T_1 = U G_2 T_2$ simultaneously unitarizes L_1 and L_2.

If L_1 and L_2 are both in the odd case, similarly we can show that $k_i = \mathrm{e}^{\alpha_i}$ with an odd $\alpha_i \in \mathcal{W}^+$ and $G_i := T_{*,i}^{-1} K_i T_{*,i} \in \Lambda^+_{\boldsymbol{B}} \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ for $i = 1$ and 2, and $U := T_{*,1}^{-1} V T_{*,2} = G_1 \cdot T_1 T_2^{-1} \cdot G_2^{-1} \in \Lambda \boldsymbol{G}_{\mathrm{t}}$. Hence, L_1 and L_2 can be simultaneously unitarized via dressing using $G_1 T_1 = U G_2 T_2$.

Moreover, k_1 and k_2 are uniquely determined by (3.16), $k_1(0) > 0$ and $k_2(0) > 0$. Hence, in any case, G_1 and G_2 in $\Lambda^+_{\boldsymbol{B}} \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ are also unique. Therefore, the unique element of $\Lambda^+_{\boldsymbol{B}} \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ that simultaneously unitarizes L_1 and L_2 is the $\Lambda^+_{\boldsymbol{B}} \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$-part in the Iwasawa splitting of $G_1 T_1$, i.e., that of $G_2 T_2$. □

Remark 3.21. In the case of the above theorem where L_1 and L_2 do not commute, up to a unitary factor on the left, the only element of $\Lambda \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ simultaneously unitarizing L_1 and L_2 is

$$G_1 T_1 = T_{*,1}^{-1} \begin{pmatrix} \big(-\frac{cd}{\overline{ab}}\big)^{1/4} & 0 \\ 0 & \big(-\frac{\overline{ab}}{cd}\big)^{1/4} \end{pmatrix} T_{*,1} T_1, \tag{3.22}$$

where $T_{*,1}$ is determined by L_1. The $\Lambda^+_{\boldsymbol{B}} \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$-part of the diagonal matrix in (3.22) is K_1 in the above proof. Such an element of $\Lambda \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ can also be given by

$$G_2 T_2 = T_{*,2}^{-1} \begin{pmatrix} \big(-\frac{\overline{ac}}{bd}\big)^{1/4} & 0 \\ 0 & \big(-\frac{bd}{\overline{ac}}\big)^{1/4} \end{pmatrix} T_{*,2} T_2, \tag{3.23}$$

where $T_{*,2}$ is determined by L_2. The $\Lambda^+_{\boldsymbol{B}} \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$-part of the diagonal matrix in (3.23) is K_2 in the above proof.

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Simultaneous unitarization of two loop matrices via r-dressing

Next, we deal with the simultaneous unitarization of two loop matrices via r-dressing with $r < 1$ and, when they do not commute, find a unique element of $\Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ to do so.

THEOREM 3.24. *Let $L_1, L_2 \in \Lambda\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ be holomorphic on a neighborhood of $\mathbb{S}^1$ and satisfy* (2.3). *Assume that for $i = 1$ and 2, the two holomorphic eigenvalues of L_i are not always equal on $\mathbb{S}^1$. For $i = 1$ and 2, let T_i be the matrix given by* (2.7) *or* (2.8) *or* (2.9) *or* (2.12) *or* (2.13) *for L_i, and define $\widehat{T}_i$ by the above convention. Moreover, set*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \widehat{T}_1 \widehat{T}_2^{-1}. \tag{3.25}$$

Then, L_1 and L_2 can be simultaneously unitarized via r-dressing for some $r \in (0,1)$ sufficiently close to 1 if and only if either $b = c = 0$, or $a = d = 0$, or as meromorphic functions on a neighborhood of $\mathbb{S}^1$,

(3.26)

$abcd \not\equiv 0$, and $\dfrac{d}{\overline{a}}$, $-\dfrac{c}{\overline{b}}$ are real and non-negative on $\mathbb{S}^1$ (away from poles).

In the first two cases, L_1 and L_2 commute; while in the last case, there is a unique element of $\Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ simultaneously unitarizing L_1 and L_2, and we can choose r such that it can be replaced by any number in $(r, 1)$.

Proof. Fix an $r \in (0,1)$ such that L_1, L_2 and their eigenvalues are holomorphic on $\mathcal{N} := \mathcal{D}_{1/s} \setminus \overline{\mathcal{D}_s}$ for some $s \in (0, r)$, T_1 and T_2 are holomorphic on $\mathcal{N} \setminus \mathbb{S}^1$, and each of a, b, c and d is either identically zero or never zero on $\mathcal{N} \setminus \mathbb{S}^1$. Since this r can be replaced by any number between r and 1, we only need to consider this r.

By Proposition 3.1, L_1 and L_2 can be simultaneously conjugated into $\Lambda^{\mathrm{d}}_r\boldsymbol{G}_{\mathrm{t}}$ or into $\Lambda^{\mathrm{a}}_r\boldsymbol{G}_{\mathrm{t}}$ if and only if $\widehat{T}_1\widehat{T}_2^{-1}$ is either diagonal or off-diagonal, i.e., if and only if L_1 and L_2 commute. So, for the rest of this proof, we can assume that $\widehat{T}_1\widehat{T}_2^{-1}$ is neither diagonal nor off-diagonal.

For $i = 1$ and 2, let $T_{*,i} = I$ if L_i is in the even case and $T_{*,i}$ equals the matrix in (2.24) if L_i is in the odd case.

By Parts iii) and iv) of Theorem 2.16, L_1 and L_2 can be simultaneously unitarized via r-dressing using an $S \in \Lambda_r\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ if and only if there are $U_1, U_2 \in \Lambda_r\boldsymbol{G}_{\mathrm{t}}$ and $G_1, G_2 \in \Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ such that each of G_1 and G_2 is diagonal or

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has the form given in (2.17), and $S = U_1G_1T_1 = U_2G_2T_2$. The latter is equivalent to $S = V_1K_1\widehat{T}_1 = V_2K_2\widehat{T}_2$, where $V_i = U_i(T_{*,i})^{-1}$ and $K_i = T_{*,i}G_i(T_{*,i})^{-1}$ for $i = 1$ and 2. Note that K_1 and K_2 are diagonal by Remark 2.30, and that V_1, V_2, K_1, K_2, $\widehat{T}_1$ and $\widehat{T}_2$ are only in the untwisted r-loop group $\Lambda_r\boldsymbol{G}^{\mathbb{C}}$ in general, and V_1 and V_2 are in $\Lambda_r\boldsymbol{G}$.

Now, assume that L_1 and L_2 can be simultaneously unitarized via r-dressing. Then, using the notation of the last paragraph, $K_1\widehat{T}_1\widehat{T}_2^{-1}K_2^{-1} = V_1^{-1}V_2 \in \Lambda_r\boldsymbol{G}^{\mathbb{C}}$ is r-unitary. Define k_1 and k_2 by (3.8), then (3.9) holds. Thus, the entries of $K_1\widehat{T}_1\widehat{T}_2^{-1}K_2^{-1}$ have holomorphic extensions to $\mathcal{D}_{1/r}\setminus\overline{\mathcal{D}_r}$ and extend continuously to $\mathbb{S}_{1/r}$, and for $\lambda \in \overline{\mathcal{D}_{1/r}}\setminus\mathcal{D}_r$,

$$(dk_2/k_1)(\lambda) = \overline{(ak_1/k_2)(1/\overline{\lambda})}, \quad (c/(k_1k_2))(\lambda) = -\overline{(bk_1k_2)(1/\overline{\lambda})}. \tag{3.27}$$

So, $abcd \not\equiv 0$ on $\mathbb{S}^1$, k_1k_2 and k_1/k_2 have meromorphic extensions to $\mathcal{D}_{1/r}\setminus\overline{\mathcal{D}_r}$ and extend continuously to $\mathbb{S}_{1/r}$, and for $\lambda \in \overline{\mathcal{D}_{1/r}}\setminus\mathcal{D}_r$,

$$\Big(\frac{k_1}{k_2}\Big)(\lambda)\cdot\overline{\Big(\frac{k_1}{k_2}\Big)(1/\overline{\lambda})} = \frac{d(\lambda)}{\overline{a(1/\overline{\lambda})}}, \quad (k_1k_2)(\lambda)\cdot\overline{(k_1k_2)(1/\overline{\lambda})} = -\frac{c(\lambda)}{\overline{b(1/\overline{\lambda})}}. \tag{3.28}$$

On $\mathbb{S}^1$ (with a finite number of points taken away), (3.28) can be rewritten as

$$\Big(\frac{k_1}{k_2}\Big)\cdot\overline{\Big(\frac{k_1}{k_2}\Big)} = \frac{d}{\overline{a}}, \quad k_1k_2\cdot\overline{k_1k_2} = -\frac{c}{\overline{b}}, \tag{3.29}$$

and the expressions in (3.28) are the unique meromorphic extensions of the ones here. By (3.29), as meromorphic functions, $d/\overline{a}$ and $-c/\overline{b}$ are real and non-negative on $\mathbb{S}^1$. Thus, (3.26) is necessary.

Conversely, to show sufficiency, assume that (3.26) is true. Then, each zero or pole of $d(\lambda)/\overline{a(1/\overline{\lambda})}$ on $\mathbb{S}^1$ has an even order. So,

$$\frac{d(\lambda)}{\overline{a(1/\overline{\lambda})}} = f(\lambda)\frac{\prod_{i=1}^{m_2}(1-\lambda/\lambda_i^+)(1-\lambda_i^+/\lambda)}{\prod_{j=1}^{m_3}(1-\lambda/\lambda_j^-)(1-\lambda_j^-/\lambda)} \quad \text{on } \mathcal{N}, \tag{3.30}$$

where $m_2, m_3 \in \mathbb{N}\cup\{0\}$, $\lambda_1^+, \lambda_2^+, \ldots, \lambda_{m_2}^+, \lambda_1^-, \lambda_2^-, \ldots, \lambda_{m_3}^- \in \mathbb{S}^1$ (it is possible that some of them repeat) such that $\lambda_i^+ \neq \lambda_j^-$ for any i and j, and f is holomorphic and never vanishes on $\mathcal{N}$. By the Birkhoff decomposition on $\mathbb{S}^1$,

$$f(\lambda) = \frac{1}{m_1}\lambda^m f_-(\lambda)f_+(\lambda) \quad \text{on } \mathcal{N}, \tag{3.31}$$

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where m_1 is a non-zero constant, $m \in \mathbb{Z}$, $f_- \in \mathcal{W}_-$ never vanishes on $\overline{\mathcal{E}} = \{|\lambda| \geq 1\} \cup \{\infty\}$ and satisfies $f_-(\infty) = 1$, while $f_+ \in \mathcal{W}_+$ never vanishes on $\overline{\mathcal{D}}$ and satisfies $f_+(0) = 1$. The reality of $d(\lambda)/\overline{a(1/\overline{\lambda})}$ on $\mathbb{S}^1$ is equivalent to that

$$m_1 \in \mathbb{R}, \quad m = 0, \quad \overline{f_+(1/\overline{\lambda})} = f_-(\lambda) \text{ on } \mathcal{N}. \tag{3.32}$$

In total, we have that

$$\frac{d(\lambda)}{\overline{a(1/\overline{\lambda})}} = \frac{1}{m_1}\overline{f_+(1/\overline{\lambda})} f_+(\lambda) \frac{\prod_{i=1}^{m_2}(1-\lambda/\lambda_i^+)(1-\lambda_i^+/\lambda)}{\prod_{j=1}^{m_3}(1-\lambda/\lambda_j^-)(1-\lambda_j^-/\lambda)} \quad \text{on } \mathcal{N}. \tag{3.33}$$

Since $d/\overline{a}$ is non-negative on $\mathbb{S}^1$, we obtain that $m_1 > 0$. Similarly,

$$-\frac{c(\lambda)}{\overline{b(1/\overline{\lambda})}} = n_1\overline{g_+(1/\overline{\lambda})} g_+(\lambda) \frac{\prod_{i=1}^{n_2}(1-\lambda/\nu_i^+)(1-\nu_i^+/\lambda)}{\prod_{j=1}^{n_3}(1-\lambda/\nu_j^-)(1-\nu_j^-/\lambda)} \quad \text{on } \mathcal{N}, \tag{3.34}$$

where $n_1 > 0$, $n_2, n_3 \in \mathbb{N} \cup \{0\}$, $\nu_1^+, \nu_2^+, \ldots, \nu_{n_2}^+, \nu_1^-, \nu_2^-, \ldots, \nu_{n_3}^- \in \mathbb{S}^1$ such that $\nu_i^+ \neq \nu_j^-$ for any i and j, and g_+ is a never-vanishing holomorphic function on $\mathcal{D}_{1/s}$ satisfying $g_+(0) = 1$. Set

$$k_1(\lambda) = \Big(\frac{n_1}{m_1}\Big)^{1/4} \left[\frac{f_+(\lambda) g_+(\lambda) \prod_{i=1}^{m_2}(1-\lambda/\lambda_i^+) \prod_{k=1}^{n_2}(1-\lambda/\nu_k^+)}{\prod_{j=1}^{m_3}(1-\lambda/\lambda_j^-) \prod_{l=1}^{n_3}(1-\lambda/\nu_l^-)}\right]^{1/2}, \tag{3.35}$$

$$k_2(\lambda) = (m_1 n_1)^{1/4} \left[\frac{g_+(\lambda) \prod_{j=1}^{m_3}(1-\lambda/\lambda_j^-) \prod_{k=1}^{n_2}(1-\lambda/\nu_k^+)}{f_+(\lambda) \prod_{i=1}^{m_2}(1-\lambda/\lambda_i^+) \prod_{l=1}^{n_3}(1-\lambda/\nu_l^-)}\right]^{1/2} \tag{3.36}$$

on $\mathcal{D}$. Then, k_1 and k_2 belong to $\mathcal{W}_r^+$, never vanish on $\mathcal{D}$, have continuous extensions to $\mathbb{S}^1$ with at most a finite number of points taken away, and satisfy (3.28) on $\mathcal{D} \setminus \mathcal{D}_r$. Thus, we have (3.27) on $\mathcal{D} \setminus \mathcal{D}_r$. Note that $k_1(0) > 0$ and $k_2(0) > 0$. Define K_1 and K_2 by (3.8), and consider $V := K_1 \widehat{T}_1 \widehat{T}_2^{-1} K_2^{-1} \in \Lambda_r \boldsymbol{G}^{\mathbb{C}}$. Since only $k_1 k_2$ and k_1/k_2 occur in V, it is meromorphic on $\mathcal{N}$. By (3.27), V is unitary and hence stays uniformly bounded at its continuous points on $\mathbb{S}^1$. Thus, the finitely many singularities of V on $\mathbb{S}^1$ are removable. Therefore, V is r-unitary. Moreover, (3.28) then holds on $\mathcal{N}$, and hence

$$k_1^2(\lambda)\overline{k_1^2(1/\overline{\lambda})} = -\frac{(cd)(\lambda)}{\overline{(ab)(1/\overline{\lambda})}}, \quad k_2^2(\lambda)\overline{k_2^2(1/\overline{\lambda})} = -\frac{\overline{(ac)(1/\overline{\lambda})}}{(bd)(\lambda)} \quad \text{on } \mathcal{N}. \tag{3.37}$$

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Next, we proceed in three non-overlapping situations separately. Since the remaining arguments are similar to the corresponding ones in the proof of Theorem 3.5, we will only sketch them.

If L_1 and L_2 are both in the even case, then $cd/\overline{ab}$ and $\overline{ac}/(bd)$ are even in λ. By (3.37) and Lemma 1.3, k_1 and k_2 are also even in λ. Hence, $G_i := K_i \in \Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ for $i = 1$ and 2, and $U := V = K_1T_1T_2^{-1}K_2^{-1} \in \Lambda_r\boldsymbol{G}_{\mathrm{t}}$. So, L_1 and L_2 can be simultaneously unitarized via r-dressing using $G_1T_1 = UG_2T_2$.

If one L_1 and L_2 is in the even case and the other is in the odd case, without loss of generality, we can assume that L_1 is in the even case and L_2 is in the odd case. So, $cd/\overline{ab}$ is even in λ, and

$$\Big(\frac{\overline{ac}}{bd}\Big)(-\lambda) = \Big(\frac{bd}{\overline{ac}}\Big)(\lambda) \quad \text{on } \mathbb{S}^1. \tag{3.38}$$

Thus, by (3.37) and Lemma 1.3, k_1 is even in λ, and $k_2 = \mathrm{e}^{\alpha_2}$ for some odd $\alpha_2 \in \mathcal{W}^+ \subset \mathcal{W}^+_r$. Then, we deduce that $G_1 := K_1 \in \Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$, $G_2 := (T_{*,2})^{-1}K_2T_{*,2} \in \Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$, and $U := VT_{*,2} = G_1 \cdot T_1T_2^{-1} \cdot G_2^{-1} \in \Lambda_r\boldsymbol{G}_{\mathrm{t}}$. Thus, $G_1T_1 = UG_2T_2$ simultaneously unitarizes L_1 and L_2.

If L_1 and L_2 are both in the odd case, then $G_i := T_{*,i}^{-1}K_iT_{*,i} \in \Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ for $i = 1$ and 2, and $U := T_{*,1}^{-1}VT_{*,2} = G_1 \cdot T_1T_2^{-1} \cdot G_2^{-1} \in \Lambda_r\boldsymbol{G}_{\mathrm{t}}$. Hence, L_1 and L_2 can be simultaneously unitarized via r-dressing using $G_1T_1 = UG_2T_2$.

Moreover, k_1 and k_2 are uniquely determined by (3.37), $k_1(0) > 0$ and $k_2(0) > 0$. Hence, in any case, G_1 and G_2 in $\Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ are also unique. Therefore, the unique element of $\Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ that simultaneously unitarizes L_1 and L_2 is the $\Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$-part in the Iwasawa splitting of G_1T_1, i.e., that of G_2T_2. ∎

Remark 3.39. In the case of the above theorem where L_1 and L_2 do not commute, up to a unitary factor on the left, the meromorphic extension to $\mathbb{S}^1$ of the only element of $\Lambda_r\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ simultaneously unitarizing L_1 and L_2 is given by (3.22) with $T_{*,1}$ determined by L_1. The $\Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$-part of the diagonal matrix in (3.22) is K_1 in the above proof. The meromorphic extension to $\mathbb{S}^1$ of such an element of $\Lambda_r\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ can also be given by (3.23) with $T_{*,2}$ determined by L_2. The $\Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$-part of the diagonal matrix in (3.23) is K_2 in the above proof.

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Remark 3.40. We note that our above proof of Theorem 3.24 actually shows the following fact: when L_1 is diagonal or anti-diagonal, there exists an $r \in (0,1)$, as close to 1 as one wants, such that $\Lambda_r^+ \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ contains an element S which commutes with L_1 and unitarizes L_2, i.e.,

$$SL_1S^{-1} = L_1, \quad SL_2S^{-1} \text{ is } r\text{-unitary.} \tag{3.41}$$

In fact, if L_1 is diagonal or anti-diagonal, then $T_1 = I$ and we can take $S = T_*^{-1}K_1T_*$, where T_* is determined by L_1. A similar statement is true when L_1 is an s-unitary conjugate of its diagonal or anti-diagonal form for some $s \in [r,1)$.

Remark 3.42. From (3.35) and (3.36) we see that in general, the K_1 and K_2 produced in the above proof can not be meromorphically extended to the whole $\mathbb{S}^1$. Thus, in general, the loop matrix simultaneously unitarizing L_1 and L_2 is *not* meromorphic on any neighborhood of $\mathbb{S}^1$.

Some examples

Now, we apply Theorems 3.5 and 3.24 to several special subcases. We start with the subcase where the two loop matrices to be simultaneously unitarized are both lower triangular, while the subcase where the two loop matrices are both upper triangular can be treated very similarly. Note that now we are in the even case in the sense of Section 2.

COROLLARY 3.43. *Consider*

$$L_1 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}f} & 0 \\ p & \mathrm{e}^{-\mathrm{i}f} \end{pmatrix}, \quad L_2 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}g} & 0 \\ q & \mathrm{e}^{-\mathrm{i}g} \end{pmatrix} \tag{3.44}$$

in $\Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ *with real-valued* f *and* g *in* $\mathcal{W}$. *Assume that* f, g, p *and* q *are holomorphic on a neighborhood of* $\mathbb{S}^1$, *and none of* $\sin f$ *and* $\sin g$ *is identically* 0 *on* $\mathbb{S}^1$.

i) *The matrices* L_1 *and* L_2 *can be simultaneously unitarized via dressing if and only if* $p/\sin f$ *and* $q/\sin g$ *are always finite on* $\mathbb{S}^1$ *and* L_1 *and* L_2 *commute, if and only if*

$$\frac{p}{\sin f} \text{ and } \frac{q}{\sin g} \text{ are always finite and equal on } \mathbb{S}^1. \tag{3.45}$$

In particular, if $p = 0$ *and* $q \neq 0$ *in* $\mathcal{W}$, *then* L_1 *and* L_2 *can not be simultaneously unitarized via dressing.*

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ii) *The matrices L_1 and L_2 can be simultaneously unitarized via r-dressing for some $r \in (0,1)$ sufficiently close to 1 if and only if they commute, i.e., if and only if*

$$\frac{p}{\sin f} = \frac{q}{\sin g} \quad \text{on } \mathbb{S}^1 \tag{3.46}$$

as meromorphic functions. In this case, there is a unique element of $\Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ simultaneously unitarizing L_1 and L_2, and we can choose r such that it can be replaced by any number in $(r,1)$. In particular, if furthermore $p = 0$ and $q \neq 0$ in $\mathcal{W}$, then L_1 and L_2 can not be simultaneously unitarized via r-dressing for any $r \in (0,1)$ sufficiently close to 1.

Proof. Our assumptions and Proposition 2.32 imply that if L_1 and L_2 are unitarizable via dressing, then $p/\sin f$ and $q/\sin g$ are always finite on $\mathbb{S}^1$. So, following the proof of Theorem 2.6, in this subcase we can take

$$T_1 = \begin{pmatrix} 1 & 0 \\ \frac{\mathrm{i}p}{2\sin f} & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 \\ \frac{\mathrm{i}q}{2\sin g} & 1 \end{pmatrix}. \tag{3.47}$$

Then,

$$T_1 T_2^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{\mathrm{i}p}{2\sin f} - \frac{\mathrm{i}q}{2\sin g} & 1 \end{pmatrix}. \tag{3.48}$$

Therefore, our results follow directly from Theorems 3.5 and 3.24. ∎

Next, we examine the subcase where one of the two loop matrices is lower triangular and the other one is upper triangular. This subcase is also motivated by some situation in the construction of constant mean curvature trinoids. Note that we are again in the even case.

COROLLARY 3.49. *Consider*

$$L_1 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}f} & 0 \\ p & \mathrm{e}^{-\mathrm{i}f} \end{pmatrix}, \quad L_2 = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}g} & q \\ 0 & \mathrm{e}^{\mathrm{i}g} \end{pmatrix} \tag{3.50}$$

in $\Lambda\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ with real-valued f and g in $\mathcal{W}$. Assume that f, g, p and q are holomorphic on a neighborhood of $\mathbb{S}^1$, and none of $\sin f$, $\sin g$, p and q is identically 0 on $\mathbb{S}^1$.

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i) *The matrices L_1 and L_2 can be simultaneously unitarized via dressing if and only if*

(3.51)
$$\frac{p}{\sin f},\ \frac{q}{\sin g}\ \text{are always finite,}\ \ pq\ \text{is real,}\ \ -4<\frac{pq}{\sin f\sin g}<0\ \ \text{on}\ \mathbb{S}^1.$$

In this case, there is a unique element of $\Lambda^+_{\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ simultaneously unitarizing L_1 and L_2.

ii) *The matrices L_1 and L_2 can be simultaneously unitarized via r-dressing for some $r\in(0,1)$ sufficiently close to 1 if and only if*

$$4\sin f\sin g+pq\not\equiv 0,\ \ pq\ \text{is real,}\ \ -4\le\frac{pq}{\sin f\sin g}\le 0\ \ \text{on}\ \mathbb{S}^1 \tag{3.52}$$

as meromorphic functions. In this case, there is a unique element of $\Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ simultaneously unitarizing L_1 and L_2, and we can choose r such that it can be replaced by any number in $(r,1)$.

Note that the inequalities in (3.52) imply, for example, that the zeros of $\sin f\sin g$ on $\mathbb{S}^1$ are among the ones of pq on $\mathbb{S}^1$.

Proof. i) By our assumptions and Proposition 2.32, if L_1 and L_2 are unitarizable via dressing, then $p/\sin f$ and $q/\sin g$ are always finite on $\mathbb{S}^1$. So, following the proof of Theorem 2.6, in this subcase we can take

$$T_1=\begin{pmatrix}1 & 0\\ \frac{\mathrm{i}p}{2\sin f} & 1\end{pmatrix},\quad T_2=\begin{pmatrix}1 & \frac{\mathrm{i}q}{2\sin g}\\ 0 & 1\end{pmatrix}. \tag{3.53}$$

Then,

$$T_1T_2^{-1}=\begin{pmatrix}1 & -\frac{\mathrm{i}q}{2\sin g}\\ \frac{\mathrm{i}p}{2\sin f} & \frac{pq}{4\sin f\sin g}+1\end{pmatrix}. \tag{3.54}$$

Thus, by Theorem 3.5, L_1 and L_2 can be simultaneously unitarized via dressing if and only if

(3.55)
$$\frac{p}{\sin f},\frac{q}{\sin g}\ \text{are always finite,}\ \ -\frac{p\sin g}{\overline{q}\sin f},\frac{pq}{4\sin f\sin g}+1\in(0,+\infty)\ \ \text{on}\ \mathbb{S}^1.$$

Assume that $p/\sin f$ and $q/\sin g$ are always finite on $\mathbb{S}^1$. Since

$$\frac{p\sin g}{\overline{q}\sin f}=\frac{pq}{\sin f\sin g}\Big/\frac{q\overline{q}}{\sin^2 g}, \tag{3.56}$$

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$p \sin g/(\overline{q} \sin f)$ is real and negative on $\mathbb{S}^1$ if and only if pq is real on $\mathbb{S}^1$ and

$$\frac{pq}{\sin f \sin g} < 0 \quad \text{on } \mathbb{S}^1. \tag{3.57}$$

Hence, L_1 and L_2 can be simultaneously unitarized via dressing if and only if the conditions of (3.51) are satisfied.

ii) The proof of this part is similar to that of the above part, and the only difference is that here we use Theorem 3.24 instead of Theorem 3.5. □

We remark that the conditions (3.51) and (3.52) appear in different forms in [7] and [1] under the additional assumption that $L_1 L_2$ is also unitarizable.

§4. Simultaneous unitarization of three or more loop matrices

In this section, we present conditions for the simultaneous unitarizability of three or more loop matrices and show that under these conditions, there is a unique element in $\Lambda^+_{r,\boldsymbol{B}} \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ unitarizing them, unless the loop matrices all commute. As pointed out in the introduction, this uniqueness result will be used in Section 5 to deal with the closing conditions.

Simultaneous unitarization of three loop matrices

First, we consider the simultaneous unitarization of three loop matrices via dressing or r-dressing with $r \in (0, 1)$. We will see that this case is crucial for the case of four or more loop matrices.

THEOREM 4.1. *Let $L_1, L_2, L_3 \in \Lambda \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}} \setminus \{\pm I\}$ be holomorphic on a neighborhood of $\mathbb{S}^1$. Suppose that neither L_1 and L_2 nor L_2 and L_3 commute.*

i) *For $i = 1$, 2 and 3, assume that $T_i \in \Lambda \boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ conjugates L_i into $\Lambda^{\mathrm{d}} \boldsymbol{G}_{\mathrm{t}} \cup \Lambda^{\mathrm{a}} \boldsymbol{G}_{\mathrm{t}}$, and define $\widehat{T}_i$ from T_i as in Section 3. Set*

$$\widehat{T}_1 \widehat{T}_2^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \widehat{T}_2 \widehat{T}_3^{-1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}. \tag{4.2}$$

Then, L_1, L_2 and L_3 can be simultaneously unitarized via dressing if and only if L_i and L_{i+1} can be simultaneously unitarized via dressing for $i = 1$ and 2 and

$$\overline{acef} = bdgh \quad \text{on } \mathbb{S}^1. \tag{4.3}$$

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In this case, there is a unique element of $\Lambda^+_{\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ *simultaneously unitarizing* L_1, L_2 *and* L_3.

ii) *For* $i = 1$, 2 *and* 3, *assume that the two holomorphic eigenvalues of* L_i *are not always equal on* $\mathbb{S}^1$, *let* T_i *be given by* (2.7) *or* (2.8) *or* (2.9) *or* (2.12) *or* (2.13) *for* L_i, *and define* $\widehat{T}_i$ *from* T_i *as in Section* 3. *Then,* L_1, L_2 *and* L_3 *can be simultaneously unitarized via* r*-dressing for some* $r \in (0,1)$ *sufficiently close to* 1 *if and only if* L_i *and* L_{i+1} *can be simultaneously unitarized via* r*-dressing for* $i = 1$ *and* 2 *and* (4.3) *is satisfied by the meromorphic functions* $a, b, \dots, h$ *defined by* (4.2). *In this case, there is a unique element of* $\Lambda^+_{r,\boldsymbol{B}}\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ *simultaneously unitarizing* L_1, L_2 *and* L_3, *and we can choose* r *such that it can be replaced by any number in* $(r, 1)$.

Proof. i) Assume that L_i and L_{i+1} can be simultaneously unitarized via dressing for $i = 1$ and 2. Then, by Theorem 3.5 and our assumptions,

$$bdef \not\equiv 0, \quad \frac{\overline{ac}}{bd} \text{ and } \frac{gh}{\overline{ef}} \text{ are always finite, real and negative} \quad \text{on } \mathbb{S}^1. \tag{4.4}$$

By Remark 3.21, the elements of $\Lambda\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ simultaneously unitarizing L_1 and L_2 have the form

$$U_{\mathrm{L}} T_*^{-1} \begin{pmatrix} \left(-\frac{\overline{ac}}{bd}\right)^{1/4} & 0 \\ 0 & \left(-\frac{bd}{\overline{ac}}\right)^{1/4} \end{pmatrix} T_* T_2 \quad \text{with } U_{\mathrm{L}} \in \Lambda\boldsymbol{G}_{\mathrm{t}}, \tag{4.5}$$

and the elements of $\Lambda\boldsymbol{G}^{\mathbb{C}}_{\mathrm{t}}$ simultaneously unitarizing L_2 and L_3 have the form

$$U_{\mathrm{R}} T_*^{-1} \begin{pmatrix} \left(-\frac{gh}{\overline{ef}}\right)^{1/4} & 0 \\ 0 & \left(-\frac{\overline{ef}}{gh}\right)^{1/4} \end{pmatrix} T_* T_2 \quad \text{with } U_{\mathrm{R}} \in \Lambda\boldsymbol{G}_{\mathrm{t}}, \tag{4.6}$$

where in both cases T_* is determined by L_2. Since the matrices given by (4.5) and (4.6) both unitarize L_2 and the two diagonal matrices in them have positive diagonal entries, the uniqueness of such splitting implies that L_1, L_2 and L_3 can be simultaneously unitarized via dressing if and only if the two diagonal matrices are equal, i.e., if and only if (4.3) holds.

ii) The proof of this part is similar to that of the above part, but uses Theorem 3.24 and Remark 3.39 in place of Theorem 3.5 and Remark 3.21. ∎

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We would like to point out that the case where L_1 and L_3 commute and $T_3 = T_1$ is a special case of the above theorem. In this case, (4.3) follows directly from (3.7) or (3.26).

The following example shows that in general, the simultaneous unitarizability via dressing of every pair of at least three loop matrices does not imply the simultaneous unitarizability via dressing of all the loop matrices.

EXAMPLE 4.7. Let $r \in (0,1]$, $k > 1$, $l \in (1/k, 1) \cup (1, k)$ and $t \in (0, \pi)$. Define $T_2 = I$,

$$T_1(\lambda) = \frac{1}{\sqrt{1+k^2}} \begin{pmatrix} k & \mathrm{i}\lambda \\ \mathrm{i}/\lambda & k \end{pmatrix}, \quad T_3(\lambda) = \frac{1}{\sqrt{1+k^2}} \begin{pmatrix} k/l & -\mathrm{i}l\lambda \\ -\mathrm{i}/l/\lambda & kl \end{pmatrix}. \tag{4.8}$$

Consider

$$L_1 = T_1^{-1} L_2 T_1, \quad L_2 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}t} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}t} \end{pmatrix}, \quad L_3 = T_3^{-1} L_2 T_3. \tag{4.9}$$

Then, for each $i \in \{1,2,3\}$, T_i and L_i are in $\Lambda_r \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$, T_i diagonalizes L_i into $\Lambda_r \boldsymbol{G}_{\mathrm{t}}$ (hence L_i is in the even case), and the difference of the holomorphic eigenvalues of L_i does not have any zero on $\mathbb{S}^1$. By Theorems 3.5 and 3.24, from

$$T_1 T_2^{-1} = T_1, \tag{4.10}$$

$$T_2 T_3^{-1} = \frac{1}{\sqrt{1+k^2}} \begin{pmatrix} kl & \mathrm{i}l\lambda \\ \mathrm{i}/l/\lambda & k/l \end{pmatrix}, \tag{4.11}$$

$$T_1 T_3^{-1} = \frac{1}{1+k^2} \begin{pmatrix} k^2 l - 1/l & \mathrm{i}k(l+1/l)\lambda \\ \mathrm{i}k(l+1/l)/\lambda & k^2/l - l \end{pmatrix} \tag{4.12}$$

and our assumptions we see that any two of L_1, L_2 and L_3 can be simultaneously unitarized via r-dressing. However, since

$$\frac{\overline{k \cdot \mathrm{i}/\lambda}}{\mathrm{i}\lambda \cdot k} = -1, \quad \frac{\mathrm{i}/l/\lambda \cdot k/l}{\overline{kl \cdot \mathrm{i}l\lambda}} = -\frac{1}{l^4} \neq -1 \quad \text{on } \mathbb{S}^1, \tag{4.13}$$

L_1, L_2 and L_3 can *not* be simultaneously unitarized via r-dressing by Theorem 4.1.

Simultaneous unitarization of four or more loop matrices

Finally, we consider the simultaneous unitarization of four or more loop matrices via dressing or r-dressing with $r \in (0,1)$.

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THEOREM 4.14. *Let $m \geq 4$ be an integer. Assume that $L_1, L_2, \ldots, L_m \in \Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}} \setminus \{\pm I\}$ are holomorphic on a neighborhood of $\mathbb{S}^1$.*

i) *If $L_1, L_2, \ldots$ and L_m can be individually conjugated into $\Lambda^{\mathrm{d}} \boldsymbol{G}_{\mathrm{t}} \cup \Lambda^{\mathrm{a}} \boldsymbol{G}_{\mathrm{t}}$ by elements of $\Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$, then they can be simultaneously unitarized via dressing if and only if each consecutive triple L_i, L_{i+1} and L_{i+2} can be simultaneously unitarized via dressing. In this case, if not all of $L_1, L_2, \ldots$ and L_m commute, then there is a unique element of $\Lambda^+_{\boldsymbol{B}} \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ simultaneously unitarizing $L_1, L_2, \ldots$ and L_m.*

ii) *Suppose that for each $i \in \{1, 2, \ldots, m\}$, the two holomorphic eigenvalues of L_i are not always equal on $\mathbb{S}^1$. Then, $L_1, L_2, \ldots$ and L_m can be simultaneously unitarized via r-dressing for some $r \in (0, 1)$ sufficiently close to 1 if and only if each consecutive triple L_i, L_{i+1} and L_{i+2} can be simultaneously unitarized via r-dressing. In this case, if not all of $L_1, L_2, \ldots$ and L_m commute, then there is a unique element of $\Lambda^+_{r,\boldsymbol{B}} \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ simultaneously unitarizing $L_1, L_2, \ldots$ and L_m, and we can choose r such that it can be replaced by any number in $(r, 1)$.*

Proof. i) The necessity is trivial. So, we only need to prove the sufficiency and the uniqueness statement.

We say that two loop matrices L_i and L_j are equivalent if they commute. Then, this defines an equivalence relation among $L_1, L_2, \ldots$ and L_m. By Proposition 3.1, it suffices to consider only one representative from each equivalence class. Thus, we can assume further that every pair of the given loop matrices do not commute. Now assume that $S \in \Lambda^+_{\boldsymbol{B}} \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ unitarizes L_1, L_2 and L_3, while $R \in \Lambda^+_{\boldsymbol{B}} \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ unitarizes L_2, L_3 and L_4. Then, the uniqueness statement of Theorem 3.5 yields that $R = S$, since both R and S unitarize L_2 and L_3. Hence, L_1, L_2, L_3 and L_4 can be simultaneously unitarized via $R = S$. An iteration of this argument proves the sufficiency and the uniqueness statement of the theorem.

ii) The proof of this part is similar to that of the above part, and the only difference is that here we use Theorem 3.24 instead of Theorem 3.5. □

§5. Pointwise simultaneous unitarization, and closing conditions

In this section, summarizing our results in Sections 3 and 4, we first state a relation between the simultaneous unitarization of loop matrices via r-dressing and the pointwise simultaneous unitarization of the matrices in $\mathrm{SL}(2, \mathbb{C})$. Then, we prove a result concerning the closing conditions to be defined below.

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In Theorems 3.24, 4.1 and 4.14, the conditions for the simultaneous unitarization of two or more loop matrices via r-dressing with $r < 1$ sufficiently close to 1 are the same as the conditions for the pointwise simultaneous unitarization on $\mathbb{S}^1$ of the corresponding matrices in $\mathrm{SL}(2,\mathbb{C})$. Thus, we have the following result.

THEOREM 5.1. *Let $m \geq 2$ be an integer. Assume that for each $i \in \{1, 2, \dots, m\}$, $L_i \in \Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}} \setminus \{\pm I\}$ is holomorphic on a neighborhood of $\mathbb{S}^1$, and the two holomorphic eigenvalues of L_i are not always equal on $\mathbb{S}^1$. Then, $L_1, L_2, \dots$ and L_m can be simultaneously unitarized by r-dressing for some $r \in (0,1)$ sufficiently close to 1 if and only if for almost every $\lambda \in \mathbb{S}^1$, the matrices $L_1(\lambda), L_2(\lambda), \dots$ and $L_m(\lambda)$ can be simultaneously unitarized in $\mathrm{SL}(2,\mathbb{C})$. In this case, we can choose r such that it can be replaced by any number in $(r, 1)$.*

We note that in [9], a result very similar to Theorem 5.1 is proved by Nick Schmitt using a quite different method.

Let $0 < r \leq 1$. Motivated by the construction of CMC surfaces from holomorphic potentials, an r-unitary loop matrix M is said to satisfy the *closing conditions* if

$$M(\lambda = 1) = \pm I, \quad M'(\lambda = 1) = 0. \tag{5.2}$$

We have the following theorem concerning these conditions.

THEOREM 5.3. *Let $m \geq 2$ be an integer. Assume that $L_1, L_2, \dots, L_m \in \Lambda \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}} \setminus \{\pm I\}$ are holomorphic on a neighborhood of $\mathbb{S}^1$, not all of $L_1, L_2, \dots$ and L_m commute, there is an $r \in (0,1)$ sufficiently close to 1 and a $T_+ \in \Lambda_r^+ \boldsymbol{G}_{\mathrm{t}}^{\mathbb{C}}$ simultaneously unitarizing $L_1, L_2, \dots$ and L_m, and the two holomorphic eigenvalues v_i and $1/v_i$ of each L_i are not always equal on $\mathbb{S}^1$. Then, for any $i \in \{1, 2, \dots, m\}$,*

i) *$T_+ L_i T_+^{-1}$ satisfies the closing conditions if and only if*

$$v_i(1) = \pm 1, \quad v_i'(1) = 0; \tag{5.4}$$

ii) *when v_i can be written as $\mathrm{e}^{\mathrm{i}f_i}$ with $f_i \in \mathcal{W}$ being real-valued, $T_+ L_i T_+^{-1}$ satisfies the closing conditions if and only if*

$$f_i(1) \in \mathbb{Z}\pi, \quad f_i'(1) = 0. \tag{5.5}$$

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Proof. i) From the proofs and the *uniqueness parts* of our simultaneous unitarization results in Theorems 3.24, 4.1 and 4.14,

$$T_+ L_i T_+^{-1} = U_i T_{*,i}^{-1} \begin{pmatrix} v_i & 0 \\ 0 & 1/v_i \end{pmatrix} T_{*,i} U_i^{-1} \tag{5.6}$$

for some $U_i \in \Lambda_r \boldsymbol{G}_{\mathrm{t}}$, where $T_{*,i}$ is determined by L_i as usual. This implies our claim by the definition (5.2) of the closing conditions.

ii) This part directly follows from Part i). □

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