ON A CHARACTERISATION OF CONFORMALLY-FLAT RIEMANNIAN SPACES OF CLASS ONE

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In a previous paper [1] we considered those conformally-flat Riemannian spaces which satisfy the tensorial characterisation

$$R_{hijk} = E(R_{hj}R_{ik} - R_{hk}R_{ij}) + F(g_{hj}g_{ik} - g_{hk}g_{ij}) \tag{1.1}$$

where, as usual, g_{ij}, R_{hijk}, R_{ij} are the fundamental tensor, the curvature tensor, the Ricci tensor and $E \neq 0$, F are certain scalars. The tensor g_{ij} is always supposed to be real and analytic. A special form of the metrics of these spaces was seen to be

$$ds^2 = \sum_i (dx^i)^2/[f(\theta)]^2, \quad \theta = \sum_i (x^i)^2, \tag{1.2}$$

where f is any real analytic function, subject to a restriction, of the argument θ. Writing $f, f', f'', \cdots$ for $f(\theta), df/d\theta, d^2f/d\theta^2, \cdots$ the quantities E, F and the scalar curvature R of the type of spaces (1.2) were seen to be

$$\begin{aligned} \frac{1}{E} &= 4(n-2)\{(n-1)ff' + \theta ff'' - (n-1)\theta f'^2\} \neq 0, \\ F &= -\frac{4}{n-2}(ff' + \theta ff'' - \theta f'^2), \\ R &= -4(n-1)(nff' + 2\theta ff'' - n\theta f'^2) \\ &= -\frac{n-1}{(n-2)E} + (n-1)(n-2)F. \end{aligned} \tag{1.3}$$

The following property was also seen in the paper referred to above: If the space (1.2) happens to be symmetric in the sense of Cartan, then

$$\text{either} \quad f'' = 0 \quad \text{or} \quad f - 2\theta f' = 0.$$

In the first case the space is of constant curvature, say $l^2 \neq 0$, and (1.3) reduces to

$$E = -\frac{n}{(n-2)R}, \quad F = \frac{R}{n(n-1)(n-2)}, \quad R = -n(n-1)l^2.$$

In the second case $f = k\theta^{\frac{1}{2}}$, where k is a non-zero constant, and so (1.2) reduces to

$$(1.4) \qquad ds^2 = \sum_i (dx^i)^2/k^2\theta, \quad k = \text{constant} \neq 0, \quad \theta = \sum_i (x^i)^2,$$

and (1.3) reduces to

$$E = -\frac{n-1}{(n-2)R}, \quad F = 0, \quad R = -(n-1)(n-2)k^2.$$

In what has been stated above and in what follows we have supposed, for the sake of definiteness, that the flat space is Euclidean and that the dimension n of the spaces is greater than 3.

The space (1.4) was studied by J. Levine [2] from the point of view of parallel vectors in conformally-flat Riemannian spaces. He took into consideration the other canonical form of the metric of the space as a product space as given by Y. Wong [3], namely

$$(1.5) \qquad ds^2 = (dx^1)^2 + \sum_\alpha (dx^\alpha)^2 \Big/ \left[1 + \frac{k^2}{4}\sum_\alpha (x^\alpha)^2\right]^2, \quad (\alpha = 2, \cdots, n),$$

and showed that the space (1.5), or its equivalent (1.4), is of class one. By 'an n-dimensional Riemannian space V_n of class one' we mean here a V_n which can be embedded in a Euclidean space E_{n+1} of $n+1$ dimension.

Later L. L. Verbickii [4] studied the geometry of conformally-flat Riemannian spaces of class one. He showed that such a space, other than a space of constant curvature, is characterised by the property that all its principal normal curvatures, say ρ, except one, say $\bar{\rho}$, are equal to one another. For the space (1.4), $\rho = -k$ and $\bar{\rho} = 0$.

In this paper we have established a necessary and sufficient condition for a conformally-flat Riemannian space to be of class one, from which it is seen that the spaces (1.2) form a special type of such spaces. The general type is also obtained. It must however be understood clearly that this work is entirely local and by 'space' we mean an n-cell.

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In order then to establish the proposed condition stated above, we first consider the general conformally-flat Riemannian space which satisfies the equations (1.1). To find such a space we proceed as we have done in the previous paper [1], referred to in § 1, by taking the fundamental tensor of a conformally-flat Riemannian space in the form $g_{ii} = 1/\phi^2$, $g_{ij} = 0$, $(i \neq j)$, and looking for the general form of ϕ for which the equations (1.1) would be satisfied. It would then be seen by the straightforward method as adopted in the previous paper that the desired general space would be given by

$$
\begin{aligned}
& ds^2 = \sum_i (dx^i)^2/[f(U)]^2, \text{ where } U = \sum_i (X^i)^2+c, \quad \text{and} \\
& X^i \equiv ax^i+b^i, \text{ with } a \neq 0,\ b^i,\ c \text{ constants.}
\end{aligned} \tag{2.1}
$$

Evidently the special case (1.2) arises when $a = 1$, $b^i = c = 0$. Writing $f' = df/dU$ etc., the equations (1.3) are now replaced by

$$
\begin{aligned}
1/E &= 4a^2(n-2)\{(n-1)ff'+(U-c)ff''-(n-1)(U-c)f'^2 \neq 0, \\
F &= -\frac{4a^2}{n-2}\{ff'+(U-c)ff''-(U-c)f'^2\}, \\
R &= -4a^2(n-1)\{nff'+2(U-c)ff''-n(U-c)f'^2\}.
\end{aligned} \tag{2.2}
$$

Now put

$$
\begin{aligned}
\rho^2 &= 4a^2f'\{f-(U-c)f'\}, \\
\rho\bar{\rho} &= 4a^2\{ff'+(U-c)ff''-(U-c)f'^2\}.
\end{aligned} \tag{2.3}
$$

The scalar curvature R of the space (2.1), as given by (2.2), can be written as

$$R = -4a^2(n-1)[(n-2)f'\{f-(U-c)f'\}+2\{ff'+(U-c)ff''-(U-c)f'^2\}]$$

So, by (2.3), we get

$$R = -(n-1)\{(n-2)\rho^2+2\rho\bar{\rho}\}. \tag{2.4}$$

In the same way it is seen that the scalars F, F, as defined by (2.2), can be expressed in terms of ρ^2 and $\rho\bar{\rho}$ as

$$1/E = (n-2)\{(n-2)\rho^2+\rho\bar{\rho}\}, \quad F = -\rho\bar{\rho}/(n-2). \tag{2.5}$$

Accordingly, the equations (1.1) can be written as

$$R_{hijk} = \frac{R_{hj}R_{ik}-R_{hk}R_{ij}}{(n-2)\{(n-2)\rho^2+\rho\bar{\rho}\}} - \frac{\rho\bar{\rho}}{n-2}(g_{hj}g_{ik}-g_{hk}g_{ij}). \tag{2.6}$$

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We now prove that the space (2.1), which satisfies the equations (1.1), where E, F are given by (2.2), is of class one. We shall do this in a straightforward manner by obtaining for the space the Gauss-Codazzi equations.

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For this purpose define a tensor Ω_{ij} in the space as[1]

$$(3.1)\qquad \Omega_{ij} = -\frac{1}{n-2}\left[\frac{R_{ij}}{\rho} + \bar{\rho} g_{ij}\right].$$

Then

$$\Omega_{hj}\Omega_{ik} - \Omega_{hk}\Omega_{ij} = \frac{1}{(n-2)^2}\left[\frac{1}{\rho^2}(R_{hj}R_{ik} - R_{hk}R_{ij}) + \bar{\rho}^2(g_{hj}g_{ik} - g_{hk}g_{ij}) + \frac{\bar{\rho}}{\rho}(R_{hj}g_{ik} + R_{ik}g_{hj} - R_{hk}g_{ij} - R_{ij}g_{hk})\right].$$

Since the space (2.1) is conformally-flat, its conformal tensor vanishes. So

$$\begin{aligned} R_{hj}g_{ik} + R_{ik}g_{hj} - R_{hk}g_{ij} - R_{ij}g_{hk} &= -(n-2)R_{hijk} + \frac{R}{n-1}(g_{hj}g_{ik} - g_{hk}g_{ij}) \\ &= -\left[\frac{R_{hj}R_{ik} - R_{hk}R_{ij}}{(n-2)\rho^2 + \rho\bar{\rho}} + \{(n-2)\rho^2 + \rho\bar{\rho}\}(g_{hj}g_{ik} - g_{hk}g_{ij})\right], \end{aligned}$$

by (2.4) and (2.6).

Therefore

$$\begin{aligned} \Omega_{hj}\Omega_{ik} - \Omega_{hk}\Omega_{ij} &= \frac{1}{(n-2)^2\rho^2}\left\{1 - \frac{\rho\bar{\rho}}{(n-2)\rho^2 + \rho\bar{\rho}}\right\}(R_{hj}R_{ik} - R_{hk}R_{ij}) \\ &\quad + \frac{\bar{\rho}^2}{(n-2)^2}\left[1 - \frac{1}{\rho\bar{\rho}}\{(n-2)\rho^2 + \rho\bar{\rho}\}\right](g_{hj}g_{ik} - g_{hk}g_{ij}) \\ &= \frac{R_{hj}R_{ik} - R_{hk}R_{ij}}{(n-2)\{(n-2)^2\rho^2 + \rho\bar{\rho}\}} - \frac{\rho\bar{\rho}}{n-2}(g_{hj}g_{ik} - g_{hk}g_{ij}) \end{aligned}$$

which is the right-hand side of (2.6). Hence finally

$$(3.2)\qquad R_{hijk} = \Omega_{hj}\Omega_{ik} - \Omega_{hk}\Omega_{ij}.$$

This shows that there exists a tensor Ω_{ij} in the space (2.1), defined by (3.1), which satisfies (3.2), where R_{hijk} is the curvature tensor of the space.

Again, suppose for the moment $i \neq j \neq k$. Then we obtain for the space (2.1) the following expressions:

$$(3.3)\qquad \begin{aligned} R_{ij} &= -4a^2(n-2)\frac{f''}{f}X^iX^j, \\ R_{ii} &= -\frac{4a^2}{f}\left[(n-1)f' + \{(n-2)(X^i)^2 + (U-c)\}f'' - (n-1)(U-c)\frac{f'^2}{f}\right]. \end{aligned}$$

Taking covariant derivatives we get

[1] Suggested from Verbicku's paper. loc. cit.

$$R_{ij,k} = -\frac{8a^3(n-2)}{f^2}(ff'''+3f'f'')X^iX^jX^k,$$

$$(3.4)\qquad R_{ij,i} = -\frac{4a^3(n-2)}{f^2}[f''\{f-2(U-c)f'\}+2(ff'''+3f'f'')(X^i)^2]X^j,$$

$$R_{ii,j} = -\frac{8a^3}{f^2}[\{(n-2)(X^i)^2+(U-c)\}(ff'''+3f'f'') + nf''\{f-2(U-c)f'\}]X^j.$$

Now taking the covariant derivatives of (3.1) and subtracting we have

$$\Omega_{ij,k}-\Omega_{ik,j} = -\frac{1}{n-2}\left[\frac{1}{\rho}(R_{ij,k}-R_{ik,j})-\frac{1}{\rho^2}(R_{ij}\rho_{,k}-R_{ik}\rho_{,j})+(g_{ij}\bar{\rho}_{,k}-g_{ik}\bar{\rho}_{,j})\right].$$

Writing $2a\,(d\rho/dU)X^l$ for $\rho_{,l}$ and similarly for $\bar{\rho}_{,l}$ and multiplying throughout by ρ^3 we obtain

$$(3.5)\qquad \rho^3(\Omega_{ij,k}-\Omega_{ik,j}) = -\frac{1}{n-2}\left[\rho^2(R_{ij,k}-R_{ik,j})-2a\rho\frac{d\rho}{dU}(R_{ij}X^k-R_{ik}X^j) + 2a\rho^3\frac{d\bar{\rho}}{dU}(g_{ij}X^k-g_{ik}X^j)\right].$$

Also from (2.3) we find by differentiation

$$(3.6)\qquad \rho\frac{d\rho}{dU} = 2a^2f''\{f-2(U-c)f'\},\qquad \frac{d}{dU}(\rho\bar{\rho}) = 4a^2\{2ff''-(U-c)f'f''+(U-c)ff'''\}.$$

So,

$$(3.7)\qquad \begin{aligned}\rho^3\frac{d\bar{\rho}}{dU} &= \rho^2\frac{d}{dU}(\rho\bar{\rho})-(\rho\bar{\rho})\rho\frac{d\rho}{dU}\\ &= 8a^4[2f'\{f-(U-c)f'\}\{2ff''-(U-c)f'f''+(U-c)ff'''\}\\ &\quad -f''\{f-2(U-c)f'\}\{ff'+(U-c)ff''-(U-c)f'^2\}].\end{aligned}$$

Therefore substituting in (3.5) from (2.3), (3.3), (3.4), (3.6) and (3.7) we find that when $i \neq j \neq k$, the right-hand side of (3.5) vanishes identically. And when $k = i$, the equation (3.5) becomes

$$\begin{aligned}&\frac{\rho^3f^2}{16a^5}(\Omega_{ii,j}-\Omega_{ij,i})\\ &= X^j[f'\{f-(U-c)f'\}f''\{f-2(U-c)f'\}\\ &\quad -f''\{f-2(U-c)f'\}f'\{f-(U-c)f'\}] \equiv 0.\end{aligned}$$

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Accordingly, for all values of i, j, k, the equations

$$\Omega_{ij,k} - \Omega_{ik,j} = 0 \tag{3.8}$$

are satisfied. Hence, it follows from (3.2) and (3.3) that the space (2.1) is of class one in which Ω_{ij}, as defined by (3.1), is taken as the second fundamental tensor.

It can be seen that Ω_{ij} can also be expressed as

$$\Omega_{ij} = \rho g_{ij} + (\bar{\rho} - \rho) \frac{X^i X^j}{(U-c)f^2}.$$

It can therefore be easily verified that the equation

$$|\Omega_{ij} - r g_{ij}| = 0 \text{ reduces to } (r-\bar{\rho})(r-\rho)^{n-1} = 0. \tag{3.9}$$

This shows that $\bar{\rho}$ is the simple principal normal curvature and ρ is the principal normal curvature of multiplicity $n-1$ of the space (2.1).

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We now come to the main proposition of the paper. In what follows an n-dimensional Riemannian space will be denoted, as usual, by V_n, a conformally-flat V_n by C_n and a C_n of class one by C_n^1. We have proved in the last section that the space (2.1), which is the general C_n satisfying equations of the form (1.1), is a C_n^1. This result may be put in the form of the following lemma:

LEMMA 1. *If a C_n satisfies equations of the form* (1.1), *it is a C_n^1.*

Let us now prove the following lemma:

LEMMA 2. *If a V_n is a C_n^1, it satisfies equations of the form* (1.1).

PROOF. If the V_n is a space of constant curvature, it is a C_n^1. Also it is an Einstein space, so that $nR_{ij} = Rg_{ij}$. Accordingly, is satisfies equations of the form (1.1).

Again, by definition, a C_n^1 satisfies the equations

$$\begin{aligned} R_{hijk} &= -\frac{1}{n-2}\left[R_{hj}g_{ik} + R_{ik}g_{hj} - R_{hk}g_{ij} - R_{ij}g_{hk} - \frac{R}{n-1}(g_{hj}g_{ik} - g_{hk}g_{ij})\right] \\ &= \Omega_{hj}\Omega_{ik} - \Omega_{hk}\Omega_{ij}, \end{aligned} \tag{4.1}$$

where Ω_{ij} is the second fundamental tensor of the C_n^1. From (4.1) we find that [5]

$$n(K_{ij}\Omega_{kl} - K_{kl}\Omega_{ij}) + \Omega(g_{ij}K_{kl} - g_{kl}K_{ij}) = 0, \tag{4.2}$$

where

$$\Omega = g^{ij}\Omega_{ij}, \quad K_{ij} = \frac{1}{n-2}[nR_{ij} - Rg_{ij}]. \tag{4.3}$$

It is at once seen that a C_n^1 is of constant curvature if and only if $K_{ij} = 0$. The case for space of constant curvature has already been considered. So suppose the C_n^1 is not of constant curvature. Then it can be seen that det $|K_{ij}| \neq 0$ [5], and so the conjugate K^{ij} exists. Multiplying (4.2) by K^{kl}, summing for k, l and applying (4.3) we find that Ω_{ij} can be expressed as a linear combination of R_{ij} and g_{ij}, that is

$$\Omega_{ij} = aR_{ij} + bg_{ij}, \tag{4.4}$$

where a and b are scalars. Therefore, just as (2.6) is obtained from (3.1) in § 3, so a C_n^1 satisfies equations of the from (1.1). Hence the lemma.

Combining lemmas 1 and 2 and considering the properties given in § 2, we may state the following theorem:

THEOREM. *A C_n is a C_n^1 if and only if it satisfies equations of the form* (1.1), *namely*

$$R_{hijk} = E(R_{hj}R_{ik} - R_{hk}R_{ij}) + F(g_{hj}g_{ik} - g_{hk}g_{ij}),$$

where, as stated in § 1, $E \neq 0$, F *are two scalars which are determined by the C_n chosen. For a C_n^1, other than a space of constant curvature, the quantities E, F as well as the scalar curvature R can be expressed in terms of the principal normal curvature ρ of multiplicity $n-1$ and the simple principal normal curvature $\bar{\rho}$ by the equations* (2.4) *and* (2.5). *A canonical form of the line element of the general C_n^1 is given by* (2.1) *in which case ρ^2 and $\rho\bar{\rho}$ have the values* (2.3).

References

[1] R. N. Sen, 'On a type of Riemannian space conformal to a flat space', *Journ. Ind. Math. Soc.* 21 (1957), 104–114.

[2] J. Levine, 'Fields of parallel vectors in conformally flat spaces', *Duke Math. Journ.* 17 (1950), 15–20.

[3] Y. Wong, 'Family of totally umbilical hypersurfaces in an Einstein space', *Annals of Math.* (2) 44 (1943), 271–297.

[4] L. L. Verbickii, 'Geometry of conformal Euclidean spaces of class one', *Transactions of the seminar of vector and tensor analysis* (translated from Russian), IX (1952), 146–182.

[5] M. Matsumoto, 'Conformally flat Riemannian spaces of class one', *Journ. Math. Soc. of Japan*, 3 (1951), 306–309.

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