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Lyapunov Stability and Attraction Under Equivariant Maps

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Abstract. Let M and N be admissible Hausdorff topological spaces endowed with admissible families of open coverings. Assume that $\mathcal{S}$ is a semigroup acting on both M and N. In this paper we study the behavior of limit sets, prolongations, prolongational limit sets, attracting sets, attractors, and Lyapunov stable sets (all concepts defined for the action of the semigroup $\mathcal{S}$) under equivariant maps and semiconjugations from M to N.

1 Introduction

In this paper we study the behavior of limit sets, prolongations, prolongational limit sets, attractors, domains of attraction, and Lyapunov stable sets under equivariant maps and semiconjugations (all defined for semigroup actions of topological spaces).

Lyapunov stable sets and attractors were first studied for dynamical and semi-dynamical systems by Bathia and Hajek [5] and Bathia and Szegö [6, 7].

Afterwards, these concepts were generalized for dynamical polysystems by Tsinias, Kalouptsidis, Bacciotti, and Mazzi [4, 32].

Ellis and Nerurkar [19] introduced the notion of homomorphism of semigroup actions to explore the fine structure of recurrence by using the algebraic structure of compactifications of the acting semigroup. Following this line of investigation, Lyapunov stable sets were also generalized for semigroup actions by Braga Barros, Rocha, and Souza [11]. Also, Braga Barros and Souza [8, 9] introduced the concepts of attractor and chain recurrence for semigroup actions and studied the behavior of Morse decomposition for semigroup actions on principal bundles and their associated bundles. If $\mathcal{S}$ is a semigroup acting on the topological spaces M and N, with M compact, and $p: M \to N$ is an open and continuous equivariant map, then an attractor, a finest Morse decomposition, or a chain transitive set in M projects respectively onto an attractor, a finest Morse decomposition, or a chain transitive set in N ([9, Propositions 3.5, 4.6, 5.3]). This paper also follows this line of investigation by dealing with the notions of Lyapunov stability and attraction for semigroup actions under equivariant maps.

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Equivariant maps, also called homomorphisms of transformation groups, are very important concepts in algebra and topological dynamics. Theorems about the existence of equivariant maps were proved by Gleason [22], Heller [24], Mostow [26], Copeland and De Groot [13], and Kister and Mann [25]. Dynamical concepts such as minimal sets and their homomorphisms were extensively studied by Gottschalk and Hedlund [23], Ellis and Gottschalk [18], Ellis [14–18], Auslander [1–3], and Furstenberg [20, 21].

Recently, Cheban [12] studied global attractors under homomorphisms of dynamical systems, and Souza [30] investigated Lyapunov stability and attraction of semiflows on equivariant fiber bundles.

It is well known that any intrinsic property of transformation groups is preserved under surjective isomorphisms (see [18, 19, 23]). We show that isomorphic (conjugated) transformation semigroups preserve stability, dynamical objects, and asymptotic behavior (Theorem 4.23).

The text is organized as follows. In the first section (Section 2) we introduce the standard notations for semigroup actions necessary for the paper. We also recall some definitions and results of the theories of admissible spaces and Lyapunov stability for semigroup actions. In Section 3, we define and present the main properties of domains of attraction and attractors. In Section 4, we study the behavior of limit sets, prolongations, prolongational limit sets, domains of attraction, attractors, and Lyapunov stable sets under equivariant maps and semiconjugations.

We now discuss applications of the results presented here. Since the projections on principal and associated bundles are equivariant maps (see [9, Section 2] for the semigroup action and details on fiber bundles), the results of this paper can be applied to fiber bundles. The concept of fiber bundle is used to describe physical situations in the most complex theories. For instance, gauge theory involves a principal fiber bundle, called the bundle of frames, in which the fiber at each point of the base space consists of possible coordinate bases for use when describing the values of objects at that point. One must choose a particular coordinate basis at each point (a local section of the fiber bundle) and express the values of the objects of the theory (usually "fields" in the physicist's sense) using this basis. If a semigroup acts on the physical system and the projection of the fiber bundle is equivariant with respect this action, then each element of the semigroup maps a possible coordinate basis for a point in the base space to a possible coordinate basis for the mapping of that point.

The results of this paper can also be applied to the study of attraction and Lyapunov stability for equivariant maps between phase spaces of n-time dynamical systems. An n-time dynamical system on a topological phase space M is an action of the Euclidean space $\mathbb{R}^n$ on M (see [8, Example 2.6] for details). For $n = 1$, we recover the definition of flow or dynamical system and the direction for limit behavior is determined by the *Fréchet filter* $\mathcal{F} = \{(t, +\infty) : t > 0\}$. For $n > 1$, there are several possibilities of directions for limit behavior. For instance, consider a maximal cone

$$\mathcal{S} = \{t = (t_1, \ldots, t_n) : t_i \geq 0\}$$

and define the filter basis $\mathcal{F} = \{\mathcal{S} + t : t \in \mathcal{S}\}$ of translates of $\mathcal{S}$. The limit behavior with respect to the family $\mathcal{F}$ means the limit behavior on the direction of the i-vector (*e.g.*, "time-like" direction in spacetime).

Interesting n-time dynamical systems appear naturally in Lie group actions. Let G be a finite-dimensional real Lie group with Lie algebra $\mathfrak{g}$ and a manifold M. Assume that G acts on the right on M with smooth action $\alpha: M \times G \to M$. Choose n right invariant vector fields $X_1, \dots, X_n$ on the center of $\mathfrak{g}$ and define n vector fields $\mathrm{X}^1, \dots, \mathrm{X}^n$ on M by

$$\mathrm{X}^i(x) = \frac{d}{dt}(x \exp tX_i)\mid_{t=0}, \quad x \in M.$$

The trajectory of X^i through x is given by $\mathrm{X}^i_t(x) = x\exp(tX_i)$. Then we have

$$\mathrm{X}^i_t \circ \mathrm{X}^j_s = \mathrm{X}^j_s \circ \mathrm{X}^i_t$$

for all i, j and $t, s \in \mathbb{R}$. For $(t_1, \dots, t_n) \in \mathbb{R}^n$ and $x \in M$, we define

$$\mu\big((t_1, \dots, t_n), x\big) = \mathrm{X}^1_{t_1} \circ \cdots \circ \mathrm{X}^n_{t_n}(x),$$

which is an n-time dynamical system on M. Moreover, for each $g \in G$ the map $\alpha_g = \alpha\mid_{M\times\{g\}}: M \to M$ is a μ-topological conjugation.

Finally, it is possible to apply the results of this paper to equivariant maps between the phase spaces of affine control systems. We can consider only control systems where the solutions of the systems are given by orbits of the "system semigroup" (for instance, control systems with piecewise constant controls). The limit behavior of the system is determined by a family of subsets of the system semigroup satisfying the hypothesis of this paper (see [8, Section 5.2] for details).

2 Preliminaries

In this section we give the standard notations for semigroup actions on topological spaces. We also recall some definitions and results of the theories of admissible spaces and Lyapunov stability for semigroup actions. We refer to [8–10, 19, 27, 28, 31] for the theory of semigroup actions on topological spaces and to [5–7, 11] for the theory of Lyapunov stability.

Throughout this section we assume that M is a Hausdorff space and $\mathcal{S}$ is a semigroup.

2.1 Semigroup Actions

We start with some standard notations of semigroup actions. An *action* (or a *left action*) of $\mathcal{S}$ on M is a mapping

$$\begin{array}{rccc} \mu\colon & \mathcal{S}\times M & \longrightarrow & M \\ & (s,x) & \longmapsto & \mu(s,x) = sx \end{array}$$

satisfying $s(ux) = (su)x$ for all $x \in M$ and $u, s \in \mathcal{S}$. In this case we say that $\mathcal{S}$ *acts* on M. As in [19], the action of $\mathcal{S}$ on M is also called a *generalized flow* on M. The triple $(\mathcal{S}, M, \mu)$ is called a *transformation semigroup*. We denote by $\mu_s: M \to M$ the map defined by $\mu_s(x) = \mu(s, x)$. We assume that μ_s is continuous for every $s \in \mathcal{S}$. The action μ is called *open* if μ_s is an open map, for every $s \in \mathcal{S}$.

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Now, assume that $\mathcal{S}$ is a semigroup acting on M. For subsets $X \subset M$ and $A \subset \mathcal{S}$, we define

$$AX = \{y \in M : \text{ there exist } s \in A \text{ and } x \in X \text{ with } sx = y\}$$

and

$$A^*X = \{y \in M : \text{ there exist } s \in A \text{ and } x \in X \text{ with } sy = x\}.$$

The set $\mathcal{S}x$ (resp. $\mathcal{S}^*x$) is called the *orbit* (resp. *backward orbit*) of x in M. A set X is called *forward* (respectively *backward*) *invariant* if $\mathcal{S}X \subset X$ (respectively $\mathcal{S}^*X \subset X$). A set is called *invariant* if it is both forward invariant and backward invariant.

The next definition of ω-limit set for semigroup actions was introduced in [8]. It generalizes the definition of ω-limit set for flows and semiflows (see [8, Examples 2.4 and 2.5]). We will often indicate by $\mathcal{P}(\mathcal{S})$ the set of all subsets of $\mathcal{S}$.

Definition 2.1 For $X \subset M$ and $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$, the *$\omega$-limit set* of X for the family $\mathcal{F}$ is defined as

$$\omega(X, \mathcal{F}) = \bigcap_{A \in \mathcal{F}} \operatorname{cl}(AX).$$

Now we recall some concepts of the theory of admissible families of open coverings of topological spaces (see [8, 9, 11, 27]).

Let $\mathcal{U}$ be an open covering of M. The *$\mathcal{U}$-neighborhood* of a subset $X \subset M$ is the open set

$$\begin{aligned} \mathrm{B}(X, \mathcal{U}) &= \{y \in M : \text{there exist } x \in X \text{ and } U \in \mathcal{U} \text{ such that } x, y \in U\} \\ &= \bigcup \{U \in \mathcal{U} : U \cap X \neq \varnothing\}. \end{aligned}$$

For every $x \in M$, we write $\mathrm{B}(x, \mathcal{U}) = \mathrm{B}(\{x\}, \mathcal{U})$. If $\mathcal{V}$ is another open covering of M, we say that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ (or $\mathcal{V}$ *refines* $\mathcal{U}$), and we write $\mathcal{V} \leqslant \mathcal{U}$ if for all $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subset U$. We write $\mathcal{V} \leqslant \frac{1}{2}\mathcal{U}$ if for every $V, V' \in \mathcal{V}$ with $V \cap V' \neq \varnothing$, there exists $U \in \mathcal{U}$ such that $V \cup V' \subset U$.

Definition 2.2 An *admissible family of open coverings* of a topological space M is a family $\mathcal{O}$ of open coverings of M that satisfies the following properties:

(i) for each $\mathcal{U} \in \mathcal{O}$, there exists an open covering $\mathcal{V} \in \mathcal{O}$ such that $\mathcal{V} \leqslant \frac{1}{2}\mathcal{U}$;
(ii) if V is an open set of M and K is a compact subset of M contained in V, then there exists an open covering $\mathcal{U} \in \mathcal{O}$ such that $\mathrm{B}(K, \mathcal{U}) \subset V$;
(iii) for any $\mathcal{U}, \mathcal{V} \in \mathcal{O}$, there exists $\mathcal{W} \in \mathcal{O}$ that is a refinement of both $\mathcal{U}$ and $\mathcal{V}$.

An admissible family is said to be *strong* if the compact subset in item (ii) can be replaced by a closed subset. A topological space that admits an admissible family of open coverings is called an *admissible space*.

The notion of admissible space was introduced in [27]. In general, the uniformizable spaces are admissible. In particular, Tychonoff spaces, metric spaces, compact spaces, topological groups, homogeneous spaces, and topological manifolds are admissible.

We also mention that strong admissible families were considered as a hypothesis in the main results of [27, 28].

Example 2.3 The family $\mathcal{O}_f$ of all finite open coverings of M is strong admissible if M is a compact Hausdorff space. We also have that the family $\mathcal{O}(M)$ of all open coverings of M is strong admissible if M is paracompact.

From now on, in this section, we assume that $\mathcal{O}$ is an admissible family of open coverings of M endowed with the order relation $\leqslant$ by covering refinement.

Next we recall the definitions of net and subnet.

Definition 2.4 A set Λ is a *directed set* if there is a relation $\leq$ on Λ satisfying:

(i) $\lambda \leq \lambda$ for each $\lambda \in \Lambda$,
(ii) if $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$, then $\lambda_1 \leq \lambda_3$,
(iii) if $\lambda_1, \lambda_2 \in \Lambda$, then there is some $\lambda_3 \in \Lambda$ with $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$.

The relation $\leq$ is referred to as a *direction* on Λ, or is said to *direct* Λ.

Note that the admissible family $\mathcal{O}$ is directed by the relation $\leqslant$.

Definition 2.5 A *net* in a set X is a function $P: \Lambda \to X$, where Λ is some directed set. The point $P(\lambda)$ is usually denoted x_λ, and we often speak of "the net $(x_\lambda)_{\lambda\in\Lambda}$". A *subnet* of a net $P: \Lambda \to X$ is the composition $P \circ \phi$, where $\phi: \Gamma \to \Lambda$ is an increasing cofinal function from a directed set Γ to Λ; that is,

(i) $\phi(\gamma_1) \leq \phi(\gamma_2)$ whenever $\gamma_1 \leq \gamma_2$,
(ii) for each $\lambda \in \Lambda$ there is some $\gamma \in \Gamma$ such that $\lambda \leq \phi(\gamma)$.

For $\gamma \in \Gamma$, the point $P \circ \phi(\gamma)$ is often written x_{λ_γ}, and we usually speak of "the subnet $(x_{\lambda_\gamma})_{\gamma\in\Gamma}$ of $(x_\lambda)_{\lambda\in\Lambda}$".

We have the following lemmas.

Lemma 2.6 *Let $\mathcal{U}$ and $\mathcal{V}$ be open coverings of M such that $\mathcal{V} \leqslant \frac{1}{2}\mathcal{U}$ and X is a subset in M. Then* $\mathrm{cl}(\mathrm{B}(X, \mathcal{V})) \subset \mathrm{B}(X, \mathcal{U})$.

Proof See [11, Proposition 2.5]. ∎

Lemma 2.7 *Let $K \subset M$ be compact. For each $\mathcal{V} \in \mathcal{O}$ take $x_{\mathcal{V}} \in \mathrm{B}(K, \mathcal{V})$. Then there exists a subnet $(x_{\mathcal{V}_\lambda})_{\lambda\in\Lambda}$ of $(x_{\mathcal{V}})_{\mathcal{V}\in\mathcal{O}}$ that converges to a point in K.*

Proof See [11, Proposition 2.6]. ∎

Lemma 2.8 *Suppose that $\mathcal{O}$ is an admissible family of open coverings of M. Let $(\Lambda, <)$ be a directed set and consider the following direction on $\Lambda \times \mathcal{O}$:*

$$(\lambda, \mathcal{V}) \geqslant (\mu, \mathcal{U}) \text{ if and only if } \lambda > \mu \text{ and } \mathcal{V} \leqslant \mathcal{U}.$$

Let $(x_{(\lambda,\mathcal{V})})_{(\lambda,\mathcal{V})\in\Lambda\times\mathcal{O}}$ and $(y_{(\lambda,\mathcal{V})})_{(\lambda,\mathcal{V})\in\Lambda\times\mathcal{O}}$ be two nets in M such that $y_{(\lambda,\mathcal{V})} \in \mathrm{B}(x_{(\lambda,\mathcal{V})}, \mathcal{V})$ for all $\mathcal{V} \in \mathcal{O}$ and assume that $x_{(\lambda,\mathcal{V})} \to x$ in M. Then $y_{(\lambda,\mathcal{V})} \to x$.

Proof Take an open covering $\mathcal{U} \in \mathcal{O}$ and choose $\mathcal{W} \in \mathcal{O}$ satisfying $\mathcal{W} \leqslant \frac{1}{2}\mathcal{U}$. We have that there exist $\lambda_0 \in \Lambda$ and $\mathcal{V}_0 \in \mathcal{O}$ such that $x_{(\lambda,\mathcal{V})} \in \mathrm{B}(x, \mathcal{W})$, if $(\lambda, \mathcal{V}) \geqslant (\lambda_0, \mathcal{V}_0)$.

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Now, take an open covering $\mathcal{V}_1 \in \mathcal{O}$ such that $\mathcal{V}_1 \leqslant \mathcal{V}_0$ and $\mathcal{V}_1 \leqslant \mathcal{W}$. Then, for $(\lambda, \mathcal{V}) \geqslant (\lambda_0, \mathcal{V}_1)$, we have $x_{(\lambda,\mathcal{V})} \in \mathrm{B}(x, \mathcal{W})$ and $y_{(\lambda,\mathcal{V})} \in \mathrm{B}(x_{(\lambda,\mathcal{V})}, \mathcal{V}) \subset \mathrm{B}(x_{(\lambda,\mathcal{V})}, \mathcal{W})$. Since $\mathcal{W} \leqslant \frac{1}{2}\mathcal{U}$, it follows that $y_{(\lambda,\mathcal{V})} \in \mathrm{B}(x, \mathcal{U})$, and therefore $y_{(\lambda,\mathcal{V})} \to x$. ■

We refer to [11] for more details of convergence of nets in admissible spaces.

Next, we recall the definitions of prolongations and prolongational limit sets for semigroup actions. They were introduced in [11].

Definition 2.9 Suppose $x \in M$ and $A \subset \mathcal{S}$. The *first A-forward prolongation* of x is defined by

$$\mathrm{D}(x, A) = \bigcap_{\mathcal{U} \in \mathcal{O}} \mathrm{cl}\big(A\mathrm{B}(x, \mathcal{U})\big).$$

Assume that $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$ is a family of subsets of $\mathcal{S}$. The *first forward $\mathcal{F}$-prolongational limit set* of x is defined by

$$\mathrm{J}(x, \mathcal{F}) = \bigcap_{A \in \mathcal{F}} \mathrm{D}(x, A).$$

For a subset $X \subset M$, we define

$$\mathrm{D}(X, A) = \bigcup_{x \in X} \mathrm{D}(x, A) \quad \text{and} \quad \mathrm{J}(X, \mathcal{F}) = \bigcup_{x \in X} \mathrm{J}(x, \mathcal{F}).$$

The prolongations and prolongational limit sets for flows are particular cases of the prolongations and prolongational limit sets for semigroup actions (see [11, Section 4]).

The following definition was introduced in [29] (for metric spaces) and reproduces the notion of divergent net in the semigroup $\mathcal{S}$. We recall that $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$ is a filter basis on the subsets of $\mathcal{S}$ if $\varnothing \notin \mathcal{F}$, and given $A, B \in \mathcal{F}$, there is $C \in \mathcal{F}$ with $C \subset A \cap B$.

Definition 2.10 Let $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$ be a filter basis. For a given net $(t_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{S}$, the notation $t_\lambda \to_{\mathcal{F}} \infty$ means that for each $A \in \mathcal{F}$ there is $\lambda_0 \in \Lambda$ such that $t_\lambda \in A$ for all $\lambda \geq \lambda_0$.

By considering the product direction on $\mathcal{F} \times \mathcal{O}$, that is, $(A, \mathcal{U}) \geqslant (B, \mathcal{V})$ if and only if $A \subset B$ and $\mathcal{U} \leqslant \mathcal{V}$, we can easily see that

$$\omega(X, \mathcal{F}) = \{\, x \in M : \text{there are nets } (t_\lambda) \text{ in } S \text{ and } (x_\lambda) \text{ in } X \text{ such that } t_\lambda \to_{\mathcal{F}} \infty \text{ and } t_\lambda x_\lambda \to x \,\},$$

for any subset $X \subset M$, and

$$\mathrm{D}(x, A) = \{\, y \in M : \text{there are nets } (t_\lambda) \text{ in } A \text{ and } (x_\lambda) \text{ in } M \text{ such that } x_\lambda \to x \text{ and } t_\lambda x_\lambda \to y \,\},$$

$$\mathrm{J}(x, \mathcal{F}) = \{\, y \in M : \text{there are nets } (t_\lambda) \text{ in } S \text{ and } (x_\lambda) \text{ in } M \text{ such that } t_\lambda \to_{\mathcal{F}} \infty,\ x_\lambda \to x, \text{ and } t_\lambda x_\lambda \to y \,\},$$

for any point $x \in M$.

Note that the limit sets of $(\mathcal{S}, X)$ with respect to the family $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$ are nonempty if X is a compact forward invariant subspace of M and $\mathcal{F}$ is a filter basis on the subsets of $\mathcal{S}$.The following additional hypothesis on the family $\mathcal{F}$ is necessary to have invariance of limit sets. These hypotheses have already been considered in [8, 9, 11, 29].

Definition 2.11 The family $\mathcal{F}$ is said to satisfy

(i) Hypothesis H_1 if for all $s \in S$ and $A \in \mathcal{F}$ there exists $B \in \mathcal{F}$ such that $sB \subset A$,
(ii) Hypothesis H_2 if for all $s \in S$ and $A \in \mathcal{F}$ there exists $B \in \mathcal{F}$ such that $Bs \subset A$,
(iii) Hypothesis H_3 if for all $s \in S$ and $A \in \mathcal{F}$ there exists $B \in \mathcal{F}$ such that $B \subset As$.

The next lemma will be used in the sequel.

Lemma 2.12 *Suppose that* $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$ *and take* $x \in M$ *and* $s \in \mathcal{S}$.

(i) *If* $\mathcal{F}$ *satisfies hypothesis* H_3, *then* $J(x, \mathcal{F}) \subset J(sx, \mathcal{F})$.
(ii) *If* $\mathcal{F}$ *satisfies hypothesis* H_2 *and the action of* $\mathcal{S}$ *on* M *is open, then* $J(sx, \mathcal{F}) \subset J(x, \mathcal{F})$.

Proof (i) Take $y \in J(x, \mathcal{F})$, $A \in \mathcal{F}$ and $\mathcal{U} \in \mathcal{O}$. By the continuity of s, there exists $\mathcal{V} \in \mathcal{O}$ such that $sB(x, \mathcal{V}) \subset B(sx, \mathcal{U})$. It follows from hypothesis H_3 that there exists $F \in \mathcal{F}$ such that $F \subset As$. Thus, we have

$$y \in \operatorname{cl}\big(FB(x, \mathcal{V})\big) \subset \operatorname{cl}\big(AsB(x, \mathcal{V})\big) \subset \operatorname{cl}\big(AB(sx, \mathcal{U})\big).$$

Therefore, $y \in J(sx, \mathcal{F})$.

(ii) For $y \in J(sx, \mathcal{F})$, $A \in \mathcal{F}$ and $\mathcal{U} \in \mathcal{O}$, take $\mathcal{V} \in \mathcal{O}$ such that $B(sx, \mathcal{V}) \subset sB(x, \mathcal{U})$. From hypothesis H_2, there is $F \in \mathcal{F}$ such that $Fs \subset A$. Since $y \in J(sx, \mathcal{F})$, we obtain

$$y \in \operatorname{cl}(FB(sx, \mathcal{V})) \subset \operatorname{cl}(FsB(x, \mathcal{U})) \subset \operatorname{cl}(AB(x, \mathcal{U})),$$

showing that $y \in J(x, \mathcal{F})$. ∎

Now we recall the notions of Lyapunov stability and asymptotic stability for semigroup actions (see [11]).

Definition 2.13 Let $\mathcal{F}$ be a family of subsets of $\mathcal{S}$ and let X be a subset of M.

(i) The set X is called $\mathcal{S}$-*stable* if for every $x \in X$ and every open covering $\mathcal{U} \in \mathcal{O}$ there exists $\mathcal{V} \in \mathcal{O}$ such that $\mathcal{S}B(x, \mathcal{V}) \subset B(X, \mathcal{U})$.
(ii) The set X is called $\mathcal{S}$-*uniformly stable* if for every open covering $\mathcal{U} \in \mathcal{O}$ there exists $\mathcal{V} \in \mathcal{O}$ such that $\mathcal{S}B(X, \mathcal{V}) \subset B(X, \mathcal{U})$.
(iii) The set X is called $\mathcal{S}$-*equistable* if for each $z \notin X$ there exists an open covering $\mathcal{U} \in \mathcal{O}$ such that $z \notin \operatorname{cl}(\mathcal{S}B(X, \mathcal{U}))$.
(iv) The set X is called $\mathcal{S}$-*orbitally stable* if for every open covering $\mathcal{U} \in \mathcal{O}$, there exists $\mathcal{V} \in \mathcal{O}$ such that $B(X, \mathcal{V}) \subset B(X, \mathcal{U})$ and $\mathcal{S}B(X, \mathcal{V}) \subset B(X, \mathcal{V})$.
(v) The set X is called $\mathcal{F}$-*asymptotically stable* if X is an $\mathcal{F}$-attractor (see Definition 3.1) and is $\mathcal{S}$-uniformly stable.

It follows immediately from Definition 2.13 that a $\mathcal{S}$-uniformly stable set is $\mathcal{S}$-stable. Moreover, every compact $\mathcal{S}$-stable set is $\mathcal{S}$-uniformly stable (see [11, Theorem 3.2]). We refer to [11] for the relation among the several concepts of Lyapunov stability in the setting of semigroup actions on topological spaces.

The following characterization for Lyapunov stability using prolongation will be used in the sequel.

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Proposition 2.14 *Let K be a compact subset of M. Suppose that $x \in \operatorname{cl}(\mathcal{S}x)$ for every $x \in K$. Then the set K is $\mathcal{S}$-equistable if and only if* $\mathrm{D}(K, \mathcal{S}) = K$.

Proof See [11, Corollary 3.1]. ∎

3 Attraction

In this section, we define and present the main properties of domains of attraction and attractors. From now on, we assume that $\mathcal{S}$ is a semigroup acting on an admissible Hausdorff space M endowed with an admissible family $\mathcal{O}$.

We start with the definitions of domains of attraction.

Definition 3.1 Let X be a subset of M and $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$. The *domain of weak $\mathcal{F}$-attraction* of X is the set

$$\mathfrak{A}_w(X, \mathcal{F}) = \{x \in M : Ax \cap \mathrm{B}(X, \mathcal{U}) \neq \varnothing \text{ for every } \mathcal{U} \in \mathcal{O} \text{ and } A \in \mathcal{F}\}.$$

The *domain of $\mathcal{F}$-attraction* of X is the set

$$\mathfrak{A}(X, \mathcal{F}) = \{x \in M : \text{for each } \mathcal{U} \in \mathcal{O} \text{ there is } A \in \mathcal{F} \text{ such that } Ax \subset \mathrm{B}(X, \mathcal{U})\}.$$

The *domain of uniform $\mathcal{F}$-attraction* of X is the set

$$\mathfrak{A}_u(X, \mathcal{F}) = \{x \in M : \text{for each } \mathcal{U} \in \mathcal{O} \\ \text{there exist } A \in \mathcal{F} \text{ and } \mathcal{V} \in \mathcal{O} \text{ such that } A\mathrm{B}(x, \mathcal{V}) \subset \mathrm{B}(X, \mathcal{U})\}.$$

The *domain of weak uniform $\mathcal{F}$-attraction* of X is defined as

$$\mathfrak{A}_{wu}(X, \mathcal{F}) = \{x \in M : A\mathrm{B}(x, \mathcal{V}) \cap \mathrm{B}(X, \mathcal{U}) \neq \varnothing, \text{ for every } A \in \mathcal{F} \text{ and } \mathcal{U}, \mathcal{V} \in \mathcal{O}\}.$$

The set X is called *weak $\mathcal{F}$-attractor*, *$\mathcal{F}$-attractor*, *uniform $\mathcal{F}$-attractor*, or *weak uniform $\mathcal{F}$-attractor* if there is $\mathcal{U} \in \mathcal{O}$ such that $\mathrm{B}(X, \mathcal{U}) \subset \mathfrak{A}_w(X, \mathcal{F})$, $\mathrm{B}(X, \mathcal{U}) \subset \mathfrak{A}(X, \mathcal{F})$, $\mathrm{B}(X, \mathcal{U}) \subset \mathfrak{A}_u(X, \mathcal{F})$, or $\mathrm{B}(X, \mathcal{U}) \subset \mathfrak{A}_{wu}(X, \mathcal{F})$, respectively.

In the context of flows on metric spaces, the region of attraction appears, in the study of attraction and asymptotic stability of closed sets. It is not difficult to show that the concepts of domains of attraction and attractor for semigroup actions generalize the corresponding concepts for flows on metric spaces. In fact, let ϕ be a continuous flow on a metric space M with metric d and $\mathcal{F} = \{(t, \infty) : t > 0\}$. Let $tx = \phi(t, x)$ be the action of $\mathbb{R}$ on M defined by the flow ϕ. We have that the $\mathcal{F}$-domains of attraction for the action of $\mathbb{R}$ on M are exactly the regions of attraction for the flow ϕ considered in [6, 7].

It is easily seen that $\mathfrak{A}_u(X, \mathcal{F}) \subset \mathfrak{A}(X, \mathcal{F})$. Hence, every uniform $\mathcal{F}$-attractor is an $\mathcal{F}$-attractor. If $\mathcal{F}$ is a filter basis on the subsets of $\mathcal{S}$, then $\mathfrak{A}(X, \mathcal{F}) \subset \mathfrak{A}_w(X, \mathcal{F})$. In this case, every $\mathcal{F}$-attractor is a weak $\mathcal{F}$-attractor.

In the following proposition we present sufficient conditions for the invariance of the domains of attraction.

Proposition 3.2 *Assume that $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$ and take a subset X of M. Suppose that $\mathfrak{A}_w(X, \mathcal{F})$, $\mathfrak{A}(X, \mathcal{F})$, $\mathfrak{A}_u(X, \mathcal{F})$, and $\mathfrak{A}_{wu}(X, \mathcal{F})$ are nonempty sets.*

(i) $\mathfrak{A}_w(X, \mathcal{F})$ *is forward invariant if $\mathcal{F}$ satisfies the hypothesis* H_3.

(ii) $\mathfrak{A}_w(X,\mathcal{F})$ *is backward invariant if* $\mathcal{F}$ *satisfies the hypothesis* H_2.
(iii) $\mathfrak{A}(X,\mathcal{F})$ *is forward invariant if* $\mathcal{F}$ *satisfies the hypothesis* H_2.
(iv) $\mathfrak{A}(X,\mathcal{F})$ *is backward invariant if* $\mathcal{F}$ *satisfies the hypothesis* H_3.
(v) $\mathfrak{A}_u(X,\mathcal{F})$ *is forward invariant if* $\mathcal{F}$ *satisfies the hypothesis* H_2 *and the action is open.*
(vi) $\mathfrak{A}_u(X,\mathcal{F})$ *is backward invariant if* $\mathcal{F}$ *satisfies the hypothesis* H_3.
(vii) $\mathfrak{A}_{wu}(X,\mathcal{F})$ *is forward invariant if the family* $\mathcal{F}$ *satisfies the hypothesis* H_3.
(viii) $\mathfrak{A}_{wu}(X,\mathcal{F})$ *is backward invariant if the family* $\mathcal{F}$ *satisfies the hypothesis* H_2 *and the action is open.*

Proof (i) Take $s\in\mathcal{S}$, $z\in\mathfrak{A}_w(X,\mathcal{F})$, and $A\in\mathcal{F}$. There exists $B\in\mathcal{F}$ such that $B\subset As$. For this element $B\in\mathcal{F}$ there exists a net $(t_{\mathcal{V}})_{\mathcal{V}\in\mathcal{O}}$ in B such that, for every open covering $\mathcal{U}\in\mathcal{O}$, there exists $\mathcal{V}_0\in\mathcal{O}$ that satisfies $t_{\mathcal{V}}z\in B(X,\mathcal{U})$ for $\mathcal{V}\leqslant\mathcal{V}_0$. Since $(t_{\mathcal{V}})_{\mathcal{V}\in\mathcal{O}}\subset As$, for each $\mathcal{V}\in\mathcal{O}$ there exists $s_{\mathcal{V}}\in A$ with $t_{\mathcal{V}}=s_{\mathcal{V}}s$. Thus, for every open covering $\mathcal{U}\in\mathcal{O}$ there exists $\mathcal{V}_0\in\mathcal{O}$ such that, if $\mathcal{V}\leqslant\mathcal{V}_0$, then $s_{\mathcal{V}}(sz)=t_{\mathcal{V}}z\in B(X,\mathcal{U})$. Therefore, $sz\in\mathfrak{A}_w(X,\mathcal{F})$ and $\mathfrak{A}_w(X,\mathcal{F})$ is forward invariant.

(ii) Take $y\in\mathcal{S}^*\mathfrak{A}_w(X,\mathcal{F})$. It follows that there exist $s\in\mathcal{S}$ and $z\in\mathfrak{A}_w(X,\mathcal{F})$ with $sy=z$. Fix $A\in\mathcal{F}$. From the translation hypothesis there exists $B\in\mathcal{F}$ satisfying $Bs\subset A$ and, since $sy\in\mathfrak{A}_w(X,\mathcal{F})$ there exists a net $(t_{\mathcal{V}})_{\mathcal{V}\in\mathcal{O}}$ in B such that for every open covering $\mathcal{U}\in\mathcal{O}$ there exists $\mathcal{V}_0\in\mathcal{O}$ with $t_{\mathcal{V}}(sy)\in B(X,\mathcal{U})$ for $\mathcal{V}\leqslant\mathcal{V}_0$. Since $t_{\mathcal{V}}(sy)=(t_{\mathcal{V}}s)y$ and $Bs\subset A$, we get $y\in\mathfrak{A}_w(X,\mathcal{F})$. Therefore, $\mathfrak{A}_w(X,\mathcal{F})$ is backward invariant.

(iii) Take $s\in\mathcal{S}$ and $z\in\mathfrak{A}(X,\mathcal{F})$. For an open covering $\mathcal{U}\in\mathcal{O}$ there exists $A\in\mathcal{F}$ such that $Az\subset \mathrm{B}(X,\mathcal{U})$. It follows from the translation hypothesis that there exists $B\in\mathcal{F}$ such that $Bs\subset A$. Thus we have $Bsz\subset\mathrm{B}(X,\mathcal{U})$ and $sz\in\mathfrak{A}(X,\mathcal{F})$. Therefore, $\mathfrak{A}(X,\mathcal{F})$ is forward invariant.

(iv) For $y\in\mathcal{S}^*\mathfrak{A}(X,\mathcal{F})$ there exist $s\in\mathcal{S}$ and $z\in\mathfrak{A}(X,\mathcal{F})$ such that $sy=z$. Since $sy\in\mathfrak{A}(X,\mathcal{F})$, given an open covering $\mathcal{U}\in\mathcal{O}$ there exists $A\in\mathcal{F}$ satisfying $Asy\subset\mathrm{B}(X,\mathcal{U})$. Moreover, it follows from the translation hypothesis that there exists $B\in\mathcal{F}$ with $B\subset As$. Therefore, $By\subset\mathrm{B}(X,\mathcal{U})$ and $\mathfrak{A}(X,\mathcal{F})$ is backward invariant.

(v) For $s\in\mathcal{S}$, $x\in\mathfrak{A}_u(X,\mathcal{F})$ and $\mathcal{U}\in\mathcal{O}$, there exist $A\in\mathcal{F}$ and $\mathcal{W}\in\mathcal{O}$ such that $A\mathrm{B}(x,\mathcal{W})\subset\mathrm{B}(X,\mathcal{U})$. It follows from the translation hypothesis that there exists $B\in\mathcal{F}$ such that $Bs\subset A$. We also have that there exists an open covering $\mathcal{V}\in\mathcal{O}$ satisfying $\mathrm{B}(sx,\mathcal{V})\subset s\mathrm{B}(x,\mathcal{W})$. Therefore, $B\mathrm{B}(sx,\mathcal{V})\subset\mathrm{B}(X,\mathcal{U})$, $sx\in\mathfrak{A}_u(X,\mathcal{F})$, and $\mathfrak{A}_u(X,\mathcal{F})$ is forward invariant.

(vi) Fix $y\in\mathcal{S}^*\mathfrak{A}_u(X,\mathcal{F})$. Take $s\in\mathcal{S}$ and $x\in\mathfrak{A}_u(X,\mathcal{F})$ such that $sy=x$. For an open covering $\mathcal{U}\in\mathcal{O}$ there exist $A\in\mathcal{F}$ and $\mathcal{W}\in\mathcal{O}$ such that $A\mathrm{B}(sy,\mathcal{W})\subset\mathrm{B}(X,\mathcal{U})$. By the continuity of μ_s there exists an open covering $\mathcal{V}\in\mathcal{O}$ such that $s\mathrm{B}(y,\mathcal{V})\subset\mathrm{B}(sy,\mathcal{W})$, and from the translation hypothesis there exists $B\in\mathcal{F}$ such that $B\subset As$. It follows that $B\mathrm{B}(y,\mathcal{V})\subset\mathrm{B}(X,\mathcal{U})$ and $y\in\mathfrak{A}_u(X,\mathcal{F})$. Therefore, $\mathfrak{A}_u(X,\mathcal{F})$ is backward invariant.

(vii) Take $s\in\mathcal{S}$, $z\in\mathfrak{A}_{wu}(X,\mathcal{F})$, open coverings $\mathcal{U},\mathcal{V}\in\mathcal{O}$, and $A\in\mathcal{F}$. By the continuity of μ_s there exists an open covering $\mathcal{W}\in\mathcal{O}$ such that $s\mathrm{B}(z,\mathcal{W})\subset\mathrm{B}(sz,\mathcal{V})$. The hypothesis H_3 provides an element $B\in\mathcal{F}$ satisfying $B\subset As$. Since $z\in\mathfrak{A}_{wu}(X,\mathcal{F})$, we have $B\mathrm{B}(z,\mathcal{W})\cap\mathrm{B}(X,\mathcal{U})\neq\varnothing$. Thus, $As\mathrm{B}(z,\mathcal{W})\cap\mathrm{B}(X,\mathcal{U})\neq\varnothing$ and $A\mathrm{B}(sz,\mathcal{V})\cap\mathrm{B}(X,\mathcal{U})\neq\varnothing$. Therefore, $sz\in\mathfrak{A}_{wu}(X,\mathcal{F})$.

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(viii) Take $y \in \mathcal{S}^*\mathfrak{A}_{wu}(X,\mathcal{F})$, $\mathcal{U},\mathcal{V} \in \mathcal{O}$ and $A \in \mathcal{F}$. From the hypothesis there exist $s \in \mathcal{S}$, $B \in \mathcal{F}$ and an open covering $\mathcal{W} \in \mathcal{O}$ such that $sy \in A_{\mathrm{wu}}(X,\mathcal{F})$, $Bs \subset A$ and $\mathrm{B}(sy,\mathcal{W}) \subset s\mathrm{B}(y,\mathcal{V})$. It follows that $B\mathrm{B}(sy,\mathcal{W}) \cap \mathrm{B}(X,\mathcal{U}) \neq \varnothing$. Thus, $Bs\mathrm{B}(y,\mathcal{V}) \cap \mathrm{B}(X,\mathcal{U}) \neq \varnothing$, and therefore $A\mathrm{B}(y,\mathcal{V}) \cap \mathrm{B}(X,\mathcal{U}) \neq \varnothing$, that is, $y \in \mathfrak{A}_{wu}(X,\mathcal{F})$. ■

Now, we show that under certain conditions the domains of attraction are open sets.

Proposition 3.3 *Suppose that $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$ and take a compact subset K of M.*

(i) *$\mathfrak{A}_w(K,\mathcal{F})$ is an open set containing K if it is a weak $\mathcal{F}$-attractor and $\mathcal{F}$ satisfies the hypothesis* H_2.
(ii) *$\mathfrak{A}(K,\mathcal{F})$ is an open set containing K if it is an $\mathcal{F}$-attractor and $\mathcal{F}$ satisfies the hypothesis* H_3.
(iii) *$\mathfrak{A}_u(K,\mathcal{F})$ is an open set containing K if it is a uniform $\mathcal{F}$-attractor and $\mathcal{F}$ satisfies the hypothesis* H_3.

Proof Take an open covering $\mathcal{W} \in \mathcal{O}$ such that $\mathrm{B}(K,\mathcal{W}) \subset \mathfrak{A}_w(K,\mathcal{F})$ and fix $x \in \mathfrak{A}_w(K,\mathcal{F})$. It follows from definition that there exists $s \in \mathcal{S}$ such that $sx \in \mathrm{B}(K,\mathcal{W})$. Since $\mathrm{B}(K,\mathcal{W})$ is an open set we can choose a neighborhood V of sx in M contained in $\mathrm{B}(K,\mathcal{W})$. For an element $y \in \mu_s^{-1}(V)$, we have $sy \in V \subset \mathfrak{A}_w(K,\mathcal{F})$ and then $y \in \mathcal{S}^*\mathfrak{A}_w(K,\mathcal{F})$. Since $\mathfrak{A}_w(K,\mathcal{F})$ is backward invariant we get $y \in \mathfrak{A}_w(K,\mathcal{F})$. Therefore $\mathfrak{A}_w(K,\mathcal{F})$ is an open set containing K. Items (ii) and (iii) follow analogously. ■

We introduce the sets $\mathrm{Atr_w}(X,\mathcal{F})$, $\mathrm{Atr}(X,\mathcal{F})$, $\mathrm{Atr_{wu}}(X,\mathcal{F})$, and $\mathrm{Atr_u}(X,\mathcal{F})$ as

$$
\begin{aligned}
\mathrm{Atr_w}(X,\mathcal{F}) &= \{x \in M : X \cap \omega(x,\mathcal{F}) \neq \varnothing\},\\
\mathrm{Atr}(X,\mathcal{F}) &= \{x \in M : \omega(x,\mathcal{F}) \neq \varnothing \text{ and } \omega(x,\mathcal{F}) \subset X\},\\
\mathrm{Atr_{wu}}(X,\mathcal{F}) &= \{x \in M : \mathrm{J}(x,\mathcal{F}) \cap X \neq \varnothing\},\\
\mathrm{Atr_u}(X,\mathcal{F}) &= \{x \in M : \mathrm{J}(x,\mathcal{F}) \neq \varnothing \text{ and } \mathrm{J}(x,\mathcal{F}) \subset X\}.
\end{aligned}
\tag{3.1}
$$

The set $\mathrm{Atr_w}(X,\mathcal{F})$ is forward invariant if $\mathcal{F}$ satisfies hypothesis H_3, and it is backward invariant if $\mathcal{F}$ satisfies hypotheses H_2; the set $\mathrm{Atr}(X,\mathcal{F})$ is invariant if $\mathcal{F}$ satisfies both hypotheses H_2 and H_3 (see [11, Proposition 2.15]). It follows from the definition and Lemma 2.12 that $\mathrm{Atr_{wu}}(X,\mathcal{F})$ is $\mathcal{S}$-forward invariant if $\mathcal{F}$ satisfies hypothesis H_3 and $\mathrm{Atr_{wu}}(X,\mathcal{F})$ is $\mathcal{S}$-backward invariant if $\mathcal{F}$ satisfies hypothesis H_2 and the action of $\mathcal{S}$ on M is open. On the invariance of $\mathrm{Atr_u}(X,\mathcal{F})$, we have the following result.

Proposition 3.4 *Assume that the action is open and $\mathcal{F}$ satisfies hypotheses* H_2 *and* H_3*. For $X \subset M$, $\mathrm{Atr_u}(X,\mathcal{F})$ is invariant if it is nonempty.*

Proof Take $s \in \mathcal{S}$ and $z \in \mathrm{Atr_u}(X,\mathcal{F})$. Then $\mathrm{J}(z,\mathcal{F}) \neq \varnothing$ and $\mathrm{J}(z,\mathcal{F}) \subset X$. By Lemma 2.12, we have $\mathrm{J}(z,\mathcal{F}) = \mathrm{J}(sz,\mathcal{F})$, and therefore $\mathrm{Atr_u}(X,\mathcal{F})$ is forward invariant. Now, take $y \in \mathcal{S}^*\mathrm{Atr_u}(X,\mathcal{F})$. There exist $s \in \mathcal{S}$ and $z \in \mathrm{Atr_u}(X,\mathcal{F})$ such that $sy = z$. Since $z \in \mathrm{Atr_u}(X,\mathcal{F})$, we have $\mathrm{J}(sy,\mathcal{F}) \neq \varnothing$ and $\mathrm{J}(sy,\mathcal{F}) \subset X$. By Lemma 2.12, it follows that $\mathrm{J}(sy,\mathcal{F}) = \mathrm{J}(y,\mathcal{F})$, and therefore $\mathrm{Atr_u}(X,\mathcal{F})$ is backward invariant. ■

The next four theorems characterize the domains of attraction of compact sets.

Theorem 3.5 *Let $\mathcal{F}$ be a filter basis on the subsets of $\mathcal{S}$ and K a compact subset of M. Then $\mathfrak{A}_w(K,\mathcal{F}) = \mathrm{Atr}_w(K,\mathcal{F})$.*

Proof Suppose that $K \cap \omega(x,\mathcal{F}) \neq \varnothing$. Then $K \cap \mathrm{cls}(Ax) \neq \varnothing$ for all $A \in \mathcal{F}$. Hence, for every $\mathcal{U} \in \mathcal{O}$ and $A \in \mathcal{F}$, we have $Ax \cap \mathrm{B}(K,\mathcal{U}) \neq \varnothing$, and therefore $x \in \mathfrak{A}_w(K,\mathcal{F})$. As to the converse, suppose that $x \in \mathfrak{A}_w(K,\mathcal{F})$. Then for every $\mathcal{U} \in \mathcal{O}$ and $A \in \mathcal{F}$, there is $t_{(A,\mathcal{U})}x \in Ax \cap \mathrm{B}(x_{(A,\mathcal{U})},\mathcal{U})$, with $t_{(A,\mathcal{U})} \in A$ and $x_{(A,\mathcal{U})} \in K$. Since K is compact, we may assume that the net $(x_{(A,\mathcal{U})})_{(A,\mathcal{U})\in\mathcal{F}\times\mathcal{O}}$ converges to the point $y \in K$. Then the net $(t_{(A,\mathcal{U})}x)_{(A,\mathcal{U})\in\mathcal{F}\times\mathcal{O}}$ also converges to y. In fact, for a given $\mathcal{W} \in \mathcal{O}$, take $\mathcal{V} \in \mathcal{O}$ such that $\mathcal{V} \leqslant \frac{1}{2}\mathcal{W}$. There is $(A_0,\mathcal{U}_0) \in \mathcal{F}\times\mathcal{O}$ such that $x_{(A,\mathcal{U})} \in \mathrm{B}(y,\mathcal{V})$ for all $(A,\mathcal{U}) \geqslant (A_0,\mathcal{U}_0)$. Take $\mathcal{V}_0 \in \mathcal{O}$ such that $\mathcal{V}_0$ refines both $\mathcal{V}$ and $\mathcal{U}_0$. As $t_{(A,\mathcal{U})}x \in \mathrm{B}(x_{(A,\mathcal{U})},\mathcal{U})$ and $x_{(A,\mathcal{U})} \in \mathrm{B}(y,\mathcal{V})$, it follows that $(A,\mathcal{U}) \geqslant (A_0,\mathcal{V}_0)$ implies $t_{(A,\mathcal{U})}x \in \mathrm{B}(y,\mathcal{W})$. Thus, the net $(t_{(A,\mathcal{U})}x)$ converges to y. Now, as $t_{(A,\mathcal{U})} \to_{\mathcal{F}} \infty$, we have $y \in K \cap \omega(x,\mathcal{F})$, and the proof is complete. ■

Theorem 3.6 *Let $\mathcal{F}$ be a filter basis on the subsets of $\mathcal{S}$ and let K be a compact subset of M. Then $\mathfrak{A}(K,\mathcal{F}) \subset \mathrm{Atr}(K,\mathcal{F})$. The equality holds if M is locally compact and Ax is connected for all $A \in \mathcal{F}$ and $x \in M$.*

Proof Take $x \in \mathfrak{A}(K,\mathcal{F})$. By Theorem 3.5, we have $\omega(x,\mathcal{F}) \neq \varnothing$, because $\mathfrak{A}(K,\mathcal{F}) \subset \mathfrak{A}_w(K,\mathcal{F})$. Take $y \in \omega(x,\mathcal{F})$ and $\mathcal{U} \in \mathcal{O}$. Since K is compact and M is a Tychonoff space, there is $\mathcal{V} \in \mathcal{O}$ such that $\mathrm{cls}(\mathrm{B}(K,\mathcal{V})) \subset \mathrm{B}(K,\mathcal{U})$. As $x \in \mathfrak{A}(K,\mathcal{F})$, there is $A \in \mathcal{F}$ such that $Ax \subset \mathrm{B}(K,\mathcal{V})$. Hence, $y \in \mathrm{cls}(Ax) \subset \mathrm{cls}(\mathrm{B}(K,\mathcal{V}))$ and therefore $y \in \mathrm{B}(K,\mathcal{U})$. Since $\mathcal{U}$ is an arbitrary open covering in $\mathcal{O}$ and K is compact, it follows that $y \in K$. Thus, $\omega(x,\mathcal{F}) \neq \varnothing$ and $\omega(x,\mathcal{F}) \subset K$. Now, assume that M is locally compact and Ax is connected for all $A \in \mathcal{F}$ and $x \in M$. Let $N \subset M$ be a compact neighborhood of K. Suppose that the point $x \in M$ satisfies $\omega(x,\mathcal{F}) \neq \varnothing$ and $\omega(x,\mathcal{F}) \subset K$. For a given $\mathcal{U} \in \mathcal{O}$, take $\mathcal{V} \in \mathcal{O}$ such that $\mathcal{V} \leqslant \mathcal{U}$ and $\mathrm{B}(K,\mathcal{V}) \subset N$. Then $Ax \cap \mathrm{B}(K,\mathcal{V}) \neq \varnothing$ for all $A \in \mathcal{F}$. Suppose by contradiction that $Ax \nsubseteq \mathrm{B}(K,\mathcal{V})$ for every $A \in \mathcal{F}$. Since Ax is connected, there is $t_A x \in N \setminus \mathrm{B}(K,\mathcal{V})$ with $t_A \in A$. As N is compact and $t_A \to_{\mathcal{F}} \infty$, it follows that there is a point $y \in \omega(x,\mathcal{F}) \cap (N \setminus \mathrm{B}(K,\mathcal{V}))$, which is a contradiction. Hence, there is $A \in \mathcal{F}$ such that $Ax \subset \mathrm{B}(K,\mathcal{V}) \subset \mathrm{B}(K,\mathcal{U})$, and therefore $x \in \mathfrak{A}(K,\mathcal{F})$. ■

Theorem 3.7 *Let $\mathcal{F}$ be a filter basis on the subsets of $\mathcal{S}$ and K a compact subset of M. Then $\mathfrak{A}_u(K,\mathcal{F}) \subset \mathrm{Atr}_u(K,\mathcal{F})$. Equality holds if M is locally compact and Ax is connected for all $A \in \mathcal{F}$ and $x \in M$.*

Proof Take $x \in \mathfrak{A}_u(K,\mathcal{F})$. Since $\mathfrak{A}_u(K,\mathcal{F}) \subset \mathfrak{A}(K,\mathcal{F})$, we have $\omega(y,\mathcal{F}) \neq \varnothing$. Hence, $\mathrm{J}(x,\mathcal{F}) \neq \varnothing$. Take $y \in \mathrm{J}(x,\mathcal{F})$ and $\mathcal{U} \in \mathcal{O}$. Choose $\mathcal{V} \in \mathcal{O}$ such that $\mathrm{cls}(\mathrm{B}(K,\mathcal{V})) \subset \mathrm{B}(K,\mathcal{U})$. As $x \in \mathfrak{A}_u(K,\mathcal{F})$, there is $A \in \mathcal{F}$ and $\mathcal{W} \in \mathcal{O}$ such that $A\mathrm{B}(x,\mathcal{W}) \subset \mathrm{B}(K,\mathcal{V})$. Hence, $y \in \mathrm{cls}(A\mathrm{B}(x,\mathcal{W})) \subset \mathrm{cls}(\mathrm{B}(K,\mathcal{V}))$, and therefore $y \in \mathrm{B}(K,\mathcal{U})$. It follows that $y \in K$. Thus, $\mathrm{J}(x,\mathcal{F}) \neq \varnothing$ and $\mathrm{J}(x,\mathcal{F}) \subset K$. Now, assume that M is locally compact and Ax is connected for all $A \in \mathcal{F}$ and $x \in M$. Let $N \subset M$ be a compact neighborhood of K. Suppose that the point $x \in M$ satisfies $\mathrm{J}(x,\mathcal{F}) \neq \varnothing$ and $\mathrm{J}(x,\mathcal{F}) \subset K$. For a given $\mathcal{U} \in \mathcal{O}$, take $\mathcal{V} \in \mathcal{O}$ such that $\mathcal{V} \leqslant \mathcal{U}$ and $\mathrm{B}(K,\mathcal{V}) \subset N$.

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Then $A\mathrm{B}(x,\mathcal{W})\cap\mathrm{B}(K,\mathcal{V})\neq\varnothing$ for all $A\in\mathcal{F}$ and $\mathcal{W}\in\mathcal{O}$. For each $A\in\mathcal{F}$ and $\mathcal{W}\in\mathcal{O}$, define the set

$$U_{(A,\mathcal{W})}=\left\{y\in\mathrm{B}(x,\mathcal{W}):Ay\cap\mathrm{B}(K,\mathcal{V})\neq\varnothing\right\}.$$

If $y\in U_{(A,\mathcal{W})}$, then there is $s\in A$ such that $sy\in\mathrm{B}(K,\mathcal{V})$. Hence, $\mu_s^{-1}(\mathrm{B}(K,\mathcal{V}))\cap\mathrm{B}(x,\mathcal{W})$ is an open neighborhood of y contained in $U_{(A,\mathcal{W})}$. Thus, $U_{(A,\mathcal{W})}$ is an open set contained in $\mathrm{B}(x,\mathcal{W})$. There are $A\in\mathcal{F}$ and $\mathcal{W}\in\mathcal{O}$ such that $AU_{(A,\mathcal{W})}\subset\mathrm{B}(K,\mathcal{V})$. Indeed, suppose by contradiction that $AU_{(A,\mathcal{W})}\nsubseteq\mathrm{B}(K,\mathcal{V})$ for every $A\in\mathcal{F}$ and $\mathcal{W}\in\mathcal{O}$. Then there is $x_{(A,\mathcal{W})}\in U_{(A,\mathcal{W})}$ such that $Ax_{(A,\mathcal{W})}\nsubseteq\mathrm{B}(K,\mathcal{V})$. Since $Ax_{(A,\mathcal{W})}$ is connected and $Ax_{(A,\mathcal{W})}\cap\mathrm{B}(K,\mathcal{V})\neq\varnothing$, there is $t_{(A,\mathcal{W})}\in A$ such that $t_{(A,\mathcal{W})}x_{(A,\mathcal{W})}\in N\setminus\mathrm{B}(K,\mathcal{V})$. Since N is compact, we may assume that the net $(t_{(A,\mathcal{W})}x_{(A,\mathcal{W})})_{(A,\mathcal{U})\in\mathcal{F}\times\mathcal{O}}$ converges to a point $z\in N\setminus\mathrm{B}(K,\mathcal{V})$. As $U_{(A,\mathcal{W})}\subset\mathrm{B}(x,\mathcal{W})$ and $t_A\to_{\mathcal{F}}\infty$, it follows that $z\in\mathrm{J}(x,\mathcal{F})\cap(N\setminus\mathrm{B}(K,\mathcal{V}))$, which is a contradiction. Thus, we can take $(A_0,\mathcal{W}_0)\in\mathcal{F}\times\mathcal{O}$ such that $A_0U_{(A_0,\mathcal{W}_0)}\subset\mathrm{B}(K,\mathcal{V})$. If $x\in U_{(A,\mathcal{W})}$ for some $(A,\mathcal{W})\geqslant(A_0,\mathcal{W}_0)$, then the result is proved. If $x\notin U_{(A,\mathcal{W})}$ for every $(A,\mathcal{W})\geqslant(A_0,\mathcal{W}_0)$, then there is $A\in\mathcal{F}$, $A\subset A_0$, such that $Ax\subset M\setminus N$. In fact, $x\notin U_{(A,\mathcal{W})}$ implies $Ax\cap\mathrm{B}(K,\mathcal{V})=\varnothing$. If $Ax\cap N\neq\varnothing$ for all $A\in\mathcal{F}$, then there is $y\in\omega(x,\mathcal{F})\cap N\neq\varnothing$, and therefore $\varnothing\neq\omega(x,\mathcal{F})\subset\mathrm{J}(x,\mathcal{F})\subset K$. As in the proof of Theorem 3.6, it follows that there is $A\in\mathcal{F}$ such that $Ax\subset\mathrm{B}(K,\mathcal{V})$, which contradicts $Ax\cap\mathrm{B}(K,\mathcal{V})=\varnothing$. Thus, there is $A\in\mathcal{F}$ such that $Ax\subset M\setminus N$, and we can take $A\subset A_0$. It follows that $A^*(M\setminus N)$ is a neighborhood of x. Hence, there is $\mathcal{W}\in\mathcal{O}$ such that $\mathrm{B}(x,\mathcal{W})\subset A^*(M\setminus N)$, and we can take $\mathcal{W}\leqslant\mathcal{W}_0$. Thus, we have $A\mathrm{B}(x,\mathcal{W})\subset M\setminus N$ with $(A,\mathcal{W})\geqslant(A_0,\mathcal{W}_0)$. In particular, $AU_{(A,\mathcal{W})}\subset M\setminus N$ with $(A,\mathcal{W})\geqslant(A_0,\mathcal{W}_0)$, and we again have a contradiction. ∎

Theorem 3.8 *Let $\mathcal{F}$ be a filter basis on the subsets of $\mathcal{S}$ and K a compact subset of M. Then $\mathfrak{A}_{wu}(K,\mathcal{F})=\mathrm{Atr}_{\mathrm{wu}}(K,\mathcal{F})$.*

Proof Take $x\in\mathrm{Atr}_{\mathrm{wu}}(K,\mathcal{F})$. It follows that there exists $k\in K\cap\mathrm{J}(x,\mathcal{F})$. Thus, $\mathrm{B}(k,\mathcal{U})\cap A\mathrm{B}(x,\mathcal{V})\neq\varnothing$ for all $A\in\mathcal{F}$ and $\mathcal{U},\mathcal{V}\in\mathcal{O}$. Since $\mathrm{B}(k,\mathcal{U})\subset\mathrm{B}(K,\mathcal{U})$, we get $x\in\mathfrak{A}_{wu}(K,\mathcal{F})$. Conversely, take $x\in\mathfrak{A}_{wu}(K,\mathcal{F})$. For $\mathcal{U}\in\mathcal{O}$ and $A\in\mathcal{F}$, we can choose $t_{(A,\mathcal{U})}\in A$ and $x_{(A,\mathcal{U})},k_{(A,\mathcal{U})}\in M$ such that $k_{(A,\mathcal{U})}\in K$, $x_{(A,\mathcal{U})}\in\mathrm{B}(x,\mathcal{U})$ and $t_{(A,\mathcal{U})}x_{(A,\mathcal{U})}\in\mathrm{B}(k_{(A,\mathcal{U})},\mathcal{U})$. By the compactness of K we can assume that $k_{(A,\mathcal{U})}\to k\in K$. Lemma 2.8 implies that $x_{(A,\mathcal{U})}\to x$. Thus, we apply Lemma 2.8 to obtain $t_{(A,\mathcal{U})}x_{(A,\mathcal{U})}\to k$. Since $t_{(A,\mathcal{U})}\to_{\mathcal{F}}\infty$, it follows that $k\in\mathrm{J}(x,\mathcal{F})$, and therefore $x\in\mathrm{Atr}_{\mathrm{wu}}(K,\mathcal{F})$. ∎

4 Behavior Under Equivariant Maps

In this section we study the behavior of limit sets, prolongations, prolongational limit sets, domains of attraction, attractors, and Lyapunov stable sets under equivariant maps and semiconjugations.

Throughout this section we assume that M and N are admissible Hausdorff spaces endowed with admissible families of open coverings $\mathcal{O}$ and $\mathcal{O}'$, respectively. We also suppose that $\mathcal{S}$ is a semigroup acting on both M and N.

4.1 Equivariant Maps

We start with the following definition.

Definition 4.1 A map $p: M \to N$ is said to be $\mathcal{S}$-*equivariant* if $p(sx) = sp(x)$ for every $s \in \mathcal{S}$ and $x \in M$. A continuous and surjective $\mathcal{S}$-equivariant map is called an $\mathcal{S}$-*topological semiconjugation*. An $\mathcal{S}$-equivariant homeomorphism is called an $\mathcal{S}$-*topological conjugation*.

Note that a map $p: M \to N$ is $\mathcal{S}$-equivariant if and only if

$$p(AX) = Ap(X) \text{ for every } A \subset \mathcal{S} \text{ and } X \subset M.$$

If $p: M \to N$ is an $\mathcal{S}$-topological semiconjugation and $x \in M$, then it is easily seen that

$$p(\omega(x, \mathcal{F})) \subset \omega(p(x), \mathcal{F}) \quad \text{and} \quad p(\mathrm{J}(x, \mathcal{F})) \subset \mathrm{J}(p(x), \mathcal{F}).$$

Moreover, if $p: M \to N$ is a bijective $\mathcal{S}$-equivariant map, then $p^{-1}: N \to M$ is also an $\mathcal{S}$-equivariant map. Hence, if $p: M \to N$ is an $\mathcal{S}$-topological conjugation, then $p^{-1}: N \to M$ is an $\mathcal{S}$-topological conjugation, and therefore

$$p(\omega(x, \mathcal{F})) = \omega(p(x), \mathcal{F}) \quad \text{and} \quad p(\mathrm{J}(x, \mathcal{F})) = \mathrm{J}(p(x), \mathcal{F}).$$

Let us present some examples of equivariant maps.

Example 4.2 Let G be a Hausdorff topological group. The right and left translations on G are G-topological conjugations.

Example 4.3 (Right invariant flows on homogeneous spaces) Let G be a Hausdorff topological group. Let H be a closed subgroup of G and consider the homogeneous space G/H. Denote by $\pi: G \to G/H$ the canonical projection. A *right invariant flow* on G is a continuous flow ϕ that commutes with the right translations of G, that is, $\phi_t(hg) = \phi_t(h)g$ for every $t \in \mathbb{R}$ and $g, h \in G$. Right invariant flows appear in Lie theory: if G is a Lie group and X if a right invariant vector field on G, it is known that the flow X_t associated with X satisfies $X_t(hg) = X_t(h)g$ for every $g, h \in G$ and $t \in \mathbb{R}$

From now on, let ϕ be a right invariant flow on G. We have that

$$\begin{aligned} \phi: \mathbb{R} \times G/H &\longrightarrow G/H \\ (t, gH) &\longmapsto \phi(t, gH) = \pi(\phi(t, g)) \end{aligned} \tag{4.1}$$

is a continuous flow on G/H. We have seen in Section 2 that G and G/H admit admissible families of open coverings. Consider the action of $\mathbb{R}$ on G/H defined by the flow ϕ, that is, for $t \in \mathbb{R}$ and $x \in G/H$ we define $tx = \phi(t, x)$.

In the following we present examples of $\mathbb{R}$-equivariant maps in this context.

(i) Given $g \in G$, the map

$$\begin{aligned} g: G/H &\longrightarrow G/H \\ xH &\longmapsto g(xH) = xgH \end{aligned}$$

is the map induced in G/H by the canonical projection π. Thus, g is a homeomorphism of G/H and $g \circ \pi = \pi \circ R_g$, where R_g denotes the right translation by g. If X is a subset of G/H, we simply denote $Xg = g(X)$ for every $g \in G$.

The flow ϕ commutes with the homeomorphisms g, that is, $\phi_t(xg) = \phi_t(x)g$ for every $t \in \mathbb{R}$, $g \in G$ and $x \in G/H$. In other words, each homeomorphism g as above is a $\mathbb{R}$-topological conjugation.

(ii) Let $H_1 \subset H_2$ be two closed subgroups of G with H_1 normal in H_2 and denote by $\pi_1 : G \to G/H_1$ and $\pi_2 : G \to G/H_2$ the respective canonical projections. Consider the *equivariant fibration*

$$\rho : G/H_1 \longrightarrow G/H_2$$
$$gH_1 \longmapsto gH_2.$$

Let ϕ^1 and ϕ^2 be the continuous flows induced in G/H_1 and G/H_2 by the canonical projections π_1 and π_2 and ϕ as in (4.1), respectively. Since $\rho \circ \pi_1 = \pi_2$, it is easily seen that ρ is a continuous and open $\mathbb{R}$-equivariant map.

Example 4.4 [Bitransformation semigroups] A *bitransformation semigroup* is a pair of transformation semigroups $(\mathcal{S}, M)$ and $(M, \mathcal{T})$ with the same phase space M such that $s(xt) = (sx)t$ for all $x \in M$, $s \in \mathcal{S}$ and $t \in \mathcal{T}$. The notation $(\mathcal{S}, X, \mathcal{T})$ signifies the bitransformation semigroup constituted by the pair $(\mathcal{S}, M)$ and $(M, \mathcal{T})$. The theory of bitransformation semigroup has several applications in the theories of control systems and semigroup actions on principal and associated bundles. We refer to [14] for the theory of bitransformation semigroups.

Now, let $(\mathcal{S}, X, \mathcal{T})$ be a bitransformation semigroup. It follows from the definition that, for each $t \in \mathcal{T}$, the map $t : M \to M$ defined by $t(x) = xt$ is a continuous $\mathcal{S}$-equivariant map. Therefore, if $\mathcal{T}$ is a group, then each application t is indeed a $\mathcal{S}$-topological conjugation.

We need the following definition of uniformly continuous map.

Definition 4.5 A map $f : M \to N$ is said to be *uniformly continuous* with respect to $\mathcal{O}$ and $\mathcal{O}'$ if for each $\mathcal{U} \in \mathcal{O}'$, there is $\mathcal{V} \in \mathcal{O}$ such that $x, y \in V$ for some $V \in \mathcal{V}$ implies $f(x), f(y) \in U$ for some $U \in \mathcal{U}$, that is, $f(\mathrm{B}(x, \mathcal{V})) \subset \mathrm{B}(f(x), \mathcal{U})$ for every $x \in M$. If f is a bijective uniformly continuous map such that $f^{-1} : N \to M$ is uniformly continuous with respect to $\mathcal{O}'$ and $\mathcal{O}$, then f is called a *uniform isomorphism*. Furthermore, if f is an $\mathcal{S}$-topological conjugation and a uniform isomorphism, then it is called a *uniform $\mathcal{S}$-conjugation*.

This notion of uniform continuity was introduced in [33] for uniform spaces. It is easily seen that every uniformly continuous map is continuous. We also have that continuous maps on compact spaces are uniformly continuous.

Let $f : M \to N$ be a continuous map. For each $\mathcal{U} \in \mathcal{O}'$, define the open covering $f^{-1}\mathcal{U}$ of M by $f^{-1}\mathcal{U} = \{f^{-1}(U) : U \in \mathcal{U}\}$. The map f is uniformly continuous with respect to $\mathcal{O}$ and $\mathcal{O}'$if for each $\mathcal{U} \in \mathcal{O}'$ there is $\mathcal{V} \in \mathcal{O}$ such that $\mathcal{V} \leqslant f^{-1}\mathcal{U}$. In fact, if $y \in \mathrm{B}(x, \mathcal{V})$, then $y, x \in V$ for some $V \in \mathcal{V}$. Take $U \in \mathcal{U}$ such that $V \subset f^{-1}(U)$. It follows that $f(y), f(x) \in U$.

Now, suppose that $\mathcal{O}(M)$ is the family of all open coverings of M and $f : M \to N$ is a continuous map. For each $\mathcal{U} \in \mathcal{O}'$, we have $f^{-1}\mathcal{U} \in \mathcal{O}(M)$. Thus, f is uniformly continuous with respect to $\mathcal{O}(M)$ and $\mathcal{O}'$. In particular, f is a uniform isomorphism if f is a homeomorphism and $\mathcal{O}' = \mathcal{O}(N)$.

Example 4.6 Let (M, d) be a pseudometric space. The family $\mathcal{O}_{\mathrm{d}}$ of the coverings $\mathcal{U}_\varepsilon = \{\mathrm{B}(x, \varepsilon) : x \in M\}$ by ε-balls, for $\varepsilon > 0$, is admissible. If (N, d') is another pseudometric space, a function $f: M \to N$ is uniformly continuous with respect d and d′ if and only if it is uniformly continuous with respect to $\mathcal{O}_{\mathrm{d}}$ and $\mathcal{O}_{\mathrm{d}'}$.

Example 4.7 Let G be a Hausdorff topological group with $\mathfrak{V}$ a base of symmetric open neighborhoods at the identity. Define the family $\mathcal{O}_R$ of open coverings of the form $\mathcal{R}_V = \{Vg : g \in G\}$ for $V \in \mathfrak{V}$. This family $\mathcal{O}_R$ is given by a diagonal uniformity on G (see [33, Problem 35F]). Hence, $\mathcal{O}_R$ is a base for a covering uniformity on G and is therefore admissible. It is easily seen that the right translations of G are uniform G-conjugations with respect to $\mathcal{O}_R$. Analogously, we define the family $\mathcal{O}_L$ of open coverings of the form $\mathcal{L}_V = \{gV : g \in G\}$ for $V \in \mathfrak{V}$. In this case, the left translations of G are uniform G-conjugations with respect to $\mathcal{O}_L$.

4.2 Uniformly Continuous Equivariant Maps

In this section, we present some results on the behavior of attractors, domains of attraction, and Lyapunov stable sets under uniformly continuous equivariant maps.

The next result establishes the behavior of domains of attraction under uniformly continuous equivariant maps.

Proposition 4.8 *Let $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$ and $p: M \to N$ be a uniformly continuous $\mathcal{S}$-equivariant map. For $X \subset M$, one has*

$$p\big(\mathfrak{A}_w(X, \mathcal{F})\big) \subset \mathfrak{A}_w\big(p(X), \mathcal{F}\big), \quad p\big(\mathfrak{A}_{wu}(X, \mathcal{F})\big) \subset \mathfrak{A}_{wu}\big(p(X), \mathcal{F}\big) \quad \text{and}$$
$$p\big(\mathfrak{A}(X, \mathcal{F})\big) \subset \mathfrak{A}\big(p(X), \mathcal{F}\big).$$

If p is open, then $p(\mathfrak{A}_u(X, \mathcal{F})) \subset \mathfrak{A}_u(p(X), \mathcal{F})$.

Proof Let $x \in \mathfrak{A}_w(X, \mathcal{F})$, $A \in \mathcal{F}$, and $\mathcal{U} \in \mathcal{O}'$. Take an open covering $\mathcal{V} \in \mathcal{O}$ such that $p(\mathrm{B}(y, \mathcal{V})) \subset \mathrm{B}(p(y), \mathcal{U})$ for every $y \in M$. Since $Ax \cap \mathrm{B}(X, \mathcal{V}) \neq \varnothing$, it follows that

$$\varnothing \neq p\big(Ax \cap \mathrm{B}(X, \mathcal{V})\big) \subset p(Ax) \cap p\big(\mathrm{B}(X, \mathcal{V})\big) \subset Ap(x) \cap \mathrm{B}\big(p(X), \mathcal{U}\big).$$

Hence, $p(\mathfrak{A}_w(X, \mathcal{F})) \subset \mathfrak{A}_w(p(X), \mathcal{F})$. For the weak domain of uniform attraction, take $x \in \mathfrak{A}_{wu}(X, \mathcal{F})$, $A \in \mathcal{F}$ and open coverings $\mathcal{U}', \mathcal{V}' \in \mathcal{O}'$. By the uniform continuity of p there exist open coverings $\mathcal{U}, \mathcal{V} \in \mathcal{O}$ satisfying $p(\mathrm{B}(x, \mathcal{V})) \subset B(p(X), \mathcal{V}')$ and $p(\mathrm{B}(x, \mathcal{U})) \subset B(p(X), \mathcal{U}')$ for every $x \in X$. Since $A\mathrm{B}(x, \mathcal{V}) \cap \mathrm{B}(X, \mathcal{U}) \neq \varnothing$, we obtain

$$\begin{aligned}\varnothing \neq p\big(A\mathrm{B}(x, \mathcal{V}) \cap \mathrm{B}(X, \mathcal{U})\big) &\subset Ap\big(\mathrm{B}(x, \mathcal{V})\big) \cap p\big(\mathrm{B}(X, \mathcal{U})\big) \\ &\subset A\mathrm{B}\big(p(x), \mathcal{V}'\big) \cap \mathrm{B}\big(p(X), \mathcal{U}'\big).\end{aligned}$$

Therefore, $p(x) \in \mathfrak{A}_{wu}(p(X), \mathcal{F})$. Now, pick $x \in \mathfrak{A}(X, \mathcal{F})$ and $\mathcal{U} \in \mathcal{O}'$. Take an open covering $\mathcal{V} \in \mathcal{O}$ such that $p(\mathrm{B}(y, \mathcal{V})) \subset \mathrm{B}(p(y), \mathcal{U})$ for every $y \in M$. There exists $A \in \mathcal{F}$ such that $Ax \subset \mathrm{B}(X, \mathcal{V})$. Hence, $Ap(x) = p(Ax) \subset \mathrm{B}(p(X), \mathcal{U})$ and, therefore,

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$p(\mathfrak{A}(X,\mathcal{F})) \subset \mathfrak{A}(p(X),\mathcal{F})$. Finally, suppose that p is open and take $x \in \mathfrak{A}_u(X,\mathcal{F})$ and $\mathcal{U} \in \mathcal{O}'$. Take an open covering $\mathcal{V} \in \mathcal{O}$ such that $p(\mathrm{B}(y,\mathcal{V})) \subset \mathrm{B}(p(y),\mathcal{U})$ for every $y \in M$. There exist $A \in \mathcal{F}$ and $\mathcal{W} \in \mathcal{O}$ such that $A\mathrm{B}(x,\mathcal{W}) \subset \mathrm{B}(X,\mathcal{V})$. Hence, $Ap(\mathrm{B}(x,\mathcal{W})) \subset \mathrm{B}(p(X),\mathcal{U})$. As $p(\mathrm{B}(x,\mathcal{W}))$ is a neighborhood of $p(x)$, there exists an open covering $\mathcal{W}' \in \mathcal{O}'$ such that $\mathrm{B}(p(x),\mathcal{W}') \subset p(\mathrm{B}(x,\mathcal{W}))$. Hence, $A\mathrm{B}(p(x),\mathcal{W}') \subset \mathrm{B}(p(X),\mathcal{U})$ and, therefore, $p(\mathfrak{A}_u(X,\mathcal{F})) \subset \mathfrak{A}_u(p(X),\mathcal{F})$. ∎

The following result is an immediate consequence of Proposition 4.8.

Corollary 4.9 *Let $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$ and $p: M \to N$ be an uniform $\mathcal{S}$-conjugation. For a given set $X \subset M$, one has*

$$p\big(\mathfrak{A}_w(X,\mathcal{F})\big) = \mathfrak{A}_w\big(p(X),\mathcal{F}\big), \quad p\big(\mathfrak{A}_{wu}(X,\mathcal{F})\big) = \mathfrak{A}_{wu}\big(p(X),\mathcal{F}\big),$$
$$p\big(\mathfrak{A}(X,\mathcal{F})\big) = \mathfrak{A}\big(p(X),\mathcal{F}\big), \quad \text{and} \quad p\big(\mathfrak{A}_u(X,\mathcal{F})\big) = \mathfrak{A}_u\big(p(X),\mathcal{F}\big).$$

Now we present a result on the behavior of attractors under uniformly continuous equivariant maps.

Proposition 4.10 *Let $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$ and $p: M \to N$ be a uniformly continuous $\mathcal{S}$-equivariant open map. Assume that $X \subset M$ is a weak $\mathcal{F}$-attractor, $\mathcal{F}$-attractor, weak uniform $\mathcal{F}$-attractor, or uniform $\mathcal{F}$-attractor. Suppose that one of the following conditions is satisfied:*

(i) *$p(X)$ is compact in N,*
(ii) *$p(X)$ is closed in N and $\mathcal{O}'$ is a strong admissible family.*

Then $p(X)$ is a weak $\mathcal{F}$-attractor, $\mathcal{F}$-attractor, weak uniform $\mathcal{F}$-attractor, or uniform $\mathcal{F}$-attractor, respectively.

Proof Suppose that $p(X)$ is compact in N. If X is a weak $\mathcal{F}$-attractor, then there exists $\mathcal{V} \in \mathcal{O}$ such that $\mathrm{B}(X,\mathcal{V}) \subset \mathfrak{A}_w(X,\mathcal{F})$. By Proposition 4.8, it follows that $p(\mathrm{B}(X,\mathcal{V})) \subset \mathfrak{A}_w(p(X),\mathcal{F})$. Since $p(X)$ is compact and $p(\mathrm{B}(X,\mathcal{V}))$ is a neighborhood of $p(X)$, there exists an open covering $\mathcal{U} \in \mathcal{O}'$ such that $\mathrm{B}(p(X),\mathcal{U}) \subset p(\mathrm{B}(X,\mathcal{V}))$. Hence, $\mathrm{B}(p(X),\mathcal{U}) \subset \mathfrak{A}_w(p(X),\mathcal{F})$, and therefore $p(X)$ is a weak $\mathcal{F}$-attractor. The cases of attractor, weak uniform attractor, and uniform attractor are proved in the same way. The result is similarly proved by assuming that $p(X)$ is closed in N and $\mathcal{O}'$ is a strong admissible family. ∎

As a consequence of Proposition 4.10 we have the following corollary.

Corollary 4.11 *Let $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$ and $p: M \to N$ be a uniformly $\mathcal{S}$-conjugation. Take a subset $X \subset M$. Suppose that either X is compact or X is closed in M and $\mathcal{O}'$ is a strong admissible family. Then the set X is is a weak $\mathcal{F}$-attractor, $\mathcal{F}$-attractor, weak uniform $\mathcal{F}$-attractor, or uniform $\mathcal{F}$-attractor if and only if $p(X)$ is respectively a weak $\mathcal{F}$-attractor, $\mathcal{F}$-attractor, weak uniform $\mathcal{F}$-attractor, or uniform $\mathcal{F}$-attractor.*

Now, we discuss stability under uniformly continuous equivariant maps.

Proposition 4.12 *Let $p: M \to N$ be a uniformly continuous $\mathcal{S}$-equivariant open map. If $X \subset M$ is $\mathcal{S}$-stable, then $p(X)$ is $\mathcal{S}$-stable.*

Proof Take $x \in X$ and $\mathcal{U} \in \mathcal{O}'$. By uniform continuity, there exists an open covering $\mathcal{W} \in \mathcal{O}$ such that $p(\mathrm{B}(X, \mathcal{W})) \subset \mathrm{B}(p(X), \mathcal{U})$. Take open coverings $\mathcal{W}' \in \mathcal{O}$ and $\mathcal{V} \in \mathcal{O}'$ such that $\mathcal{S}\mathrm{B}(x, \mathcal{W}') \subset \mathrm{B}(X, \mathcal{W})$ and $\mathrm{B}(p(x), \mathcal{V}) \subset p(\mathrm{B}(x, \mathcal{W}'))$. It follows that

$$\mathcal{S}\mathrm{B}\big(p(x), \mathcal{V}\big) \subset \mathcal{S}p\big(\mathrm{B}(x, \mathcal{W}')\big) \subset p\big(\mathrm{B}(X, \mathcal{W})\big) \subset \mathrm{B}\big(p(X), \mathcal{U}\big).$$

Therefore, $p(X)$ is $\mathcal{S}$-stable. ∎

As a consequence, we have the following corollary.

Corollary 4.13 *Let $p: M \to N$ be a uniform $\mathcal{S}$-conjugation. Then a subset $X \subset M$ is $\mathcal{S}$-stable if and only if $p(X)$ is $\mathcal{S}$-stable.*

Concerning uniform stability, we have the following result.

Proposition 4.14 *Let $p: M \to N$ be a uniformly continuous $\mathcal{S}$-equivariant open map. Assume that $X \subset M$ is $\mathcal{S}$-uniformly stable. Suppose that one of the following conditions is satisfied:*

(i) *$p(X)$ is compact in N,*
(ii) *$p(X)$ is closed in N and $\mathcal{O}'$ is a strong admissible family.*

Then $p(X)$ is $\mathcal{S}$-uniformly stable.

Proof Suppose that $X \subset M$ is $\mathcal{S}$-uniformly stable and $p(X)$ is compact in N. For an open covering $\mathcal{U} \in \mathcal{O}'$ there exists $\mathcal{W} \in \mathcal{O}$ such that $p(\mathrm{B}(X, \mathcal{W})) \subset \mathrm{B}(p(X), \mathcal{U})$. Take open coverings $\mathcal{W}' \in \mathcal{O}$ and $\mathcal{V} \in \mathcal{O}'$ such that $\mathcal{S}\mathrm{B}(X, \mathcal{W}') \subset \mathrm{B}(X, \mathcal{W})$ and $\mathrm{B}(p(X), \mathcal{V}) \subset p(\mathrm{B}(X, \mathcal{W}'))$. It follows that

$$\mathcal{S}\mathrm{B}\big(p(X), \mathcal{V}\big) \subset \mathcal{S}p\big(\mathrm{B}(X, \mathcal{W}')\big) \subset p\big(\mathrm{B}(X, \mathcal{W})\big) \subset \mathrm{B}\big(p(X), \mathcal{U}\big).$$

Hence, $p(X)$ is $\mathcal{S}$-uniformly stable. The result is similarly proved by assuming that $p(X)$ is closed in N and $\mathcal{O}'$ is a strong admissible family. ∎

In the following proposition, we show that uniform conjugations preserve uniform stability.

Proposition 4.15 *Let $p: M \to N$ be a uniform $\mathcal{S}$-conjugation. The set $X \subset M$ is $\mathcal{S}$-uniformly stable if and only if $p(X)$ is $\mathcal{S}$-uniformly stable.*

Proof Suppose that $X \subset M$ is $\mathcal{S}$-uniformly stable. For a given $\mathcal{U} \in \mathcal{O}'$ there exist open coverings $\mathcal{V}, \mathcal{W} \in \mathcal{O}$ such that $p(\mathrm{B}(X, \mathcal{V})) \subset \mathrm{B}(p(X), \mathcal{U})$ and $\mathcal{S}\mathrm{B}(X, \mathcal{W}) \subset \mathrm{B}(X, \mathcal{V})$. Since p^{-1} is uniformly continuous, there exists an open covering $\mathcal{U}' \in \mathcal{O}'$ such that $\mathrm{B}(p(X), \mathcal{U}') \subset p(\mathrm{B}(X, \mathcal{W}))$. Hence,

$$\mathcal{S}\mathrm{B}\big(p(X), \mathcal{U}'\big) \subset \mathcal{S}p\big(\mathrm{B}(X, \mathcal{W})\big) \subset p\big(\mathrm{B}(X, \mathcal{V})\big) \subset \mathrm{B}\big(p(X), \mathcal{U}\big).$$

Therefore, $p(X)$ is $\mathcal{S}$-uniformly stable. The converse is clear. ∎

Corollary 4.11 and Proposition 4.15 imply the following result on asymptotic stability.

Corollary 4.16 *Let $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$ and $p: M \to N$ be a uniform $\mathcal{S}$-conjugation. Take a subset $X \subset M$. Suppose that either X is compact or X is closed in M and $\mathcal{O}'$ is a strong admissible family. Then a subset $X \subset M$ is $\mathcal{F}$-asymptotically stable if and only if $p(X)$ is $\mathcal{F}$-asymptotically stable.*

4.3 Topological Semiconjugations

Now, we discuss the behavior of prolongations and prolongational limit sets under topological semiconjugations. Let $p: M \to N$ be an $\mathcal{S}$-topological semiconjugation. Assume that M is a compact space and $\mathcal{F}$ is a filter basis on the subsets of $\mathcal{S}$. Theorem 3.1 from [9] assures that

$$p\big(\omega(X,\mathcal{F})\big) = \omega\big(p(X),\mathcal{F}\big)$$

for all $X \subset M$. We can prove an analogous result for prolongational limit sets.

Theorem 4.17 *Suppose that M is a compact space. Assume that $\mathcal{F}$ is a filter basis on the subsets of $\mathcal{S}$, $p: M \to N$ is an open $\mathcal{S}$-topological semiconjugation. For a given nonempty subset X of M, we have*

$$p\big(\mathrm{J}(X,\mathcal{F})\big) = \mathrm{J}\big(p(X),\mathcal{F}\big).$$

Proof Let $x \in M$. We claim that

$$\mathrm{J}\big(p(x),\mathcal{F}\big) = \bigcap_{\substack{\mathcal{U}\in\mathcal{O},\\ A\in\mathcal{F}}} \mathrm{cl}\big(Ap(\mathrm{B}(x,\mathcal{U}))\big) = \bigcap_{\substack{\mathcal{U}\in\mathcal{O},\\ A\in\mathcal{F}}} p\big(\mathrm{cl}(A\mathrm{B}(x,\mathcal{U}))\big). \tag{4.2}$$

In fact, the second equality follows immediately from the assumptions on p. Now, take $y \in \mathrm{J}(p(x),\mathcal{F})$, $A \in \mathcal{F}$ and $\mathcal{W} \in \mathcal{O}$. Since p is an open map, it follows that $p(\mathrm{B}(x,\mathcal{W}))$ is a neighborhood of $p(x)$ in N. Hence, there is $\mathcal{V} \in \mathcal{O}'$ such that $\mathrm{B}(p(x),\mathcal{V}) \subset p(\mathrm{B}(x,\mathcal{W}))$. Thus,

$$y \in \mathrm{cl}\big(A\mathrm{B}(p(x),\mathcal{V})\big) \subset \mathrm{cl}\big(Ap(\mathrm{B}(x,\mathcal{W}))\big).$$

On the other hand, fix $y \in M$ such that $y \in \mathrm{cl}(Ap(\mathrm{B}(x,\mathcal{U})))$, for every $\mathcal{U} \in \mathcal{O}$ and $A \in \mathcal{F}$. We fix $A \in \mathcal{F}$ and $\mathcal{W} \in \mathcal{O}'$. By the continuity of p and admissibility of $\mathcal{O}$ there exists $\mathcal{V} \in \mathcal{O}$ such that $p(\mathrm{B}(x,\mathcal{V})) \subset \mathrm{B}(p(x),\mathcal{W})$. Therefore,

$$y \in \mathrm{cl}\big(Ap(\mathrm{B}(x,\mathcal{V}))\big) \subset \mathrm{cl}\big(A\mathrm{B}(p(x),\mathcal{W})\big)$$

and $y \in \mathrm{J}(p(x),\mathcal{F})$. Now, we show that $\mathrm{J}(p(x),\mathcal{F}) \subset p(\mathrm{J}(x,\mathcal{F}))$. Take $y \in \mathrm{J}(p(x),\mathcal{F})$. For each $A \in \mathcal{F}$ and $\mathcal{U} \in \mathcal{O}$, we have from (4.2) that $y = p(z_{(A,\mathcal{U})})$, where $z_{(A,\mathcal{U})} \in \mathrm{cl}(A\mathrm{B}(x,\mathcal{U}))$. Since M is compact, there exists a subnet $(z_{(A_\lambda,\mathcal{U}_\lambda)})_{\lambda\in\Lambda}$ of $(z_{(A,\mathcal{U})})_{(A,\mathcal{U})\in\mathcal{F}\times\mathcal{O}}$ such that $z_{(A_\lambda,\mathcal{U}_\lambda)} \to z$ in M. Fix $A \in \mathcal{F}$ and $\mathcal{U} \in \mathcal{O}$ and take $\lambda_0 \in \Lambda$ such that $A_\lambda \subset A$ and $\mathcal{U}_\lambda \leqslant \mathcal{U}$, for $\lambda > \lambda_0$. Thus, for $\lambda > \lambda_0$, one has

$$z_{(A_\lambda,\mathcal{U}_\lambda)} \in \mathrm{cl}\big(A_\lambda\mathrm{B}(x,\mathcal{U}_\lambda)\big) \subset \mathrm{cl}\big(A\mathrm{B}(x,\mathcal{U})\big),$$

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and we get $z \in \mathrm{cl}(A\mathrm{B}(x,\mathcal{U}))$ and $z \in \mathrm{J}(x,\mathcal{F})$. Since $p(z_{(A_\lambda,\mathcal{U}_\lambda)}) = y$, for all $\lambda \in \Lambda$, we conclude that $y = p(z) \in p(\mathrm{J}(x,\mathcal{F}))$. On the other hand, if $z \in \mathrm{J}(x,\mathcal{F})$, it follows from (4.2) that

$$p(z) \in p\big(\mathrm{J}(x,\mathcal{F})\big) \subset \bigcap_{\substack{\mathcal{U}\in\mathcal{O} \\ A\in\mathcal{F}}} p\big(\mathrm{cl}(A\mathrm{B}(x,\mathcal{U}))\big) = \mathrm{J}\big(p(x),\mathcal{F}\big),$$

and we obtain $p(\mathrm{J}(x,\mathcal{F})) = \mathrm{J}(p(x),\mathcal{F})$. Therefore, for a subset X of M one has

$$p\big(\mathrm{J}(X,\mathcal{F})\big) = \bigcup_{x\in X} p\big(\mathrm{J}(x,\mathcal{F})\big) = \bigcup_{x\in X} \mathrm{J}\big(p(x),\mathcal{F}\big) = \mathrm{J}\big(p(X),\mathcal{F}\big).$$ ∎

For prolongations we have the following result.

Corollary 4.18 *Suppose that M is a compact space. Let $p\colon M \to N$ be an open $\mathcal{S}$-topological semiconjugation. For nonempty subsets $A \subset \mathcal{S}$ and $X \subset M$, we have*

$$p\big(\mathrm{D}(X,A)\big) = \mathrm{D}\big(p(X),A\big).$$

Proof The family $\{A\}$ is a filter basis of subsets of $\mathcal{S}$ and $\mathrm{D}(X,A) = \mathrm{J}(X,\{A\})$. Therefore, the result follows immediately from Theorem 4.17. ∎

For compact sets, the uniform continuity in Proposition 4.8 can be omitted.

Lemma 4.19 *Let $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$ and $p\colon M \to N$ an $\mathcal{S}$-topological semiconjugation. For a given compact set $X \subset M$, one has*

$$p\big(\mathfrak{A}_w(X,\mathcal{F})\big) \subset \mathfrak{A}_w\big(p(X),\mathcal{F}\big), \quad p\big(\mathfrak{A}_{wu}(X,\mathcal{F})\big) \subset \mathfrak{A}_{wu}\big(p(X),\mathcal{F}\big) \quad \text{and}$$
$$p\big(\mathfrak{A}(X,\mathcal{F})\big) \subset \mathfrak{A}\big(p(X),\mathcal{F}\big).$$

If p is open, then $p(\mathfrak{A}_u(X,\mathcal{F})) \subset \mathfrak{A}_u(p(X),\mathcal{F})$.

Proof Let $x \in \mathfrak{A}_w(X,\mathcal{F})$, $A \in \mathcal{F}$, and $\mathcal{U} \in \mathcal{O}'$. Since X is compact and $X \subset p^{-1}(\mathrm{B}(p(X),\mathcal{U}))$, there exists an open covering $\mathcal{V} \in \mathcal{O}$ such that $\mathrm{B}(X,\mathcal{V}) \subset p^{-1}(\mathrm{B}(p(X),\mathcal{U}))$. Then $Ax \cap \mathrm{B}(X,\mathcal{V}) \neq \varnothing$ and therefore

$$\varnothing \neq p\big(Ax \cap \mathrm{B}(X,\mathcal{V})\big) \subset Ap(x) \cap \mathrm{B}\big(p(X),\mathcal{U}\big).$$

Thus, $p(x) \in \mathfrak{A}_w(p(X),\mathcal{F})$, and $p(\mathfrak{A}_w(X,\mathcal{F})) \subset \mathfrak{A}_w(p(X),\mathcal{F})$. Now, take $x \in \mathfrak{A}_{wu}(K,\mathcal{F})$, $A \in \mathcal{F}$ and $\mathcal{U}',\mathcal{V}' \in \mathcal{O}'$. Take open coverings $\mathcal{U},\mathcal{V} \in \mathcal{O}$ such that $\mathrm{B}(x,\mathcal{V}) \subset p^{-1}(\mathrm{B}(p(x),\mathcal{V}'))$ and $\mathrm{B}(K,\mathcal{U}) \subset p^{-1}(\mathrm{B}(p(K),\mathcal{U}'))$. We have that $A\mathrm{B}(x,\mathcal{V}) \cap \mathrm{B}(K,\mathcal{U}) \neq \varnothing$. Thus, it follows that

$$\varnothing \neq p\big(A\mathrm{B}(x,\mathcal{V}) \cap \mathrm{B}(K,\mathcal{U})\big) \subset A\mathrm{B}\big(p(x),\mathcal{V}'\big) \cap \mathrm{B}\big(p(K),\mathcal{U}'\big)$$

and therefore $p(x) \in \mathfrak{A}_{wu}(p(K),\mathcal{F})$. Now, let $x \in \mathfrak{A}(X,\mathcal{F})$ and $\mathcal{U} \in \mathcal{O}'$. Take an open covering $\mathcal{V} \in \mathcal{O}$ such that $\mathrm{B}(X,\mathcal{V}) \subset p^{-1}(\mathrm{B}(p(X),\mathcal{U}))$. Now choose $A \in \mathcal{F}$ satisfying $Ax \subset \mathrm{B}(X,\mathcal{V})$. It follows that $Ap(x) \subset \mathrm{B}(p(X),\mathcal{U})$, and therefore $p(x) \in \mathfrak{A}(p(X),\mathcal{F})$. Finally, suppose that p is open, $x \in \mathfrak{A}_u(X,\mathcal{F})$, and $\mathcal{U} \in \mathcal{O}'$. Take an open covering $\mathcal{W} \in \mathcal{O}$ such that $\mathrm{B}(X,\mathcal{W}) \subset p^{-1}(\mathrm{B}(p(X),\mathcal{U}))$. Then there exist $A \in \mathcal{F}$, $\mathcal{W}' \in \mathcal{O}$, and $\mathcal{V} \in \mathcal{O}'$ such that $A\mathrm{B}(x,\mathcal{W}') \subset \mathrm{B}(X,\mathcal{W})$ and $\mathrm{B}(p(x),\mathcal{V}) \subset p(\mathrm{B}(x,\mathcal{W}'))$. Hence, $A\mathrm{B}(p(x),\mathcal{V}) \subset \mathrm{B}(p(X),\mathcal{U})$, and therefore $p(\mathfrak{A}_u(K,\mathcal{F})) \subset \mathfrak{A}_u(p(K),\mathcal{F})$. ∎

As a consequence of Lemma 4.19 one has the following proposition.

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Proposition 4.20 *Let $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$ and $p\colon M \to N$ be an open $\mathcal{S}$-topological semiconjugation. Assume that $X \subset M$ is a compact set. If X is a weak $\mathcal{F}$-attractor, weak uniform $\mathcal{F}$-attractor, $\mathcal{F}$-attractor, or uniform $\mathcal{F}$-attractor, then $p(X)$ is respectively a weak $\mathcal{F}$-attractor, weak uniform $\mathcal{F}$-attractor, $\mathcal{F}$-attractor, or uniform $\mathcal{F}$-attractor.*

Proof If X is a weak $\mathcal{F}$-attractor, then there exists an open covering $\mathcal{V} \in \mathcal{O}$ such that $\mathrm{B}(X,\mathcal{V}) \subset \mathfrak{A}_w(X,\mathcal{F})$. By Lemma 4.19, it follows that $p(\mathrm{B}(X,\mathcal{V})) \subset \mathfrak{A}_w(p(X),\mathcal{F})$. Since $p(X)$ is compact and $p(\mathrm{B}(X,\mathcal{V}))$ is a neighborhood of $p(X)$, there exists an open covering $\mathcal{U} \in \mathcal{O}'$ such that $\mathrm{B}(p(X),\mathcal{U}) \subset p(\mathrm{B}(X,\mathcal{V}))$. Hence, $\mathrm{B}(p(X),\mathcal{U}) \subset \mathfrak{A}_w(p(X),\mathcal{F})$, and therefore $p(X)$ is a weak $\mathcal{F}$-attractor. The cases for attractors and uniform attractors are proved in the same way. ∎

The next proposition describes the behavior of the sets given in (3.1) under continuous equivariant maps.

Proposition 4.21 *Let $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$ and $p\colon M \to N$ an $\mathcal{S}$-topological semiconjugation. For a given set $X \subset M$, one has $p(\mathrm{Atr_w}(X,\mathcal{F})) \subset \mathrm{Atr_w}(p(X),\mathcal{F})$ and $p(\mathrm{Atr_{wu}}(X,\mathcal{F})) \subset \mathrm{Atr_{wu}}(p(X),\mathcal{F})$. Furthermore, if M is compact and p is open, then $p(\mathrm{Atr}(X,\mathcal{F})) \subset \mathrm{Atr}(p(X),\mathcal{F})$ and $p(\mathrm{Atr_u}(X,\mathcal{F})) \subset \mathrm{Atr_u}(p(X),\mathcal{F})$.*

Proof Suppose that $y \in \mathrm{Atr_w}(X,\mathcal{F})$ and take $x \in \omega(y,\mathcal{F}) \cap X$. Then $p(x) \in \omega(p(y),\mathcal{F}) \cap p(X)$. Hence, $p(y) \in \mathrm{Atr_w}(p(X),\mathcal{F})$, and therefore $p(\mathrm{Atr_w}(X,\mathcal{F})) \subset \mathrm{Atr_w}(p(X),\mathcal{F})$. The proof for the inclusion $p(\mathrm{Atr_{wu}}(X,\mathcal{F})) \subset \mathrm{Atr_{wu}}(p(X),\mathcal{F})$ follows in the same way. Now, assume that M is compact and p is open. Since $p(\omega(x,\mathcal{F})) = \omega(p(x),\mathcal{F})$ for every $x \in M$, we have the inclusion $p(\mathrm{Atr}(X,\mathcal{F})) \subset \mathrm{Atr}(p(X),\mathcal{F})$. By Theorem 4.17, $p(\mathrm{J}(x,\mathcal{F})) = \mathrm{J}(p(x),\mathcal{F})$ for every $x \in M$, and therefore we have the inclusion $p(\mathrm{Atr_u}(X,\mathcal{F})) \subset \mathrm{Atr_u}(p(X),\mathcal{F})$. ∎

On the behavior of Lyapunov stability under continuous equivariant maps, we have the next theorem.

Theorem 4.22 *Let $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$ and $p\colon M \to N$ be an open $\mathcal{S}$-topological semiconjugation. For a given compact set $K \subset M$, one has*

(i) *$p(K)$ is $\mathcal{S}$-stable if K is $\mathcal{S}$-stable.*
(ii) *$p(K)$ is $\mathcal{S}$-equistable if K is $\mathcal{S}$-equistable, $x \in \mathrm{cl}(\mathcal{S}x)$ for all $x \in K$ and M is compact.*
(iii) *$p(K)$ is $\mathcal{F}$-asymptotically stable if K is $\mathcal{F}$-asymptotically stable.*

Proof (i) Take $\mathcal{U} \in \mathcal{O}'$. By the continuity of p and the admissibility of $\mathcal{O}$ there exists $\mathcal{U}' \in \mathcal{O}$ such that $\mathrm{B}(K,\mathcal{U}') \subset p^{-1}(\mathrm{B}(p(K),\mathcal{U}))$. It follows from $\mathcal{S}$-stability of K that there exists $\mathcal{U}'' \in \mathcal{O}$ satisfying $\mathcal{S}\mathrm{B}(K,\mathcal{U}'') \subset p^{-1}(\mathrm{B}(p(K),\mathcal{U}))$. Thus $\mathcal{S}p(\mathrm{B}(K,\mathcal{U}'')) \subset \mathrm{B}(p(K),\mathcal{U})$. Since $p(K)$ is compact and $p(\mathrm{B}(K,\mathcal{U}''))$ is an open set it follows from the admissibility of $\mathcal{O}'$ that there exists $\mathcal{V} \in \mathcal{O}'$ which satisfies $\mathrm{B}(p(K),\mathcal{V}) \subset p(\mathrm{B}(K,\mathcal{U}''))$. Therefore $\mathcal{S}\mathrm{B}(p(K),\mathcal{V}) \subset \mathrm{B}(p(K),\mathcal{U})$ and $p(K)$ is $\mathcal{S}$-stable.

(ii) First we observe that if $x \in \mathrm{cl}(\mathcal{S}x)$ for all $x \in K$, then $p(x) \in \mathrm{cl}(\mathcal{S}p(x))$ for all $x \in K$. Now, suppose that K is a compact and $\mathcal{S}$-equistable and M is compact.

We have from Proposition 2.14 that $\mathrm{D}(K,\mathcal{S}) = K$. Thus, Corollary 4.18 implies that $\mathrm{D}(p(K),\mathcal{S}) = p(K)$ and we conclude that $p(K)$ is $\mathcal{S}$-equistable.

(iii) It follows immediately from the first item above and Proposition 4.20. ∎

For topological conjugations, we have the following result.

Theorem 4.23 *Let $\mathcal{F} \subset \mathcal{P}(\mathcal{S})$ and let $p\colon M \to N$ be an $\mathcal{S}$-topological conjugation. For subsets X and K of M, with K compact, one has the following:*

(i) $p(\omega(X,\mathcal{F})) = \omega(p(X),\mathcal{F})$ *and* $p(\mathrm{J}(X,\mathcal{F})) = \mathrm{J}(p(X),\mathcal{F})$.
(ii) *If* $A \subset \mathcal{S}$ *then* $p(\mathrm{D}(X,A)) = \mathrm{D}(p(X),A)$.
(iii) $p(\mathrm{Atr_w}(X,\mathcal{F})) = \mathrm{Atr_w}(p(X),\mathcal{F})$, $p(\mathrm{Atr}(X,\mathcal{F})) = \mathrm{Atr}(p(X),\mathcal{F})$, *and* $p(\mathrm{Atr_u}(X,\mathcal{F})) = \mathrm{Atr_u}(p(X),\mathcal{F})$.
(iv) $p(\mathfrak{A}_w(K,\mathcal{F})) = \mathfrak{A}_w(p(K),\mathcal{F})$, $p(\mathfrak{A}(K,\mathcal{F})) = \mathfrak{A}(p(K),\mathcal{F})$ *and* $p(\mathfrak{A}_u(K,\mathcal{F})) = \mathfrak{A}_u(p(K),\mathcal{F})$.
(v) *The set K is an $\mathcal{F}$-weak attractor (respectively $\mathcal{F}$-attractor, $\mathcal{F}$-uniform attractor) if and only if $p(X)$ is a $\mathcal{F}$-weak attractor (respectively $\mathcal{F}$-attractor, $\mathcal{F}$-uniform attractor).*
(vi) *The set K is $\mathcal{S}$-stable if and only if $p(K)$ is $\mathcal{S}$-stable.*
(vii) *The set K is $\mathcal{S}$-equistable if and only if $p(K)$ is $\mathcal{S}$-equistable.*
(viii) *The set K if $\mathcal{F}$-asymptotically stable if and only if $p(K)$ is $\mathcal{F}$-asymptotically stable.*

Proof We know that $p(\omega(X,\mathcal{F})) \subset \omega(p(X),\mathcal{F})$ holds without compactness of M. Thus, since p^{-1} is an $\mathcal{S}$-equivariant map, one has

$$\omega\big(p(X),\mathcal{F}\big) = p\big(p^{-1}(\omega(p(X),\mathcal{F}))\big) \subset p\big(\omega(X,\mathcal{F})\big). \tag{4.3}$$

Analogously, the inclusion $p(\mathrm{J}(X,\mathcal{F})) \subset \mathrm{J}(p(X),\mathcal{F})$ does not need compactness of M. In fact, for $z \in \mathrm{J}(x,\mathcal{F})$, with $x \in X$, $\mathcal{U} \in \mathcal{O}'$ and $A \in \mathcal{F}$, there exists an open covering $\mathcal{V} \in \mathcal{O}$ satisfying $p(\mathrm{B}(x,\mathcal{V})) \subset \mathrm{B}(p(x),\mathcal{U})$. Hence,

$$p(z) \in \mathrm{cl}\big(p(A\mathrm{B}(x,\mathcal{V}))\big) \subset \mathrm{cl}\big(A\mathrm{B}(p(x),\mathcal{U})\big).$$

Thus, the equality $p(\mathrm{J}(X,\mathcal{F})) = \mathrm{J}(p(X),\mathcal{F})$ follows as in (4.3). In particular, for every $A \subset \mathcal{S}$, we have $p(\mathrm{D}(X,A)) = \mathrm{D}(p(X),A)$. The statements for domains of attraction and attractors can be proved analogously as was done in (4.3). For stability and asymptotic stability, we can argue analogously. Now, suppose that K is $\mathcal{S}$-equistable and fix $y \notin p(K)$. Take $x \in M$ such that $p(x) = y$. One has $x \notin K$. By the equistability of K there exists $\mathcal{U} \in \mathcal{O}$ such that $x \notin \mathrm{cl}(\mathcal{S}\mathrm{B}(K,\mathcal{U}))$. Thus we have $y \notin p(\mathrm{cl}(\mathcal{S}\mathrm{B}(K,\mathcal{U})))$ and $y \notin \mathrm{cl}(\mathcal{S}\mathrm{B}(p(K),p(\mathcal{U})))$, where $p(\mathcal{U}) = \{p(U) : U \in \mathcal{U}\}$. Since $p(K)$ is compact there exists $\mathcal{V} \in \mathcal{O}'$ such that $\mathrm{B}(p(K),\mathcal{V}) \subset \mathrm{B}(p(K),p(\mathcal{U}))$. Therefore, $y \notin \mathrm{cl}(\mathcal{S}\mathrm{B}(p(K),\mathcal{V}))$ and $p(K)$ is $\mathcal{S}$-equistable. Finally, $p^{-1}(p(K)) = K$ is $\mathcal{S}$-equistable if $p(K)$ is $\mathcal{S}$-equistable. ∎

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