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On the Chow Groups of Supersingular Varieties

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Abstract. We compute the rational Chow groups of supersingular abelian varieties and some other related varieties, such as supersingular Fermat varieties and supersingular $K3$ surfaces. These computations are concordant with the conjectural relationship, for a smooth projective variety, between the structure of Chow groups and the coniveau filtration on the cohomology.

0 Introduction

It is well known that the structure of the Chow groups of a smooth projective variety X over an algebraically closed field is intimately related to the coniveau filtration on its cohomology [8]; for example, the Mumford-Roitman theorem says that if $N^1H^i(X) \neq H^i(X)$ for some $i > 1$ then the group of zero cycles modulo rational equivalence is infinite dimensional (over a universal domain). Over the complex numbers the coniveau filtration is reflected in the Hodge structure on the singular cohomology of X, and many results have been obtained which show that if the Chow groups of a variety are "simple", for example finite dimensional or representable, then the Hodge structure must also be "simple", (*op. cit.* and the references therein). However, whether or not the converse is true is still unknown, the prototypical question being Bloch's conjecture for surfaces: $p_g = 0 \Rightarrow CH_0(X)_{\deg 0}$ is representable.

In positive characteristics, if one thinks of the slopes of the crystalline cohomology as providing a (partial) substitute for the Hodge numbers, then the cohomology is simplest when X is a supersingular variety (*i.e.*, all the Newton polygons have constant slope). In this case the Tate conjecture implies that all the even dimension cohomology groups are generated by the classes of algebraic cycles; as a generalisation of Bloch's conjecture for surfaces it is reasonable to expect that *all* the Chow groups are finite dimensional.

In this note we (almost) determine the structure of the Chow groups of supersingular abelian varieties and some other related varieties, over algebraically closed fields k of characteristic $p > 0$. For such varieties we show that for each d there exists an abelian variety $Ab^d(X)$ which is an algebraic representative for $A^d(X)$ (see [11]), where $A^d(X) \subset CH^d(X)$ is the subgroup of cycles which are algebraically equivalent to zero. Moreover, we show that $\dim Ab^d(X) = \dim H^{2d-1}_{\text{et}}(X, \mathbb{Q}_l)/2$, $l \neq p$, and the kernel of the (surjective) map $A^d(X) \to Ab^d(X)(k)$ is finite. As a consequence of our results it follows that the conjecture of Beauville [2] on the eigenvalues occurring in the decomposition of the Chow groups are true for supersingular abelian varieties.

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Our method is essentially the Bloch-Srinvas method [5] in reverse—for a supersingular abelian variety X we obtain an explicit decomposition of the diagonal in $X \times X$ and then use it to obtain information about the Chow groups. In fact, our method applies to any generalised cohomology theory which is functorial for the action of correspondences. Roughly speaking, we reduce the computation of the generalised cohomology groups of X to those of a supersingular elliptic curve. For example, we show that if k is the algebraic closure of a finite field then the higher K groups of X are torsion. Using correspondences we extend our results to varieties which are products of supersingular curves of any genus and supersingular Fermat varieties. The results for zero cycles were obtained earlier by Maruyama and Suwa [10].

In the last section we prove Bloch's conjecture for supersingular $K3$ surfaces by an extension of the method of Bloch, Kas and Lieberman [4].

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1

Let X be an abelian variety of dimension g. Beauville [2] has constructed a canonical decomposition

$$CH^d(X)_{\mathbb{Q}} = \bigoplus_{s=d-g}^{s=d} CH^d_s(X) \tag{1}$$

where $CH^d_s(X)$ is the subspace of $CH^d(X)_{\mathbb{Q}}$ on which m^* (resp. m^*) acts by multiplication by m^{2d-s} (resp. $m^{2g-2d+s}$).

Let $B^d(X)$ be the subspace of $CH^d_0(X)$ generated by the classes of abelian subvarieties of X of codimension d.

Lemma 1 *$\bigoplus_d B^d(X)$ is a subring of $CH^{\cdot}(X)_{\mathbb{Q}}$.*

Proof This follows from the fact that if two abelian subvarieties of an abelian variety do not meet properly then their intersection is 0 in the Chow ring. Note that since we are working with $\mathbb{Q}$-coefficients, the class of a translate of an abelian subvariety by a torsion point also lies in $\bigoplus_d B^d(X)$. ■

Now suppose X is a supersingular abelian variety over an algebraically closed field of characteristic p. By a theorem of Oort [14], X is isogenous to a product of supersingular elliptic curves. Since all our results will be invariant under isogeny, without loss of generality we may assume that $X = E^g$ where E is a supersingular elliptic

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curve. Since $\mathrm{End}(E)$ is of rank 4, it follows that $B^1(X) = CH^1_0(X)$ and the map $\phi_d \colon B^1(X)^{\otimes d} \to B^d(X)$ induced by the intersection product is surjective.

Let $l \neq p$ be a prime. We fix an isomorphism $\mathbb{Q}_l(1) \cong \mathbb{Q}_l$ and from now on ignore Tate twists.

Lemma 2 *The cycle class map $CH^d(X) \to H^{2d}_{\mathrm{et}}(X, \mathbb{Q}_l)$ induces an isomorphism $B^d(X) \otimes \mathbb{Q}_l \xrightarrow{\cong} H^{2d}_{\mathrm{et}}(X, \mathbb{Q}_l)$.*

Proof The proof is by induction on d. For $d = 1$ we have

$$B^1(X) \otimes \mathbb{Q}_l = CH^1_0(X) \otimes \mathbb{Q}_l \xrightarrow{\cong} H^2_{\mathrm{et}}(X, \mathbb{Q}_l) \tag{2}$$

because E is supersingular.

Let G be the group of units of norm 1 of $\mathrm{End}(X)_{\mathbb{Q}}$, regarded as an algebraic group over $\mathbb{Q}$. It is a form of SL_{2g} and acts on $\bigoplus_d B^d(X)$ as follows: Given $\gamma \in G(\mathbb{Q})$ there exists $n \in \mathbb{Z}$ such that $n \cdot \gamma^{-1}$ is an isogeny. We let γ act on $B^d(X)$ by $1/n^{2d} \cdot (n \cdot \gamma^{-1})^*$. It is easy to see that this gives a well defined action and that the intersection product $\bigoplus_d B^d(X) \otimes \bigoplus_d B^d(X) \to \bigoplus_d B^d(X)$ is G equivariant. By using the well known description of the Neron-Severi group in terms of the endomorphism ring and the remark preceding the lemma, it follows that the representation is algebraic. Consideration of the action of $G_{\mathbb{Q}_l}$ on $H^1_{\mathrm{et}}(X, \mathbb{Q}_l)$ and the isomorphism (2) show that $B^d(X)$ is an irreducible G module with highest weight λ_{2g-2} (in the usual notation for representations of SL_n).

Now consider the map $\psi_2 \colon \mathrm{Sym}^2 B^1(X) \to B^2(X)$ induced by the intersection product. $\mathrm{Sym}^2 B^1(X)$ is the sum of two absolutely irreducible representations of G with highest weights λ_{2g-4} and $2\lambda_{2g-2}$. ψ_2 is not injective because $[Y] \cdot [Y] = 0$, where Y is any abelian subvariety of codimension one of X. It follows that the map $B^2(X) \otimes \mathbb{Q}_l \to H^4(X, \mathbb{Q}_l)$ is an isomorphism.

Now suppose $d \geq 2$. The map ϕ_{d+1} can be factored as

$$B^1(X)^{\otimes d+1} \xrightarrow{\mathrm{Id} \otimes \phi_d} B^1(X) \otimes B^d(X) \longrightarrow B^{d+1}(X)$$

and also as

$$B^1(X)^{\otimes d+1} \xrightarrow{\phi_d \otimes \mathrm{Id}} B^d(X) \otimes B^1(X) \longrightarrow B^{d+1}(X).$$

By induction, $B^d(X)$ is an absolutely irreducible representation of G with highest weight λ_{2g-2d}, so there exists a canonical G equivariant section σ_d of ϕ_d. By Pieri's formula (see for example [6]), $B^d(X) \otimes B^1(X)$ and $B^1(X) \otimes B^d(X)$ are sums of three absolutely irreducible representations of G. It is then easy to see that $(\mathrm{Id} \otimes \phi_d) \circ (\sigma_d \otimes \mathrm{Id})\big(B^d(X) \otimes B^1(X)\big)$ consists of a single absolutely irreducible representation. This shows that $B^{d+1}(X)$ is also absolutely irreducible, and hence $B^{d+1}(X) \otimes \mathbb{Q}_l \xrightarrow{\cong} H^{2(d+1)}(X, \mathbb{Q}_l)$. ∎

The next lemma gives a decomposition of the diagonal in $X \times X$ which is the key to the proof of Proposition 1.

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Lemma 3 *Let $g > 1$. For each $\alpha \in B^g(X \times X)$ there exist g-dimensional abelian subvarieties Y_k of $X \times X$ such that $\alpha = \sum_k c_k[Y_k]$ for some $c_k \in \mathbb{Q}$, and for each k at least one of the projections from $X \times X$ to X restricted to Y_k is not dominant.*

Proof Consider the class β of α in $H^{2g}(X \times X, \mathbb{Q}_l)$. By the Künneth formula we may write β as a sum of β_i's, with each $\beta_i = \bigotimes_{j=1}^{2g} \delta_{i,j}$, where $\delta_{i,j}$ is of pure degree in $H^{\cdot}(E, \mathbb{Q}_l)$ with $\sum_{j=1}^{2g} \deg(\delta_{i,j}) = 2g$. Note that for a given i there must be an even number of j's with $\delta_{i,j} = 1$.

Fix i. Since we must have $\sum_{j=1}^{g} \deg(\delta_{i,j}) \geq g$ or $\sum_{j=g+1}^{2g} \deg(\delta_{i,j}) \geq g$, up to switching the factors of $X \times X$ and permutation of the factors of $X = E^g$, we may assume that there exist non-negative integers l_0, l_1, l_2 with

(a) $l_0 + l_1 + l_2 = g$,
(b) $l_1 + 2l_2 \geq g$,

and such that the first l_0 δ's are of degree 0, the next l_1 δ's are of degree 1 and the last l_2 δ's are of degree 2. Now $H^2(E \times E, \mathbb{Q}_l)$ is generated by classes of codimension 1 abelian subvarieties, so by pairing the j's in $\{1, 2, \ldots, 2g\}$ with $\delta_{i,j} = 1$ in such a way that there is at most one pair $\{j_1, j_2\}$ with $j_1 \leq g < j_2$, we see that we may write each β_i as a linear combination of classes of abelian subvarieties of $X \times X$, such that the image under the projection to the first factor of each such subvariety is of dimension at most $l_0 + (l_1 + 1)/2$. This is less than g because of conditions (a) and (b).

By taking the sum over all i's we obtain a similar expression for β, and by applying the previous lemma, for α as well. Note that in the expressions for the β_i's we may need to use $\mathbb{Q}_l$ coefficients, but we may choose the coefficients c_k in the expression for α to be in $\mathbb{Q}$. This is because a system of linear equations over a field has a solution over an extension field if and only if it has a solution over the base field. ■

The following proposition implies that the conjectures of Beauville [2] and Murre [12, Conjectures B and D] are true for supersingular abelian varieties.

Proposition 1 *$CH^{\cdot}(X)_{\mathbb{Q}}$ is generated as a module over $B^{\cdot}(X)$ by 1 and $\mathrm{Pic}^0(X)$. Consequently $CH^d_s(X) = 0$ if $s \neq 0, 1$ and $CH^d_0(X) = B^d(X)$.*

For Y any abelian variety, let $C^{\cdot}(Y)$ be the $B^{\cdot}(Y)$ submodule of $CH^{\cdot}(Y)_{\mathbb{Q}}$ generated by 1 and $\mathrm{Pic}^0(Y)$. If $f\colon Z \to Y$ is a homomorphism of abelian varieties then it follows from the definitions that $f^*\big(C^{\cdot}(Y)\big) \subset C^{\cdot}(Z)$. If f is also a finite morphism, then using Poincaré's complete reducibility theorem one easily sees that $f_*\big(C^{\cdot}(Z)\big) \subset C^{\cdot}(Y)$.

We now proceed by induction on g, the case $g = 1$ being trivial. Assume $g > 1$ and apply the previous lemma with $\alpha = [\Delta]$, the class of the diagonal. An abelian subvariety of a supersingular abelian variety is also supersingular, so by the condition on the Y_k's we see that $CH^{\cdot}(X)$ is generated by the images of the following two types of maps

(a) $\iota_*\colon CH^{\cdot}(Z)_{\mathbb{Q}} \to CH^{\cdot}(X)_{\mathbb{Q}}$, where ι is the inclusion of a proper abelian subvariety Z of X and

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(b) $g_* \circ f^* \colon CH^{\cdot}(Z)_{\mathbb{Q}} \to CH^{\cdot}(X)_{\mathbb{Q}}$ where Z is a supersingular abelian variety with $\dim(Z) < g$, Y is a supersingular abelian variety with $\dim(Y) = g$, $f \colon Y \to Z$ is a homomorphism and $g \colon Y \to X$ is an isogeny.

The main claim of the proposition follows by induction and the remarks of the previous paragraph. The second statement follows from the multiplicativity of Beauville's decomposition and the fact that $CH^1_1(X) = \mathrm{Pic}^0(X)_{\mathbb{Q}}$.

Remark 1 If we work with $CH^d(X)$ instead of $CH^d(X)_{\mathbb{Q}}$ then Lemma 3 gives us a decomposition of $n \cdot [\Delta]$ for some positive integer n. The proof of Proposition 1 then shows that for each d, $1 \le d \le g$, there exist finitely many abelian subvarieties X_i of X of dimension $g - d + 1$, such that the cokernel of the pushforward map $\bigoplus_i CH^1(X_i) \to CH^d(X)$ is killed by some positive integer. Since the cokernels of the restriction maps $CH^1(X) \to CH^1(X_i)$ are also killed by some positive integer, it follows that the quotient of the Chow ring modulo the subring generated by divisors, *a fortiori* the Griffiths group of any codimension, is also killed by some positive integer.

Recall (*cf.* [11, p. 223]) that if X is a smooth projective variety over an algebraically closed field k, a homomorphism $\rho \colon A^d(X) \to Y(k)$, where Y is an abelian variety over k, is said to be *regular* if for any smooth pointed variety (T, t_0) over k and $\eta \in CH^d(T \times X)$ the map $T(k) \to Y(k)$, given by $t \mapsto \rho(\eta_t - \eta_{t_0})$ for all $t \in T(k)$, is induced by a morphism of varieties $T \to Y$.

Theorem 1 *Let X be a supersingular abelian variety over an algebraically closed field k of characteristic p. For each d, $1 \le d \le \dim X$ there exists an abelian variety $Ab^d(X)$ and a surjective homomorphism $\nu_d \colon A^d(X) \to Ab^d(X)(k)$ which is universal for regular maps from $A^d(X)$ to abelian varieties. Furthermore $\dim Ab^d(X) = \dim H^{2d-1}_{\mathrm{et}}(X, \mathbb{Q}_l)$, $l \neq p$, and the kernel of ν_d is finite.*

Proof To show that $Ab^d(X)$ exists we use the following result of H. Saito (see [16] or [11]): $Ab^d(X)$ exists if there exists a constant c such that for every surjective regular homomorphism $\rho \colon A^d(X) \to Y(k)$, Y an abelian variety, we have $\dim Y \le c$. Remark 1 immediately shows that this is true in our case. For, if $\rho \colon A^d(X) \to Y(k)$ is any surjective regular homomophism, then by composition of the pushforward maps with ρ we get a surjective homomorphism $\bigoplus_i \mathrm{Pic}^0(X_i)(k) \to Y(k)$. Since ρ is regular, this must be induced from a morphism $\prod_i \mathrm{Pic}^0(X_i) \to Y$. Thus we may take c to be $n_i \cdot (g - d + 1)$.

Let $S^1(X)$ be the subgroup of $CH^1(X)$ consisting of the symmetric divisors and let $S^d(X)$ be the image of $\mathrm{Sym}^d S^1(X)$ in $CH^d(X)$ under the intersection product. Note that $S^d(X)$ is a finitely generated abelian group of rank equal to $\dim CH^d_0(X)$. Let $\overline{S^d}(X)$ be $S^d(X)$ modulo torsion. Since k is algebraically closed $\mathrm{Pic}^0(X)(k)$ is divisible, hence the intersection product induces a well defined homomorphism $\pi_d \colon \overline{S^{d-1}}(X) \otimes \mathrm{Pic}^0(X)(k) \to A^d(X)$ which is surjective by Proposition 1 and the remark following it (recall that $A^d(X)$ is divisible). Note that $C^d(X) := \overline{S^{d-1}}(X) \otimes \mathrm{Pic}^0(X)$ has a natural structure of an abelian variety and π_d induces a surjective morphism $C^d(X) \to Ab^d(X)$. Any isogeny γ of X induces an isogeny γ^* of $C^d(X)$ by applying γ^* to both factors, and this is compatible via π_d with γ^* on $A^d(X)$. Then this also induces an action of $G_{\mathbb{Q}_l}$ on $W_d := H^1_{\text{et}}\big(C^d(X), \mathbb{Q}_l\big)$.

Let β be the class of $E \times E \times \cdots \times E \times 0 \times \cdots \times 0 \subset X$ in $\overline{S^{d-1}}(X)$, where we have $g - d + 1$ nonzero factors. Let $F \subset \operatorname{Pic}^0(X)$ be the elliptic curve obtained by pullback via the projection from X to the last factor. Then $\beta \otimes F$ is an abelian subvariety of $C^d(X)$ and it is clear that $\pi_d\big(\beta \otimes F(k)\big) = 0$. Since π_d is compatible with the action of isogenies of X, it follows that if Z is the abelian subvariety of $C^d(X)$ generated by $\gamma^*(F)$, where γ runs over all isogenies of X, then $\pi_d\big(Z(k)\big) = 0$. Z is clearly preserved by all isogenies as above, hence V_d, the kernel of the restriction map $W_d \to H^1_{\text{et}}(Z, \mathbb{Q}_l)$, is invariant under the action of $G_{\mathbb{Q}_l}$. By the universal property of $Ab^d(X)$, isogenies of X also induce isogenies of $Ab^d(X)$, so we also get an action of $G_{\mathbb{Q}_l}$ on $H^1_{\text{et}}\big(Ab^d(X), \mathbb{Q}_l\big)$ such that $(\nu_d \circ \pi_d)^* \colon H^1_{\text{et}}\big(Ab^d(X), \mathbb{Q}_l\big) \to W_d(X)$ is a map of $G_{\mathbb{Q}_l}$ modules. As a $G_{\mathbb{Q}_l}$ module W_d is the tensor product ot two irreducible modules with highest weights λ_1 and λ_{2d-2}. Using Pieri's formula again, one sees that W_d is the direct sum of two absolutely irreducible representations with highest weights λ_{2d-1} and $\lambda_1 + \lambda_{2d-2}$. This implies that V_d absolutely irreducible. Clearly $(\nu_d \circ \pi_d)^*$ is an injection on H^1 and so it follows that $H^1_{\text{et}}\big(Ab^d(X), \mathbb{Q}_l\big) \to V_d$ is an isomorphism (it is obvious that $Ab^d(X) \neq 0$, $1 \leq d \leq g$). Thus we see that the induced map $C^d(X)/Z \to Ab^d(X)$ is an isogeny and hence the kernel of ν_d must be finite.

Since $Ab^g(X)$ is equal to X, we see that V_g is isomorphic to the representation with highest weight λ_{2g-1}. This implies that for all d, V_d is isomorphic to the representation with highest weight λ_{2d-1}, the reason being that the representation with highest weight λ_{2g-1} does not occur as a component of the tensor product of the representations with highest weights $\lambda_1 + \lambda_{2d-2}$ and λ_{2g-2d}. ■

Remark 2 It seems likely that the kernel of ν_d is always 0. Our proof does show that it has no p-torsion.

2

It will be convenient to use the language of motives in order to formulate a generalisation of the results of the previous section. We shall use the paper of Manin [9] as a reference; the intersection theory that we use is the Chow theory with $\mathbb{Q}$-coefficients.

Definition 1 A smooth projective variety X over an algebraically closed field k is called strongly supersingular if the motive of X, $\tilde{X}$ is a direct summand of the motive of a supersingular abelian variety.

Of course, strongly supersingular varieties are supersingular. It would be interesting to know if the converse is true. Examples of such varieties include products of supersingular curves of arbitrary genus [9] and supersingular Fermat varieties ([17], along with the results in [9]). Theorem 1 can be extended to all strongly supersingular varieties, *i.e.*, we have:

Theorem 2 *Let X be a strongly supersingular variety over an algebraically closed field k of characteristic p. For each d, $1 \leq d \leq \dim X$, there exists an abelian variety $Ab^d(X)$ and a surjective homomorphism $\nu_d \colon A^d(X) \to Ab^d(X)(k)$ which is universal for regular maps from $A^d(X)$ to abelian varieties. Furthermore $\dim Ab^d(X) = \dim H^{2d-1}_{\text{et}}(X, \mathbb{Q}_l)$, $l \neq p$, and the kernel of ν_d is finite.*

Proof The existence of $Ab^d(X)$ can be seen as follows: If M is a supersingular abelian variety such that $\tilde{X}$ is a direct summand of $\tilde{M}$ then we get a surjective map from $C^d(M)(k)$ (*cf.* proof of Theorem 1) to $A^d(X)$. We then apply the theorem of H. Saito [16].

If we had used motives with integral coefficients in our definition of strongly supersingular, it would be immediate from Theorem 1 that the kernel of ν_d was finite. However, by "clearing denominators" we may assume in any case that we have integral correspondences from X to M and M to X which "almost" give a splitting. Since $A^d(X)$ is divisible, this does not cause any problems. We leave the details to the reader.

To show that $\dim Ab^d(X) = \dim H^{2d-1}_{\text{et}}(X, \mathbb{Q}_l)$, $l \neq p$, we shall need to use the Bloch map $\lambda^d \colon CH^d(X)(l) \to H^{2d-1}_{\text{et}}(X, \mathbb{Q}_l/\mathbb{Z}_l)$ ([3]; we are ignoring twists). N. Suwa [19] has shown that if X is a supersingular abelian variety then λ^d is surjective for all d, $l \neq p$, and from the results of Section 1 it follows that the kernel of λ^d is killed by some power of l. Since λ^d is functorial for the action of correspondences [3], it follows that the same is true for any strongly supersingular variety. The fact that the kernel of ν_d is finite completes the proof. ■

Let $K_j(X)$ denote the j-th Quillen K-theory of X and $\mathcal{K}_j$ the associated Zariski sheaf. The following theorem improves some of the results of Soulé [18] in the special case of strongly supersingular varieties.

Theorem 3 *Let X be a strongly supersingular variety over the algebraic closure of a finite field. Then $K_j(X)$ and $H^r(X, \mathcal{K}_s)$ are torsion for all $j > 0$ and $r \neq s$.*

Proof The Brown-Gersten-Quillen spectral sequence shows that it is enough to prove the result for $H^r(X, \mathcal{K}_s)$. It has been proved by Gillet [7, Appendix] that the $H^r(X, \mathcal{K}_s)$'s form part of a generalised cohomology theory and are functorial for the action of correspondences. By the definition of strongly supersingular, it follows that it is enough to prove the result for supersingular abelian varieties. We then proceed as in the proof of Proposition 1; so by descending induction on the dimension of X we are reduced to the case of a supersingular elliptic curve. Since the theorem is in fact known to hold for all curves (*cf.* [18]), we are done. ■

3

Let X be a $K3$ surface with rank $NS(X) = 22$, over an algebraically closed field k of characteristic p and let $\sigma_0(X)$ denote the Artin invariant of X [1]. It is known that X has an elliptic pencil and if $\sigma_0(X) < 10$ then X has an elliptic pencil with a section. We shall use a bootstrapped version of the method of Bloch, Kas and Lieberman [4] to prove that $CH_0(X) = \mathbb{Z}$ if $p \geq 5$. Since Rudakov and Shafarevich have proved that if $p = 2$ then X is unirational, the result is also true for $p = 2$. The reader may consult the article [15] for a nice survey of the known results concerning $K3$ surfaces in characteristic $p > 0$.

For an elliptic fibration $\pi\colon X \to \mathbb{P}^1$ we denote by $J(X) \to \mathbb{P}^1$ the corresponding Jacobian fibration.

Lemma 4 *Let $\pi\colon X \to \mathbb{P}^1$ be an elliptic fibration with X a supersingular $K3$ surface which has a section σ. Then there exists a scheme $\mathcal{X}$ and a smooth proper morphism $\mathcal{X} \to \operatorname{Spec} k[[t]]$ such that the special fibre is isomorphic to X, the elliptic pencil lifts to $\mathcal{X}$, $J(\mathcal{X}_{\overline{k((t))}}) \cong X_{\overline{k((t))}}$ and $\sigma_0(J(\mathcal{X}_{\overline{k((t))}}) = \sigma_0(X) + 1$.*

Proof Let k_1 be an algbraically closed field of transcendence degree one over k. It follows from Artin's computation in [1] of the Brauer group of an elliptic supersingular $K3$ surface that the natural map $\operatorname{Br}(X) \to \operatorname{Br}(X_{k_1})$ is not surjective. Since $\operatorname{Br}(X)$ is canonically isomorphic to the Shafarevich-Tate group of the generic fibre of π it follows that there exists a supersingular $K3$ surface Y over k_1 and an elliptic pencil $Y \to \mathbb{P}^1_{k_1}$ such that $J(Y) \cong X_{k_1}$ and Y is not isomorphic to the base change of a surface defined over k. By choosing a suitable subfield of k_1, finitely generated over k, we obtain a family of elliptic surfaces $\mathcal{Y} \to C \times \mathbb{P}^1 \to C$, where C is a smooth connected curve over k, with geometric generic fibre isomorphic to Y and $J(\mathcal{Y}_c) \cong J(X)$ for all closed points c of C. By twisting the family with the negative of the class in the Shafarevich-Tate group represented by some closed fibre if necessary, we may assume that there exists a closed point o in C such that $\mathcal{Y}_o \cong X$ has an elliptic fibration. Let $\mathcal{X}$ be the restriction of the family to $\operatorname{Spec} \widehat{\mathcal{O}_{C,o}} \cong \operatorname{Spec} k[[t]]$.

It remains to show that $\sigma_0\big(J(\mathcal{X}_{\overline{k((t))}})\big) = \sigma_0(X) + 1$. By construction, $\mathcal{X}_{\overline{k((t))}}$ corresponds to a nontrivial element of $\operatorname{Br}(X_{\overline{k((t))}})$ and hence the section σ does not lift to $\mathcal{X}_{\overline{k((t))}}$. Thus $\sigma_0\big(J(\mathcal{X}_{\overline{k((t))}})\big) \geq \sigma_0(X) + 1$. To show equality it suffices to show that all D in $\operatorname{Pic}(X)$, such that p divides $D \cdot E$, lift to $\operatorname{Pic}(\mathcal{X}_{\overline{k((t))}})$, where E is the class of a closed fibre of π. It is known that $p \cdot D$ lifts for all D in $\operatorname{Pic}(X)$ (see [1]), hence in particular $p \cdot [\sigma(\mathbb{P}^1)]$ lifts. Using the fact that $J(\mathcal{X}_{\overline{k((t))}}) \cong X_{\overline{k((t))}}$, it follows from Tsen's theorem that for all D in $\operatorname{Pic}(X)$ such that p divides $D \cdot E$ there exists $\tilde{D}$ in $\operatorname{Pic}(\mathcal{X}_{\overline{k((t))}})$ such that the specialization of $\tilde{D}$ and D are rationally equivalent when restricted to the generic fibre of π. This completes the proof since it is clear by construction that all vertical divisors lift. ∎

Theorem 4 *Let X be a $K3$ surface with $\rho = 22$ over an algebraically closed field k of characteristic $p \geq 5$. Then $CH_0(X) = \mathbb{Z}$.*

Proof Before proving the theorem we first recall a result of Ogus [13]: Let N be a $K3$-lattice with discriminant $-p^{20}$. Let S_N be the moduli space of N-marked $K3$ surfaces and let M_N be the corresponding period space. Ogus has proved that the period map $S_N \to M_N$ is étale and surjective and all N-marked $K3$ surfaces corresponding to points in a geometric fibre are isomorphic as (unmarked) surfaces.

Let X_1 be the Kummer surface associated to a product of supersingular elliptic curves over k. It is known that $\sigma_0(X_1) = 1$. Applying the lemma inductively to X_1 we obtain surfaces X_i, $1 < i \leq 10$, each of which carries an elliptic pencil with the property that $J(X_i) \cong X_{i-1}$, all these surfaces being defined over some algebraically closed extension field K of k. By construction, X_i can be specialized to X_{i-1}, hence under the period map the moduli point of X_{10} (with any N-marking) maps to the generic point of M_N (over k). Shioda has proved that X_1 is unirational, hence $CH_0(X_{1L}) = \mathbb{Z}$ for any algebraically closed extension field L of k. By repeated application of Proposition 4 of [4], it follows that $CH_0(X_{10}) = \mathbb{Z}$. Using the theorem of Ogus, we conclude the proof by specialization. ■

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