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ON SOME NEW MOCK THETA FUNCTIONS

NANCY S. S. GU and LI-JUN HAO✉

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Abstract

In 1991, Andrews and Hickerson established a new Bailey pair and combined it with the constant term method to prove some results related to sixth-order mock theta functions. In this paper, we study how this pair gives rise to new mock theta functions in terms of Appell–Lerch sums. Furthermore, we establish some relations between these new mock theta functions and some second-order mock theta functions. Meanwhile, we obtain an identity between a second-order and a sixth-order mock theta functions. In addition, we provide the mock theta conjectures for these new mock theta functions. Finally, we discuss the dual nature between the new mock theta functions and partial theta functions.

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1. Introduction

Throughout this paper, let q denote a complex number with $|q| < 1$. Here and in what follows, we adopt the standard q-series notation [12]. For any positive integer n,

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),$$
$$(a_1,a_2,\ldots,a_m;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

For convenience, we use $(a)_n$ to denote $(a;q)_n$.

Jacobi's triple product identity is stated as

$$j(x;q) := (x)_\infty(q/x)_\infty(q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{\binom{n}{2}} x^n.$$

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Let m and n be integers with m positive. Define

$$J_{a,m} := j(q^a; q^m), \quad \overline{J}_{a,m} := j(-q^a; q^m), \quad J_m := \prod_{i\geq 1}(1-q^{mi}),$$
$$j(b_1, b_2, \dots, b_m; q) := j(b_1; q) j(b_2; q) \cdots j(b_m; q).$$

Recall that mock theta functions are q-series which were introduced by Ramanujan in his last letter to G. H. Hardy on January 12, 1920. In that letter, Ramanujan listed seventeen mock theta functions and divided them into four classes: one class of third-order, two of fifth-order and one of seventh-order. However, Ramanujan neither rigorously defined a mock theta function nor the order of a mock theta function. Over the years, mock theta functions have received a great deal of attention [1, 11, 15, 16, 30, 31]. Until 2002, it was not known how these functions fit into the theory of modular forms. A new chapter in the study of mock theta functions was opened due to the work of Zwegers [33] and Bringmann and Ono [8, 9]. We now know that each of Ramanujan's mock theta functions is the holomorphic part of a weight $1/2$ harmonic weak Maass form $f(\tau)$, where $q := e^{2\pi i\tau}$ and $\tau = x + iy \in \mathbb{H}$. Following Zagier [32], the holomorphic part of any weight k harmonic weak Maass form f is called a mock theta modular form of weight k. If $k = 1/2$ and the image of f under the operator $\xi_k := 2iy^k(\overline{\partial/\partial\overline{\tau}})$ is a unary theta function, then the holomorphic part of f is called a mock theta function. Hickerson and Mortenson [17] defined Appell–Lerch sums as follows.

Definition 1.1. Let $x, z \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}$ with neither z nor zq an integer power of q. Then

$$m(x, q, z) := \frac{-z}{j(z; q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r+1}{2}} z^r}{1 - q^r xz}.$$

Specializations of the Appell–Lerch sums are perhaps the most important class of mock theta functions. In other words, for any function $f(z)$, if we could express $f(z)$ as Appell–Lerch sums up to the addition of a weakly holomorphic modular form, then the function $f(z)$ is a mock theta function. Hickerson and Mortenson [17] studied the properties of Appell–Lerch sums and established the representations of mock theta functions in terms of Appell–Lerch sums. For more on mock theta functions, their remarkable history and modern developments, see [13, 27, 32].

Recently, many new mock theta functions have been found [7, 14, 18–21]. McIntosh [22] defined three second-order mock theta functions which were given by

$$A(q) = \sum_{n=0}^{\infty} \frac{q^{n+1}(-q^2; q^2)_n}{(q; q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q; q^2)_n}{(q; q^2)^2_{n+1}},$$
$$B(q) = \sum_{n=0}^{\infty} \frac{q^n(-q; q^2)_n}{(q; q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q^2; q^2)_n}{(q; q^2)^2_{n+1}},$$
$$\mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q; q^2)_n}{(-q^2; q^2)^2_n}. \tag{1.1}$$

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Notice that $B(q)$ and $\mu(q)$ appeared in Ramanujan's 'Lost' notebook [28]. In this paper, we define the following new mock theta functions which can be represented as the form of Appell–Lerch sums.

$$R_1(q) := \sum_{n=0}^{\infty} \frac{q^n(1+q)(-1;q^2)_n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_{n+1}}{(q;q^2)_{n+1}^2}, \tag{1.2}$$

$$R_2(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n}(1+q)(-q;q^2)_n}{(-q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n}(1+q)(q;q^2)_n}{(-q^2;q^2)_n(-q^2;q^2)_{n+1}}. \tag{1.3}$$

Notice that

$$R_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(1+q^{2n+1})(-q;q^2)_n}{(q;q^2)_{n+1}^2} = R_1'(q) + A(q),$$

where

$$R_1'(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_{n+1}^2}.$$

Based on the following theorems, $R_1'(q)$ is also a mock theta function.

THEOREM 1.2. *We have*

$$R_1(q) = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{j=-\infty}^{\infty} \frac{(-1)^j q^{2j^2+j}(1+q^{2j+1})}{1-q^{2j+1}} = -4m(q,q^4,q) + 1, \tag{1.4}$$

$$R_2(q) = \frac{2(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{j=-\infty}^{\infty} \frac{q^{2j^2+3j}}{1+q^{2j}} = -4m(-q,q^4,-q) - \frac{J_1^5}{J_2^4} + 2. \tag{1.5}$$

Additionally, we establish some identities involving these mock theta functions and some second-order mock theta functions. Meanwhile, we find a new identity between $\mu(q)$ and a sixth-order mock theta function $\psi(q)$ that appeared in [28, page 13] as follows.

$$\psi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n+1}(q;q^2)_n}{(-q;q)_{2n+1}}.$$

THEOREM 1.3. *We have*

$$R_1(q) - 4A(q) = 1, \tag{1.6}$$

$$R_2(q) + \mu(q) = 2, \tag{1.7}$$

$$\mu(q^3) - 2\psi(q) = \frac{J_1^2 J_4 J_6^5}{J_2^3 J_3 J_{12}^3}. \tag{1.8}$$

In [15], Hickerson introduced the universal mock theta function

$$g_3(z;q) := z^{-1}\left(-1 + \sum_{n=0}^{\infty} \frac{q^{n^2}}{(z)_{n+1}(qz^{-1})_n}\right) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(z)_{n+1}(qz^{-1})_{n+1}}.$$

Then Gordon and McIntosh [13] defined a universal mock theta function as

$$g_2(z;q) := \sum_{n=0}^{\infty} \frac{(-q)_n q^{\binom{n+1}{2}}}{(z)_{n+1}(qz^{-1})_{n+1}}.$$

Another universal mock theta function $K(z;q)$ given by McIntosh [23] was

$$K(z;q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(q^2z;q^2)_n(q^2z^{-1};q^2)_n}. \tag{1.9}$$

The two universal mock theta functions $g_2(z;q)$ and $K(z;q)$ are related by modular transformation [23]. Notice that all the fifth-order mock theta functions can be expressed by $g_3(z;q)$ in [15]. In addition, Hickerson [16] showed that Ramanujan's seventh-order mock theta functions can be expressed as specializations of $g_3(z;q)$. Later, Gordon and McIntosh [13] gave the expressions in terms of $g_3(z;q)$ for the third-order mock theta functions. For the expressions of the other mock theta functions in terms of universal mock theta functions, see [13, 24].

Finally, we present the mock theta conjectures for $R_1(q)$ and $R_2(q)$.

Theorem 1.4. *We have*

$$R_1(-q) = -K(-1;q) + \frac{J_1^5}{J_2^4} + 1, \tag{1.10}$$

$$R_2(q) = -K(-1;q) + 2. \tag{1.11}$$

This paper is organized as follows. In Section 2, by applying a Bailey pair given by Andrews and Hickerson [6], we prove Theorem 1.2. In Section 3, by means of some properties of Appell–Lerch sums, we prove Theorems 1.3 and 1.4. In the final section, based on the dual nature between Appell–Lerch sums and partial theta functions introduced by Mortenson [25], we study the dual nature of $R_1(q)$ and $R_2(q)$.

2. The proof of Theorem 1.2

In this section, combining the Bailey pair given by Andrews and Hickerson [6] and the properties of Appell–Lerch sums collected by Hickerson and Mortenson [17], we give the representations for $R_1(q)$ and $R_2(q)$ in terms of Appell–Lerch sums.

Definition 2.1. The sequences (α_n, β_n) are called a Bailey pair relative to (a, q) if (α_n, β_n) satisfy

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}.$$

Bailey's lemma says that if (α_n, β_n) is a Bailey pair relative to a, then

$$\sum_{n\geq 0}(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\beta_n = \frac{(aq/\rho_1)_\infty(aq/\rho_2)_\infty}{(aq)_\infty(aq/\rho_1\rho_2)_\infty} \sum_{n\geq 0} \frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n}{(aq/\rho_1)_n(aq/\rho_2)_n}\alpha_n, \tag{2.1}$$

provided both sums converge absolutely.

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Lemma 2.2 [6, Theorem 2.3]. *Let a, b, c and q be complex numbers with $a \neq 1$, $b \neq 0$, $c \neq 0$, $q \neq 0$ and with none of a/b, a/c, qb and qc of the form q^{-k} with $k \geq 0$. For $n \geq 0$, define*

$$A_n'(a,b,c,q) = \frac{q^{n^2}(bc)^n(1-aq^{2n})(a/b)_n(a/c)_n}{(1-a)(qb)_n(qc)_n} \sum_{j=0}^{n} \frac{(-1)^j(1-aq^{2j-1})(a)_{j-1}(b)_j(c)_j}{q^{j(j-1)/2}(bc)^j(q)_j(a/b)_j(a/c)_j}$$

and

$$B_n'(a,b,c,q) = \frac{1}{(qb)_n(qc)_n}.$$

Then the sequences $\{A_n'(a,b,c,q)\}$ and $\{B_n'(a,b,c,q)\}$ form a Bailey pair relative to a.

Next, we review some properties for Appell–Lerch sums. Following [17], the term 'generic' means that the parameters do not cause poles in the Appell–Lerch sums or in the quotients of theta functions.

Proposition 2.3 [17]. *For generic $x, z, z_0, z_1 \in \mathbb{C}^*$,*

$$m(x,q,z) = m(x,q,qz), \tag{2.2}$$
$$m(x,q,z) = x^{-1}m(x^{-1},q,z^{-1}), \tag{2.3}$$
$$m(qx,q,z) = 1 - xm(x,q,z), \tag{2.4}$$
$$m(x,q,z) = m(x,q,x^{-1}z^{-1}). \tag{2.5}$$

Notice that the equality of the two series in the definition of $R_1(q)$ can be obtained by replacing q, a, b, d, e by q^2, q^2, $-q^3$, q^3, q^3 and then letting c tend to ∞ in the Sears–Thomae transformation formula [12, Equation (III.9)].

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n(q)_n}\left(\frac{de}{abc}\right)^n = \frac{(e/a)_\infty(de/bc)_\infty}{(e)_\infty(de/abc)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n(d/b)_n(d/c)_n}{(d)_n(de/bc)_n(q)_n}\left(\frac{e}{a}\right)^n.$$

Similarly, the equality of the two series in the definition of $R_2(q)$ can also be obtained by replacing q, a, b, d, e by q^2, q^2, q, $-q^2$, $-q^4$ and then letting c tend to ∞ in the above identity.

Proof of Theorem 1.2. Based on Lemma 2.2, we obtain

$$A_n'(q^2,q,q,q^2) = \frac{q^{2n^2+2n}(1-q)(1+q^{2n+1})}{(1+q)(1-q^{2n+1})} \sum_{j=-n}^{n} (-1)^j q^{-j^2-j}, \tag{2.6}$$

$$B_n'(q^2,q,q,q^2) = \frac{1}{(q^3;q^2)_n^2}. \tag{2.7}$$

Replacing q, a, ρ_1 by q^2, q^2, $-q^3$ and setting $\rho_2 \to \infty$ in (2.1), and then applying (2.6) and (2.7), we conclude that

$$\sum_{n=0}^{\infty} q^{n^2}(-q^3;q^2)_n B_n'(q^2,q,q,q^2) = \frac{(-q;q^2)_\infty}{(q^4;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}(1+q^{2n+1})}{1+q} A_n'(q^2,q,q,q^2).$$

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That is,

$$\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_{n+1}}{(q;q^2)_{n+1}^2}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\frac{(-1)^jq^{3n^2+2n-j^2-j}(1+q^{2n+1})^2}{1-q^{2n+1}}.$$

Combining (1.2) and the above identity yields

$$\begin{aligned}
R_1(q)&=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\bigg(\sum_{n=0}^{\infty}\sum_{j=0}^{n}+\sum_{n=1}^{\infty}\sum_{j=-n}^{-1}\bigg)\frac{(-1)^jq^{3n^2+2n-j^2-j}(1+q^{2n+1})^2}{1-q^{2n+1}}\\
&=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\bigg(\sum_{j=0}^{\infty}\sum_{n=j}^{\infty}+\sum_{j=-\infty}^{-1}\sum_{n=-j}^{\infty}\bigg)\frac{(-1)^jq^{3n^2+2n-j^2-j}(1+q^{2n+1})^2}{1-q^{2n+1}}\\
&=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\bigg(\sum_{j=0}^{\infty}\sum_{n=j}^{\infty}\frac{(-1)^jq^{3n^2+2n-j^2-j}(1+q^{2n+1})^2}{1-q^{2n+1}}\\
&\qquad-\sum_{j=0}^{\infty}\sum_{n=j+1}^{\infty}\frac{(-1)^jq^{3n^2+2n-j^2-j}(1+q^{2n+1})^2}{1-q^{2n+1}}\bigg)\\
&=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\bigg(\sum_{j=0}^{\infty}\frac{(-1)^jq^{2j^2+j}(1+q^{2j+1})^2}{1-q^{2j+1}}\bigg)\\
&=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\bigg(\sum_{j=0}^{\infty}\frac{(-1)^jq^{2j^2+j}(1+2q^{2j+1}+q^{4j+2})}{1-q^{2j+1}}\bigg)\\
&=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\bigg(\sum_{j=-\infty}^{\infty}\frac{(-1)^jq^{2j^2+j}}{1-q^{2j+1}}+\sum_{j=-\infty}^{\infty}\frac{(-1)^jq^{2j^2+3j+1}}{1-q^{2j+1}}\bigg),
\end{aligned}\tag{2.8}$$

which implies the first equation in (1.4). Furthermore, based on the definition of Appell–Lerch sums and (2.8),

$$\begin{aligned}
R_1(q)&=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\bigg(\sum_{j=-\infty}^{\infty}\frac{(-1)^jq^{2j^2+j}(1+q^{2j+1})}{1-q^{4j+2}}+\sum_{j=-\infty}^{\infty}\frac{(-1)^jq^{2j^2+3j+1}(1+q^{2j+1})}{1-q^{4j+2}}\bigg)\\
&=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\bigg(\sum_{j=-\infty}^{\infty}\frac{(-1)^jq^{2j^2+j}}{1-q^{4j+2}}+3\sum_{j=-\infty}^{\infty}\frac{(-1)^jq^{2j^2+3j+1}}{1-q^{4j+2}}\bigg)\\
&=m(q^3,q^4,q^{-1})-3m(q,q^4,q)\\
&=1-4m(q,q^4,q)\quad\text{(by (2.3) and (2.4))},
\end{aligned}$$

which gives the last equation in (1.4).

Next, according to Lemma 2.2, we have the following Bailey pair.

$$A_n'(q^2,-1,-q^2,q^2)=\frac{2q^{2n^2+2n}(1+q^2)(1-q^{4n+2})}{(1-q^2)(1+q^{2n})(1+q^{2n+2})}\sum_{j=-n}^{n}(-1)^jq^{-j^2-j},$$

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$$B'_n(q^2,-1,-q^2,q^2)=\frac{1}{(-q^2;q^2)_n(-q^4;q^2)_n}.$$

By inserting the above Bailey pair into (2.1), replacing q, a, ρ_1 by q^2, q^2, q and letting $\rho_2\to\infty$,

$$\sum_{n=0}^{\infty}(-1)^n q^{n^2+2n}(q;q^2)_n B'_n(q^2,-1,-q^2,q^2)$$
$$=\frac{(q;q^2)_\infty}{(q^4;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+2n}}{1-q^{2n+1}}A'_n(q^2,-1,-q^2,q^2).$$

It follows from (1.3) that

$$\begin{aligned}R_2(q)&=\frac{2(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\frac{(-1)^{n+j}q^{3n^2+4n-j^2-j}(1+q)(1+q^{2n+1})}{(1+q^{2n})(1+q^{2n+2})}\\
&=\frac{2(q;q^2)_\infty}{(q^2;q^2)_\infty}\bigg(\sum_{j=0}^{\infty}\sum_{n=j}^{\infty}+\sum_{j=-\infty}^{-1}\sum_{n=-j}^{\infty}\bigg)\frac{(-1)^{n+j}q^{3n^2+4n-j^2-j}(1+q)(1+q^{2n+1})}{(1+q^{2n})(1+q^{2n+2})}\\
&=\frac{2(q;q^2)_\infty}{(q^2;q^2)_\infty}\bigg(\sum_{j=0}^{\infty}\sum_{n=j}^{\infty}\frac{(-1)^{n+j}q^{3n^2+4n-j^2-j}(1+q)(1+q^{2n+1})}{(1+q^{2n})(1+q^{2n+2})}\\
&\qquad-\sum_{j=0}^{\infty}\sum_{n=j+1}^{\infty}\frac{(-1)^{n+j}q^{3n^2+4n-j^2-j}(1+q)(1+q^{2n+1})}{(1+q^{2n})(1+q^{2n+2})}\bigg)\\
&=\frac{2(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{j=0}^{\infty}\frac{q^{2j^2+3j}(1+q)(1+q^{2j+1})}{(1+q^{2j})(1+q^{2j+2})}\\
&=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{j=-\infty}^{\infty}\frac{q^{2j^2+3j}(1+q)(1+q^{2j+1})}{(1+q^{2j})(1+q^{2j+2})}.\end{aligned}$$

Observing that

$$\frac{(1+q)(1+q^{2j+1})}{(1+q^{2j})(1+q^{2j+2})}=\frac{1}{1+q^{2j}}+\frac{q}{1+q^{2j+2}},$$

we deduce that

$$\begin{aligned}R_2(q)&=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\bigg(\sum_{j=-\infty}^{\infty}\frac{q^{2j^2+3j}}{1+q^{2j}}+\sum_{j=-\infty}^{\infty}\frac{q^{2j^2+3j+1}}{1+q^{2j+2}}\bigg)\\
&=\frac{2(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{j=-\infty}^{\infty}\frac{q^{2j^2+3j}}{1+q^{2j}},\end{aligned}$$

which is the first equation in (1.5). Finally, we need the following result from [4, Equation (12.2.7)].

$$\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{(n+1)(2n+1)}(1+q^{2n+1})}{(1+aq^{2n+1})(1+q^{2n+1}/a)}=\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-aq;q^2)_{n+1}(-q/a;q^2)_{n+1}}.\tag{2.9}$$

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Additionally, Andrews and Berndt [4, Equation (12.3.3)] provided that

$$\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=-\infty}^{\infty}\frac{q^{2n^2+n}}{1+aq^{2n}}-\left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty}\frac{q^{(n+1)(2n+1)}(1+q^{2n+1})}{(1+aq^{2n+1})(1+q^{2n+1}/a)}\right)$$
$$=\frac{(q;q^2)_\infty(q;q)_\infty^2}{(-a;q)_\infty(-q/a;q)_\infty(q^2;q^2)_\infty}. \tag{2.10}$$

Then using (2.9) and (2.10) and setting $a = q$,

$$\begin{aligned}
R_2(q) &= \frac{2(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{2n^2+n}}{1+q^{2n+1}}-\frac{2(q;q^2)_\infty(q;q)_\infty}{(-1;q)_\infty(-q;q)_\infty^2}\\
&= \frac{2(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{2n^2+n}(1-q^{2n+1})}{1-q^{4n+2}}-(q;q)_\infty(q;q^2)_\infty^4\\
&= \frac{2(q;q^2)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=-\infty}^{\infty}\frac{q^{2n^2+n}}{1-q^{4n+2}}-\sum_{n=-\infty}^{\infty}\frac{q^{2n^2+3n+1}}{1-q^{4n+2}}\right)-(q;q)_\infty(q;q^2)_\infty^4\\
&= 2m(-q^3,q^4,-q^{-1})-2m(-q,q^4,-q)-(q;q)_\infty(q;q^2)_\infty^4\\
&= -2q^{-3}m(-q^{-3},q^4,-q)-2m(-q,q^4,-q)-(q;q)_\infty(q;q^2)_\infty^4 \quad \text{(by (2.3))}\\
&= 2(1-m(-q,q^4,-q))-2m(-q,q^4,-q)-(q;q)_\infty(q;q^2)_\infty^4 \quad \text{(by (2.4))}\\
&= 2-4m(-q,q^4,-q)-(q;q)_\infty(q;q^2)_\infty^4.
\end{aligned}$$

Hence, we finish the proof of (1.5). □

Notice that one can prove identity (1.5) of Theorem 1.2 using identity (2.15) of [25, Proposition 2.6] (see [25, Section 4.3]).

3. Proofs of Theorems 1.3 and 1.4

In this section, with the aid of the results in Section 2 and the properties of Appell–Lerch sums, we establish some identities related to $R_1(q)$, $R_2(q)$, $A(q)$, $\mu(q)$ and $\psi(q)$. Meanwhile, we provide the mock theta conjectures for $R_1(q)$ and $R_2(q)$.

First, we review the definition of Hecke-type double sums.

Definition 3.1. Let $x, y \in \mathbb{C}^*$ and define $sg(r) := 1$ for $n \geq 0$ and $sg(r) := -1$ for $n < 0$. Then

$$f_{a,b,c}(x,y,q) := \sum_{sg(r)=sg(s)} sg(r)(-1)^{r+s}x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}.$$

Proof of Theorem 1.3. First, it can be seen from (2.2) and (2.5) that

$$m(q,q^4,q) = m(q,q^4,q^2). \tag{3.1}$$

Then by combining (1.4), (3.1) and the relation [17, Equation (5.1)]

$$A(q) = -m(q,q^4,q^2), \tag{3.2}$$

we derive (1.6).

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In addition, Andrews [3, Equation (3.28)] gave that

$$4A(-q) + \mu(q) = \frac{J_1^5}{J_2^4}. \tag{3.3}$$

Then by combining (1.5), (3.1) and (3.2),

$$R_2(q) = 4A(-q) - \frac{J_1^5}{J_2^4} + 2.$$

Hence, examining (3.3) and the above identity yields (1.7).

Furthermore, based on the identity [6, Equation (3.2)]

$$2\overline{J}_{1,4}\psi(q) = \sum_{sg(r)=sg(s)} (-1)^r sg(r) q^{rs+\binom{r+s+2}{2}},$$

we have

$$\begin{aligned}
2\psi(q) &= \frac{q(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{\substack{sg(r)=sg(s)\\ r\equiv s \pmod 2}} (-1)^r sg(r) q^{((r^2+s^2)/2)+2rs+((3r+3s)/2)} \\
&= \frac{q(q;q^2)_\infty}{(q^2;q^2)_\infty}(f_{1,2,1}(-q^5,-q^5,q^4) - q^6 f_{1,2,1}(-q^{11},-q^{11},q^4)) \\
&= \frac{2q(q;q^2)_\infty}{(q^2;q^2)_\infty} f_{1,2,1}(-q^5,-q^5,q^4),
\end{aligned}$$

where, in the last line, we use the identity [17, Proposition 6.2]

$$f_{a,b,c}(x,y,q) = -\frac{q^{a+b+c}}{xy} f_{a,b,c}(q^{2a+b}/x, q^{2c+b}/y, q).$$

On the other hand, Hickerson and Mortenson [17, Equation (1.7)] gave that

$$\begin{aligned}
f_{1,2,1}(x,y,q) = j(y;q)m(q^2x/y^2,q^3,-1) + j(x;q)m(q^2y/x^2,q^3,-1) \\
- \frac{yJ_3^3 j(-x/y;q) j(q^2xy;q^3)}{\overline{J}_{0,3} j(-qy^2/x,-qx^2/y;q^3)}.
\end{aligned}$$

Thus,

$$\psi(q) = 2m(-q^3,q^{12},-1) - \frac{J_{12}^3 \overline{J}_{0,4} J_{6,12}}{\overline{J}_{1,4}\overline{J}_{0,12} J_{3,12}^2}. \tag{3.4}$$

On the other hand, Hickerson and Mortenson [17, Equation (5.3)] established that

$$\mu(q) = 4m(-q,q^4,-1) - \frac{J_{2,4}^4}{J_1^3}.$$

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Applying (3.4) and the above identity, we deduce that

$$2\psi(q) - \mu(q^3) = \frac{J_{6,12}^4}{J_3^3} - \frac{2J_{12}^3\overline{J}_{0,4}J_{6,12}}{\overline{J}_{1,4}\overline{J}_{0,12}J_{3,12}^2}.$$

Simplifying the above identity gives

$$2\psi(q) - \mu(q^3) = \frac{J_6^8}{J_3^3J_{12}^4} - \frac{2J_1J_6^4J_8^2J_{12}}{J_2^2J_3^2J_4J_{24}^2}. \tag{3.5}$$

In addition, applying the standard computational techniques for modular forms, we can prove the identity

$$\frac{2\eta(z)\eta(3z)\eta^4(6z)\eta^2(8z)\eta(12z)}{\eta^2(2z)\eta(4z)\eta^2(24z)} - \frac{\eta^8(6z)}{\eta^4(12z)} = \frac{\eta^2(z)\eta^2(3z)\eta(4z)\eta^5(6z)}{\eta^3(2z)\eta^3(12z)}, \tag{3.6}$$

where $\eta(z) := q^{1/24}(q;q)_\infty$ and $q := e^{2\pi iz}$. Since (3.6) is an equality between holomorphic modular forms of weight two on $\Gamma_0(24)$ with a certain character, its truth is established by verifying that the q-expansions of both sides agree up to q^5. (Those unfamiliar with this method might consult [26].) Then by (3.5) and (3.6), we arrive at (1.8). Therefore, we complete the proof. □

In the following, we show the mock theta conjectures for $R_1(q)$ and $R_2(q)$.

PROOF OF THEOREM 1.4. According to (1.1) and (1.9),

$$\mu(q) = K(-1;q). \tag{3.7}$$

Then applying (1.6) and (3.3) yields

$$R_1(-q) = -\mu(q) + \frac{J_1^5}{J_2^4} + 1.$$

Hence, combining (3.7) and the above identity, we derive (1.10).

Finally, applying (1.7) and (3.7), we prove (1.11). □

4. Concluding remarks

Partial theta functions are sums of the form

$$\sum_{n=0}^{\infty} q^{An^2+Bn}x^n.$$

In 2014, Mortenson [25] introduced the dual nature between Appell–Lerch sums and partial theta functions. In this sense, Appell–Lerch sums and partial theta functions appear to be dual to each other. Recently, Chen [10] provided some further results.

In the same manner, replacing q by q^{-1} in the second series in (1.2) and (1.3), respectively, we obtain

$$\begin{aligned} S_1(q) &:= \sum_{n=0}^{\infty} \frac{q^{2n+1}(-q;q^2)_{n+1}}{(q;q^2)_{n+1}^2}, \\ S_2(q) &:= \sum_{n=0}^{\infty} \frac{q^{2n+1}(1+q)(q;q^2)_n}{(-q^2;q^2)_n(-q^2;q^2)_{n+1}}. \end{aligned} \tag{4.1}$$

Then we derive some identities related to partial theta functions.

THEOREM 4.1. *We have*

$$\begin{aligned} S_1(q) = &-2\sum_{n=0}^{\infty}(-1)^n q^{2n^2+3n+1}(1-q^{2n+2}) \\ &-2\frac{J_2^4}{J_1^3J_4}\sum_{n=0}^{\infty} q^{3n^2+5n+2}(1-q^{2n+2}) + \frac{J_2^4}{J_1^3J_4} - 1, \end{aligned} \tag{4.2}$$

$$S_2(q) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n^2+3n+2)/2} - \frac{J_1J_2}{J_4^2}\sum_{n=0}^{\infty} q^{3n^2+4n+1}(1-q^{4n+4}). \tag{4.3}$$

PROOF. First, Andrews [2, Theorem 1] gave that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^n(B)_n(-Abq)_n}{(-aq)_n(-bq)_n} = &-\frac{(B)_\infty(-Abq)_\infty}{a(-aq)_\infty(-bq)_\infty}\sum_{n=0}^{\infty}\frac{(1/A)_n}{(-B/a)_{n+1}}\Big(\frac{Abq}{a}\Big)^n \\ &+(1+b)\sum_{n=0}^{\infty}\frac{(-1/a)_{n+1}(-ABq/a)_n}{(-B/a)_{n+1}(Abq/a)_{n+1}}(-b)^n. \end{aligned}$$

Replacing q, A, B, a and b by q^2, -1, 0, $-q$ and $-q$, respectively, in the above identity, we derive

$$\sum_{n=0}^{\infty}\frac{q^{2n}(-q^3;q^2)_n}{(q^3;q^2)_n^2} = \frac{(-q^3;q^2)_\infty}{q(q^3;q^2)_\infty^2}\sum_{n=0}^{\infty}(-1)^n q^{2n}(-1;q^2)_n + (1-q)\sum_{n=0}^{\infty}\frac{q^n(q^{-1};q^2)_{n+1}}{(-q^2;q^2)_{n+1}}.$$

That is,

$$\begin{aligned} S_1(q) &= \frac{(-q;q^2)_\infty}{(q;q^2)_\infty^2}\sum_{n=0}^{\infty}(-1)^n q^{2n}(-1;q^2)_n - (1+q)\sum_{n=0}^{\infty}\frac{q^n(q;q^2)_n}{(-q^2;q^2)_{n+1}} \\ &= \frac{(-q;q^2)_\infty}{(q;q^2)_\infty^2}\Big(1+\sum_{n=1}^{\infty}(-1)^n q^{2n}(-1;q^2)_n\Big) - (1+q)\sum_{n=0}^{\infty}\frac{q^n(q;q^2)_n}{(-q^2;q^2)_{n+1}} \\ &= \frac{(-q;q^2)_\infty}{(q;q^2)_\infty^2}\Big(1-2\sum_{n=0}^{\infty}(-1)^n q^{2n+2}(-q^2;q^2)_n\Big) - (1+q)\sum_{n=0}^{\infty}\frac{q^n(q;q^2)_n}{(-q^2;q^2)_{n+1}} \\ &= -\frac{2(-q;q^2)_\infty}{(q;q^2)_\infty^2}\sum_{n=0}^{\infty}(-1)^n q^{2n+2}(-q^2;q^2)_n + \frac{(-q;q^2)_\infty}{(q;q^2)_\infty^2} \end{aligned}$$

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$$- (1+q)\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^n}{(-q^2;q^2)_{n+1}}. \tag{4.4}$$

Additionally, the Rogers–Fine identity [29, page 334, Equation (1)] is stated as

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\beta)_n}\tau^n = \sum_{n=0}^{\infty}\frac{\beta^n\tau^n q^{n^2-n}(1-\alpha\tau q^{2n})(\alpha)_n(\alpha\tau q/\beta)_n}{(\beta)_n(\tau)_{n+1}}. \tag{4.5}$$

Then replacing q, α, β and τ by $q^2, -q^2, 0$ and $-q^2$, respectively, in (4.5), we arrive at

$$\sum_{n=0}^{\infty}(-1)^n q^{2n+2}(-q^2;q^2)_n = \sum_{n=0}^{\infty} q^{3n^2+5n+2}(1-q^{2n+2}). \tag{4.6}$$

Furthermore, by replacing q, α, β and τ by $q^2, q, -q^4$ and q, respectively, in (4.5),

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{q^n(q;q^2)_n}{(-q^4;q^2)_n} &= \sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2+3n}(1-q^{4n+2})(q;q^2)_n(-1;q^2)_n}{(q;q^2)_{n+1}(-q^4;q^2)_n} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2+3n}(1+q^{2n+1})(-1;q^2)_n}{(-q^4;q^2)_n} \\
&= 1+q+2(1+q^2)\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n^2+3n}(1+q^{2n+1})}{(1+q^{2n})(1+q^{2n+2})} \\
&= 1+q-2(1+q^2)\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2+7n+5}(1+q^{2n+3})}{(1+q^{2n+2})(1+q^{2n+4})} \\
&= 1+q-\frac{2(1+q^2)}{1+q}\sum_{n=0}^{\infty}(-1)^n q^{2n^2+7n+5}\Big(\frac{1}{1+q^{2n+2}}+\frac{q}{1+q^{2n+4}}\Big) \\
&= 1+q-\frac{2(1+q^2)}{1+q}\sum_{n=0}^{\infty}\Big(\frac{(-1)^n q^{2n^2+7n+5}}{1+q^{2n+2}}+\frac{(-1)^n q^{2n^2+7n+6}}{1+q^{2n+4}}\Big) \\
&= 1+q-\frac{2(1+q^2)}{1+q}\Big(\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2+7n+5}}{1+q^{2n+2}}-\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n^2+3n+1}}{1+q^{2n+2}}\Big) \\
&= 1+q-\frac{2(1+q^2)}{1+q}\Big(\frac{q^5}{1+q^2}-\sum_{n=1}^{\infty}(-1)^n q^{2n^2+3n+1}(1-q^{2n+2})\Big) \\
&= 1+q-\frac{2q^5}{1+q}+\frac{2(1+q^2)}{1+q}\sum_{n=1}^{\infty}(-1)^n q^{2n^2+3n+1}(1-q^{2n+2}) \\
&= \frac{1+q^2}{1+q}+\frac{2(1+q^2)}{1+q}\sum_{n=0}^{\infty}(-1)^n q^{2n^2+3n+1}(1-q^{2n+2}).
\end{aligned} \tag{4.7}$$

Examining (4.4), (4.6) and (4.7), we deduce (4.2).

Next, Andrews and Berndt [5, entry 6.3.6] proved that

$$\Big(1+\frac{1}{a}\Big)\sum_{n=0}^{\infty}\frac{q^{2n+1}(q;q^2)_n}{(-aq;q^2)_{n+1}(-q/a;q^2)_{n+1}} = \sum_{n=0}^{\infty}(-1)^n a^n q^{n(n+1)/2} - \frac{(q;q)_\infty}{j(-aq;q^2)}\sum_{n=0}^{\infty} a^{3n}q^{3n^2+n}(1-a^2q^{4n+2}).$$

Then setting $a = q$ in the above identity yields

$$\Big(1+\frac{1}{q}\Big)\sum_{n=0}^{\infty}\frac{q^{2n+1}(q;q^2)_n}{(-q^2;q^2)_{n+1}(-1;q^2)_{n+1}} = \sum_{n=0}^{\infty}(-1)^n q^{n(n+3)/2} - \frac{(q;q)_\infty}{j(-1;q)}\sum_{n=0}^{\infty} q^{3n^2+4n}(1-q^{4n+4}),$$

which implies (4.3) by (4.1). Therefore, we finish the proof. □

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NANCY S. S. GU, Center for Combinatorics, LPMC,
Nankai University, Tianjin 300071, PR China
e-mail: gu@nankai.edu.cn

LI-JUN HAO, Center for Combinatorics, LPMC,
Nankai University, Tianjin 300071, PR China
e-mail: haolijun152@163.com