Canad. Math. Bull. Vol. **57** (1), 2014 pp. 132–140
http://dx.doi.org/10.4153/CMB-2012-013-7

Twisted Conjugacy Classes in Abelian Extensions of Certain Linear Groups

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Abstract. Given a group automorphism $\phi\colon \Gamma \to \Gamma$, one has an action of Γ on itself by ϕ-twisted conjugacy, namely, $g.x = gx\phi(g^{-1})$. The orbits of this action are called ϕ-twisted conjugacy classes. One says that Γ has the R_∞-property if there are infinitely many ϕ-twisted conjugacy classes for every automorphism ϕ of Γ. In this paper we show that $\mathrm{SL}(n,\mathbb{Z})$ and its congruence subgroups have the R_∞-property. Further we show that any (countable) abelian extension of Γ has the R_∞-property where Γ is a torsion free non-elementary hyperbolic group, or $\mathrm{SL}(n,\mathbb{Z})$, $\mathrm{Sp}(2n,\mathbb{Z})$ or a principal congruence subgroup of $\mathrm{SL}(n,\mathbb{Z})$ or the fundamental group of a complete Riemannian manifold of constant negative curvature.

1 Introduction

Let Γ be a finitely generated infinite group and let $\phi\colon \Gamma \to \Gamma$ be an endomorphism. One has an action of Γ on itself defined as $g.x = gx\phi(g^{-1})$. This is just the conjugation action when ϕ is identity. The orbits of this action are called the ϕ-twisted conjugacy classes; the ϕ-twisted conjugacy class containing $x \in \Gamma$ is denoted $[x]_\phi$ or simply $[x]$ when ϕ is clear from the context. If x and y are in the same ϕ-twisted conjugacy class, we write $x \sim_\phi y$. The set of all ϕ-twisted conjugacy classes is denoted by $\mathcal{R}(\phi)$. The cardinality $R(\phi)$ of $\mathcal{R}(\phi)$ is called the Reidemeister number of ϕ. One says that Γ has the R_∞-property for automorphisms (more briefly, R_∞-property) if there are infinitely many ϕ-twisted conjugacy classes for every automorphism ϕ of Γ. If Γ has the R_∞-property, we shall call Γ an R_∞-group. All these notions make sense for any group, not necessarily finitely generated.

The notion of twisted conjugacy originated in Nielson–Reidemeister fixed point theory and also arises in other areas of mathematics such as representation theory, number theory, and algebraic geometry. See [4] and the references therein. The problem of determining which classes of groups have R_∞-property is an area of active research initiated by Fel'shtyn and Hill [6]. We now state the main result of this paper.

Theorem 1.1 *Let Λ be an extension of a group Γ by an arbitrary countable abelian group A. Then Λ has the R_∞-property in case any one of the following holds:*

(i) Γ *is a torsion-free non-elementary hyperbolic group;*
(ii) $\Gamma = \mathrm{SL}(n,\mathbb{Z}), \mathrm{PSL}(n,\mathbb{Z}), \mathrm{GL}(n,\mathbb{Z}), \mathrm{PGL}(n,\mathbb{Z}), \mathrm{Sp}(2n,\mathbb{Z}), PSp(2n,\mathbb{Z})$, $n \geq 2$;
(iii) Γ *is a normal subgroup of* $\mathrm{SL}(n,\mathbb{Z})$, $n > 2$, *not contained in the centre;*
(iv) Γ *is the fundamental group of a complete Riemannian manifold of constant negative sectional curvature and finite volume.*

Received by the editors September 30, 2011.
Published electronically May 17, 2012.
AMS subject classification: **20E45**.
Keywords: twisted conjugacy classes, hyperbolic groups, lattices in Lie groups.

Our proofs involve straightforward arguments, using well-known results concerning the group Γ in each case. More precisely, in each of the cases, we show that A or a bigger subgroup $N \subset \Lambda$ in which A has finite index is *characteristic* in Λ. Proof of this requires some facts concerning normal subgroups of Γ. In the cases (ii), (iii), and (iv) we invoke the normal subgroup theorem of Margulis [21, Chapter 8]; in case (i) we use the quasi-convexity property of infinite cyclic subgroups of Γ. Using the fact that Γ is hopfian, the R_∞-property for Λ is then deduced from the R_∞-property for Γ. That Γ has the R_∞-property when it is a torsion-free non-elementary hyperbolic group is due to [9]. This result was extended to arbitrary non-elementary hyperbolic groups by [3]. The R_∞-property for the groups $\mathrm{Sp}(2n,\mathbb{Z})$ was established by Fel'shtyn and Gonçalves [5]. The R_∞-property for $\mathrm{SL}(n,\mathbb{Z})$ and $\mathrm{PGL}(n,\mathbb{Z})$ is established in Section 3. We show, in Section 3, the R_∞-property for non-central normal subgroups of $\mathrm{SL}(n,\mathbb{Z}), n > 2$, using the Mostow–Margulis strong rigidity theorem and the congruence subgroup property of $\mathrm{SL}(n,\mathbb{Z})$. The proof of the main theorem is given in Section 4.

2 Preliminaries

Let G be a group and H a subgroup of G. Recall that a subgroup H is said to be *characteristic* in G if $\phi(H) = H$ for every automorphism ϕ of G. We will call G *hopfian* (resp. *co-hopfian*) if every surjective (resp. injective) endomorphism of G is an automorphism of G. One says that G is *residually finite* if, given any $g \in G$, there exists a finite index subgroup H in G such that $g \notin H$.

We shall recall here some facts concerning the R_∞-property. Let

$$1 \longrightarrow N \overset{j}{\hookrightarrow} \Lambda \overset{\eta}{\longrightarrow} \Gamma \longrightarrow 1 \tag{2.1}$$

be an exact sequence of groups.

Lemma 2.1 *Suppose that N is characteristic in Λ and that Γ has the R_∞-property, then Λ also has the R_∞-property.*

Proof Let $\phi\colon \Lambda \to \Lambda$ be any automorphism. Since N is characteristic, $\phi(N) = N$ and so ϕ induces an automorphism $\bar{\phi}\colon \Gamma \to \Gamma$. Since $R(\bar{\phi}) = \infty$, it follows that $R(\phi) = \infty$. ■

Lemma 2.2 *Suppose that N is a characteristic subgroup of Λ.*

(i) *If N is finite and Λ has the R_∞-property, then Γ also has the R_∞ -property.*
(ii) *If Γ is finite and N has the R_∞-property, then Λ has the R_∞- property.*

Proof (i) Any automorphism $\phi\colon \Lambda \to \Lambda$ maps N isomorphically onto itself and hence induces an automorphism $\bar{\phi}\colon \Gamma \to \Gamma$ (where $\Gamma = \Lambda/N$).

It is readily seen that $x \sim_\phi y$ implies $\eta(x) \sim_{\bar{\phi}} \eta(y)$ for any $x, y \in \Lambda$. Therefore η induces a surjection $\widetilde{\eta}\colon \mathcal{R}(\phi) \to \mathcal{R}(\bar{\phi})$ where $\widetilde{\eta}([x]_\phi) = [\eta(x)]_{\bar{\phi}}$. We need only show that the fibres of $\widetilde{\eta}$ are finite.

Suppose the contrary and let $x_k \in \Lambda, k \geq 0$, be such that $[x_k]_\phi \neq [x_l]_\phi$ for $k \neq l$ and that $[\eta(x_k)]_{\bar{\phi}} = [\eta(x_0)]_{\bar{\phi}}$. For each $k \geq 1$, there exists $g_k \in \Lambda$ such that

$$\eta(x_0) = \eta(g_k)\eta(x_k)\bar{\phi}\big(\eta(g_k^{-1})\big) = \eta\big(g_k x_k \phi(g_k^{-1})\big).$$

https://doi.org/10.4153/CMB-2012-013-7 Published online by Cambridge University Press

Therefore there exists an $h_k \in N$ such that $x_0 h_k = g_k x_k \phi(g_k)^{-1}$. That is, for any $k \geq 1$, we have $x_k \sim_\phi x_0 h_k$ for some $h_k \in N$. Since N is finite, it follows that $x_k \sim_\phi x_l$ for some $k \neq l$, a contradiction.

(ii) Let $\phi\colon \Lambda \to \Lambda$ be an automorphism and let $\theta = \phi|N$. Let $\widetilde{j}\colon \mathcal{R}(\theta) \to \mathcal{R}(\phi)$ be the map defined as $[x]_\theta \mapsto [x]_\phi$. Suppose that $R(\phi) < \infty$ but that $R(\theta) = \infty$. Then there exist elements $x_k \in N, k \geq 0$, such that $[x_k]_\theta \neq [x_l]_\theta, k \neq l$, but $x_k \sim_\phi x_0$ for all $k \geq 0$. Choose $g_k \in \Lambda$ such that $x_k = g_k x_0 \phi(g_k^{-1}), k \geq 1$. Since $\Gamma = \Lambda/N$ is finite, there exist distinct positive integers k, l such that $h := g_k g_l^{-1} \in N$. Now

$$x_k = g_k x_0 \phi(g_k^{-1}) = g_k g_l^{-1} x_l \phi(g_l) \phi(g_k^{-1}) = h x_l \theta(h^{-1}),$$

and so $[x_k]_\theta = [x_l]_\theta$, a contradiction. This completes the proof. ■

Lemma 2.3 *Suppose that there is no non-trivial homomorphism from N to Γ and that either Γ is hopfian or N is co-hopfian. If Γ has the R_∞-property, then so does Λ.*

Proof Let $\phi\colon \Lambda \to \Lambda$ be any automorphism. Consider the homomorphism $f\colon N \to \Gamma$ defined as $f = \eta \circ \phi|N$, where $\eta\colon \Lambda \to \Gamma$ is the quotient map as in (2.1). By our hypothesis f is trivial, and so it follows that $\phi(N) \subset \ker(\eta) = N$. If N is co-hopfian then $\phi(N) = N$ and so N is characteristic. In any case ϕ defines a homomorphism $\bar{\phi}\colon \Gamma \to \Gamma$, where $\bar{\phi}(xN) = \phi(x)N,\ x \in \Lambda$. It is clear that $\bar{\phi}$ is surjective with kernel $\phi^{-1}(N)/N$. If Γ is hopfian, $\bar{\phi}$ is an isomorphism and it follows that $\phi(N) = N$. Thus our hypothesis implies that N is characteristic in Λ, and the lemma now follows from Lemma 2.1. ■

We conclude this section with the following observation.

Proposition 2.4 *Let Γ be a countably infinite residually finite group. Then $R(\phi) = \infty$ for any inner automorphism ϕ of Γ.*

Proof Let $\phi = \iota_\gamma$ and let $x \sim_\phi y$. Thus $y = gx\gamma g^{-1}\gamma^{-1}$. Equivalently $x\gamma$ is conjugate to $y\gamma$. Hence it suffices to show that Γ has infinitely many conjugacy classes.

Since Γ is infinite and since Γ is residually finite, there exist finite quotients $\bar{\Gamma}$ of Γ having *arbitrarily large* (finite) order. It is a classical result of R. Brauer [1] (see also [15]) that the number of conjugacy classes of a finite group of order n is bounded below by $\log\log n$. Since Γ has at least as many conjugacy classes as any of its quotients, it follows that Γ has infinitely many conjugacy classes. ■

Remark 2.5 (i) Recall that finitely generated residually finite groups are hopfian. A well-known class of residually finite groups is the class of finitely generated subgroups of $\mathrm{GL}(n, K)$ where K is any field. See [10]. This class includes, in particular, all lattices in linear Lie groups. An important unsolved problem is whether or not hyperbolic groups are residually finite. It has been shown by Sela [18] that torsion-free hyperbolic groups are hopfian.

(ii) It is known that there are countably infinite groups with only finitely many conjugacy classes. (See [19, §1.4] or [10, Chapter 4, §3].) Finitely generated examples have been constructed by S. Ivanov. Recently D. Osin [14] has constructed a finitely generated infinite group that has exactly one non-trivial conjugacy class.

3 The R_∞-property for Special Linear Groups

In this section we give new examples of R_∞-groups that are subgroups of finite index in $\mathrm{SL}(n,\mathbb{Z})$. Recall that $\mathrm{Sp}(2n;\mathbb{Z})$ has been shown to have the R_∞-property by Fel'shtyn and Gonçalves [5].

Our first result is the following theorem.

Theorem 3.1 *The groups* $\mathrm{SL}(n,\mathbb{Z}), \mathrm{PSL}(n,\mathbb{Z}), \mathrm{GL}(n,\mathbb{Z})$*, and* $\mathrm{PGL}(n,\mathbb{Z})$ *have the* R_∞*-property for all* $n \geq 2$.

Proof It follows from Lemma 2.2(ii) that the R_∞-property for $\mathrm{SL}(n,\mathbb{Z})$ implies the R_∞ property for $\mathrm{GL}(n,\mathbb{Z})$. Also the R_∞-property for $\mathrm{SL}(n,\mathbb{Z})$ (resp. $\mathrm{GL}(n,\mathbb{Z})$) implies the R_∞ for $\mathrm{PSL}(n,\mathbb{Z})$ (resp. $\mathrm{PGL}(n,\mathbb{Z})$) in view of Lemma 2.2(i). Therefore we need only prove the theorem for $\mathrm{SL}(n,\mathbb{Z})$.

The group $\mathrm{SL}(2,\mathbb{Z})$ is non-elementary hyperbolic group and hence, by [3], has the R_∞-property. Let $n \geq 3$ and set $\Gamma := \mathrm{SL}(n,\mathbb{Z})$. Since Γ is residually finite, $R(\phi) = \infty$ for any inner automorphism by Proposition 2.4. In this case we can see this more directly: the set $\{tr(A) \mid A \in \Gamma\}$ is infinite and so there are infinitely many conjugacy classes in Γ.

It remains only to show that $\mathcal{R}(\phi)$ is infinite for a set of representatives of (the non-trivial) elements of the group $\mathrm{Out}(\Gamma)$ of all outer automorphisms of Γ. It is known from the work of Hua and Reiner [8] and of O'Meara [13] that $\mathrm{Out}(\Gamma) \cong \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ according as n is odd or even.

The group $\mathrm{Out}(\Gamma)$ is generated by a set S where $S = \{\tau\}$ when n is odd and when n is even, $S = \{\sigma, \tau\}$, where $\tau\colon \Gamma \to \Gamma$ is defined as $X \mapsto {}^tX^{-1}$, and, when n is even, the involution $\sigma\colon \Gamma \to \Gamma$ is defined as $X \mapsto JXJ^{-1} = JXJ$ where J is the diagonal matrix $\mathrm{diag}(1,\dots,1,-1)$. Thus $X \sim_\tau Y$ (resp. $X \sim_\sigma Y$) if and only if there exists a Z such that $Y = ZX({}^tZ)$ (resp. $Y = ZXJZ^{-1}J$)).

First we consider τ-twisted conjugacy classes. Let $k \geq 1$ and let $A(k)$ be the block diagonal matrix $A(k) = \mathrm{diag}(B(k), I_{n-2})$, where $B(k) = \left(\begin{smallmatrix} 1 & 0 \\ k & 1 \end{smallmatrix}\right)$. We shall show that $A(k) \sim_\tau A(l)$ implies $k = l$. This clearly implies that $R(\tau) = \infty$.

Let $X = (x_{ij}) \in \Gamma$ be such that

$$X.A(k).{}^tX = A(l). \tag{3.1}$$

We shall denote the i-th row and i-th column of X by r_i and c_i respectively. A straightforward computation shows that $X.A(k).{}^tX = X.{}^tX + kc_2.{}^tc_1$. Comparing the $(2,1)$-entries on both sides of (3.1) we get $r_2.{}^tr_1 + kx_{22}x_{11} = l$, whereas comparing the $(1,2)$-entries gives $r_1.{}^tr_2 + kx_{12}x_{21} = 0$. Therefore $r_2.{}^tr_1 = r_1.{}^tr_2 = -kx_{12}x_{21}$ and so $l = k(x_{11}x_{22} - x_{12}x_{21})$. Since $x_{i,j} \in \mathbb{Z}$, we obtain that $k|l$. Interchanging the roles of k, l we get $l|k$, and so we must have $k = l$, since $k, l \geq 1$.

Now consider σ-twisted conjugacy classes. Since $A \sim_\sigma B$ if and only if $AJ = X(BJ)X^{-1}$ for some $X \in \mathrm{SL}(n,\mathbb{Z})$. We need only show that the set $\{\mathrm{tr}(AJ) \mid A \in \mathrm{SL}(n,\mathbb{Z})\}$ is infinite. Let $A' \in \mathrm{SL}(n-1,\mathbb{Z})$ and let $A = \mathrm{diag}(A', 1)$, where $A' \in \mathrm{SL}(n-1,\mathbb{Z})$. Then $AJ = \mathrm{diag}(A', -1)$. Therefore $\mathrm{tr}(A) = \mathrm{tr}(A') - 1$. Since $n > 2$, the set $\{\mathrm{tr}(A') \mid A' \in \mathrm{SL}(n-1,\mathbb{Z})\}$ is infinite, and we conclude that $R(\sigma) = \infty$.

The proof of that $R(\sigma\tau) = \infty$ is similar, so we omit the details. ∎

It is possible to give a more direct proof of the R_∞-property for $\mathrm{SL}(2,\mathbb{Z})$ as for $\mathrm{SL}(n,\mathbb{Z}), n > 2$, given above, using the description of the (outer) automorphism group of $\mathrm{SL}(2,\mathbb{Z})$ given in [8, Theorem 2].

It is known that the R_∞-property is not inherited by finite index subgroups in general. For example, the infinite dihedral group, which contains the infinite cyclic group as an index 2 subgroup, has the R_∞ -property (whereas $R(-id_{\mathbb{Z}}) = 2$). (See [7].) However we have the following result.

Let Γ_m denote the principal level m congruence subgroup of $\mathrm{SL}(n,\mathbb{Z})$; thus Γ_m is the kernel of the surjection $\mathrm{SL}(n,\mathbb{Z}) \to \mathrm{SL}(n,\mathbb{Z}/m\mathbb{Z})$ induced by the surjection $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$.

Theorem 3.2 *Let $n \geq 3$. Let Λ be a non-central normal subgroup of* $\mathrm{SL}(n,\mathbb{Z})$. *Then Λ has the R_∞-property.*

Proof Let $\Gamma = \mathrm{SL}(n,\mathbb{Z})$. We shall use the notations introduced in the proof of Theorem 3.1. It is known that any finite index subgroup of $\mathrm{SL}(n,\mathbb{Z})$ contains Γ_m for some $m \geq 2$. This is the congruence subgroup property for $\mathrm{SL}(n,\mathbb{Z}), n > 2$. See [20, §4.4].

Let $M = (m_{i,j}) \in \mathrm{SL}(n,\mathbb{Z})$ and let $\phi := \phi_M$ be the restriction to Λ of the inner automorphism ι_M of Γ. Then $X \sim_\phi Y$ if and only if $XM = Z(YM)Z^{-1}$ for some $Z \in \Lambda$. In particular $\mathrm{tr}(XM) = \mathrm{tr}(YM)$. To show that $R(\phi) = \infty$ we need only show that the set $\{\mathrm{tr}(AM) \mid A \in \Lambda\}$ is infinite for any $M \in SL(n,\mathbb{Z})$. There are two cases to consider: (1) $m_{ii} \neq 0$ for some i, (2) $m_{ii} = 0$ for all i.

Case (1): Without loss of generality we may assume that $m_{11} \neq 0$. Let $k > 1$ and let $X(k)$ be the block diagonal matrix $X(k) = \mathrm{diag}(C(k), I_{n-2})$ where $C(k) = \begin{pmatrix} k^2+1 & k \\ k & 1 \end{pmatrix}$. A straightforward computation shows that

$$\mathrm{tr}(X(k)M) = (k^2+1)m_{11} + k(m_{12} + m_{21}) + \sum_{2 \leq j \leq n} m_{jj}.$$

Therefore $\mathrm{tr}(X(k)M) = \mathrm{tr}(X(l)M)$ if and only if $(k+l)m_{11} + m_{12} + m_{21} = 0$. Choose $k_0 > (m_{12} + m_{21})/m_{11}$. Then $X(mk), k \geq k_0$, belong to pairwise distinct ϕ-twisted conjugacy classes in this case.

Case (2): Without loss of generality assume that $m_{12} \neq 0$. Let $A(k)$ be as in the proof of Theorem 3.1. Then $\mathrm{tr}(A(k)M) = km_{1,2}$. Therefore $\mathrm{tr}(A(k)M) = \mathrm{tr}(A(l)M)$ if and only if $k = l$. Since $A(mk) \in \Gamma_m \subset \Lambda$ for all k, it follows that $R(\phi) = \infty$ in this case as well.

Suppose that $\tau(\Lambda) = \Lambda$, where $\tau(X) = {}^tX^{-1}$ as in the proof of Theorem 3.1. We see that $R(\tau|\Lambda) = \infty$ arguing as we did to establish that $R(\tau) = \infty$ in the proof of Theorem 3.1 by considering the set of elements $A(mk) \in \Gamma_m \subset \Lambda, k \geq 1$. Similarly, we show that $R(\theta|\Lambda) = \infty$ for each representative θ of the outer automorphisms of Γ, which leaves Λ invariant.

To complete the proof, we need only show that every automorphism of Λ extends to an automorphism of Γ. For this purpose we observe that the $\mathbb{R}$-rank of the semi simple Lie group $G := \mathrm{SL}(n,\mathbb{R})$ equals $n-1 \geq 2$. Let $\theta\colon \Lambda \to \Lambda$ be any automorphism. By the Mostow–Margulis strong rigidity theorem [21, Chapter 5], θ extends to an automorphism $\widetilde{\theta}\colon \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$. By a result of Newman [12, Lemma 2] we have $N_G(\Lambda) = \Gamma$. So $\widetilde{\theta}$ restricts to an automorphism $\bar{\theta}$ of Γ. Thus θ is the restriction of an automorphism of Γ, namely $\bar{\theta}$. This completes the proof. ■

Remark 3.3 Recall that Fel'shtyn and Gonçalves [5] have shown that $\mathrm{Sp}(2n,\mathbb{Z})$ has the R_∞- property. One could also establish this result along the same lines as for $\mathrm{SL}(n,\mathbb{Z})$ given above. We assume that $n \geq 2$ as $\mathrm{Sp}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})$. To fix notations, regard $\mathrm{Sp}(2n,\mathbb{Z})$ as the subgroup of $\mathrm{SL}(2n,\mathbb{Z})$ that preserves the skew symmetric form $\beta\colon \mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \to \mathbb{Z}$ defined as

$$\beta(e_{2i}, e_{2j}) = 0 = \beta(e_{2i-1}, e_{2j-1}),\ \beta(e_{2i-1}, e_{2j}) = \delta_{ij}, 1 \leq i \leq j \leq n$$

(Kronecker δ). Equivalently $\mathrm{Sp}(2n,\mathbb{Z}) = \{X \in \mathrm{SL}(2n,\mathbb{Z}) \mid {}^tXJ_0X = J_0\}$, where $J_0 = \mathrm{diag}(j_0,\dots,j_0)$, $j_0 := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is the matrix of β. Infinitude of (untwisted) conjugacy classes follows from the residual finiteness of $\mathrm{Sp}(2n,\mathbb{Z})$. (*cf.* Proposition 2.4). Alternatively, observe that $X(k) \in \mathrm{Sp}(2n,\mathbb{Z})$, where $X(k)$ is as in the proof of Theorem 3.2. This shows that the trace function is unbounded on $\mathrm{Sp}(2n,\mathbb{Z})$.

To complete the proof, we need only verify that $R(\phi) = \infty$ for representatives of the elements of $\mathrm{Out}(\mathrm{Sp}(2n,\mathbb{Z}))$. One knows from [17] that the outer automorphism group of $\mathrm{Sp}(2n,\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if $n > 2$ and is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ when $n = 2$.

The generators of the outer automorphism groups may be described as follows. Let θ be the automorphism of $\mathrm{Sp}(2n,\mathbb{Z})$, which is conjugation by

$$J := \mathrm{diag}(I', I_{2n-2}) \in \mathrm{GL}(2n,\mathbb{Z}),$$

where $I' = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Let ϕ be the automorphism of $\mathrm{Sp}(4,\mathbb{Z})$ defined as $\phi(X) = \chi(X)X$, where $\chi\colon \mathrm{Sp}(4,\mathbb{Z}) \to \{1,-1\}$ is the (non-trivial) central character. Then $\mathrm{Out}(\mathrm{Sp}(2n,\mathbb{Z})) = \langle\theta\rangle$, $n > 2$, and $\mathrm{Out}(\mathrm{Sp}(4,\mathbb{Z})) = \langle\theta,\phi\rangle$.

To see that $R(\theta) = \infty$ we note that $\mathrm{tr}(X(k)J) = 2k + (2n-2)$. Therefore the $X(k)$, $k \geq 1$, belong to pairwise distinct θ-twisted conjugacy classes.

As already observed in [5, Lemma 3.1], any ϕ-twisted conjugacy class of X is a union of the (untwisted) conjugacy class of X and of $-X$. Since the number of conjugacy classes in $\mathrm{Sp}(4,\mathbb{Z})$ is infinite, it follows that $R(\phi) = \infty$. Proof that $R(\theta\phi) = \infty$ is similar and thus omitted. This completes the proof.

It is an interesting problem to determine which (irreducible) lattices in semi simple Lie groups have the R_∞-property. We shall address this question in a sequel to this paper.

4 Proof of the Main Theorem

We now proceed to the proof of the main theorem. Let $j\colon A \hookrightarrow \Lambda$ be the inclusion and $\eta\colon \Lambda \to \Gamma$ the canonical quotient map so that $1 \to A \hookrightarrow \Lambda \to \Gamma \to 1$ is an exact sequence of groups.

Proof of Theorem 1.1 Let $\phi\colon \Lambda \to \Lambda$ be any automorphism and let $f\colon A \to \Gamma$ be the composition $\eta \circ \phi \circ j$. Note that since A is normal in Λ, $\phi(A)$ is normal in Λ and hence $f(A)$ is normal in Γ.

(i) In this case we claim that f is trivial. Suppose that $f(A)$ is not the trivial subgroup. Since Γ is a non-elementary group, it does not contain a free abelian group of rank 2. Since Γ is torsion free, the centralizer of any non-trivial element of Γ is infinite cyclic. By [2, Corollary 3.10, Chapter III.Γ] $f(A)$ is quasi-convex. Hence by [2, Proposition 3.16, Chapter III.Γ] the subgroup $f(A)$ has finite index in its normalizer, which is Γ. This contradicts the assumption that Γ is non-elementary. Therefore $f(A)$ must be trivial. This means that $\phi(A) \subset A$, and we have the following diagram, in which the top horizontal sequence is exact:

$$\begin{array}{rcccc} A & \hookrightarrow & \Lambda & \longrightarrow & \Gamma \\ \phi|A \downarrow & & \phi \downarrow & & \downarrow \bar{\phi} \\ A & \hookrightarrow & \Lambda & \longrightarrow & \Gamma. \end{array}$$

Now $\bar{\phi}$ is a surjection since $\eta \circ \phi$ is. Since Γ is assumed to be torsion-free, by Sela's theorem [18], Γ is hopfian and so $\bar{\phi}$ is an isomorphism. Therefore $\phi(A) = A$. Hence A is characteristic in Λ. Since Γ has the R_∞-property by [9] (*cf.* [3]), Lemma 2.1 now implies that Λ has the R_∞-property.

(ii) The group Γ is a lattice in one of the simple linear Lie groups

$$G = \mathrm{SL}(n,\mathbb{R}), \quad \mathrm{PGL}(n,\mathbb{R}), \quad \mathrm{Sp}(2n,\mathbb{R}), \quad \mathrm{PSp}(2n,\mathbb{R}).$$

These Lie groups have as centre a group of order at most 2. Also, Γ is hopfian. First we consider the case

$$\Gamma = \mathrm{SL}(n,\mathbb{Z}), \mathrm{PSL}(n,\mathbb{Z}), \mathrm{PGL}(n,\mathbb{Z}), n > 2, \quad \text{or} \quad \mathrm{Sp}(2n,\mathbb{Z}), \mathrm{PSp}(2n,\mathbb{Z}), n > 1,$$

so that the corresponding Lie group G has real rank at least 2. By the normal subgroup theorem of Margulis [21, Chapter 8], the subgroup $f(A)$ being normal in Γ is either of finite index or is contained in the centre of G. Since A is abelian, $f(A)$ cannot be of finite index in Γ. Hence $f(A) \subset Z(\Gamma)$ the centre of Γ which is of order at most 2. First assume that $f(A)$ is trivial. Then we have $\phi(A) \subset A$. Using the fact that Γ is hopfian, we conclude as above, that A is characteristic. Now Γ has the R_∞-property by Theorem 3.1 in the case of $\mathrm{SL}(n,\mathbb{Z}), \mathrm{PSL}(n,\mathbb{Z}), \mathrm{PGL}(n,\mathbb{Z})$ and by the work of Fel'shtyn-Gonçalves [5] in the case of $\mathrm{Sp}(2n,\mathbb{Z}), \mathrm{PSp}(2n,\mathbb{Z})$ (*cf.* Lemma 2.2(i)). It follows as in case (i) that Λ also has the R_∞-property. Now assume that $f(A) = Z(\Gamma) \cong \mathbb{Z}/2\mathbb{Z}$. Set $\bar{\Gamma} = \Gamma/Z(\Gamma)$, which is the lattice $\mathrm{PSL}(n,\mathbb{Z})$ or $\mathrm{PSp}(2n,\mathbb{Z})$ in the corresponding Lie group of adjoint type. Let $N = \eta^{-1}(Z(\Gamma))$. Clearly $N/A \cong Z(\Gamma)$. Now we have the exact sequence

$$N \overset{\tilde{j}}{\hookrightarrow} \Lambda \xrightarrow{\bar{\eta}} \bar{\Gamma},$$

where $\bar{\eta}$ is the canonical quotient map. Now we claim that N is characteristic. Indeed, let $\tilde{f}\colon N \to \bar{\Gamma}$ be defined as $\bar{\eta}\circ\phi\circ\tilde{j}$. Again using Margulis' normal subgroup theorem, the fact that N is virtually abelian forces $\widetilde{f}(N)$ to be contained in the centre of $\bar{\Gamma}$. Since $\bar{\Gamma}$ has trivial centre, we must have $\widetilde{f}(N) \subset N$. Now $\bar{\Gamma}$ is again hopfian (being finitely generated and linear). As before, we conclude that N is characteristic. By Lemma 2.1 applied to $\bar{\Gamma}$ we conclude that Λ has the R_∞-property.

We now consider the case $\mathrm{SL}(2,\mathbb{Z}) \cong \mathrm{Sp}(2,\mathbb{Z})$. Proceeding as above we see that $f(A)$ is a normal abelian subgroup of $\mathrm{SL}(2,\mathbb{Z})$. We need only show that $f(A) \subset \{I, -I\}$. Let $F \subset \mathrm{SL}(2,\mathbb{Z})$ be a free group of finite index that is normal. Then $F \cap f(A)$ is trivial, since any normal subgroup of F is a non-abelian free group. Hence $f(A)$ is finite as it imbeds in the finite group $\mathrm{SL}(2,\mathbb{Z})/F$. Let C be the image of $f(A)$ in $\mathrm{PSL}(2,\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ under the natural quotient map. Any element of finite order is conjugate to the generator of $\mathrm{PSL}(2,\mathbb{Z})$ of order 2 or that of order 3 (see [10, Theorem 2.7, Chapter IV]). Since C is normal and finite, it follows easily that C is trivial. Hence $f(A) \subset \{I, -I\}$.

(iii) By Theorem 3.2 the R_∞-property holds for Γ. The rest of the proof is as in case (ii) above and hence omitted.

(iv) If M is compact, then Γ is a torsion-free hyperbolic group, and our statement follows from part (i). In any case, Γ is a lattice in G, the group of orientation preserving isometries of the universal cover of M. Thus G is a simple Lie group with trivial centre and real rank 1. In particular, G is linear and so Γ is residually finite. Indeed G is the identity component of the real points $\mathbf{G}_\mathbb{R}$ of the complex linear algebraic group $\mathbf{G}$ of adjoint type whose Lie algebra equals $\mathrm{Lie}(G) \otimes_\mathbb{R} \mathbb{C}$.

If M is non-compact, then Γ is relatively hyperbolic (with respect to the family of stabilizers of the cusps of M). Fel'shtyn [4, Theorem 3.3] has established the R_∞-property for such groups Γ.

Next we show that $f(A)$ is trivial. Let $Z \subset \mathbf{G}_\mathbb{R}$ be the Zariski closure of $f(A)$ and let H be the normalizer of Z in $\mathbf{G}_\mathbb{R}$. Then H is an algebraic subgroup that contains Γ. Since Γ is Zariski dense in $\mathbf{G}_\mathbb{R}$ by the Borel density theorem [16], it follows that $H = \mathbf{G}_\mathbb{R}$ and so Z is normal in $\mathbf{G}_\mathbb{R}$. Since Z is abelian and since G is simple, it follows that Z is finite and is contained in the centre of $\mathbf{G}_\mathbb{R}$. Therefore $f(A)$ equals $Z \cap G$ and is contained in the centre of G. Since the centre of G is trivial, we conclude that $f(A) = \{1\}$. The rest of the proof is as in the previous cases above. ■

We conclude this paper with the following remarks.

Remark 4.1 (i) Theorem 1.1 contains as special cases the direct product $A \times \Gamma$ as well as the the restricted wreath product $C \wr \Gamma = (\oplus_{\gamma\in\Gamma} C_\gamma) \ltimes \Gamma$, where $C_\gamma = C$ is any cyclic group.

(ii) Let P be any set of primes containing 2; thus any homomorphism $A(P) \to \mathbb{Z}/2\mathbb{Z}$ is trivial. Let $A(P) = \mathbb{Z}[1/p \mid p \in P] \subset \mathbb{Q}$. Note that $i\colon A(P) \to A(Q)$ is any non-trivial homomorphism, then $P \subset Q$. Set $\Lambda(P) := A(P) \wr \Gamma$, where Γ is as in Theorem 1.1. Suppose that $\theta\colon \Lambda(P) \to \Lambda(Q)$ is an isomorphism. Then, as in the proof of Theorem 1.1, the composition

$$\oplus_{\gamma\in\Gamma} A(P) \hookrightarrow \Lambda(P) \xrightarrow{\theta} \Lambda(Q) \longrightarrow \Gamma$$

is trivial. It follows that $\theta(\oplus_{\gamma\in\Gamma} A(P)) \subset \oplus_{\gamma\in\Gamma} A(Q)$ and so $P \subset Q$. Similarly $Q \subset P$ and so $P = Q$. It follows that there are $2^{\aleph_0}$ many pairwise non-isomorphic count-

https://doi.org/10.4153/CMB-2012-013-7 Published online by Cambridge University Press

able groups Λ satisfying the R_∞-property for each Γ as in Theorem 1.1. The same conclusion can also be arrived at by considering the groups $A(P) \times \Gamma$.

Note Recently T. Nasybullov [11] established the R_∞-property for $\mathrm{SL}(n, K)$ and $GL(n, K)$, where K is any infinite integral domain such that either (i) characteristic of K is zero and $\mathrm{Aut}(K)$ is torsion, or, (ii) K has arbitrary characteristic and $\mathrm{Aut}(K)$ is the trivial group.

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