K. Yamamoto
Nagoya Math. J.
Vol. 116 (1989), 25–42

ELASTIC WAVES IN TWO SOLIDS AS PROPAGATION OF SINGULARITIES PHENOMENON

KAZUHIRO YAMAMOTO

In this paper we shall study elastic waves in two isotropic media with different densities and Lamè's constants. In seismology elastic waves are studied when the border of two media is a hyperplane, however there are no results on elastic waves in the non-flat border case. First in Section 1 we shall show an existence theorem of the solutions of an initial boundary value problem which is satisfied by the displacements of two media. Next we shall discuss about propagation of singularities of the solutions, for Hörmander and Lax-Nirenberg showed that an appearance of propagation of singularities is similar to one of propagation of waves.

As first part on singularities in Section 3 we shall show existence of Stoneley waves as propagation of singularities, which is explained as follows: if the boundary values of the initial data of the solutions have singularities, then there exist singularities of the solutions which start from the singularities of the initial data and propagate in the elliptic region of the border of two media according to the passage of time.

Second part on singularities is to study relations between incident waves and reflected and refracted waves, which are stated in Sections 4, 5. Under various conditions on the incident angle of singularities corresponding to the fast waves or the slow waves we have interesting refractive phenomena. For example if we assume that densities and Lamè's constants satisfy some conditions, and that the incident angle of singularities corresponding to the fast waves is not sharp, then the solutions have only refracted singularities corresponding to the slow waves (see Theorem 4.1). If the singularities corresponding to the slow waves make incidence, the solutions have both refracted singularities corresponding to the fast and slow waves or they have only refracted singularities cor-

Received September 22, 1987.

responding to the slow waves (see Theorems 5.1, 5.2).

§ 1. Existence of solutions

In this section we shall show existence of the solutions of initial boundary value problems in two solids. We suppose that there exist two simply connected open subsets O_i $(i = 1, 2)$ of R^3 with smooth boundaries ∂O_i such that ∂O_i is compact, or ∂O_i is equal to a hyperplane in $\{x: |x| > R\}$ for some positive constant R. Then domains Ω_1, Ω_2 of two solids are defined as follows; $\Omega_1 = O_1 \cap O_2$ and $\Omega_2 = O_1 \setminus \overline{O}_2$. The strain tensors $\varepsilon_{jk}(u)$ $(j, k = 1, 2, 3)$ are $(\partial u_j/\partial x_k + \partial u_k/\partial x_j)/2$ and in the stress tensors $\sigma_{jk}^{(i)}(u)$ are $\lambda_i(\operatorname{div} u)\delta_{jk} + 2\mu_i\varepsilon_{jk}(u)$, where $u = {}^t(u_1, u_2, u_3)$ and λ_i and μ_i are Lamè constants such that μ_i, $3\lambda_i + 2\mu_i$ and $\lambda_i + \mu_i$ are positive. If the solids are isotropic, then the displacements $u_i(x, t) = {}^t(u_{i1}, u_{i2}, u_{i3})$ in Ω_i satisfy the following boundary value problem:

$$\rho_i\partial^2 u_{ij}/\partial t^2 - \sum_{k=1}^{3} \partial(\sigma_{jk}^{(i)}(u_i))/\partial x_k = 0 \qquad \text{in } \Omega_i \times R, \tag{1.1}$$

$$u_1 = u_2 \qquad \text{on } \Gamma \times R, \tag{1.2}$$

$$\sum_j n_j(x)\sigma_{jk}^{(1)}(u_1) = \sum_j n_j(x)\sigma_{jk}^{(2)}(u_2) \qquad \text{on } \Gamma \times R, \tag{1.3}$$

$$\sum_j n_j(x)\sigma_{jk}^{(i)}(u_i) = 0 \qquad \text{on } \Gamma_i \times R, \tag{1.4}$$

where $\rho_i > 0$ is the density of Ω_i, $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$, $\Gamma_i = \partial\Omega_i \setminus \Gamma$, and $n(x) = {}^t(n_1(x), n_2(x), n_3(x))$ is the unit normal vector of Γ or Γ_j $(j = 1, 2)$.

We consider an initial boundary value problem (1.1) to (1.4) with data

$$u_i(x, 0) = f_i(x), \qquad (\partial u_i/\partial t)(x, 0) = g_i(x) \qquad \text{in } \Omega_i. \tag{1.5}$$

We introduce a Hilbert space $\mathscr{H}$ whose elements are equal to $(L^2(O_1))^3$, with the inner product $(f, g)_{\mathscr{H}} = \rho_1(f, g)_{L^2(\Omega_2)} + \rho_2(f, g)_{L^2(\Omega_2)}$, and a subspace D of $\mathscr{H}$ such that f belongs to D if $f \in (H^1(O_1))^3$ and the distributions $L_i(f) = (\sum_k \partial(\sigma_{jk}^{(i)}(f))/\partial x_k)|_{\Omega_i})_{j=1,2,3}$ belong to $(L^2(\Omega_i))^3$. The boundary conditions (1.3) and (1.4) are represented as follows:

$$\begin{aligned}\int_{\Omega_1} \{L_1(f)\cdot v + \sum_{j,k} \sigma_{kj}^{(1)}(f)\partial v_k/\partial x_j\}dx \\ + \int_{\Omega_2} \{L_2(f)\cdot v + \sum_{j,k} \sigma_{kj}^{(2)}(f)\partial v_k/\partial x_j\}dx = 0\end{aligned} \tag{1.6}$$

for any $v = {}^t(v_1, v_2, v_3) \in (H^1(O_1))^3$. The operator A on $D_N = \{f \in D: f$ satisfies (1.6)$\}$ is defined by $Af|_{\Omega_i} = -L_i(f)/\rho_i$. We have the following

THEOREM 1.1. *A is a self-adjoint operator on* $\mathscr{H}$.

Proof. Since a set of $C_0^\infty(O_1)$ functions which vanish in a neighbourhood of $\partial\Omega_1 \cup \partial\Omega_2$ is a dense subset of $\mathscr{H}$, D_N is clearly a dense subset of $\mathscr{H}$. If f belongs to D_N, from (1.6) we have that

$$(Af, g)_{\mathscr{H}} = \sum_{i=1}^{i} \int_{\Omega_i} \{\sum_{i=1} (\lambda_i \delta_{jk} + 2\mu_i)\varepsilon_{jk}(f)\varepsilon_{jk}(\bar{g})\}dx \tag{1.7}$$

for any $g \in (H^1(O_1))^3$. From the above equality and the assumption $3\lambda_i + 2\mu_i > 0$ it follows that $A \subset A^*$ and $A \geqq 0$. Thus in order to prove self-adjointness of A we have to show that the range of $I + A$ is equal to $\mathscr{H}$. Let $\mathscr{H}^1(O_1) = (H^1(O_1))^3$ be a Hilbert space with the inner product $(f, g)_1$ defined by $(f, g)_{\mathscr{H}}$ plus the right hand side of (1.7). By Korn's inequality on O_1 (see p. 110 of [1]) it follows that $\|f\|^2_{H^1(O_1)} \leqq C(f, f)_1$. This fact and Riesz's theorem imply that for any $g \in \mathscr{H}$ there exists $f \in \mathscr{H}^1(O_1)$ such that $(g, v)_{\mathscr{H}} = (f, v)_1$ for any $v \in H^1(O_1)$. From (1.6) this implies that $f \in D_N$ and $(I + A)f = g$. The proof is completed.

Since the domain $D(A)$ of A is a dense subset of $\mathscr{H}^1(O_1)$ and the domain $D(A^{1/2})$ of $A^{1/2}$ with the graph norm, we have the following

PROPOSITION 1.2. *The domain of* $A^{1/2}$ *is equal to* $\mathscr{H}^1(O_1)$ *and*

$$\|A^{1/2}f\|^2 = \sum_{i=1}^{2} (\sum_{j,k} (\lambda_i \delta_{jk} + 2\mu_i)\|\varepsilon_{jk}(f)\|^2_{L^2(\Omega_i)}) .$$

In (1.5) *we assume that* $f = (f_1, f_2) \in D_N$ *and* $g = (g_1, g_2) \in \mathscr{H}^1(O_1)$, *then* $u^{(i)}(x, t) = u(x, t)|_{\Omega_i}$, *where* $u(x, t) = (\cos tA^{1/2})f + A^{-1/2}(\sin tA^{1/2})g$, *satisfies* (1.1) *and* (1.5) *in the distribution sense. On* (1.2) (1.3) *and* (1.4) *we have the following*

PROPOSITION 1.3. *We put* $\Gamma_0 = \Gamma \backslash (\bar{\Gamma}_1 \cup \bar{\Gamma}_2)$. *Then* $u_i(x, t)|_{\Gamma_0 \times R} \in C(R_t\colon H^{1/2}_{\mathrm{loc}}(\Gamma_0))$, $\sum_j n_j(x)\sigma^{(i)}_{jk}(u_i)|_{\Gamma_h \times R} \in C(R_t\colon H^{-1/2}_{\mathrm{loc}}(\Gamma_h))$ *where* $h = 0, i$ $(i = 1, 2)$, *and these satisfy* (1.2), (1.3) *and* (1.4).

Proof. Since $u_i(x, t)$ belongs to $C(R_t\colon H^1(\Omega_i))$, it follows that $u_i(x, t)|_{\Gamma_0 \times R} \in C(R_t\colon H^{1/2}_{\mathrm{loc}}(\Gamma_0))$ and $u_1(x, t) = u_2(x, t)$ on $\Gamma_0 \times R$. Let x_0 be a point of Γ_0 and U be an open neighbourhood of x_0 such that $U \subset O_1$ and that there exists a diffeomorphism κ from U to $\{y \in R^3\colon |y| < \delta\}$ which maps $U \cap \Omega_1$ to $\{y \in R^3\colon |y| < \delta, y_3 > 0\}$. For any $\phi(x) \in C_0^\infty(U)$ we put $v_i(y, t) = (\phi u_i)(\kappa^{-1}(y), t)$. Then from Theorem 4.3.1 of [2] we see that $v_1(y, t) \in C(R_t\colon H_{(2,-1)}(\bar{R}^3_+))$, where $H_{(m,s)}(\bar{R}^3_+)$ is a function space denoted in Definition 2.5.1 of [2]. By

https://doi.org/10.1017/S0027763000001677 Published online by Cambridge University Press

the trace theorem (see Theorem 2.5.6) in [2] it follows that the trace of $\sum_j n_j(x)\sigma_{jk}^{(1)}(u_1)$ on $\Gamma_0 \times R$ belongs to $C(R_t: H_{\mathrm{loc}}^{-1/2}(\Gamma_0))$. Put $v_{1\varepsilon}(y, t) = \varepsilon^{-2}\int \rho((y' - z')/\varepsilon)v_1(z', y_3, t)dz'$, where a non-negative function $\rho(y')$ belongs to $C_0^\infty(\{y' \in R^2: |y'| < 1\})$ and $\int \rho(y')dy' = 1$. Then using Theorem 2.5.4 of [2], we can easily prove that $v_{1\varepsilon}(y, t) \in C(R_t: H_{(2,0)}(\bar{R}_+^3))$, supp $v_{1\varepsilon} \subset \{y \in R^3: |y| < \delta\}$ if ε is sufficiently small, and $v_{1\varepsilon}(y, t)$ converges to $v_1(y, t)$ in the topology of $C(R_t: H_{(2,-1)}(\bar{R}_+^3))$. These facts imply that the divergence theorem is valid for $(\sum_j (\sigma_{jk}^{(1)}(u_1)v_j))_{k=1,2,3}$, where $v = {}^t(v_1. v_2, v_3) \in (H^1(O_1))^3$. Similarly we can prove the same fact for ϕu_2. By (1.6) it follows that

$$\int_{\Gamma_0} \sum_{j,k} n_j(x)\{\sigma_{jk}^{(1)}(u_1) - \sigma_{jk}^{(2)}(u_2)\}v_k(x)dx = 0$$

for any $v(x) \in (C_0^\infty(U))^3$. The (1.3) is valid. Similarly we can prove (1.4). The proof is completed.

Remark 1.4. The arguments used in this section are easily extended for finite number of media which are not isotropic, whose displacements satisfy the similar boundary value problem to (1.1) to (1.5).

§ 2. Reduction to first order systems and definition of rays

In this section in order to study propagation of singularities to the solutions of (1.1) to (1.3) we shall reduce the considered boundary value problem to the first order system. After that we shall define an incident rays, a reflected ray, a transferred reflected ray, a refracted ray and a transferred refracted ray, which are half null bicharacteristics of $\tau^2 - \alpha_j^2|\xi|^2$ or $\tau^2 - \beta_j^2 - \beta_j^2|\xi|^2$.

Let us consider a solution $u_i(x, t)$ of (1.1). Hereafter we assume that u_i is an extensible distribution, that is, there exists a distribution $U_i(x, t)$ on R^4 such that $U_i = u_i$ on $\Omega_i \times R$. Thus by Theorem 4.3.1 of [2] the traces of $u_i|_{\Gamma_0\times R}$ and $\sigma_{jk}^{(i)}(u_i)|_{\Gamma_0\times R}$ are distributions on $\Gamma_0 \times R$, and we can suppose these distributions satisfy the conditions (1.2) and (1.3). From now on we assume that $n(x)$ appearing in (1.3) is the unit outer normal vector of Ω_1 at Γ_0. Since the boundary value problem (1.1) to (1.3) is rotation free, we may assume that the origin of R^3 belongs to Γ_0 and $n(0) = {}^t(0, 0, -1)$. In a neighbourhood U_0 of 0 Γ_0 is defined by $x_3 = g(x')$, where $x' = (x_1, x_2)$. Making use of the coordinate transform κ; $y' = x'$, $y_3 = x_3 - g(x')$ such that $\Omega_1 \cap U_0$ is tranformed into $\{y: y_3 > 0\}$ and putting

$U_i(y,t) = {}^t(\Lambda(D_{y'}, D_t)\,{}^t u_i, D_{y_3}{}^t u_i)$, where Λ is a pseudo-differential operator with the symbol $\Lambda_1(\eta',\tau) = (|\eta'|^2 + \tau^2 + 1)^{1/2}$, the problem (1.1) to (1.3) is reduced to the following boundary value problem (1.1) to (1.3) is reduced to the following boundary value problem (see section 1.1 of [9])

$$(2.1)\qquad \begin{cases} D_{y_3}U_i = M_i(y', D_{y'}, D_t)U_i & \text{in } (-1)^{i+1}y_3 > 0, \\ (I_3, 0)U_1 = (I_3, 0)U_2 & \text{on } y_3 = 0, \\ B_1(y', D_{y'}, D_t)U_1 = B_2(y', D_{y'}, D_t)U_2 & \text{on } y_3 = 0, \end{cases}$$

where I_3 is the 3×3 identity matrix and the principal symbol $(B_{i1}, B_{i2})(y', \eta', \tau)$ of $B_i = (B_{i1}, B_{i2})(y', D_{y'}, D_t)$ is

$$(2.2)\qquad \begin{cases} B_{i1}(y',\eta',\tau) = (\lambda_i G^t\bar\eta + \mu_i \bar\eta^t G + \mu_i G\cdot\bar\eta)\Lambda_1^{-1}, \\ B_{i2}(y',\eta',\tau) = (\lambda_i + \mu_i)G^tG + \mu_i|G|^2 I_3 \end{cases}$$

with $G = {}^t(-\nabla g(y'), 1)$ and $\bar\eta = {}^t(\eta_1, \eta_2, 0)$. Here the principal symbol $M_{i1}(y',\eta',\tau)$ of $M_{i1}(y', D_{y'}, D_t)$ satisfies that $\det(\eta_3 I_6 - M_{i1}) = ((\eta_3 - a)^2 + p_i)((\eta_3 - a)^2 + s_i)^2$, where $a(y',\eta') = \eta'\cdot\nabla g(y')/|G|^2$, $s_i(y',\eta',\tau) = (|\eta'|^2 - \tau^2/\alpha_i^2 - (\eta'\cdot\nabla g)/|G|^2)/|G|^2$ and $p_i(y',\eta',\tau) = (|\eta'|^2 - \tau^2/\beta_i^2 - (\eta'\cdot\nabla g)/|G|^2)/|G|^2$ with $\alpha_i^2 = \mu_i/\rho_i$ and $\beta_i^2 = (\lambda_i + 2\mu_i)/\rho_i$.

We shall use notions of wave front set $WF(G)$ for $G(y',t) \in \mathscr{D}'(R^3_{y',t})$ defined in [3] and micro-local smoothness of $F \in C^\infty([0,\delta]: \mathscr{D}'(R^3_{y',t})) \cup C^\infty([-\delta, 0]: \mathscr{D}'(R^3_{y',t}))$ at $\rho \in T^*(R^3_{y',t})\backslash 0$, which means that there exists a properly supported pseudo-differential operator $A(y', t, D_{y'}, D_t)$ such that A is elliptic at ρ and $(AF)(y,t) \in C^\infty([0, \pm\varepsilon] \times R^3_{y',t})$ for some $\varepsilon > 0$.

Let us consider a point $(0, \eta'_{0,}, \tau_0)$ such that $\tau_0|\eta_0|(p_1p_2s_1s_2)(0, \eta'_0, \tau_0) \neq 0$. Put $\tilde a_i^\pm(y',\eta',\tau) = a \pm (-1)^{i+1}(-s_i)^{1/2}$ if $s_i(0,\eta'_0,\tau_0) > 0$ and put $\tilde a_i^\pm(y',\eta',\tau) = a \pm (-1)^{i+1}\varepsilon(-s_i)^{1/2}$ if $s_i(0,\eta'_0,\tau_0) < 0$ and $\varepsilon = \operatorname{sgn}\tau_0$, where the branch of $(-s_i)^{1/2}$ is taken like that $(-1)^{1/2}$ is the imaginary unit. Similarly making use of p_i instead of s_i, we define $\tilde b_i^\pm(y',\eta',\tau)$. Then by Lemma 1.1 of [9] $\tilde a_i^\pm$ and $\tilde b_i^\pm$ are eigen values of $M_{i1}(y',\eta',\tau)$ and there exist eigen vectors $s_{ik}^\pm$ $(k = 1, 2)$ and $s_{i3}^\pm$ of $\tilde a_i^\pm$ and $\tilde b_i^\pm$, respectively, which are linearly independent. By the argument of Sections 1.2 and 1.3 of [9] (see also Section 2 of [8]) we can reduce the boundary value problem (1.2) as follows: There exists an elliptic pseudo-differential operator $S_i(y', D_{y'}, D_t)$ of order 0 defined in a conic neighbourhood of $\rho_0 = (0, t_0, \eta'_0, \tau_0) \in T^*(R^3_{y',t})\backslash 0$ with the principal symbol $(s_{11}^+, s_{12}^+, s_{13}^+, s_{11}^-, s_{12}^-, s_{13}^-)$ such that the boundary value problem (2.1) is micro-locally reduced to the following

$$
(2.3)\qquad \begin{cases} D_{y_3}V_i - \begin{pmatrix} A_i^+ & & & 0 \\ & b_i^+ & & \\ 0 & & A_i^- & \\ & & & b_i^- \end{pmatrix} V_i = F_i, & \text{in } (-1)^{i+1}y_3 > 0, \\ C_1V_1 - C_2V_2 = G & \text{on } y_3 = 0, \end{cases}
$$

where $V_i = S_i^{-1}U_i$, $C_i(y', D_{y'}, D_t) = \begin{pmatrix} I_3 & 0 \\ B_{i1} & B_{i2} \end{pmatrix} S_i$, $\rho_0 \notin WF(G)$ and F_i is smooth at ρ. Moreover the principal symbol of $A_i^\pm(y', S_{y'}, D_t)$ is the diagonal matrix $\tilde{a}_i^\pm(y', \eta', \tau)I_2$ and the principal symbol of $b_i^\pm(y', D_{y'}, D_t)$ is $\tilde{b}_i^\pm(y', \eta', \tau)$. At $y = 0$ the principal symbols of C_1 and C_2 are simplified as follows:

LEMMA 2.1. *If $\tau_0|\eta_0'|(p_1p_2s_1s_2)(0, \eta_0', \tau_0) \neq 0$, then we may assume that at $y' = 0$ the principal symbol $C_i = (C_i^+, C_i^-)$ of $C_i(y', D_{y'}, D_t)$ is given by $C_i^\pm = {}^t({}^tC_{i1}^\pm, {}^tC_{i2}^\pm)$, where*

$$
(2.4)\qquad C_{i1}^\pm = \begin{pmatrix} a_i^\pm|\eta'|^2\Lambda_1^{-3} & 0 & |\eta'|^2\Lambda_1^{-2} \\ 0 & |\eta'|^2\Lambda_1^{-2} & 0 \\ -|\eta'|^2\Lambda_1^{-2} & 0 & b_i^\pm\Lambda_1^{-1} \end{pmatrix},
$$

$$
C_{i2}^\pm = \begin{pmatrix} (\rho_i\tau^2 - 2\mu_i|\eta'|^2)|\eta'|^2\Lambda_1^{-4} & 0 & 2\mu_ib_i^\pm|\eta'|^2\Lambda_1^{-3} \\ 0 & \mu_ia_i^\pm|\eta'|^2\Lambda_1^{-3} & 0 \\ -2\mu_ia_i^\pm|\eta'|^2\Lambda_1^{-3} & 0 & (\rho_i\tau^2 - 2\mu_i|\eta'|^2)\Lambda_1^{-2} \end{pmatrix}
$$

with $a_i^\pm(\eta', \tau) = \tilde{a}_i^\pm(0, \eta', \tau)$ and $b_i^\pm(\eta, \tau) = \tilde{b}_i^\pm(0, \eta', \tau)$.

Proof. In order to simplify the principal symbol of $C_i(y', D_{y'}, D_t)$ we use an elliptic pseudo-differential operator $D_0(D_{y'}, D_t) = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}$ of order 0, where the principal symbol $d_{ij}(\eta', \tau)$ of the components of 3×3 matrix $D(D_{y'}, D_t)$ are $d_{11} = d_{22} = \eta_1\Lambda_1^{-1}(\eta', \tau)$, $d_{12} = -d_{21} = \eta_2\Lambda_1^{-1}(\eta', \tau)$, $d_{33} = 1$ and $d_{13} = d_{23} = d_{31} = d_{32} = 0$. By (1.8) of [9] we can take the principal symbol of S_i as follows; $s_{i1}^\pm = {}^t({}^tw_{i1}^\pm, \tilde{a}_i^\pm\Lambda_1^{-1}\,{}^tw_{i1})$ with ${}^tw_{i1}^\pm = (\tilde{a}_i^\pm(\eta' - \tilde{a}_i^\pm\nabla g), -|\eta' - \tilde{a}_i^\pm\nabla g|^2)\Lambda_1^{-2}$, $s_{i2}^\pm = {}^t(w_{i2}^\pm, \tilde{a}_i^\pm\Lambda_1^{-1}\,{}^tw_{i2}^\pm)$ with ${}^tw_{i2}^\pm = (-(\eta_2 - \tilde{a}_i^\pm\partial g/\partial y_2), \eta_1 - \tilde{a}_i^\pm\partial g/\partial y_1, 0)\Lambda_1^{-1}$ and $s_{i3}^\pm = {}^t({}^tw_{i3}^\pm, \tilde{b}_i^\pm\Lambda_1^{-1}w_{i3}^\pm)$ with ${}^tw_{i3}^\pm = (\eta' - b_i\nabla g, b_i^\pm)\Lambda_1^{-1}$. Since $\nabla g = 0$ at $y' = 0$, making use of (2.2), we can compute the principal symbol $D_0C_i(y', D_{y'}, D_t)$, which is given by (2.4) at $y' = 0$. The proof is completed.

Let us consider the incident P ray $P_i(\omega)$ in Ω_1 hitting on $(0, t_0)$ with a direction $\omega = (\omega_1, \omega_2, \omega_3) \in S^2$ such that $0 < n(0)\cdot\omega < 1$, which is the half connected null bicharacteristic $\{(-\beta^1\omega t, t_0 - t, -\varepsilon\omega, \varepsilon\beta_1) \in T^*(\Omega_1 \times R): t > 0\}$ of $\tau^2 - \beta_1^2|\xi|^2$ passing through $\rho_0 = (0, t_0, -\varepsilon\omega, \varepsilon\beta_1)$, where $\varepsilon^2 = 1$. Since the outer unit normal vector of Ω_1 at 0 is $(0, 0, -1)$, the reflected P ray $P_r(\omega)$ of $P_i(\omega)$ is given by the connected ray $\{(\beta_1\omega_r t, t + t_0, -\varepsilon\omega_r, \varepsilon\beta_1) \in T^*(\Omega_1 \times R)\colon t > 0\}$, where $\omega_r = (\omega_1, \omega_2, -\omega_3)$. Hereafter we say that a ray $\gamma(t)$ in Ω_i parametrized by time t is outgoing (incoming) if $(-1)^i(dx_3/dt)(0) < 0$ (>0), where $x_3(t)$ is the x_3 component of $\gamma(t)$. The half connected outgoing null bicharacteristic $S_{\mathrm{tr}}(\omega)$ of $\tau^2 - \alpha_1^2|\xi|^2$ in $T^*(\Omega_1 \times R)$ passing through ρ with $\pi(\rho) = \pi(\rho_0)$ is called the transferred reflected ray of $P_i(\omega)$, where π is the projection from $T^*(R^4_{x,t})$ to $T^*(\partial\Omega \times R)$. If there exists the half connected outgoing null bicharacteristic $\tilde{P}_r(\omega)$ $(\tilde{S}_{\mathrm{tr}}(\omega))$ of $\tau^2 - \beta_2^2|\xi|^2$ $(\tau^2 - \alpha_2^2|\xi|^2)$ in $T^*(\Omega_2 \times R)$ passing through with $\pi(\rho) = \pi(\rho_0)$, we call it the refracted P ray (the transferred refracted S ray) of $P_i(\omega)$. Similarly for the incident S ray $S_i(\omega)$ passing through $\rho_1 = (0, t_0, -\varepsilon\omega, \varepsilon\alpha_1)$, the reflected S ray $S_r(\omega)$, the transferred reflected P ray $P_{\mathrm{tr}}(\omega)$, the refracted S ray $\tilde{S}_r(\omega)$ and the transferred refracted P ray $\tilde{P}_{\mathrm{tr}}(\omega)$ are defined, if these rays exist. The rays are concretely denoted as follows:

LEMMA 2.2. i) *Put* $\omega_{\mathrm{tr}}^{\pm} = (\omega', \pm(\beta_1^2/\alpha_1^2 - 1 + (n(0)\cdot\omega)^2)^{1/2})$, *then* $S_{\mathrm{tr}}(\omega) = \{(\alpha_1^2\omega_{\mathrm{tr}}^{+}t/\beta_1, t + t_0, -\varepsilon\omega_{\mathrm{tr}}^{+}, \varepsilon\beta_1) \in T^*(\Omega_1 \times R)\colon t > 0\}$. *A similar statement is valid for* $P_{\mathrm{tr}}(\omega)$, *if* $1 - \alpha_1^2/\beta_1^2 < (n(0)\cdot\omega)^2$.

ii) *Put* $\tilde{\omega}_r^{\pm} = (\omega', \pm(\beta_1^2/\beta_2^2 - 1 + (n(0)\cdot\omega)^2)^{1/2})$, *if* $1 - \beta_1^2/\beta_2^2 < (n(0)\cdot\omega)^2$; *then* $\tilde{P}_r(\omega) = \{(\beta_2^2\tilde{\omega}_r^{-}t/\beta_1, t + t_0, -\varepsilon\tilde{\omega}_r^{-}, \varepsilon\beta_1) \in T^*(\Omega_2 \times R)\colon t > 0\}$. *Similarly* $\tilde{S}_r(\omega)$ *is defined, if* $1 - \alpha_1^2/\alpha_2^2 < (n(0)\cdot\omega)^2$.

iii) *Put* $\tilde{\omega}_{\mathrm{tr}}^{\pm} = (\omega', \pm(\beta_1^2/\alpha_2^2 - 1 + (n(0)\cdot\omega)^2)^{1/2})$, *if* $1 - \beta_1^2/\alpha_2^2 < (n(0)\cdot\omega)^2$, *then* $\tilde{S}_{\mathrm{tr}}(\omega) = \{(\alpha_2^2\tilde{\omega}_{\mathrm{tr}}^{-}t/\beta_1, t + t_0, -\varepsilon\tilde{\omega}_{\mathrm{tr}}^{-}, \varepsilon\beta_1) \in T^*(\Omega_2 \times R)\colon t > 0\}$. *Similarly* $\tilde{P}_{\mathrm{tr}}(\omega)$ *is defined, if* $1 - \alpha_1^2/\beta_2^2 < (n(0)\cdot\omega)^2$.

The proof of Lemma 2.2 is easily derived from the definitions of the rays. For the incident P ray $P_i(\omega)$ we also denote by $S_{\mathrm{in}}(\omega)$, $\tilde{P}_{\mathrm{in}}(\omega)$ and $\tilde{S}_{\mathrm{in}}(\omega)$ half connected incoming null bicharacteristics of $\tau^2 - \alpha_1^2|u|^2$, $\tau^2 - \alpha_2^2|\xi|$ and $\tau^2 - \beta_2^2|\xi|^2$ passing through $(0, t_0, -\varepsilon\omega_{\mathrm{tr}}^{-}, \varepsilon\beta_1)$, $(0, t_0, -\varepsilon\tilde{\omega}_r^{+}, \varepsilon\beta_1)$ and $(0, t_0, -\varepsilon\tilde{\omega}_{\mathrm{tr}}^{+}, \varepsilon\beta_1)$, repsectively, if these rays exist. Similarly for the incident S ray $S_i(\omega)P_{\mathrm{in}}(\omega)$, $\tilde{P}_{\mathrm{in}}(\omega)$ and $\tilde{S}_{\mathrm{in}}(\omega)$ are defined.

§ 3. Singularities corresponding to Stonly waves

In this section we analyze singularities to a solution of (1.1) to (1.3) near an elliptic point $(0, t_0, \eta_0', \tau_0)$, that is, $s_i(0, \eta_0', \tau_0) > 0$ for $i = 1, 2$. In [7] he proved that if Γ_0 is a hyperplane of R^3, there are surface waves satisfying (1.1) to (1.3) and propagating on the boundary $\Gamma_0 \times R$. In this

https://doi.org/10.1017/S0027763000001677 Published online by Cambridge University Press

section without assuming the flatness of Γ_0, we shall show that there exist rays belonging to the wave front set of a solution of (1.1) to (1.3) which propagates in the elliptic region of the boundary $\Gamma_0 \times R$.

We shall consider the Lopatinski matrix of the boundary value problem (2.3) in the elliptic region $\{(y', t, \eta', \tau)\colon s_i(y', \eta', \tau) > 0\ i = 1, 2\}$, where $p_i(y', \eta', \tau) > 0$ $(i = 1, 2)$. We remark that if $y' = 0$, then the elliptic region is $\{(0, t, \eta', \tau)\colon \min(\alpha_1^2, \alpha_2^2)|\eta'|^2 > \tau^2\}$.

LEMMA 3.1. *We assume that* $\tau_0 \neq 0$ *and* $\min(\alpha_1^2, \alpha_2^2)|\eta_0'|^2 > \tau_0^2$. *Then the necessary and sufficient condition that the determinant* (C_1^+, C_2^+) *is zero is given by* $F(\tau^2/|\eta'|^2) = 0$, *where*

$$\begin{aligned} F(s) = \{&(\rho_2 a_1^+(s) - \rho_1 a_2^+(s))(\rho_2 b_1^+(s) - \rho_1 b_2^+(s)) + (\rho_1 - \rho_2)s^2 \\ &- 4(\mu_1 - \mu_2)\{\rho_1(a_2^+ b_2^+)(s) - \rho_2(a_1^+ b_1^+)(s) + \rho_1 - \rho_2\}s \\ &+ 4(\mu_1 - \mu_2)^2\{(a_1^+ b_1^+ a_2^+ b_2^+)(s) + (a_2^+ b_2^+)(s) + (a_2^+ b_2^+)(s) + 1\} \end{aligned}$$

with $a_j^+(s) = (-1)^{j+1} i(1 - s/\alpha_j^2)^{1/2}$ *and* $b_j^+(s) = (-1)^{j+1} i(1 - s/\beta_j^2)^{1/2}$.

Proof. By (2.4) the second and fifth column vectors of (C_1^+, C_2^+) are linearly independent to the other column vectors, and these two vectors are linearly independent. Thus the condition $\det(C_1^+, C_2^+) = 0$ is equivalent to the condition $\det M_1 = 0$, where M_1 is the 4×4 square matrix generated by eliminating the second and fifth column and line vectors of $(C_1^+, -C_2^+)$. By simple calculations we can show that $\det M_1 = C|\eta'|^6 F(\tau^2/|\eta'|^2)$, where C is a non zero constant. The proof is completed.

If we assume the Wiechert condition in seismology, that is, $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, then we can get informations on the roots of $F(s) = 0$.

LEMMA 3.2. *We assume* $\alpha_1 = \alpha_2$ *and* $\beta_1 = \beta_2$, *and put* $\alpha = \alpha_i$ *and* $\beta = \beta_i$. *Then we have the following statements.*

i) *The roots of* $F(s) = 0$ *in* $(0, \alpha^2)$ *are also the roots of* $f(x) = 0$ *in* $(0, 1)$ *where* $x = s/\alpha^2$ *and* $f(x) = \gamma x^4 - \{\gamma + 1 + 8\gamma M\}x^3 + \{1 + 24\gamma M + 8M + 8\gamma M^2 - M^2 - 16\gamma^2 M^2\}x^2 - 8\{\gamma M^2 - 4\gamma^2 M^2 + 3M + 2\gamma M\}x + 16(M + \gamma^2 M)$ *with* $\gamma = \alpha/\beta$ *and* $M = (\rho_1 - \rho_2)^2/(\rho_1 + \rho_2)^2$.

ii) $f(x)$ *has at least one root in* $(0, 1)$.

iii) *If all roots* $f(x) = 0$ *in* $(0, 1)$ *are simple, then the set* $\{y', t, \eta', \tau)\colon \det(C_1^+, C_2^+)(y', \eta', \tau) = 0\}$ *is locally given by* $\tau = h(y', \eta')$, *where* $h(y', \eta')$ *is a positively homogeneous function of degree* 1.

Proof. In $F(s)$ put $s/\alpha^2 = x$, then $F(s)$ is equal to $\alpha^4[(\rho_1 - \rho_2)^2\{x^2 +$

$4(1-x)(2-\gamma x)\} - \{(\rho_1+\rho_2)^2x^2 + 4(\rho_1-\rho_2)^2(2-x)\}(1-\gamma x)^{1/2}(1-x)^{1/2}] = \alpha^4[g_1(x) - g_2(x)]$, where $g_i(x)$ is defined by the last equality. Since $g_1(x)$ and $g_2(x)$ are positive if $x \in (0, 1)$, the condition $F(s) = 0$ is equivalent to $g_2^2(x) - g_1^2(x) = (\rho_1+\rho_2)^2x^2f(x) = 0$. The statement ii) is clear, for $f(0) = 16M(1-\gamma^2) > 0$ and $f(1) = (\rho_1+\rho_2)^{-4}\{g_2^2(1) - g_1^2(1)\} = -M^2 < 0$. The statement iii) is a consequence of the implicit function theorem. The proof is completed.

We shall check conditions that the polynomial $f(x)$ has only simple roots in $(0, 1)$.

Remark 3.3. i) One of the equivalent conditions that a polynomial $f(x)$ has a double roots is that the discriminant of $f(x)$ is zero. In this case the discriminant of $(\rho_1+\rho_2)^2\beta^2f(x)$ is a polynomial with respect to ρ_1, ρ_2, α and β. Thus for almost everywhere $(\rho_1, \rho_2, \alpha, \beta)$ the equation $f(x) = 0$ has only simple roots.

ii) A simple condition of simplicity of the roots in $(0, 1)$ of $f(x) = 0$ is given as follows: If $f''(1) \geqq 0$ and $(\gamma+1+8M)/4\gamma \geqq 1$, where $x = (\gamma+1+8M)/4\gamma$ is the symmetric axis of $f''(x) = 0$, then $f''(x) > 0$ in $[0, 1]$. This condition implies that $f(x) = 0$ has only one root in $(0, 1)$. Thus if $3\gamma + 8M(\gamma M + 1) \geqq (16\gamma^2+1)M^2 + 2$ and $8\gamma M + 1 \geqq 3\gamma$, then $f(x) = 0$ has only one simple root in $(0, 1)$.

Let us consider the Lopatinski determinant of the boundary value problem (2.3) in a conic neighbourhood of $\rho_0 = (0, t_0, \eta_0', 0)$.

LEMMA 3.4. *If $\tau_0 = 0$, then the Lopatinski determinant of* (2.3), *that is, the determinat of the principal symbol (C_1^+, C_2^+) is not zero at ρ_0.*

Proof. In the case $\tau_0 = 0$, in Section 1.2 of [9] $S_k(0, \eta_0', 0)$ $(k = 1, 2)$ is given as follows: Put $S_k(0, \eta_0', 0) = (s_{k1}^-, s_{k2}^+, s_{k3}^+, s_{k1}^-, s_{k2}^-, s_{k3}^-)$, then

$$
\begin{aligned}
{}^ts_{k1}^\pm &= (\pm(-1)^k\,{}^t\eta_0', \pm(-1)^k|\eta_0'|, (3\beta_k-\alpha_k)\,{}^t\eta_0', \pm(-1)^{k+1}i(\beta_k-3\alpha_k)|\eta_0'|)\Lambda_1^{-1},\\
{}^ts_{k2}^\pm &= ({}^tw_{k2}, \pm(-1)^{k+1}i\,{}^tw_{k2}) \quad \text{with } {}^tw_{k2} = (-\eta_{02}, \eta_{01}, 0)\Lambda_1^{-1}, \quad \text{and}\\
{}^ts_{k3}^\pm &= ({}^tw_{k3}, \pm(-1)^{k+1}i\,{}^tw_{k3}) \quad \text{with } {}^tw_{k3} = ({}^s\eta_0', \pm i|\eta_0'|)\Lambda_1^{-1}.
\end{aligned}
$$

Using $S_k(0, \eta_0', 0)$ and (2.2) we can easily compute $\det(C_1^+, C_2^+)(0, \eta_0', 0)$, which is equal to $C(\mu_1+\mu_2) \times \{\mu_1(\lambda_1+2\mu_1) + \mu_2(\lambda_1+3\mu_1)\}\{\mu_2(\lambda_2+2\mu_2) + \mu_1(\lambda_2+3\mu_2)\}/\rho_1\rho_2$ with a non zero constant C. That is not zero. The proof is completed.

For the solution u_i of (1.1) we define $WF_b(u_i) \subset (T^*(\Omega_i \times R)\backslash 0) \cup (T^*(\Gamma_0 \times R)\backslash 0)$ as follows: i) $\rho \in T^*(\Omega_i \times R)$ belongs to $WF_b(u_i)$, if ρ belongs to $WF(u_i|_{\Omega_i})$, ii) $\rho \in T^*(\Gamma_0 \times R)$ does not belong to $WF_b(u_i)$, if $u_i(\kappa^{-1}(y), t)$ has the property of micro-local smoothness at $\pi(\kappa^*(\rho))$, where π is the projection from $T^*(R^4_{y,t})$ to $T^*(\{y_3 = 0\} \times R)$ and κ^* is the diffeomorphism from $T^*(U_0 \times R)$ to $T^*(R^4_{y,t})$ induced from κ, that is,

$$\kappa^*(x, t, \xi, \tau) = (x', x_3 - g(x'), t, \xi' + (\nabla g)(x')\xi_3, \xi_3, \tau).$$

This definition of $WF_b(u_i)$ is invariant by the diffeomorphism κ (see Proposition 1.2 in [6]). Let Σ'_e be an elliptic region on the boundary, i.e., $\Sigma'_e = \{(y', t, \eta', \tau) \in T^*(R^3_{y',t})\backslash 0\colon s_i(y', \eta', \tau) > 0,\ i = 1, 2\}$, and Σ'_0 be a subset of Σ'_e such that $\det(C_1^+, C_2^+)(y', \eta', \tau) = 0$. Put $\Sigma_e = (\pi \circ \kappa^*)^{-1}(\Sigma'_e)$ and $\Sigma_0 = (\pi \circ \kappa^*)^{-1}(\Sigma'_0)$; Then we have the following

THEOREM 3.5. *We assume that* $\alpha_1 = \alpha_2$ *and* $\beta_1 = \beta_2$ *and that* $f(x)$ *of Lemma* 3.2 *has only simple roots in* $(0, 1)$. *Then* $WF_b(u_i) \cap \Sigma_e \subset \Sigma_0$, *where* Σ_0 *is locally given by* $\tau - h(x, \xi)$ *with* C^∞ *homogeneous function* $h(x, \xi)$ *on* $T^*(\Gamma_0)$ 0 *of order* 1, *and* $WF_b(u_1) \cup WF_b(u_2)$ *is invariant under the Hamilton vector field* $H_{\tau-h}$ *on* $T^*(\Gamma_0 \times R)$ 0.

Proof. In (2.3) we denote $V_i = {}^t(V_i^+, {}^tV_i^-)$. Then since V_1^- and V_2^- satisfy backward parabolic equations, ρ_0 does not belong to $WF(V_1^-|_{y_3=0}) \cup WF(V_2^-|_{y_3=0})$. It follows that $WF(C_1^+V_1^+|_{y_3=0} - C_2^+V_2^+|_{y_3=0})$ does not contain ρ_0. By Hörmander's theorem on propagation of singularities (see Theorem 6.1.1 of [4] we have the desired statement.

Remark 3.6. i) Let $\{f_i, g_i\}$ be the initial data of the solution $u(x, t)$ of (1.1) to (1.5). Assume that $(x_0, \xi_0) \in WF(f_i|_{\Gamma_0}) \cup WF(g_i|_{\Gamma_0})$, Then by Theorem 2.5.11′ of [3] there exists τ_0 such that an element $\rho_0 = (x_0, 0, \xi_0, \tau_0)$ of $T^*(\Gamma_0 \times R)$ belongs to $WF(u|_{\Gamma_0 \times R})$. If ρ_0 is an elliptic point, then by Theorem 3.5 there exists a ray belonging to $WF_b(u_1) \cup WF_b(u_1)$, which starts at ρ_0 and propagates on the border $\Gamma_0 \times R$.

ii) From the form of $F(s)$, the null points of $F(s) = 0$ are roots of some polynomial of degree 22 whose coefficients are polynomial of $(\rho_1, \lambda_1, \mu_1, \rho_2, \lambda_2, \mu_2)$. Thus for almost all $(\rho_1, \lambda_1, \mu_1, \rho_2, \lambda_2, \mu_2)$ with $(F(s)/s)^2|_{s=0}$ $F(\min(\alpha_1^2, \alpha_2^2)) < 0$ Theorem 3.5 holds.

§ 4. Incident P singularities

In the case that $\alpha_2 < \beta_1 < \beta_2$ and $\alpha_1 \neq \alpha_2$ there exist interesting reflective and refractive phenomena. Thus in this section we assume the above condition. We shall consider incident P singularities and show

https://doi.org/10.1017/S0027763000001677 Published online by Cambridge University Press

the following theorems on reflective and refractive phenomena of singularities.

THEOREM 4.1. i) *We assume* $\beta_1^2 < \beta_2^2(1 - (n(0)\cdot\omega)^2)$. *Then* $\tilde{P}_{\mathrm{in}}(\omega)$ *and* $\tilde{P}_r(\omega)$ *do not exist. Furthermore we suppose that* $S_{\mathrm{in}}(\omega) \cap WF(u_1) = \tilde{S}_{\mathrm{in}}(w) \cap WF(u_2) = \phi$ *and* $P_i(\omega) \subset WF(u_1)$. *Then* $P_r(\omega) \cup S_{\mathrm{tr}}(\omega) \subset WF(u_1)$ *and* $\tilde{S}_{\mathrm{tr}}(\omega) \subset WF(u_2)$.

ii) *We assume* $\beta_1^2 > \beta_2^2(1 - (n(0)\cdot\omega)^2)$. *Then there exists a function* $G_1(s)$ *whose null points are at most* 30 *such that if* $G_1(\beta_1^2/(1 - n(0)\cdot\omega)^2)) \neq 0$, $S_{\mathrm{in}}(\omega) \cap WF(u_1) = (\tilde{P}_{\mathrm{in}}(\omega) \cup \tilde{S}_{\mathrm{in}}(\omega)) \cap WF(u_2) = \phi$ *and* $P_i(\omega) \subset WF(u_1)$, *then* $P_r(\omega) \cup S_{\mathrm{tr}}(\omega) \subset WF(u_1)$ *and* $\tilde{P}_r(\omega) \cup \tilde{S}_{\mathrm{tr}}(\omega) \subset WF(u_2)$.

The concrete form of $G_1(s)$ is given in Lemma 4.3. The idea of proving the above theorem is as follows: First we shall look for an elliptic pseudodifferential operator A such that some components of AC_i^- $(i = 1, 2)$ vanish. After that making use of the assumptions of $WF(u_i)$, we shall check the conditions to the wave front sets of the components of $V_i|_{y_3=0}$ $(i = 1, 2)$, which derive the statements of the theorem.

Let ρ_0 be $(0, t_0, -\varepsilon\omega, \varepsilon\beta_1)$, where $\varepsilon^2 = 1$, $0 < n(0)\cdot\omega < 1$. Then the projected point ρ_1 of ρ_0 to $T^*(\partial\Omega \times R)$ is $(0, t_0, -\varepsilon\omega', \varepsilon\beta_1)$. In a conic neighbourhood Γ_1 of ρ_1 in $T^*(R^3_{y',t})\backslash 0$ we may assume that at $y = 0$ the principal symbols of $A_i^\pm$, $b_i^\pm$ in (2.3) are $a_1^\pm(\eta', \tau)I_2 = (\pm\varepsilon(\tau^2/\alpha_1^2 - |\eta'|^2)^{1/2})I_2$, $a_2^\pm(\eta', \tau)I_2 = (\mp\varepsilon(\tau^2/\alpha_2^2 - |\eta'|^2)^{1/2})I_2$, $b_1^\pm(\eta', \tau) = \pm\varepsilon(\tau^2/\beta_1^2 - |\eta'|^2)^{1/2}$, and $b_2^\pm(\eta', \tau)$ is $\mp i(|\eta'|^2 - \tau^2/\beta_2^2)^{1/2}$ if $\beta_1^2 < \beta_2^2(1 - (n(0)\cdot\omega)^2)$ and is $\pm\varepsilon(\tau^2/\beta_2^2 - |\eta'|^2)^{1/2}$ if $\beta_1^2 > \beta_2^2(1 - (n(0)\cdot\omega)^2)$. The boundary operator $C_i(y', D_{y'}, D_t)$ of (2.3) is also defined by using these notations. We say that a pseudodifferential operator $P(y', t, D_{y'}, D_t)$ belongs to $L^{-\infty}(\Gamma)$, where Γ is a conic open set of $T^*(R^3_{y',t})$, if the symbol of P is rapidly decreasing with respect to (η', τ) in Γ. We have the following

LEMMA 4.2. *There exists a pseudo-differential operator* $A(y', D_{y'}, D_t)$ *of order* 0 *defined in a conic neighbourhood* Γ_1 *of* ρ_1 *such that the principal symbol* $A_0(y', \eta', \tau)$ *of* A *is the identity matrix* I_6 *at* $y' = 0$, *and that the* (1, 2), (2, 1), (2, 3), (3, 2), (4, 2), (5, 1), (5, 3) *and* (6, 2) *components of* $(AC_i^-)(y', D_{y'}, D_t)$ $(i = 1, 2)$ *are* 0 *modulo* $L^{-\infty}(\Gamma_1)$.

Proof. Put $C_1^-(y', D_{y'}, D_t) = (c_1, c_2, c_3)(y', D_{y'}, D_t)$, $C_2^-(y', D_{y'}, D_t) = (c_4, c_5, c_6)(y', D_{y'}, D_t)$ and denote by $a_j(y', D_{y'}, D_t)$ the j-th line vector of $A(y', D_{y'}, D_t)$. Then the required conditions are

(4.1) $\qquad a_j\cdot c_2 = a_j\cdot c_5 = 0 \qquad \text{modulo } L^{-\infty}(\Gamma_1)\ (j = 1, 3, 4, 6)$

(4.2) $\qquad a_j\cdot c_1 = a_j\cdot c_3 = a_j\cdot c_4 = a_j\cdot c_6 = 0 \qquad \text{modulo } L^{-\infty}(\Gamma_1)\ (j = 2, 5).$

We denote the symbol of a_j by $\sum_{k=0}^{\infty} a_{jk}(y', \eta', \tau)$, where a_{jk} is of order $-k$, and the principal symbol of c_j by $c_{j0}(y', \eta', \tau)$. Put $a_{10} = f_1 + x_{12}c_{20} + x_{15}c_{50}$, where ${}^t f_1 = (1, 0, \cdots, 0) \in R^6$. Then the condition $\langle a_{10}, c_{20}\rangle = \langle a_{10}, c_{50}\rangle = 0$ is equivalent to

$$\begin{pmatrix} \langle c_{20}, c_{20}\rangle & \langle c_{50}, c_{20}\rangle \\ \langle c_{20}, c_{50}\rangle & \langle c_{50}, c_{50}\rangle \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{15}\end{pmatrix} = -\begin{pmatrix} \langle f_1, c_{20}\rangle \\ \langle f_1, c_{15}\rangle \end{pmatrix}. \tag{4.3}$$

Since from (2.4) c_{20} and c_{50} are linearly independent, we can solve the equation (4.3) and x_{12} and x_{15} are zero at $y' = 0$. Similarly if we put $a_{1k} = x_{k2}c_{20} + x_{k5}c_{50}$, where $x_{k2}(y', \eta'\tau)$ and $x_{k5}(y', \eta', \tau)$ are of order $-k$, we can decide the required $a_{1k}(y', \eta', \tau)$. Similarly in order to construct $a_j(y', D_{y'}, D_t)$ which satisfies (4.2) we only check that c_{10}, c_{30}, c_{40} and c_{60} are linearly independent. This condition is equivelent to the condition $\det M(\eta', \tau) \neq 0$, where $M(\eta', \tau)$ is the 4×4 square matrix which is generated by eliminating the second and fourth column and line vectors of $(C_1^-, C_2^-)(0, \eta', \tau)$. We have

$$\begin{aligned} \det M(\eta', \tau) = [\{&\rho_2^2 a_1^- b_1^- + \rho_1^2 a_2^- b_2^- - \rho_1\rho_2(a_1^- b_2^- + a_2^- b_1^-) + (\rho_1 - \rho_2)^2|\eta'|^2\}\tau^4 \\ &+ 4(\mu_1 - \mu_2)\{\rho_2 a_1^- b_1^- - \rho_1 a_2^- b_2^- - (\rho_1 - \rho_2)|\eta'|^2\}|\eta'|^2\tau^2 \\ &+ 4(\mu_1 - \mu_2)^2\{a_1^- b_1^- a_2^- b_2^- + (a_1^- b_1^- + a_2^- b_2^-)|\eta'|^2 + |\eta'|^4\}|\eta'|^2]\,|\eta'|^4 \Lambda_1^{-10}. \end{aligned} \tag{4.4}$$

If $b_2^-(\eta', \tau)$ is real valued, then $|\eta'|^{-4}\Lambda_1^{10} \times (4.4)$ is equal to $\{(\rho_1 - \rho_2)\tau^2 - 2(\mu_1 - \mu_2)|\eta'|^2\}^2|\eta'|^2 + \{\rho_2\tau^2 + 2(\mu_1 - \mu_2)|\eta'|^2\}^2 a_1^- b_1^- + \{\rho_1\tau^2 - 2(\mu_1 - \mu_2)|\eta'|^2\}^2 a_2^- b_2^- - \rho_1\rho_2(a_1^- b_2^- + a_2^- b_1^-)\tau^4 + 4(\mu_1 - \mu_2)^2 a_1^- b_1^- a_2^- b_2^- |\eta'|^2$, which is positive because $a_1^- b_1^-$ and $a_2^- b_2^-$ are positive and $a_1^- b_2^-$ and $a_2^- b_1^-$ are negative. If $b_2^-(\eta', \tau)$ is pure imaginary, then $\operatorname{Re}(\det M(\eta', \tau))$ is also positive. Thus c_{10}, c_{30}, c_{40} and c_{60} are linearly independent. The proof is completed.

Next we shall compute the principal symbol of $(C^-)^{-1}C_1^+$, where $C^- = (C_1^-, C_2^-)$, which is elliptic from the proof of Lemma 4.2, Define $a_i(s) = (s/\alpha_i^2 - 1)^{1/2}$ and $b_i(s) = (s/\beta_i^2 - 1)^{1/2}$ $(i = 1, 2)$ and put

$$\begin{aligned} g_1(s) &= (\rho_2 s + 2(\mu_1 - \mu_2))((\rho_2 - \rho_1)s + 2(\mu_1 - \mu_2)) \\ &\qquad - 2(\mu_1 - \mu_2)(\rho_1 s - 2(\mu_1 - \mu_2))a_2(s)b_2(s), \\ g_3(s) &= \rho_1\rho_2(a_1 b_2 - b_1 a_2)(s)s^2 + (\rho_2 s + 2(\mu_1 - \mu_2))^2(a_1 b_1)(s) \\ &\qquad - (\rho_1 s - 2(\mu_1 - \mu_2))^2(a_2 b_2)(s) + 4(\mu_1 - \mu_2)^2(a_1 b_1 a_2 b_2)(s) \\ &\qquad\qquad - ((\rho_1 - \rho_2)s - 2(\mu_1 - \mu_2))^2, \end{aligned}$$

https://doi.org/10.1017/S0027763000001677 Published online by Cambridge University Press

$$g_4(s) = \{(\rho_1 - \rho_2)^2 - 4(\mu_1 - \mu_2)^2/(\alpha_1^2\beta_2^2)\}s$$
$$- 4(\mu_1 - \mu_2)\{(\rho_1 - \rho_2) - (\mu_1 - \mu_2)(\alpha_1^{-2} + \beta_2^{-2})\},$$

$$g_6(s) = (\rho_2^1/\alpha_1^2 - \rho_1^2/\alpha_2^2)s^2 + (\rho_1 + \rho_2)\{(\rho_1 - \rho_2) + 4(\mu_1 - \mu_2)(\alpha_1^{-2} + \alpha_2^{-2})\}s$$
$$+ 4(\mu_1 - \mu_2)\{(\mu_1 - \mu_2)(\alpha_1^{-2} - \alpha_2^{-2}) - (\rho_1 + \rho_2)\}.$$

Then we have

LEMMA 4.3. i) *If* $\rho_1 \neq \rho_2$, $\mu_1 \neq \mu_2$ *and* $\beta_1^2 < \beta_2^2(1 - (n(0)\cdot\omega)^2)$, *then the* (1.3), (3.3) *and* (4.3) *component of* $(C^-)^{-1}C_1^+$ *are elliptic at* ρ_1, *where* $C^- = (C_1^-, -C_2^-)$.

ii) *If* $\beta_1^2 > \beta_2^2(1 - (n(0\cdot\omega)^2)$ *and* $g_j(\beta_1^2/(1 - (n(0)\cdot\omega)^2)) \neq 0$, *then the* $(j, 3)$ *component of* $(C^-)^{-1}C_1^+$ *is elliptic at* ρ_1, *where* $j = 1, 3, 4, 6$.

iii) *If* $\alpha_1 \neq \alpha_2$, *then* $G_1(s) = (g_1g_3g_4g_6)(s)$ *has at most* 30 *null points in* (β_1^2, ∞).

Proof. Let us denote by $M_1(y', \eta', \tau) = (c_1, c_2, c_3, c_4)$ the 4×4 square matrix which is generated by eliminating the second and fourth column and line vectors of the principal symbol of $C^-(y', D_{y'}, D_t)$ and put $c(y, \eta', \tau)$ to be the column vector which is generated by eliminating the second and fourth components of the principal symbol of the third column vector of $C_1^+(y', D_{y'}, D_t)$. Then from Lemma 4.2 and Cramer's formula we may check that the determinant of $(c, c_{i_1}, c_{i_2}, c_{i_3})$, where $1 \leqq i_j \leqq 4$ and $i_j \neq i_k$ if $j \neq k$, is not zero at ρ_1. From (2.4) it follows that

(4.5) $$\det(c, c_2, c_3, c_4)(0, \eta', \tau) = A_1[(\rho_2\tau^2 + 2(\mu_1 - \mu_2)|\eta'|^2)$$
$$\times ((\rho_1 - \rho_2)\tau^2 - 2(\mu_1 - \mu_2)|\eta'|^2) + 2(\mu_1 - \mu_2)(\rho_1\tau^2 - 2(\mu_1 - \mu_2)|\eta'|^2)a_2^-b_2^-],$$

(4.6) $$\det(c_1, c, c_3, c_4)(0, \eta', \tau) = A_2[\rho_1\rho_2(a_1^-b_2^- + a_2^-b_1^+)\tau^4$$
$$- (\rho_2\tau^2 + 2(\mu_1 - \mu_2)|\eta'|^2)^2a_1^-b_1^+ - (\rho_2\tau^2 - 2(\mu_1 - \mu_2)|\eta'|^2)^2a_2^-b_2^-$$
$$- ((\rho_1 - p_2)\tau^2 - 2(\mu_1 - \mu_2)|\eta'|^2)^2|\eta'|^2],$$

(4.7) $$\det(c_1, c_2, c, c_4)(0, \eta', \tau) = A_3\{(\rho_1 - \rho_2)\tau^2 - 2(\mu_1 - \mu_2)(|\eta'|^2 - a_1^-b_2^-)\},$$

(4.8) $$\det(c_1, c_2, c_3, c)(0, \eta', \tau) = A_4\{(\rho_2\tau^2 + 2(\mu_1 - \mu_2)|\eta'|^2)a_1^-$$
$$- (\rho_1\tau^2 - 2(\mu_1 - \mu_2)|\eta'|^2)a_2^-\},$$

where $A_j(\eta', \tau)$ is not zero, if $\eta' \neq 0$ and $\tau \neq 0$. If $\rho_1 \neq \rho_2$, $\mu_1 \neq \mu_2$ and b_2^- is pure imaginary, then the real part and the imaginary part of (4.5) do not vanish at the same point and the imaginary parts of (4.6) and (4.7) are not zero. It follows that the statement i) holds. (4.5) and (4.6) clearly implies that the statement ii) holds for $j = 1, 3$, $g_4(s)$ and $g_6(s)$ are equal

https://doi.org/10.1017/S0027763000001677 Published online by Cambridge University Press

to $[((\rho_1-\rho_2)s-2(\mu_1-\mu_2))^2-4(\mu_1-\mu_2)^2(a_1^-b_2^-)^2]/s$ and $[\{(\rho_2 s+2(\mu_1-\mu_2))a_1^-\}^2-\{(\rho_1 s-2(\mu_1-\mu_2))a_2^-\}^2/s$. Thus when $\rho_1\neq\rho_2$ and $\mu_1\neq\mu_2$, $g_1(s)$, $g_4(s)$ and $g_6(s)$ are not identically zero and one of the equivalent conditions that $g_1(s)=0$ is $[\{(\rho_2 s+2(\mu_1-\mu_2))((\rho_1-\rho_2)s+2(\mu_1-\mu_2)\}^2-4(\mu_1-\mu_2)^2\{(\rho_1 s--2(\mu_1-\mu_2))(a_2b_2)(s)\}^2]/s=0$. It follows that the null points of $(g_1g_4g_6)(s)$ are at most 6. By the form of $g_3(s)$ the null points of $g_3(s)$ are roots of some polynomial of degree 24. These show that the null points of $G_1(s)$ in (β_1^2,∞) are at most 30. The proof is completed.

The proof of Theorem 4.1. The proof of the statement i) is similar to that of the statement ii). Thus we only prove the statement ii). In (2.3) we put ${}^tV_i={}^t({}^tV_i^+,{}^tV_i^-)$ with $V_i^\pm={}^t(v_{i1}^\pm,v_{i2}^\pm,v_{i3}^\pm)$. The assumption implies that $\rho_1\notin WF(v_{i1}^+|_{y_3=0})\cup WF(v_{i2}^+|_{y_3=0})\cup WF(v_2^+|_{y_3=0})$ and $\rho_1\in WF(v_{i3}^+|_{y_3=0})$, where $\rho_1=(0,t_0,-\varepsilon\omega',\varepsilon\beta_1)$. Thus the boundary condition in (2.3) is reduced to the following

$$ {}^t({}^tV_1^-,{}^tV_2^-)=-((C^-)^{-1}c_{13}^+)v_{13}^++\tilde{G}, \tag{4.9}$$

where $c_{13}^+(y',D_{y'},D_t)$ is the third column vector of $C_1^+(y',D_{y'},D_t)$ and ρ_1 does not belong to $WF(\tilde{G})$. From Lemma 4.3 and (4.9) we see the $\rho_1\in WF(v_{11}^-)\cap WF(v_{13}^-)\cap WF(v_{21}^-)\cap WF(v_{22}^-)$. Using Theorem 2.5.11′ of [3], we have the desired conclusions. The proof is completed.

§ 5. Incident S singularities

Let us consider incident S singularities. So all functions and pseudo-differential operators are defined in Γ_2, where Γ_2 is a conic neighbourhood of $\rho_1=(0,t_0,-\varepsilon\omega',\varepsilon\alpha_1)$. Under the assumption that $\alpha_2<\beta_1<\beta_2$ and $\alpha_1\neq\alpha_2$ we shall show the following

THEOREM 5.1. *We assume that $\beta_1^2<\alpha_1^2/(1-(n(0)\cdot\omega)^2)<\beta_2^2$. Then $\tilde{P}_{\mathrm{in}}(\omega)$ and $\tilde{P}_r(\omega)$ do not exist and there exists a function $H_1(s)$ whose null points are at most* 48 *such that if $H_1(\alpha_1^2/(1-(n(0)\cdot\omega)^2))\neq0$, $S_i(\omega)\subset WF(u_1)$ and $P_{\mathrm{in}}(w)\cap WF(u_1)=\tilde{S}_{\mathrm{in}}(\omega)\cap WF(u_2)=\phi$, then one of the following two cases occurs*; a) *$S_r(\omega)\cup P_{\mathrm{tr}}(\omega)\subset WF(u_1)$ and $\tilde{S}_r(\omega)\subset WF(u_2)$,* b) *$S_r(\omega)\subset WF(u_1)$, $\tilde{S}_r(\omega)\subset WF(u_1)$ and $P_{\mathrm{tr}}(\omega)\cap WF(u_1)=\phi$.*

THEOREM 5.2. *We assume that $\beta_2^2<\alpha_1^2/(1-(n(0)\cdot\omega)^2)$. There exists a function $H_2(s)$ whose null points are at most* 49 *such that if $H_2(\alpha_1^2/(1-(n(0)\cdot\omega)^2))\neq0$, $P_{\mathrm{in}}(\omega)\cap WF(u_1)=(\tilde{P}_{\mathrm{in}}(\omega)\cup\tilde{S}_{\mathrm{in}}(\omega))\cap WF(u_2)=\phi$ and $S_i(\omega)\subset WF(u_1)$, then one of the following two cases occurs*: a) *$S_r(\omega)\subset WF(u_1)$ and*

$\tilde{S}_r(\omega) \cup \tilde{P}_{\mathrm{tr}}(\omega) \subset WF(u_2)$. b) $S_r(\omega) \subset WF(u_1)$, $\tilde{S}_r(\omega) \subset WF(u_2)$ *and* $\tilde{P}_{\mathrm{tr}}(\omega) \cap WF(u_2) = \phi$. *In the above statement is not complete in the following sense: If we suppose that the assumptions mentioned in the above hold and in a small neighbourhood of* 0 *the border* Γ_0 *is equal to a hyperplane in* R^3, *then we have one of the following two cases;* a′) $(S_r(\omega) \cup P_{\mathrm{tr}}(\omega)) \subset WF(u_1)$ *and* $(\tilde{S}_r(\omega) \cup \tilde{P}_{\mathrm{tr}}(\omega)) \subset WF(u_2)$, b′) $S_r(\omega) \subset WF(u_1)$, $\tilde{S}_r(\omega) \subset WF(u_2)$ *and* $P_{\mathrm{tr}}(\omega) \cap WF(u_1) = \tilde{P}_{\mathrm{tr}}(\omega) \cap WF(u_2) = \phi$.

In order to prove the above theorems we need to change the components of V_i in (2.3) corresponding to S waves. Put

$$\begin{aligned} h_1(s) = {} & ((\rho_1 - \rho_2)s - 2(\mu - \mu_2))(\rho_2 s + 2(\mu_1 - \mu_2)) \\ & + 2(\mu_1 - \mu_2)(\rho_1 s - 2(\mu_1 - \mu_2))(a_2 b_2)(s)\,, \\ h_2(s) = {} & \{(\rho_1 - \rho_2)^2 - 4(\mu_1 - \mu_2)^2/(\beta_1^2 \alpha_2^2)]s \\ & - 4(\mu_1 - \mu_2)\{(\rho_1 - \rho_2) - (\mu_1 - \mu_2)(\beta_1^{-2} + \alpha_2^{-2})\}\,, \end{aligned}$$

where $a_2 = (s/\alpha_2^2 - 1)^{1/2}$ and $b_2 = (s/\beta_2^2 - 1)^{1/2}$, then we have the following

LEMMA 5.3. i) *If* $\beta_1^2 < \alpha_1^2/(1 - (n(0)\cdot\omega)^2) < \beta_2^2$, *then there exists a pseudo-differential operator* $a(y', D_{y'}, D_t)$ *of order* 0 *such that the principal symbol of* a *is zero at* $y' = 0$ *and that the* (3, 2) *component of* $(C^-)^{-1}(C_1^+)(I_3 + A)$ *is* 0 *modulo* $L^{-\infty}(\Gamma_2)$, *where* I_3 *is the* 3×3 *identity matrix and* $A(y', D_{y'}, D_t)$ *is a* 3×3 *square matrix whose* (1, 2) *component is* a *and other components are* 0.

ii) *We assume that* $\alpha_1^2/(1 - (n(0)\cdot\omega)^2) > \beta_2^2$. *Then there exists a pseudo-differential operator* $a(y', D_{y'}, D_t)$ *such that if* $h_1(\alpha_1^2/(1 - (n(0)\cdot\omega)^2) \neq 0$, (3.2) *component of* $(C^-)^{-1}(C_1^+)(I_3 + A)$ *is zero modulo* $L^{-\infty}(\Gamma_2)$, *where* A *is a similar pseudo-differential operator to that mentioned in* i). *The similar property on the* (6, 2) *component of* $(C^-)^{-1}(C_1^+)(I_3 + A)$ *holds, if* $h_2(\alpha_1^2(1 - (n(0)\cdot\omega))^2) \neq 0$.

Proof. Define c_j $(j = 1, \cdots, 6)$ and c_j^+ $(j = 1, 2, 3)$ to be the j-th line and column vector of the principal symbol of $(C^-)^{-1}$ and C_1^+, respectively. If we can show that $X = {}^t c_3 \cdot c_1^+$ or $Y = {}^t c_6 \cdot c_1^+$ is not zero at $y' = 0$, we have the statements of Lemma 5.3, by using the calculus on symbols of pseudo-differential operators. Let f_1 be the first column vector of the principal symbol of C^-. Then $X = {}^t c_3 \cdot (c_1^+ - f_1)$ is equal to $2a_1^+ |\eta'|^2 \Lambda_1^{-3}(c_{31} - 2\mu_1 c_{36})$ at $y' = 0$ and $Y = {}^t c_6 \cdot (c_1^+ - f_1)$ is equal to $2a_1^+ |\eta'|^2 \Lambda_1^{-3}(c_{61} - 2\mu_1 c_{66})$ at $y' = 0$, where c_{ij} is the (i, j) component of the principal symbol of $(C^-)^{-1}$.

Using (2.4), we can easily derive that

$$c_{31} = |\eta'|^4 \Lambda_1^{-8}[\{(\rho_1 - \rho_2)\tau^2 - 2(\mu_1 - \mu_1)|\eta'|^2\}(\rho_2\tau^2 - 2\mu_2|\eta'|^2) - 2\mu_2 a_2^- b_2^-\{(\rho_1 - \rho_2)\tau^2 - 2(\mu_1 - \mu_2)|\eta'|^2\} - 2\rho_2 b_2^-(\mu_2 a_2^- - \mu_1 a_1^-)\tau^2],$$

$$c_{36} = -|\eta'|^4 \Lambda_1^{-8}[\{(\rho_1 - \rho_2)\tau^2 - 2(\mu_1 - \mu_2)|\eta'|^2\}(|\eta'|^2 + a_2^- b_2^-) - \rho_2 b_2^-(a_1^- - a_2^-)\tau^2],$$

$$c_{61} = -|\eta'|^4 \Lambda_1^{-8}[\{(\rho_1 - \rho_2)\tau^2 - 2(\mu_1 - \mu_2)|\eta'|^2\}(\rho_1\tau^2 - 2\mu_1|\eta'|^2) + 2\mu_1 a_1^- b_1^-\{(\rho_1 - \rho_2)\tau^2 - 2(\mu_1 - \mu_2)|\eta'|^2\} + 2\rho_1 b_1^-(\mu_2 a_2^- - \mu_1 a_1^-)\tau^2],$$

$$c_{66} = -|\eta'|^4 \Lambda_1^{-8}[\{(\rho_1 - \rho_2)\tau^2 - 2(\mu_1 - \mu_2)|\eta'|^2\}(|\eta'|^2 + a_1^- b_1^-) - \rho_1 b_1^-(a_1^- - a_2^-)\tau^2].$$

Thus if b_2^- is pure imaginary, $c_{31} - 2\mu_1 c_{36}$ does not vanish. When b_2^- is real, $c_{31} - 2\mu_1 c_{36}$ is not zero at ρ_1, if $h_1(\alpha_1^2/(1 - (n(0)\cdot\omega)^2)) \neq 0$, and $c_{61} - 2\mu_1 c_{36}$ is not zero at ρ_1, if $h_2(\alpha_1^2/(1 - (n(0)\cdot\omega)^2) \neq 0$. The proof is completed.

Next we shall check the ellipticity of the components of $(C^-)^{-1}(C_1^+)$. Put

$$h_3(s) = (\rho_1 s - 2\mu_1)(\rho_2 s - 2(\mu_1 - \mu_2))b_2(s) + \rho_1 s(\rho_2 s + 2(\mu_1 - \mu_2))b_1(s) + 2\mu_1(\rho_1 s - 2(\mu_1 - \mu_2))a_2(s) + 4\mu_1(\mu_1 - \mu_2)(a_1 b_1)(s)(b_2 - a_2)(s),$$

where $b_2(s)$ is equal to $i \in (1 - s/\beta_2^2)^{1/2}$, if $\beta_1^2 < \alpha_1^2/(1 - (n(0)\cdot\omega)^2) < \beta_2^2$ and is equal to $(s/\beta_2^2 - 1)^{1/2}$, if $\beta_2^2 < \alpha_1^2/(1 - (n(0)\cdot\omega)^2)$. Then we have the following

LEMMA 5.4. *Let $d_{ij}(y', D_{y'}, D_t)$ be the (i, j) component of $(C^-)^{-1}C_1^+$.*

i) *When $\beta_1^2 < \alpha_1^2/(1 - (n(0)\cdot\omega)^2) < \beta_2^2$, d_{31} is elliptic at $\rho_1 = (0, t_0, -\varepsilon\omega', \varepsilon\alpha_1)$ and $(d_{ij})_{i=5, j=1,2}$ is also elliptic at ρ_1, if $h_3(\alpha_1/(1 - (n(0)\cdot\omega)^2) \neq 0$.*

ii) *When $\beta_2^2 < \alpha_1^2/(1 - n(0)\cdot\omega)^2)$, d_{31} is elliptic at ρ_1, if $h_1(\alpha_1^2/(1 - (n(0)\omega)^2) \neq 0$, d_{61} is elliptic at ρ_1, if $h_1(\alpha_1^2/(1 - (n(0)\cdot\omega)^2) \neq 0$, and $(d_{ij})_{i=4,5, j=1,2}$ is elliptic at ρ_1, if $h_3(\alpha_1/(1 - (n(0)\cdot\omega)^2) \neq 0$.*

Proof. The ellipticity of d_{31} and d_{61} at ρ_1 proved in the proof of Lemma 5.3. From (2.4) it follows that the principal symbols of d_{42} and d_{51} are zero at ρ_1 and the one of d_{52} is not zero at ρ_1. Thus we may prove that the principal symbol of d_{41} is not zero. By the same way as in the proof of Lemma 5.3, one of equivalent conditions of the ellipticity of d_{41} at ρ_1 is the principal symbol of $f_{41} - 2\mu_1 f_{46}$ is not zero at ρ_1, where f_{ij} is the (i, j) component of the principal symbol of $(C^-)^{-1}$. Making use of (2.4), we can easily derive that $f_{41} - 2\mu_1 f_{46}$ is equal to $-|\eta'|^7 \Lambda_1^{-7} h_3(\tau^2/|\eta'|^2)$ at $y' = 0$. The proof is completed. To prove the theorems we need the following function

$$\begin{aligned} h_4(s) = &\{(\rho_1 - \rho_2)s - 2(\mu_1 - \mu_2)\}^2 - \{\rho_2 s + 2(\mu_1 - \mu_2)\}^2(a_1 b_1)(s) \\ &+ \{\rho_1 s - 2(\mu_1 - \mu_2)\}^2(a_2 b_2)(s) - \rho_1\rho_2\{(a_2 b_1)(s) - (a_1 b_2)(s)\}s^2 \\ &- 4(\mu_1 - \mu_2)^2(a_1 b_1 a_2 b_2)(s)\,, \end{aligned}$$

where $a_i(s) = (s/\alpha_i^2 - 1)^{1/2}$ $(i = 1, 2)$, $b_1(s) = (s/\beta_1^2 - 1)^{1/2}$ and $b_2(s)$ is $i\varepsilon(1 - s/\beta_2^2)^{1/2}$ if $\beta_1^2 < \alpha_1^2/(1 - n(0)\cdot\omega)^2) < \beta_2^2$ and is $(s/\beta_2^2 - 1)^{1/2}$ if $\beta_2^2 < \alpha_1^2/(1 - (n(0)\cdot\omega)^2)$.

Proof of Theorem 5.1. We shall use the same notations appeared in the proof of Theorem 4.1 and $H_1(s) = (h_3 h_4)(s)$. By Lemma 5.3 there exists a pseudo-differential operator $a(y', D_{y'}, D_t)$ such that the (3, 2) component of $(C^-)^{-1}C_1^+(I_3 + A)$ is essentially zero. We shall put

$$ {}^t(\tilde{v}_{11}^+, \tilde{v}_{12}^+) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{-1t} (\tilde{v}_{11}^+, \tilde{v}_{12}^+)\,. $$

Then they satisfy a hyperbolic equation $(D_{y_3} - \tilde{A}_1^+)^t(\tilde{v}_{11}^+, \tilde{v}_{12}^+) = g$ in $y_3 > 0$, where g is smooth at ρ_1 and the principal symbol $\tilde{A}_1^+(y', D_{y'}, D_t)$ is $\tilde{a}_1^+(y', \eta', \tau)I_2$. From the assumption the boundary condition in (2.3) is reduced to

$$ \begin{pmatrix} V_1^- \\ V_2^- \end{pmatrix} = F(y', D_{y'}, D_t)\begin{pmatrix} \tilde{v}_{11}^+ \\ \tilde{v}_{12}^+ \end{pmatrix} + G \qquad \text{on } y_3 = 0\,, \tag{5.1} $$

where $\rho_1 \notin WF(G)$ and the first and second column vectors of the 6×2 matrix $F = (c_{ij})$ are equal to these of $-(C^-)^{-1}C_1^+(I_3 + A)$. The assumption $S_i(\omega) \subset WF(u_1)$ is equivalent to $\rho_1 \in WF(\tilde{v}_{11}^+|_{y_3=0}) \cup WF(\tilde{v}_{12}^+|_{y_3=0})$. From Lemma 5.4 it follows that $\rho_1 \in WF(v_{21}^-|_{y_3=0}) \cup WF(v_{22}^-|_{y_3=0})$, which means $\tilde{S}_r(\omega) \subset WF(u_2)$. If we assume that $\rho_1 \in WF(\tilde{v}_{11}^+|_{y_3=0})$, then from the third component of the right hand side of (5.1) we see that $\rho_1 \in WF(v_{13}^-|_{y_3=0})$, that is $P_{\mathrm{tr}}(\omega) WF(u_1)$. On the other hand if $\rho_1 \notin WF(\tilde{v}_{11}^+|_{y_3=0})$, then by the same reason it follows that $\rho_1 \notin WF(v_{13}|_{y_3=0})$, that is $P_{\mathrm{tr}}(\omega) \cap WF(u_1) = \phi$. Finally we shall show that $\rho_1 \in WF(v_{11}^-|_{y_3=0}) \cup WF(v_{12}^-|_{y_3=0})$, if $h_4(\alpha_1^2/(1 - (n(0)\cdot\omega)^2) \neq 0$. We assume $\rho_1 \notin WF(v_{11}^-|_{y_3=0}) \cup WF(v_{12}^-|_{y_3=0})$, then from the assumptions it follows that does not belong to the wave front set of $F_1(y', D_{y'}, D_t)^t(\tilde{v}_{11}^+, \tilde{v}_{12}^+, v_{13}^-, {}^tV_2)|_{y_3=0}$, where the first and second column vectors of F_1 are equal to these of $C_1^+(I_3 + A)$, the third column vector of F_1 is equal to one of C_1^-, and the fourth, fifth and sixth column vectors of F_1 are equal to these of $-C_2$. If F_1 is elliptic at ρ_1, we have ρ_1 does not belong to $WF(\tilde{v}_{11}^+|_{y_3=0}) \cup WF(\tilde{v}_{12}^+|_{y_3=0})$. This is a contradiction. From (2.4) if $h_4(\alpha_1^2/(1 - (n(0)\cdot\omega)^2)$ is not zero, F_1 is elliptic at ρ_1. The proof is completed. Next we shall consider the case $\beta_2^2 < \alpha_1^2/(1 - (n(0)\cdot\omega)^2)$.

Proof of Theorem 5.2. We put $H_2(s) = (h_2h_3h_4)(s)$. Using Lemma 5.3, we can prove the first part by the same way as in the proof of Theorem 5.1. If we assume that Γ_0 is flat near 0, then the reduced boundary value problem (2.3) does not depend on y'. Thus the symbol of $C_i^{\pm}(D_{y'}, D_t)$ is given by (2.4). It follows that the (3, 2) and (6, 2) components of $(C^-)^{-1}C_1^+$ is both zero. Using this fact, we can prove the later part of Theorem 5.1.

In the statement of the first part of Theorem 5.2 we only consider refracted singularities, however on reflected singularities we have the following

Remark 5.5. In assumptions of Theorem 5.2 we assume $(h_1h_3h_4)(\alpha_1^2/(1-(n(0)\cdot\omega))^2) \neq 0$ instead of $(h_2h_3h_4)(\alpha_1^2/(1-(n(0)\cdot\omega))^2) \neq 0$. Then making use of the statement ii) of Lemma 5.3 we have the following two cases: a$''$) $S_r(\omega) \cup P_{\mathrm{tr}}(\omega) \subset WF(u_1)$ and $\tilde{S}_r(\omega) \subset WF(u_2)$. b$''$) $S_r(\omega) \subset WF(u_1)$, $\tilde{S}_r(\omega) \subset WF(u_2)$ and $P_{\mathrm{tr}}(\omega) \cap WF(u_1) = \phi$.

REFERENCES

[1] G. Duvaut and J. L. Lions, Inequalities in mechanics and physics, Grundlehren d. Math. Wiss., **219**, Springer-Verlag, 1976.
[2] L. Hörmander, Linear partial differential operators, Grundlehren d. Math. Wiss., **116**, Springer-Verlag, 1963.
[3] ——, Fourier integral operator I, Acta Math., **127** (1971), 79–183.
[4] ——, Fourier integral operators II, Acta Math., **128** (1972), 183–269.
[5] L. Nirenberg, Lectures on linear partial differential equations, Regional Conferences Series in Math., **17**, Providence 4, R.I., 1973.
[6] R. Melrose and J. Sjostrand, Singularities of boundary value problems I, Comm. Pure Appl. Math., **31** (1978), 593–617.
[7] R. Stoneley, Elastic waves at the surface of separation of two solids, Proc. Roy. Soc. London, **106** (1924), 416–428.
[8] M. E. Taylor, Reflection of singularities of solutions to systems of differential equations, Comm. Pure Appl. Math., **28** (1975), 457–478.
[9] K. Yamamoto, Singularities of solutions to the boundary value problems for elastic and Maxwell's equations, Japanese J. Math., **14** (1988), 119–163.
[10] ——, Reflective elastic waves at the boundary as a propagation of singularities phenomenon, (to appear).
[11] C. H. Wilcox, Scattering Theory for the d'Alembert Equation in Exterior Domains, Lecture Notes in Math., **442**, Springer-Verlag, 1975.

Department of Mathematics
Nagoya Institute of Technology
Nagoya 466, Japan