Proceedings of the Edinburgh Mathematical Society (2023) **66**, 960–970
doi:10.1017/S0013091523000482

PERRON'S CAPACITY OF RANDOM SETS

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(Received 15 March 2023)

Abstract We answer in a probabilistic setting two questions raised by Stokolos in a private communication. Precisely, given a sequence of random variables $\{X_k : k \geq 1\}$ uniformly distributed in $(0, 1)$ and independent, we consider the following random sets of directions

$$\Omega_{\text{rand,lin}} := \left\{ \frac{\pi X_k}{k} : k \geq 1 \right\}$$

and

$$\Omega_{\text{rand,lac}} := \left\{ \frac{\pi X_k}{2^k} : k \geq 1 \right\}.$$

We prove that almost surely the directional maximal operators associated to those sets of directions are not bounded on $L^p(\mathbb{R}^2)$ for any $1 < p < \infty$.

Keywords: maximal operator; differentiation of integrals; harmonic analysis; lacunary sets of finite order; real analysis

2020 Mathematics subject classification: Primary 42B25; 26B05;
Secondary 42B35

We denote by $\mathcal{R}$ the collection of all rectangles in the plane; if R belongs to $\mathcal{R}$, we denote by $\omega_R \in (0, \pi)$ the angle that its longest side makes with the Oy-axis. Without loss of generality, we will always suppose that we have actually $0 \leq \omega_R \leq \frac{\pi}{2}$.

1. Introduction

Given any set of directions $\Omega \subset \mathbb{S}^1$, one can define the directional family of rectangle $\mathcal{R}_\Omega$ as

$$\mathcal{R}_\Omega := \{R \in \mathcal{R} : \omega_R \in \Omega\}$$

and then consider the *directional maximal operator* M_Ω defined for $f : \mathbb{R}^2 \to \mathbb{R}$ locally integrable and $x \in \mathbb{R}^2$ as

$$M_\Omega f(x) := \sup_{x \in R \in \mathcal{R}_\Omega} \frac{1}{|R|} \int_R |f| \, .$$

The boundedness property of the operator M_Ω is deeply related to the geometric structure of the set of directions Ω. For example, in the case where $\Omega = \mathbb{S}^1$, the following obstruction of the Euclidean plane (which is also true in higher dimension) allows us to completely describe the boundedness property of the operator $M_{\mathbb{S}^1}$.

Theorem 1. (Kakeya blow with $\mathcal{R}$). *Given any large constant $A \gg 1$, there exists a finite family of rectangles $\{R_i : i \in I\} \subset \mathcal{R}$ such that we have*

$$\left| \bigcup_{i \in I} TR_i \right| \geq A \left| \bigcup_{i \in I} R_i \right| .$$

Here, we have denoted by TR the rectangle R translated along its longest side by its own length

The reader can find a proof of this Theorem in [4]: it follows that given any large constant $A \gg 1$, there exists a bounded set E satisfying the following estimate

$$\left| \left\{ M_{\mathbb{S}^1} \mathbb{1}_E \geq \frac{1}{2} \right\} \right| \geq A \, |E| \, .$$

It suffices to set $E = \cup_{i \in I} R_i$ and to observe that we have the following inclusion:

$$\bigcup_{i \in I} TR_i \subset \left\{ M_{\mathbb{S}^1} \mathbb{1}_E \geq \frac{1}{2} \right\} .$$

The previous estimate easily implies the following.

Theorem 2. *The operator $M_{\mathbb{S}^1}$ is not bounded on L^p for any $p < \infty$.*

Far from being exotic, Theorem 1 has deep implications in harmonic analysis: for example, it is a central part of Fefferman's work in [4] where he disproves the famous *Ball multiplier* conjecture. A natural question is the following: given a set of directions Ω, is it possible to make a Kakeya blow only with the directional family $\mathcal{R}_\Omega$? This question has been investigated by different analyst among which are [2, 5–7, 9] to cite a few. In [1], Bateman answered this question as he classified the $L^p(\mathbb{R}^2)$ behaviour of those operators according to the *geometry* of the set Ω. Precisely, he proved that the notion of *finitely lacunary* for a set of directions were the correct one to consider.

Theorem 3. (Bateman). *We have the following alternative:*

- *If Ω is finitely lacunary, then M_Ω is bounded on L^p for any $p > 1$.*
- *If Ω is not finitely lacunary, then it is possible to make a Kakeya blow with the family $\mathcal{R}_\Omega$. In particular, the operator M_Ω is not bounded on L^p for any $p < \infty$.*

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Let us define the notion of *finite lacunarity* following a nice presentation made by Kroc and Pramanik [8]: we start by defining the notion of *lacunary sequence* and then the notion of *lacunary set of finite order*. We say that a sequence of real numbers $L = \{\ell_k : k \geq 1\}$ is a lacunary sequence converging to $\ell \in \mathbb{R}$ when there exists $0 < \lambda < 1$ such that

$$|\ell - \ell_{k+1}| \leq \lambda |\ell - \ell_k|$$

for any k. For example, the sequences $\left\{\frac{1}{2^k} : k \geq 2\right\}$ and $\left\{\frac{1}{k!} : k \geq 4\right\}$ are lacunary. We define now by induction the notion of lacunary set of finite order.

Definition 4. (Lacunary set of finite order). *A lacunary set of order 0 in $\mathbb{R}$ is a set which is either empty or a singleton. Recursively, for $N \in \mathbb{N}^*$, we say that a set Ω included in $\mathbb{R}$ is a lacunary set of order at most $N+1$ – and write $\Omega \in \Lambda(N+1)$ – wether there exists a lacunary sequence L with the following properties: for any $a, b \in L$ such that $a < b$ and $(a, b) \cap L = \emptyset$, the set $\Omega \cap (a, b)$ is a lacunary set of order at most N, that is, $\Omega \cap (a, b) \in \Lambda(N)$.*

For example, the set

$$\Omega := \left\{\frac{\pi}{2^k} + \frac{\pi}{4^l} : k, l \in \mathbb{N}, l \leq k\right\}$$

is a lacunary set of order 2. In this case, observe that the set Ω cannot be re-written as a monotone sequence, since it has several points of accumulation. We can finally give a definition of a finitely lacunary set.

Definition 5. (Finitely lacunary set). *A set Ω in $[0, \pi)$ is said to be finitely lacunary if there exists a finite number of set $\Omega_1, \ldots, \Omega_M$, which are lacunary of finite order such that*

$$\Omega \subset \bigcup_{k \leq M} \Omega_k.$$

2. Can we apply Bateman's Theorem?

A classic example of set which is known to be not finitely lacunary is the set

$$\Omega_{\text{lin}} = \left\{\frac{\pi}{n} : n \in \mathbb{N}^*\right\}.$$

Indeed, the classic construction of *Perron trees* shows that it is possible to make a Kakeya blow with the family $\mathcal{R}_{\Omega_{\text{lin}}}$: hence, an application of Bateman's Theorem implies that Ω_{lin} is not finitely lacunary. The second classic example of set which is known to be finitely lacunary is the set

$$\Omega_{\text{lac}} = \left\{\frac{\pi}{2^n} : n \in \mathbb{N}^*\right\}.$$

One can see that the set Ω_{lac} is finitely lacunary by definition, and it was in [9] that Nagel, Stein and Waigner proved that the maximal operator $M_{\Omega_{\text{lac}}}$ is bounded on $L^p(\mathbb{R}^2)$ for

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any $1 < p < \infty$; their proof relies on Fourier analysis. *Also, let us say that the comprehension of the sets Ω_{lin} and Ω_{lac} is important because they are the most simple (and smallest) cases of infinite sets which yield maximal operator having different boundedness properties.*

However, even if Bateman's Theorem is extremely satisfying, it appears to be difficult to decide if a given set Ω is finitely lacunary or not. The most striking example was raised by Stokolos: at the present time, it is not known if the set

$$\Omega_{\text{sin,lac}} := \left\{ \frac{\pi \sin(n)}{2^n} : n \in \mathbb{N}^* \right\}$$

is finitely lacunary or not. The main problem of this set of directions is that we have a very poor control on the deterministic sequence $\{\sin(n) : n \geq 1\}$ and that initially, the set Ω_{lac} is finitely lacunary; hence, the perturbations are quite difficult to handle. In [3], with D'Aniello and Moonens, we were able to show that the following set

$$\Omega_{\text{sin,lin}} := \left\{ \frac{\pi \sin(n)}{n} : n \in \mathbb{N}^* \right\}$$

is not finitely lacunary (this was also a set considered by Stokolos). More precisely, we studied the maximal operator $M_{\Omega_{\text{sin,lin}}}$ associated, and improving on concrete techniques, we proved that this operator is not bounded on $L^p(\mathbb{R}^2)$ for any $1 < p < \infty$: the heart of the method relied on the introduction of the *Perron's capacity* of a set of directions. We need some notations to recall our results: given an infinite set of directions $\Omega \subset \mathbb{S}^1$ whose only point of accumulation is 0 and we denote, for notational convenience, by Ω^{-1}, the set $\left\{ \frac{\pi}{u} : u \in \Omega \right\}$, that is

$$\Omega^{-1} := \frac{\pi}{\Omega},$$

and we order Ω^{-1} as a strictly increasing sequence $\{u_k : k \in \mathbb{N}^*\}$. With those notations, we define the *Perron's factor* of Ω as

$$G(\Omega) := \sup_{\substack{k \geq 1 \\ l \leq k}} \left(\frac{u_{k+2l} - u_{k+l}}{u_{k+l} - u_k} + \frac{u_{k+l} - u_k}{u_{k+2l} - u_{k+l}} \right).$$

In [6], Hare and Ronning proved the following Theorem.

Theorem 6 (Hare and Ronning). *If we have $G(\Omega) < \infty$, then it is possible to make a Kakeya blow with the family $\mathcal{R}_\Omega$.*

It turns out that it is difficult to compute the Perron factor of the set

$$\Omega_{\text{sin,lin}} = \left\{ \frac{\pi \sin(n)}{n} : n \geq 1 \right\}$$

since the oscillation of the cosinus prevent us to obtain a good description of the increasing sequence $\{u_k : k \in \mathbb{N}^*\}$ associated to $\Omega_{\text{sin,lin}}$. Based on a careful reading of the proof of

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Theorem 6, for an arbitrary set of directions Ω included in $\mathbb{S}^1$, we define its *Perron's capacity* as

$$P(\Omega) := \liminf_{N\to\infty} \inf_{\substack{U\subset\Omega^{-1} \\ \#U=2^N}} G(\Omega) \in [2,\infty],$$

where as before

$$G(\Omega) = \sup_{\substack{k,l\geq 1 \\ k+2l\leq 2^N}} \left(\frac{u_{k+2l}-u_{k+l}}{u_{k+l}-u_k} + \frac{u_{k+l}-u_k}{u_{k+2l}-u_{k+l}} \right)$$

if $U = \{u_1 < \cdots < u_{2^N}\}$. In [3], we proved the following (in contrast with Hare and Ronning Theorem, we do not assume that the set Ω is ordered).

Theorem 7. (D'Aniello, G. and Moonens). *For any set of directions Ω, if we have*

$$P(\Omega) < \infty,$$

then it is possible to make a Kakeya blow with the family $\mathcal{R}_\Omega$. In particular, for any $p < \infty$, one has $\|M_\Omega\|_p = \infty$.

Loosely speaking, if $P(\Omega) < \infty$, then the set Ω contains arbitrary large set which are (more or less) uniformly distributed, and this geometric pattern prevents the set Ω to be finitely lacunary. The advantage of Theorem 7 is that it allows us to make *effective* computation. However, as mentioned earlier, the following case is still unsettled.

Question 1. Is the following set of direction

$$\Omega_{\sin,\text{lac}} := \left\{ \frac{\pi \sin(n)}{2^n} : n \in \mathbb{N}^* \right\}$$

finitely lacunary or not?

3. Results

Our result concerns random sets of directions which are meant to give a *generic* comprehension of the two classic examples Ω_{lin} and Ω_{lac} when they are randomly perturbated. Precisely, we consider the following random sets of slopes

$$\Omega_{\text{rand,lin}} := \left\{ \frac{\pi X_k}{k} : k \geq 1 \right\}$$

and

$$\Omega_{\text{rand,lac}} := \left\{ \frac{\pi X_k}{2^k} : k \geq 1 \right\},$$

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where $\{X_k : k \geq 1\}$ are random variables uniformly distributed in $(0, 1)$ and independent. To begin with, we prove the following Theorem.

Theorem 8. *The Perron's capacities of $\Omega_{rand,lin}$ is finite almost surely, that is, we have almost surely*

$$P(\Omega_{rand,lin}) < \infty.$$

In some sense, Theorem 8 means that if a set Ω presents structured patterns – like large uniformly distributed sequence – then a small perturbation of Ω will still exhibit those patterns. The second result reads as follow.

Theorem 9. *The Perron's capacities of $\Omega_{rand,lac}$ is finite almost surely, that is, we have almost surely*

$$P(\Omega_{rand,lac}) < \infty.$$

The proof of Theorems 8 and 9 relies on the possibility to compute effectively the Perron's capacity of the random sets $\Omega_{\text{rand,lin}}$ and $\Omega_{\text{rand,lac}}$.

4. Proof of Theorem 8

We wish to prove that the Perron's capacity of $\Omega_{\text{rand,lin}}$ is finite almost surely. We are simply going to prove that *the set $\Omega^{-1}_{rand,lin}$ contains small perturbation of arbitrarily long homogeneous sets.* We say that a set H of the form

$$H := H_{a,N} = \{ka : 1 \leq k \leq 2^N\}$$

for some integer $a \in \mathbb{N}^*$ is a *homogeneous set*. The following claim is easy.

Claim 1. *For any $a, N \in \mathbb{N}$, one has $G(H_{a,N}) = 2$.*

We wish to perturb a little a homogeneous set H into a set H' such that the Perron's factor of H' is still controlled. Precisely, fix any $a, N \in \mathbb{N}^*$ and let ϵ be an arbitrary function

$$\epsilon : H_{a,N} \to (0, \infty).$$

Define then the set $H_{a,N}(\epsilon)$ as

$$H' := H_{a,N}(\epsilon) := \{(1 + \epsilon(l))l : l \in H_{a,N}\}.$$

If the perturbation ϵ is small enough compared to the integer N, one can control uniformly $G(H')$.

Proposition 1. *With the previous notations, if we have*

$$2^N \|\epsilon\|_\infty \leq \frac{1}{2},$$

then we have $G(H_{a,N}(\epsilon)) < 6$.

We are now ready to prove Theorem 8. We fix a large integer $N \in \mathbb{N}$ and consider the following set of indices

$$E_N := \left\{k \in \mathbb{N} : |X_k - 1| \leq 2^{-N}\right\}.$$

In other words, an integer k belongs to E_N when X_k is close to 1 with precision 2^{-N}. We claim that this random set E_N contains almost surely large (with at least 2^N points) homogeneous sequences.

Claim 2. *For any* $N \geq 1$*, the set* E_N *contains a homogeneous set of cardinal* 2^N *almost surely.*

Proof. Observe that for any $a \in \mathbb{N}^*$, the following probability $\mathbb{P}(H_{a,N} \subset E_N)$ is independent of a: indeed since the random variables $\{X_k : k \geq 1\}$ are independent and uniformly distributed, we have

$$\mathbb{P}(H_{a,N} \subset E_N) = \prod_{k \in H_{a,N}} \mathbb{P}\left(|X_k - 1| \leq 2^{-N}\right) = \frac{1}{2^{N2^N}}.$$

Hence, we fix a sequence $\{a_i\}_{i\geq 1}$ satisfying for any $i \neq j$

$$H_{a_i,N} \cap H_{a_j,N} = \emptyset.$$

For example, setting $a_i = 2^{2N(i+1)}$ works since we have $a_i 2^N < a_{i+1}$ for any $i \geq 1$. In particular, this means that the events

$$\{\left(H_{a_i,N} \subset E_N\right) : i \geq 1\}$$

are independent, and since we have

$$\sum_{i\geq 1} \mathbb{P}(H_{a_i,N} \subset E_N) = \infty,$$

an application of Borel–Cantelli lemma yields the conclusion. □

We can now conclude the proof: we define a perturbation ϵ for any $n \geq 1$ as

$$1 + \epsilon(n) = X_n^{-1}.$$

We fix a large integer $N \gg 1$, and we know that almost surely there exists an integer $a \in \mathbb{N}^*$ such that $H_{a,N} \subset E_N$. Observe now that by definition one has the following

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inclusion

$$H_{a,N}(\epsilon) \subset \Omega_{\text{rand,lin}}^{-1}.$$

However, since $H_{a,N} \subset E_N$ and that it is not difficult to see that we have

$$\|\epsilon_{|H_{a,N}}\|_\infty \lesssim 2^{-N}.$$

Indeed, for $k \in H_{a,N} \subset E_N$, we have

$$|1+\epsilon(k)| = \left|\frac{1}{1+(X_k-1)}\right| \lesssim 1 + 2\,|X_k - 1| \lesssim 1 + 2^{-N}.$$

It follows that we have $P(\Omega_{\text{rand,lin}}) < 6$ almost surely applying Proposition 1.

5. Proof of Theorem 9

We wish to prove that the Perron's capacity of $\Omega_{\text{rand,lac}}$ is finite almost surely. Observe that if U is a set in $\mathbb{R}$ who is well distributed, then one can control its Perron's capacity.

Claim 3. *If we have $\delta > 0$ and a set*

$$U = \left\{u_1 < \cdots < u_{2^N}\right\}$$

such that for any $1 \leq i \leq 2^N - 1$, we have

$$\delta \leq u_{i+1} - u_i \leq 3\delta,$$

then one has $G(U) \lesssim 1$.

Proof. For any i, j such that $i + j \leq 2^N - 1$, one has $u_{i+j} - u_i \simeq j\delta$. Hence, i, j such that $i + 2j \leq 2^{N-1}$, we have

$$\frac{u_{i+2j} - u_{i+j}}{u_{i+j} - u_i} + \frac{u_{i+j} - u_i}{u_{i+2j} - u_{i+j}} \simeq \frac{2j\delta}{j\delta} + \frac{j\delta}{2j\delta} \simeq 1.$$

Hence, we obtain $G(U) \lesssim 1$ as claimed □

We are going to prove the following proposition.

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Proposition 2. *For any $N \geq 1$, there exists almost surely a scale $\delta > 0$ and a set $U \subset \Omega_{rand,lac}^{-1}$ such that*

$$U := \left\{u_1 < \cdots < u_{2N-1}\right\}$$

and for any $i \leq 2^{N-1}$, one has

$$\delta < u_{i+1} - u_i < 3\delta.$$

Theorem 9 is a consequence of Claim 3 and Proposition 2: for any N, we can exhibit almost surely a set $U \subset \Omega_{\mathrm{rand,lac}}^{-1}$ of cardinal 2^N such that $G(U) \lesssim 1$, and so we obtain

$$P(\Omega_{\mathrm{rand,lac}}) < \infty$$

as expected. The rest of the section is devoted to the proof of Proposition 2.

Proof of Proposition 2

We consider the following dyadic intervals for $d \in \mathbb{N}$

$$I_d := \left[2^d, 2^{d+1}\right].$$

We wish to obtain information on the distribution of the points of the set $\Omega_{\mathrm{rand,lac}}$ that may be in the interval I_d. We fix a large integer $N \gg 1$, and we divide each dyadic interval I_d into 2^N intervals of same length, that is, for any $1 \leq l \leq 2^N$, we set

$$I_{d,l} = \left[2^d\left(1 + \frac{l-1}{2^N}\right), 2^d\left(1 + \frac{l}{2^N}\right)\right].$$

Claim 4. *For any $d \geq 2^N + 1$ and any $1 \leq l \leq 2^N$, the probability*

$$\mathbb{P}\left(\frac{2^{d-l}}{X_{d-l}} \in I_{d,l}\right) := p_{N,l}$$

is independent of d.

Proof. By definition of $I_{d,l}$, one has $2^{d-l}X_{d-l}^{-1} \in I_{d,l}$ if and only if

$$2^{-l}\left(1 + \frac{l}{2^N}\right)^{-1} \leq X_{d-l} \leq 2^{-l}\left(1 + \frac{l-1}{2^N}\right)^{-1}.$$

One has

$$\mathbb{P}\left(2^{d-l}X_{d-l}^{-1} \in I_{d,l}\right) = 2^{-l}\left(\left(1 + \frac{l-1}{2^N}\right)^{-1} - \left(1 + \frac{l}{2^N}\right)^{-1}\right) := p_{l,N}$$

since the variable X_{d-l} is uniformly distributed in $(0,1)$. □

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We fix an extraction $\{d_s : s \geq 1\}$ satisfying the following property

$$d_{s+1} - d_s > 2^N + 1$$

for any $s \geq 1$; this growth condition will assure that the events we will consider are independent and we will be able to apply Borel–Cantelli lemma when needed. Thanks to Claim 4 and independence, one can see that for any $s \geq 1$, the following probability is independent of s

$$\mathbb{P}\left(\bigcap_{l \leq 2^N} \left(\frac{2^{d_s - l}}{X_{d_s - l}} \in I_{d_s, l}\right)\right) = \prod_{l \leq 2^N} p_{N,l} := \eta_N.$$

Now for any $d \geq 1$, we consider the following event $A_{d,N}$ defined as

$$A_{d,N} := \bigcap_{l \leq 2^N} \left\{\Omega_{\text{rand,lac}}^{-1} \cap I_{d,l} \neq \emptyset\right\}.$$

In other words, the event $A_{d,N}$ occurs when the random set $\Omega_{\text{rand,lac}}^{-1}$ fills each subintervals $\left\{I_{d,l} : 1 \leq l \leq 2^N\right\}$ with at least one point. In particular, observe that we have

$$\bigcap_{l \leq 2^N} \left(\frac{2^{d-l}}{X_{d-l}} \in I_{d,l}\right) \subset A_{d,N}.$$

We claim that the union of those events

$$B_N := \bigcup_{d \geq 1} A_{d,N}$$

occurs almost surely.

Claim 5. *One has* $\mathbb{P}\,(B_N) = 1$.

Proof. Indeed, we have

$$\sum_s \mathbb{P}\left(\bigcap_{l \leq 2^N} \left(\frac{2^{d_s - l}}{X_{d_s - l}} \in I_{d_s, l}\right)\right) = \sum_s \eta_N = \infty.$$

Using Borel–Cantelli lemma, one obtains

$$\mathbb{P}\left(\bigcup_{s \geq 1} A_{N, d_s}\right) = 1.$$

In particular, one has $\mathbb{P}(B_N) = 1$. □

We can now prove Proposition 2 since we have

$$\mathbb{P}\left(\bigcap_{N\geq 1} B_N\right) = 1.$$

Precisely, for any $N \geq 1$, the event B_N occurs almost surely, and this means that there exists a (for each N, we just need one) dyadic interval I_d such that

$$\Omega_{\text{rand,lac}}^{-1} \cap I_{d,l} \neq \emptyset$$

for any $1 \leq l \leq 2^N$. We let u_l be one point in $\Omega_{\text{rand,lac}}^{-1} \cap I_{d,l}$, and we claim that the set

$$U := \left\{u_{2l} : 1 \leq l \leq 2^{N-1}\right\}$$

satisfy the condition of Proposition 2 with $\delta \simeq 2^{d-N}$. In particular, we have $G(U) \lesssim 1$ for $U \subset \Omega_{\text{rand,lac}}^{-1}$ with arbitrary large cardinal. This yields almost surely

$$P(\Omega_{\text{rand,lac}}) \lesssim 1,$$

which concludes the proof.

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