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WHEN CAN ONE DOMAIN ENCLOSE ANOTHER IN $\mathbb{R}^3$?

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Abstract

In this paper, we give a sufficient condition (Theorem) in order that one domain D_1 bounded by a C^2-smooth boundary can be enclosed in, or enclose, another domain D_0 bounded by the same kind of boundary. A same kind of sufficient condition for convex bodies (Corollary) is also obtained.

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1. Introduction

Many mathematicians have been interested in getting sufficient conditions to insure that a given domain D_1 of surface area F_1, bounded by a piecewise smooth boundary ∂D_1, of volume V_1 may be moved 'inside' another domain D_0 of surface area F_0, bounded by a piecewise smooth boundary ∂D_0, of volume V_0. The general principle underlying this investigation can be briefly described as follows.

Let D_0, D_1 be two suitable domains in Euclidean space $\mathbb{R}^n$, for example, two convex bodies with interior points. Let G be the group of rigid motions of $\mathbb{R}^n$ and let m be its (suitably normalized) invariant measure. Then

$$(1)\quad m\{g \in G : gD_1 \subset D_0 \text{ or } gD_1 \supset D_0\} = m\{g \in G : D_0 \cap gD_1 \neq \emptyset\} - m\{g \in G : \partial D_0 \cap g\partial D_1 \neq \emptyset\}.$$

(If D_0, D_1 are not convex, one assumes that their boundaries are connected.) By integral geometric methods it is possible to estimate the measure $m\{g \in G : D_0 \cap gD_1 \neq \emptyset\}$ from below and the measure $m\{g \in G : \partial D_0 \cap g\partial D_1 \neq \emptyset\}$ from above in terms of geometric invariants of D_0 and D_1. This results in an inequality of the form

$$(2) \qquad m\{g \in G : gD_1 \subset D_0 \text{ or } gD_1 \supset D_0\} \geq f(A_0^1, \dots, A_0^k; A_1^1, \dots, A_1^k),$$

where A_i^j is a geometric invariant of D_i ($i = 0, 1$), for example, volume, surface area, total mean curvature of the boundary ∂D_i, etcetera. One can then state the following conclusion: If $f(A_0^1, \dots, A_0^k; A_1^1, \dots, A_1^k) > 0$, then there is a rigid motion g such that either gD_1 is contained in D_0 or gD_1 contains D_0.

In 1941 Hadwiger (see [9]) was the first to use the method of integral geometry to obtain some sufficient conditions in the Euclidean plane $\mathbb{R}^2$. Delin Ren (see [8]) in 1986 obtained other sufficient conditions in $\mathbb{R}^2$. But there was no general result or analogue of Hadwiger's theorem in Euclidian space $\mathbb{R}^n$ ($n \geq 3$) until the works [11, 12, 13, 14, 15] appeared, even if some very strong restrictions are put on the domains involved. (For example, the domains are supposed to be convex bodies and some topological conditions are put on their boundaries and intersection.) The situation of n-dimensional space $\mathbb{R}^n$ ($n \geq 3$) is much more complex and difficult than that of the 2-dimensional plane $\mathbb{R}^2$. All the formulas and method in $\mathbb{R}^2$ cannot be directly transferred. Moreover, the situations and techniques appropriate to $\mathbb{R}^3$ and $\mathbb{R}^{2k}$ ($k \geq 2$) are totally different due to different topological structures.

In this paper we try to obtain other analogues of Hadwiger's theorem in the space $\mathbb{R}^3$. We follow the ideas in [15] and estimate the arc length of the intersection curve $\partial D_0 \cap g\partial D_1$ of the boundaries $\partial D_0, \partial D_1$ of two domains D_0, D_1 in $\mathbb{R}^3$. By restricting the Euler-Poincaré characteristic $\chi(D_0 \cap gD_1)$ of the intersection $\partial D_0 \cap g\partial D_1$ to be at most a finite integer N_0 for each g, a rigid motion in $\mathbb{R}^3$, we obtain a sufficient condition (*Theorem*) to insure that one domain D_0 with smooth boundary ∂D_0 can contain, or be contained in, another domain D_1 with the same kind of boundary ∂D_1. This is a natural assumption: for example, when D_0 and D_1 are convex bodies, $\chi(D_0 \cap gD_1) \leq N_0 = 1$. As an easy consequence of our theorem we obtain a sufficient condition (*Corollary*) for one convex body D_1 with smooth boundary to be contained in, or to contain, another convex body D_0 with the same kind of boundary. As one would expect, the conditions are inequalities involving volumes, surface areas and curvature integrals of the boundaries. Finally, we give an application of our geometric inequality.

2. Main results and the proof

For a C^2-smooth surface Σ in Euclidean space $\mathbb{R}^3$, denote by K its Gaussian curvature and by H the mean curvature. Let $\tilde{K}$, $\tilde{H}$ and $\tilde{H}^{(2)}$, respectively, be the *total Gaussian curvature*, the *total mean curvature* and the *total square mean curvature*, that is

$$\tilde{K} = \int_\Sigma K d\sigma, \qquad \tilde{H} = \int_\Sigma H d\sigma, \qquad \tilde{H}^{(2)} = \int_\Sigma H^2 d\sigma, \tag{3}$$

where $d\sigma$ is the volume element of Σ.

In this paper, we suppose that the domains D_i ($i = 0, 1$) in $\mathbb{R}^3$ are bounded by C^2-smooth surfaces ∂D_i. Denote by V_i the volumes and by F_i the surface areas, respectively. Let $\tilde{K}_i$, $\tilde{H}_i$ and $\tilde{H}_i^{(2)}$ be the total Gaussian curvature, the total mean curvature and the total square mean curvature of ∂D_i, respectively. Denote by $\chi(\cdot)$ the Euler-Poincaré characteristic. We have the following conclusion:

THEOREM. *Let D_i ($i = 0, 1$) be domains in $\mathbb{R}^3$ with connected C^2-smooth boundaries ∂D_i such that for all $g \in G$, the group of rigid motions in $\mathbb{R}^3$, the Euler-Poincaré characteristic $\chi(D_0 \cap gD_1)$ of the intersection $\partial D_0 \cap g\partial D_1$, satisfies $\chi(D_0 \cap gD_1) \le N_0$, a finite integer. Then a sufficient condition for D_1 to be contained in, or to contain, D_0 is*

$$\begin{aligned} &8\pi(V_0\chi(D_1) + V_1\chi(D_0)) + 2(F_0\tilde{H}_1 + F_1\tilde{H}_0) \\ &\quad - N_0 \cdot \pi R\left[3(F_0\tilde{H}_1^{(2)} + F_1\tilde{H}_0^{(2)}) - 2\pi\left(F_0\chi(\partial D_1) + F_1\chi(\partial D_0)\right)\right] > 0, \end{aligned} \tag{4}$$

where R is the smaller radius of the circumscribed balls of D_0 and D_1. Moreover,

(i) *if $V_1 \ge V_0$, then D_0 can be contained in D_1;*
(ii) *if $V_0 \ge V_1$, then D_0 can contain D_1.*

Consider two domains D_i ($i = 0, 1$) with connected C^2-smooth boundaries ∂D_i in $\mathbb{R}^3$, one fixed and the other moving under the group G of rigid motions in $\mathbb{R}^3$. Let the fixed one be D_0 and the moving one be gD_1 for $g \in G$, and let dg be the kinematic density so normalized that the measure of all position about a point is $8\pi^2$. Then we have C-S. Chen's kinematic formula first proved (see [3]) in 1972 and then reproved (see [15]) by this author in 1991 by a different method:

$$\int_{\{g:\partial D_0 \cap g\partial D_1 \ne \emptyset\}} \left(\int_{C_g} \kappa_{C_g}^2 ds\right) dg = 2\pi^3(3\tilde{H}_0^{(2)} - \tilde{K}_0)F_1 + 2\pi^3(3\tilde{H}_1^{(2)} - \tilde{K}_1)F_0, \tag{5}$$

where κ_{C_g} is the curvature of the intersection curve $C_g = \partial D_0 \cap g\partial D_1$ and ds is the arc element. Generically, the intersection curve C_g is composed of several components, that is, several simple closed curves. For every rigid motion $g \in G$, let $C_g = \partial D_0 \cap g\partial D_1 = \bigcup_i^{N_g} C_i$, where N_g is finite for almost all g.

For a C^2-smooth simple closed curve C in $\mathbb{R}^3$, we have the inequality (see [1])

$$L \le R_C \int_C \kappa_C ds, \tag{6}$$

where L is the length of curve C, ds is the arc element, κ_C the curvature of C and R_C the radius of the circumscribed ball of C.

The kinematic fundamental formula of Blaschke (see [9, 4]) reads

$$(7)\ \int_{\{g:D_0\cap gD_1\neq\emptyset\}} \chi(D_0\cap gD_1)dg = 8\pi^2(V_0\chi(D_1)+V_1\chi(D_0))+2\pi(F_0\tilde{H}_1+F_1\tilde{H}_0).$$

PROOF OF THEOREM. By Hölder's inequality and (6) we have

$$(8)\quad \begin{aligned}\left(\int_{C_g}\kappa_{C_g}ds\right)^2 &\le \left(\int_{C_g}1^2\cdot ds\right)\left(\int_{C_g}\kappa_{C_g}^2 ds\right)\\ &= \left(\sum_i^{N_g} L_i\right)\left(\int_{C_g}\kappa_{C_g}^2 ds\right) \le \left(\sum_i^{N_g} R_{C_i}\int_{C_i}\kappa_{C_i}ds\right)\left(\int_{C_g}\kappa_{C_g}^2 ds\right)\\ &\le R_g\left(\sum_i^{N_g}\int_{C_i}\kappa_{C_i}ds\right)\left(\int_{C_g}\kappa_{C_g}^2 ds\right) = R_g\left(\int_{C_g}\kappa_{C_g}ds\right)\left(\int_{C_g}\kappa_{C_g}^2 ds\right),\end{aligned}$$

where R_{C_i} is the radius of the circumscribed ball of C_i, and $R_g = \max\{R_{C_1}, \cdots, R_{C_{N_g}}\}$.

Fenchel's theorem reads

$$(9)\qquad \int_C \kappa_C ds \ge 2\pi,$$

where C is a simple closed curve, with equality holding if and only if C is a plane convex curve. Using this in (8) gives

$$(10)\qquad 2\pi \le \int_{C_g}\kappa_{C_g}ds \le R_g\int_{C_g}\kappa_{C_g}^2 ds \le R\int_{C_g}\kappa_{C_g}^2 ds,$$

where R is the smaller radius of the circumscribed balls of D_0 and D_1. For a fixed $g \in G$, equality in

$$(11)\qquad 2\pi \le \int_{C_g}\kappa_{C_g}ds \le R_g\int_{C_g}\kappa_{C_g}^2 ds$$

holds if and only if C_g is a circle. If for almost all $g \in G$ the equality in

$$(12)\qquad 2\pi \le \int_{C_g}\kappa_{C_g}\,ds$$

holds, then all C_g must be plane convex curves (Fenchel). This will force D_0 and D_1 to be two balls — a result due to Goodey, see [6, 5].

If we integrate the inequality $2\pi \le R\int_{C_g}\kappa_{C_g}^2\,ds$ over $\{g : \partial D_0 \cap g\partial D_1 \neq \emptyset\}$, then by (5) we obtain

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$$\int_{\{g:\partial D_0\cap g\partial D_1\neq\emptyset\}} dg \leq \frac{R}{2\pi}\int_{\{g:\partial D_0\cap g\partial D_1\neq\emptyset\}}\left(\int_{C_g}\kappa_{C_g}^2 ds\right)dg \tag{13}$$
$$= \pi^2 R[(3\tilde{H}_0^{(2)} - \tilde{K}_0)F_1 + (3\tilde{H}_1^{(2)} - \tilde{K}_1)F_0].$$

By our supposition on D_0 and D_1 we have

$$\int_{\{g:D_0\cap gD_1\neq\emptyset\}} \chi(D_0\cap gD_1)dg \leq N_0\int_{\{g:D_0\cap gD_1\neq\emptyset\}} dg. \tag{14}$$

From (7), (13) and (14) we have the kinematic measure of one domain moving to another under the group G of rigid motions in $\mathbb{R}^3$, that is,

$$\begin{aligned}(15)\quad & m\{g\in G : gD_1\subseteq D_0 \quad\text{or}\quad gD_0\subseteq D_1\}\\ &= \int_{\{g:gD_1\subseteq D_0 \text{ or } gD_0\subseteq D_1\}} dg\\ &= \int_{\{g:D_0\cap gD_1\neq\emptyset\}} dg - \int_{\{g:\partial D_0\cap g\partial D_1\neq\emptyset\}} dg\\ &\geq \frac{1}{N_0}[8\pi^2(V_0\chi(D_1) + V_1\chi(D_0)) + 2\pi(F_0\tilde{H}_1 + F_1\tilde{H}_0)]\\ &\qquad -\pi^2 R[3(F_0\tilde{H}_1^{(2)} + F_1\tilde{H}_0^{(2)}) - (F_0\tilde{K}_1 + F_1\tilde{K}_0)]\\ &= \frac{1}{N_0}[8\pi^2(V_0\chi(D_1) + V_1\chi(D_0)) + 2\pi(F_0\tilde{H}_1 + F_1\tilde{H}_0)]\\ &\qquad -\pi^2 R[3(F_0\tilde{H}_1^{(2)} + F_1\tilde{H}_0^{(2)}) - 2\pi\,(F_0\chi(\partial D_1) + F_1\chi(\partial D_0))].\end{aligned}$$

The last equality comes from the *Gauss-Bonnet* formula $\tilde{K}_i = 2\pi\cdot\chi(\partial D_i)$ $(i = 0, 1)$. This proves the theorem.

If D_0 and D_1 are convex bodies in $\mathbb{R}^3$, we have $\chi(D_0) = \chi(D_1) = N_0 = 1$. Denote by W_2^i the *quermassintegrals* [9] of the convex bodies D_i. Then we have the following consequence.

COROLLARY. *Let D_i $(i = 0, 1)$ be two convex bodies in $\mathbb{R}^3$ with C^2-smooth boundaries ∂D_i, and denote by $\tilde{H}_i^{(2)}$ the total square of the mean curvatures of ∂D_i. Then a sufficient condition for a convex body D_0 to contain, or to be contained in, another convex body D_1, is*

$$8\pi(V_0+V_1)+6(F_0W_2^1+F_1W_2^0)-\pi R\Big[3(F_0\tilde{H}_1^{(2)}+F_1\tilde{H}_0^{(2)}) - 4\pi(F_0+F_1)\Big]>0. \tag{16}$$

Moreover,

(i) *if $V_1\geq V_0$, then D_0 can be contained in D_1;*
(ii) *if $V_0\geq V_1$, then D_0 can contain D_1.*

3. Remarks

REMARK 1. Let D_1 be a convex body with diameter $2R$, and let D_0 be a ball of radius R. Since we have neither $D_0 \subset D_1$ nor $D_1 \subset D_0$, the condition (16) cannot hold. Thus we have

$$\begin{aligned} &8\pi(V_0 + V_1) + 6(F_0 W_2^1 + F_1 W_2^0) \\ &\quad -\frac{\pi\delta_m}{2}\left[3(F_0\tilde{H}_1^{(2)} + F_1\tilde{H}_0^{(2)}) - 4\pi(F_0 + F_1)\right] \le 0, \end{aligned} \tag{17}$$

that is,

$$\tilde{H}_1^{(2)} \ge \frac{4}{9}(2 + 3\pi) + \frac{2(1-\pi)}{3R^2\pi}F_1 + \frac{2}{3R^3\pi}V_1 + \frac{2}{R\pi}W_2^1. \tag{18}$$

Formula (18) is an application of our formula (16), that is:

Let Σ be a convex surface in $\mathbb{R}^3$ and $2R$ its diameter. Denote by $d\sigma$, F, V, W and H the volume element, surface area, the volume bounded, the quermassintegral of the convex body bounded by Σ, and mean curvature, respectively. Then we have the inequality

$$\int_\Sigma H^2\,d\sigma \ge \frac{4}{9}(2 + 3\pi) + \frac{2}{R\pi}W + \frac{2(1-\pi)}{3R^2\pi}F + \frac{2}{3R^3\pi}V. \tag{19}$$

Note that the right-hand side of inequality (19) is well-defined for any convex body D, whereas the definition of H and the integral $\int_{\partial D} H^2\,d\sigma$ makes sense only if ∂D is of class C^2. But by inequality (19), for a convex body with piecewise C^2-smooth boundary ∂D, we can estimate the integral $\int_{\partial D} H^2\,d\sigma$.

B.-Y. Chen [2] and others give the following estimate:

$$\int_\Sigma H^2\,d\sigma \ge 4\pi. \tag{20}$$

One would see that (19) can give the estimates which are much bigger than 4π for some convex surfaces.

REMARK 2. It would be interesting to remove the 'smoothness' restriction on the convex bodies involved in the corollary. All the notions except $\tilde{H}^{(2)}$ here are well-defined for non-smooth convex bodies. If we could find a substitute for $\tilde{H}^2$, the consequences in this paper could be interpreted for arbitrary convex bodies. This is definitely worth further investigation.

REMARK 3. Of course, these conditions are not necessary.

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References

[1] Yu. D. Burago 1. A. Zalgaller, *Geometric inequalities* (Springer, Berlin, 1988).
[2] B-Y Chen, *Geometry of submanifolds* (Marcel Dekker, New York, 1973).
[3] C-S. Chen, 'On the kinematic formula of square of mean curvature', *Indiana Univ. Math. J.* **22** (1972-73), 1163-1169.
[4] S. S. Chern, 'On the kinematic formula in the euclidean space of n dimensions', *Amer. J. Math.* **74** (1952), 227-236.
[5] P. Goodey, 'Homothetic ellipsoids', *Math. Proc. Cambridge Philos. Soc.* **93** (1983), 25–34.
[6] P. R. Goodey, 'Connectivity and free rolling convex bodies', *Mathematika* **29** (1982), 249-259.
[7] R. Howard, 'The kinematic formula in riemannian geometry', *Mem. Amer. Math. Soc.*, to appear.
[8] Delin Ren, *Introduction to integral geometry* (Shanghai Press of Sciences and Technology, Shanghai, 1987).
[9] L. A. Santaló, *Integral geometry and geometric probability* (Addision-Wesley, Reading, 1976).
[10] Michael Spivak, *A comprehensive introduction to differential geometry (II)* (Publish or Perish, Houston, 1979).
[11] Gaoyong Zhang, 'A sufficient condition for one convex body containing another', *Chinese Ann. Math. Ser. B* **9** (1988), 447-451.
[12] Jiazu Zhou, 'Analogues of Hadwiger's theorem–sufficient conditions for one convex body to fit another in $\mathbb{R}^3$', preprint.
[13] ———, 'The sufficient condition for a convex body to fit another in $\mathbb{R}^4$', *Proc. Amer. Math. Soc.* **121** (1994), 907–913.
[14] ———, 'Analogues of Hadwiger's theorem in space $\mathbb{R}^n$ and sufficient condition for a convex domain to enclose another', *Acta. Math. Sinica*, to appear.
[15] ———, 'A kinematic formula and analogues of Hadwiger's theorem in space', *Contemp. Math.* **140** (1992), 159–167.
[16] ———, 'Kinematic formulas for total square of mean curvature of hypersurfaces', *Bull. Inst. Math. Acad. Sinica.* **22** (1994), 31–47.

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