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COMPACT SUBSEMIGROUPS OF ($\beta\mathbb{N}, +$) CONTAINING THE IDEMPOTENTS

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The space $\beta\mathbb{N}$ is the Stone–Čech compactification of the discrete space of positive integers. The set of elements of $\beta\mathbb{N}$ which are in the kernel of every continuous homomorphism from $\beta\mathbb{N}$ to a topological group is a compact semigroup containing the idempotents. At first glance it would seem a good candidate for the smallest such semigroup. We produce an infinite nested sequence of smaller such semigroups all defined naturally in terms of addition on $\mathbb{N}$.

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1. Introduction

Given a discrete semigroup $(S, \cdot)$ the operation can be extended to the Stone–Čech compactification βS of S so that $(\beta S, \cdot)$ is a compact right topological semigroup. (See **[12]** for an elementary construction of this extension, with the caution that there βS is left rather than right topological.) As a compact right topological semigroup βS has idempotents **[6**, Corollary 2.10**]**. The existence of these idempotents, especially idempotents in the smallest ideal of βS, has important combinatorial consequences (See **[11]** and **[15]**, for example).

Of special interest are the semigroups $(\mathbb{N}, +)$ and $(\mathbb{N}, \cdot)$, where $\mathbb{N}$ is the set of positive integers. Let $E=\{p\in\beta\mathbb{N}: p+p=p\}$ and let $\Gamma=clE$. It turns out that Γ is a right ideal of $(\beta\mathbb{N}, \cdot)$. This fact provided the first (and for a long time only) proof of the following result: If $\mathbb{N}$ is partitioned into finitely many cells, then there exist sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ such that $FS(\langle x_n\rangle_{n=1}^{\infty})\cup FP(\langle y_n\rangle_{n=1}^{\infty})$ is contained in one cell of the partition **[9**, Theorem 2.6**]**. (Here $FS(\langle x_n\rangle_{n=1}^{\infty})=\{\sum_{n\in F}x_n: F$ is a finite nonempty subset of $\mathbb{N}\}$ and $FP(\langle y_n\rangle_{n=1}^{\infty})=\{\prod_{n\in F}y_n: F$ is a finite nonempty subset of $\mathbb{N}\}$).

It is an intriguing fact that Γ is defined additively, is a right ideal, in particular a subsemigroup, of $(\beta\mathbb{N}, \cdot)$, and yet is not a subsemigroup of $(\beta\mathbb{N}, +)$. In fact there exist idempotents p and q in $(\beta\mathbb{N}, +)$ such that $p+q\notin\Gamma$. (See Section 3 for the easy proof of this latter assertion.) An intriguing and potentially useful problem then arises: *Characterize the smallest compact subsemigroup of* $(\beta\mathbb{N}, +)$ *which contains the set* E *of idempotents*.

We take the points of $\beta\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$. The reader is referred to **[12]**

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for background material. We will often use the fact that $A \in p+q$ if and only if $\{x \in \mathbb{N}: A-x \in q\} \in p$, where $A-x=\{y \in \mathbb{N}: y+x \in A\}$. (And similarly $A \in p \cdot q$ if and only if $\{x \in \mathbb{N}: A/x \in q\} \in p$, where $A/x=\{y \in \mathbb{N}: y \cdot x \in A\}$.)

Homomorphisms to other algebraic structures are a useful tool for investigating the algebraic structure of $\beta\mathbb{N}$. For example, such homomorphisms were used in [**13**] to show that the maximal groups in the smallest ideal of $(\beta\mathbb{N},+)$ contain copies of the free group on 2^{c} generators. Now given any continuous homomorphism from $(\beta\mathbb{N},+)$ to a compact topological group the kernel necessarily contains E. (It also must contain any element of finite order [**1**, Corollary 2.3]. Whether any such exist besides the idempotents is a difficult open problem.)

Let C be the intersection of the kernels of all continuous homomorphisms from $(\beta\mathbb{N},+)$ to arbitrary compact topological groups. (We use "C" for kernel because K standardly represents the smallest ideal.) Then C is a compact semigroup containing E and at first glance seems like a good candidate for the smallest such. This turns out to fail badly, as we shall see.

The set $\Gamma = clE$ can be characterized as follows [**11**, Lemma 2.3(a)]: Let $p \in \beta\mathbb{N}$. Then $p \in \Gamma$ if and only if for every $A \in p$ there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$. In a similar fashion we define sets $S_n \subseteq \beta\mathbb{N}$ for each $n \in \mathbb{N} \backslash \{1\}$ as follows: Let $p \in \beta\mathbb{N}$. Then $p \in S_n$ if and only if for each $A \in p$, there is a sequence $\langle x_t \rangle_{t=1}^{n}$ with $FS(\langle x_t \rangle_{t=1}^{n}) \subseteq A$. (Given an index set J, $FS(\langle x_i \rangle_{i \in J}) = \{\sum_{i \in F} x_i$: F is a finite nonempty subset of $J\}$.) In a similar vein define T and M by agreeing that, given $p \in \beta\mathbb{N}$, $p \in T$ if and only if whenever $A \in p$, there exist some a and some $\langle y_t \rangle_{t=1}^{\infty}$ with $a+FS(\langle y_t \rangle_{t=1}^{\infty}) \subseteq A$ and that $p \in M$ if and only if whenever $A \in p$ and $n \in \mathbb{N}$, there exist $\langle x_t \rangle_{t=1}^{n}$ and $\langle y_t \rangle_{t=1}^{\infty}$ such that $FS(\langle x_t \rangle_{t=1}^{n}) + FS(\langle y_t \rangle_{t=1}^{\infty}) \subseteq A$. It will be shown in Theorem 2.4 that T is the smallest closed left ideal of $(\beta\mathbb{N},+)$ containing the idempotents.

Let I be the semigroup generated by the set E of idempotents and let S_I be the smallest compact subsemigroup of $(\beta\mathbb{N},+)$ containing E. In Section 2 we investigate each of the objects defined above, show that all (except Γ and clI) are semigroups and show that the following pattern of inclusion holds:

$$\Gamma, I \subseteq clI \subseteq S_I \subseteq M \subseteq T \cap \bigcap_{n=2}^{\infty} S_n \subseteq \bigcap_{n=2}^{\infty} S_n \subseteq \ldots S_3 \subseteq S_2 \subseteq C.$$

In Section 3 we show that $\Gamma \backslash I \neq \emptyset$, $I \backslash \Gamma \neq \emptyset$, $T \backslash \bigcap_{n=1}^{\infty} S_n \neq \emptyset$, $\bigcap_{n=1}^{\infty} S_n \backslash T \neq \emptyset$, and that all but one of the inclusions displayed above (including "...") is proper. (We have been unable to decide whether $M = T \cap \bigcap_{n=2}^{\infty} S_n$.) In Section 4 we present relationships between these sets and other structures.

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We conclude this introduction by displaying some results which we will utilise later.

Lemma 1.1. (*a*) *Let* $p \in E$ *and let* $A \in p$. *There exists* $\langle x_n \rangle_{n=1}^{\infty}$ *such that* $FS(\langle x_n \rangle_{n=1}^{\infty} \subseteq A$.

(*b*) *Let* $\langle x_n\rangle_{n=1}^{\infty}$ *be a sequence in* $\mathbb{N}$. *There exist* $p\in E$ *such that for all* $m\in\mathbb{N}$, $FS(\langle x_n\rangle_{n=m}^{\infty})\in p$.

Proof. (a) This is what is shown in the Galvin–Glazer proof of the Finite Sum Theorem. See [**5**, Theorem 10.3] or [**12**].

(b) [**10**, Lemma 2.4 and Theorem 2.5]. □

Lemma 1.2. *Let* n *and* r *be in* $\mathbb{N}$. *There is some* $m\in\mathbb{N}$ *such that whenever* $\langle y_t\rangle_{t=1}^{m}$ *is a sequence in* $\mathbb{N}$ *and* $D_1, D_2, \ldots, D_r$ *are subsets of* $\mathbb{N}$ *with* $FS(\langle y_t\rangle_{t=1}^{m})\subseteq\bigcup_{i=1}^{r}D_i$, *there exist* $i\in\{1,2,\ldots,r\}$ *and* $\langle x_t\rangle_{t=1}^{n}$ *with* $FS(\langle x_t\rangle_{t=1}^{n})\subseteq D_i$.

Proof. By the finite version of the Finite Unions Theorem [**8**, p. 82] pick $m\in\mathbb{N}$ such that whenever the finite nonempty subsets of $\{1,2,\ldots,m\}$ are covered by r cells, there will exist pairwise disjont $B_1, B_2, \ldots, B_n$ with all sets of the form $\bigcup_{t\in F}B_t$ in the same cell of the cover (for $\emptyset\neq F\subseteq\{1,2,\ldots,n\}$).

Next let $\langle y_t\rangle_{t=1}^{m}$ and $\langle D_i\rangle_{i=1}^{r}$ be given with $FS(\langle y_t\rangle_{t=1}^{m})\subseteq\bigcup_{i=1}^{r}D_i$. For each $i\in\{1,2,\ldots,r\}$, let $H_i=\{F\subseteq\{1,2,\ldots,m\}$: $F\neq\emptyset$ and $\sum_{t\in F}y_t\in D_i\}$. Pick $i\in\{1,2,\ldots,r\}$ and pairwise disjoint B_1, B_2, B_n with $\bigcup_{j\in F}B_j\in H_i$ whenever $\emptyset\neq F\{1,2,\ldots,n\}$. Let $x_j=\sum_{t\in Bj}y_t$ for $j\in\{1,2,\ldots,n\}$. Then given $\emptyset\neq F\subseteq\{1,2,\ldots,n\}$, $\sum_{j\in F}x_j=\sum_{j\in F}\sum_{t\in Bj}y_t$. Since $\bigcup_{j\in F}B_j\in H_i$ one has that $\sum_{j\in F}x_t\in D_i$. □

The following lemma is apparently originally due to Frolik.

Lemma 1.3. *Let* X *and* Y *be* σ-*compact subsets of* $\beta\mathbb{N}$. *If* $clX\cap clY\neq\emptyset$, *then* $X\cap clY\neq\emptyset$ *or* $Y\cap clX\neq\emptyset$.

Proof. See [**14**, Lemma 1.1]. □

2. Inclusions among semigroups containing the idempotents

We begin by displaying the definitions of the objects we are studying. Recall that $E=\{p\in\beta\mathbb{N}: p+p=p\}$.

Definition 2.1. (a) $C=\{p\in\beta\mathbb{N}$: for any compact topological group G and any continuous homomorphism ϕ from $(\beta\mathbb{N},+)$ to G, $\phi(p)$ is the identity of $G\}$.

(b) For $n\in\mathbb{N}\backslash\{1\}$, $S_n=\{p\in\beta\mathbb{N}$: for all $A\in p$ there exists $\langle x_t\rangle_{t=1}^{n}$ such that $FS(\langle x_t\rangle_{t=1}^{n})\subseteq A\}$.

(c) $T=\{p\in\beta\mathbb{N}$: for all $A\in p$ there exist $a\in\mathbb{N}$ and $\langle y_t\rangle_{t=1}^{\infty}$ such that $a+FS(\langle y_t\rangle_{t=1}^{\infty})\subseteq A\}$.

(d) $M=\{p\in\beta\mathbb{N}$: for all $A\in p$ and all $n\in\mathbb{N}$ there exist $\langle x_t\rangle_{t=1}^{n}$ and $\langle y_t\rangle_{t=1}^{\infty}$ such that $FS(\langle x_t\rangle_{t=1}^{n})+FS(\langle y_t\rangle_{t=1}^{\infty})\subseteq A\}$.

(e) $S_I=\bigcap\{S$: S is a compact subsemigroup of $(\beta\mathbb{N},+)$ and $E\subseteq S\}$.

(f) $I=\bigcap\{S$: S is a semigroup of $(\beta\mathbb{N},+)$ and $E\subseteq S\}$.

(g) $\Gamma=\{p\in\beta\mathbb{N}$: for all $A\in p$ there exists $\langle y_t\rangle_{t=1}^{\infty}$ such that $FS(\langle y_t\rangle_{t=1}^{\infty})\subseteq A\}$.

Lemma 2.2. *Each of the objects defined in Definition 2.1 contains E and all except I are compact.*

Proof. The idempotents are contained in Γ by Lemma 1.1(a). Clearly Γ is contained in each of M, T, and S_n (for $n \in \mathbb{N}\backslash\{1\}$). The idempotents are contained in I and S_I by definition and are contained in C by elementary algebra.

That S_I and C are compact follows from elementary topology. The others all have definitions which begin "for all $A \in p$" (and refer no more to p). If a point p is not in the specified set is has a member A failing the definition. Then clA is a neighbourhood of p missing the specified set. $\square$

We will see in the next section that I is not closed when we show that the inclusion $I \subseteq clI$ is proper.

Lemma 2.3. *Each of the objects defined in Definition 2.1 except Γ is a semigroup.*

Proof. That C, I, and S_I are semigroups follows by elementary algebra.

Let $n \in \mathbb{N}\backslash\{1\}$ and let $p, q \in S_n$. To see that $p+q \in S_n$, let $A \in p+q$. Then $\{x \in \mathbb{N}: A-x \in q\} \in p$ so pick $\langle x_t \rangle_{t=1}^n$ such that $FS(\langle x_t \rangle_{t=1}^n) \subseteq \{x \in \mathbb{N}: A-x \in q\}$. Now $FS(\langle x_t \rangle_{t=1}^n)$ is finite so if $B = \bigcap\{A-a: a \in FS(\langle x_t \rangle_{t=1}^n)\}$ we have $B \in q$. Pick $\langle y_t \rangle_{t=1}^n$ such that $FS(\langle y_t \rangle_{t=1}^n) \subseteq B$. We claim $FS(\langle x_t + y_t \rangle_{t=1}^n) \subseteq A$. To see this let $\emptyset \neq F \subseteq \{1, 2, \ldots, n\}$. Then $\sum_{t \in F} y_t \in B \subseteq A - \sum_{t \in F} x_t$ so $\sum_{t \in F}(x_t + y_t) \in A$.

That T is a semigroup follows from the fact that it is a left ideal which we will present in Theorem 2.4. To see that M is a semigroup, let p, $q \in M$ and let $A \in p+q$. Let $B = \{x \in \mathbb{N}: A-x \in q\}$. Then $B \in p$ so pick $\langle x_t \rangle_{t=1}^n$ and $\langle y_t \rangle_{t=1}^\infty$ such that $FS(\langle x_t \rangle_{t=1}^n) + FS(\langle y_t \rangle_{t=1}^\infty) \subseteq B$. In particular $FS(\langle x_t + y_t \rangle_{t=1}^n) \subseteq B$. Let $D = \cap\{A-a: a \in FS(\langle x_t + y_t \rangle_{t=1}^n)\}$. Then $D \in q$ so pick $\langle z_t \rangle_{t=1}^n$ and $\langle w_t \rangle_{t=1}^\infty$ such that $FS(\langle z_t \rangle_{t=1}^n) + FS(\langle w_t \rangle_{t=1}^\infty) \subseteq D$. Then $FS(\langle z_t + x_t + y_t \rangle_{t=1}^n) + FS(\langle w_t \rangle_{t=1}^\infty) \subseteq A$. $\square$

We shall see in Theorem 2.11 that Γ is not a semigroup.

Theorem 2.4. *T is the smallest closed left ideal of $(\beta\mathbb{N}, +)$ which contains the idempotents and $T = cl\bigcup\{\beta\mathbb{N} + p: p \in E\} = cl\bigcup\{\mathbb{N} + p: p \in E\}$.*

Proof. By Lemma 2.2 T is closed and contains the idempotents. To see that S is a left ideal let $p \in \beta\mathbb{N}$ and $q \in T$. Let $A \in p+q$. Then $\{x \in \mathbb{N}: A-x \in q\} \in p$ so pick x such that $A-x \in q$. Pick a and $\langle y_t \rangle_{t=1}^\infty$ such that $a + FS(\langle y_t \rangle_{t=1}^\infty) \subseteq A-x$. Then $x+a+FS(\langle y_t \rangle_{t=1}^\infty) \subseteq A$.

As a closed left ideal containing the idempotents, $T \supseteq cl\bigcup\{\beta\mathbb{N} + p: p \in E\}$. To complete the proof, we show $T \subseteq cl\bigcup\{\beta\mathbb{N} + p: p \in E\}$. To this end let $q \in T$ and let $A \in q$. Pick a and $\langle y_t \rangle_{t=1}^\infty$ such that $a + FS(\langle y_t \rangle_{t=1}^\infty) \subseteq A$. Pick Lemma 1.1 $p \in E$ with $FS(\langle y_t \rangle_{t=1}^\infty) \in p$. Then $A \in a+p$ so $(clA) \cap (\mathbb{N}+p) \neq \emptyset$. $\square$

Theorem 2.5. (a) $\Gamma \subseteq clI$.

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(*b*) $clI \subseteq S_I$
(*c*) $S_I \subseteq M$
(*d*) $M \subseteq T \cap \bigcap_{n=2}^{\infty} S_n$
(*e*) *For each* $n \in \mathbb{N}\backslash\{1\}$, $S_{n+1} \subseteq S_n$
(*f*) $S_2 \subseteq C$.

Proof. Statements (b), (d) and (e) are trivial and (c) follows immediately from the fact that M is a compact subsemigroup of $\beta\mathbb{N}$ containing the idempotents. By [**11**, Lemma 2.3], $\Gamma = clE$ so (a) holds.

To verify (f), let $p \in S_2$ and let ϕ be a continuous homomorphism from ($\beta\mathbb{N}$, +) to a topological group (G, +) with identity 0. Suppose that $\phi(p) = a \neq 0$. Then $a \neq a + a$ so pick a neighbourhood V of a such that $V \cap (V + V) = \varnothing$. Pick $A \in p$ such that $\phi[clA] \subseteq V$ and pick x_1 and x_2 with $\{x_1, x_2, x_1 + x_2\} \subseteq A$. Then $\phi(x_1 + x_2) \in V \cap (V + V)$, a contradiction. □

The following simple result allows us to tell when a set A has closure intersecting various of our special semigroups. For example, it tells us that for $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}\backslash\{1\}$, $clA \cap S_n \neq \varnothing$ if and only if whenever F is a finite partition of A there exists $B \in F$ and $\langle x_t\rangle_{t=1}^{n}$ with $FS(\langle x_t\rangle_{t=1}^{n}) \subseteq B$. (Let $\mathscr{G} = \{FS(\langle x_t\rangle_{t=1}^{n})$: $\langle x_t\rangle_{t=1}^{n}$ is an n-term sequence in $\mathbb{N}\}$. Then $S_n = \{p \in \beta\mathbb{N}$: for each $A \in p$ there exists $G \in \mathscr{G}$ with $G \subseteq A\}$).

Theorem 2.6. *Let X be a discrete space, let $A \subseteq X$, and let $\mathscr{G} \subseteq \mathscr{P}X$. The following statements are equivalent.*

(*a*) *There exists $p \in clA$ such that for every $B \in p$ there exists $G \in \mathscr{G}$ with $G \subseteq B$.*
(*b*) *Whenever $\mathscr{F}$ is a finite partition of A there exist $B \in \mathscr{F}$ and $G \in \mathscr{G}$ with $G \subseteq B$.*
(*c*) *When $\mathscr{F}$ is finite and $\bigcup\mathscr{F} = A$, there exist $B \in \mathscr{F}$ and $G \in \mathscr{G}$ with $G \subseteq B$.*

Proof. That (a) implies (b) and (b) implies (c) is trivial.

To see that (c) implies (a), it suffices to show that $\{A\} \cup \{\mathbb{N}\backslash B$: $B \subseteq \mathbb{N}$ and for all $G \in \mathscr{G}$, $G\backslash B \neq \varnothing\}$ has the finite intersection property, since any ultrafilter p extending this family is as required by (a). But a failure of the finite intersection property would make $A = \bigcup\mathscr{F}$ where $\mathscr{F}$ is finite and for each $B \in \mathscr{F}$, one has no $G \in \mathscr{G}$ with $G \subseteq B$, contradicting (c). □

Theorem 2.7. *Let $A \subseteq \mathbb{N}$. Then $(clA) \cap \bigcap_{n=2}^{\infty} S_n \neq \varnothing$ if and only if for every $n \in \mathbb{N}$ there exists $\langle x_t\rangle_{t=1}^{n}$ with $FS(\langle x_t\rangle_{t=1}^{n}) \subseteq A$.*

Proof. The necessity is an immediate consequence of Theorem 2.6.

Sufficiency. We have by Lemma 2.2 and Theorem 2.5 that $\{(clA) \cap S_n$: $n \in \mathbb{N}\backslash\{1\}\}$ is a nested collection of closed sets so it suffices to show that $(clA) \cap S_n \neq \varnothing$ for each $n \in \mathbb{N}\backslash\{1\}$. To this end let $n \in \mathbb{N}\{1\}$ and let $\mathscr{F}$ be a finite partition of A. Let $r = |\mathscr{F}|$ and pick m as guaranteed by Lemma 1.2 for n and r. Pick $\langle y_t\rangle_{t=1}^{m}$ with $FS(\langle y_t\rangle_{t=1}^{m}) \subseteq A$. By Lemma 1.2 pick $B \in \mathscr{F}$ and $\langle x_t\rangle_{t=1}^{n}$ with $FS(\langle x_t\rangle_{t=1}^{n}) \subseteq B$. □

The following notion, used to characterize members of C, is of independent interest.

Definition 2.8. Let $A \subseteq \mathbb{N}$. Then A is a *rational approximation set* if and only if whenever F is a finite nonempty subset of $\mathbb{R}$ and $\varepsilon > 0$, there exists some $n \in A$ such that for each $x \in F$ there exists $m \in \mathbb{Z}$ with $|x - m/n| < \varepsilon/n$.

Lemma 2.9. *Let $p \in \beta\mathbb{N}$. The following statements are equivalent.*

(a) $p \in C$;

(b) for each $A \in p$, A is a rational approximation set;

(c) for each $A \in p$, each $x \in \mathbb{R}$, and each $\varepsilon > 0$ there exist $n \in A$ and $m \in \mathbb{Z}$ with $|x - m/n| < \varepsilon/n$.

Proof. To see that (a) implies (b), let $A \in p$ and let finite nonempty $F \subseteq \mathbb{R}$ be given. Write $F = \{x_1, x_2, \ldots, x_k\}$. We view the circle group $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$, denoting by $[x]$ the equivalence class $x + \mathbb{Z}$. Define h: $\mathbb{N} \to \times_{i=1}^{k} \mathbb{T}$ by $h(n) = ([nx_1], [nx_2], \ldots, [nx_k])$. Then h is a homomorphism so the continuous extension h^β: $\beta\mathbb{N} \to \times_{i=1}^{k} \mathbb{T}$ is a homomorphism, as was observed by Milnes [17]. Since $p \in C$, $h^\beta(p) = [0], [0], \ldots, [0])$ so pick $B \in p$ such that $h^\beta[clB] \subseteq \{([y_1], [y_2], \ldots, [y_k])$: for each $i \in \{1, 2, \ldots, k\}$, $-\varepsilon < y_i < \varepsilon\}$. Pick $n \in B \cap A$. Since $n \in B$, pick for each $i \in \{1, 2, \ldots, k\}$, some y_i with $-\varepsilon < y_i < \varepsilon$ such that $[nx_i] = [y_i]$. Given $i \in \{1, 2, \ldots, k\}$, pick $m_i \in \mathbb{Z}$ such that $nx_i = y_i + m_i$ then $-\varepsilon < nx_i - m_i < \varepsilon$ so $|x_i - m_i/n| < \varepsilon/n$.

That (b) implies (c) is trivial.

To see that (c) implies (a), observe that it suffices to show that given any continuous homomorphism ϕ: $\beta\mathbb{N} \to \mathbb{T}$ one has $\phi(p) = [0]$. (See for example the introduction to [1].) So let such ϕ be given and pick $x \in \mathbb{R}$ with $[x] = \phi(1)$. Suppose that $\phi(p) \neq [0]$ and pick $\varepsilon > 0$ such that $\phi(p) \notin \{[y]$: $-\varepsilon \leqq y \leqq \varepsilon\}$. Pick $A \in p$ such that $\phi[clA] \cap \{[y]$: $-\varepsilon \leqq y \leqq \varepsilon\} = \emptyset$. Pick $n \in A$ and $m \in \mathbb{Z}$ such that $|x - m/n| < \varepsilon/n$ and let $y = nx - m$. Then $\phi(n) = [y]$ and $-\varepsilon < y < \varepsilon$, a contradiction. □

Theorem 2.10. *Let $A \subseteq \mathbb{N}$. Then $clA \cap C \neq \emptyset$ if and only if A is a rational approximation set.*

Proof. *Necessity.* Pick $p \in clA \cap C$. By Lemma 2.9, A is a rational approximation set.

Sufficiency. Let $\mathcal{G} = \{B \subseteq \mathbb{N}$: B is a rational approximation set.$\}$ It is an easy consequence of the definition of rational approximation sets that whenever $\mathcal{F}$ is a finite partition of A, one has $\mathcal{F} \cap \mathcal{G} \neq \emptyset$. Thus by Theorem 2.6 there is some $p \in clA$ such that for every $B \in p$ there is some $G \in \mathcal{G}$ with $G \subseteq B$ (and hence $B \in \mathcal{G}$). Then by Lemma 2.9 $p \in C$. □

Theorem 2.11. Γ *is not a semigroup. In fact* $(E + E) \backslash \Gamma \neq \emptyset$.

Proof. Pick by Lemma 1.1 (b) idempotents p and q such that $FS(\langle 2^{2t} \rangle_{t=m}^{\infty}) \in p$ and $FS(\langle 2^{2t+1} \rangle_{t=m}^{\infty}) \in q$ for each $m \in \mathbb{N}$. Let $A = \{\sum_{t \in F} 2^{2t} + \sum_{t \in G} 2^{2t+1}$: F and G are finite nonempty subsets of $\mathbb{N}$ and $\max F < \min G\}$. We claim that $A \in p + q$. To see this it suffices to show that $FS(\langle 2^{2t} \rangle_{t=1}^{\infty}) \subseteq \{x \in \mathbb{N}$: $A - x \in q\}$ so let F be a finite nonempty subset of $\mathbb{N}$ and let $m = \max F + 1$. Then $FS(\langle 2^{2t+1} \rangle_{t=m}^{\infty}) \subseteq A - \sum_{t \in F} 2^{2t}$ so $A - \sum_{t \in F} 2^{2t} \in q$.

Now suppose $p+q\in\Gamma$. Then pick a sequence $\langle y_t\rangle_{t=1}^{\infty}$ with $FS(\langle y_t\rangle_{t=1}^{\infty})\subseteq A$. Pick F_1 and G_1 with $\max F_1<\min G_1$ such that $y_1=\sum_{t\in F_1}2^{2t}+\sum_{t\in G_1}2^{2t+1}$. Let $m=\max G_1+1$. Pick nonempty $H\subseteq\mathbb{N}\backslash\{1\}$ such that 2^{2m} divides $\sum_{t\in H}y_t$. (Take any 2^m elements with all y_t in the same congruence class mod 2^{2m}.) Pick F_2 and G_2 with $\max F_2<\min G_2$ such that $\sum_{t\in H}y_t=\sum_{t\in F_2}2^{2t}+\sum_{t\in G_2}2^{2t+1}$. Since 2^{2m} divides $\sum_{t\in H}y_t$ we have $\min F_2\geqq m$. Thus $y_1+\sum_{t\in H}y_t=\sum_{t\in F_1}2^{2t}+\sum_{t\in G_1}2^{2t+1}+\sum_{t\in F_2}2^{2t}+\sum_{t\in G_2}2^{2t+1}$ where $\max F_1<\min G_1<\max G_1<\min F_2<\max F_2<\min G_2$ so by uniqueness of binary expansions, $y_1+\sum_{t\in H}y_t\notin A$, a contradiction. □

Our proof that clI is not a semigroup is in some respects similar to the proof that Γ is not a semigroup. However, instead of the binary expansion of integers we use the factorial expansion, $x=\sum_{t\in F}a_t\cdot t!$ where each $a_t\in\{1,2,\dots,t\}$. In the proof we also utilize in an incidental fashion the semigroup $(\beta\mathbb{N},\cdot)$.

Theorem 2.12. *clI is not a semigroup. In fact $(E+\Gamma)\backslash clI\neq\varnothing$.*

Proof. Since $\Gamma\subseteq clI$, the second statement implies the first. Let $A=\{\sum_{n\in F}n!+\sum_{n\in G}k\cdot n!\colon F$ and G are finite nonempty subsets of $\mathbb{N}$ and $\max F<\min G$ and $k\in\mathbb{N}$ and $k\leqq\min G\}$. Define $g\colon\mathbb{N}\to\mathbb{N}$ by $g(x)=a_l$ where $x=\sum_{t\in F}a_t\cdot t!$, each $a_t\in\{1,2,\dots,t\}$, and $l=\max F$. That is $g(x)$ is the leftmost nonzero digit in the factorial expansion of x. Denote also by g its continuous extension from $\beta\mathbb{N}$ to $\beta\mathbb{N}$.

We claim that:

$$\text{If } q\in\bigcap_{n=1}^{\infty}cl\mathbb{N}n),\text{ then } g(p+q)=g(q)\text{ for all } p\in\beta\mathbb{N}. \tag{1}$$

To see this, suppose instead there is some $B\subseteq\mathbb{N}$ with $g(p+q)\in clB$ and $g(q)\in cl(\mathbb{N}\backslash B)$. Pick $C\in p+q$ and $D\in q$ with $g[clC]\subseteq clB$ and $g[clD]\subseteq cl(\mathbb{N}\backslash B)$. Since $C\in p+q$ pick $x\in\mathbb{N}$ with $C-x\in q$. Pick $y\in(C-x)\cap D\cap\mathbb{N}x!$. Then $y+x\in C$ so $g(y+x)\in B$. But $g(y+x)=g(y)\in\mathbb{N}\backslash B$, a contradiction.

Next we claim:

$$\text{If } q\in E\text{ and } clA\cap(\beta\mathbb{N}+q)\neq\varnothing,\text{ then } g(q)\in\mathbb{N}. \tag{2}$$

To see this suppose that $g(q)\notin\mathbb{N}$, so that for each k, $D_k=\{m\in\mathbb{N}\colon g(m)>k\}\in q$. Pick $p\in\beta\mathbb{N}$ with $p+q\in clA$. Let $B=\{m+n\colon m,n\in\mathbb{N}$ and $g(n)>g(m)>1$ and $n\in\mathbb{N}m!\}$. We show that $B\in p+q$ which will be a contradiction since $B\cap A=\varnothing$. We claim in fact that for all $x\in\mathbb{N}$, $B-x\in q$. For this, since $q=q+q$, it suffices to show that $(\mathbb{N}x!)\cap D_1\subseteq\{m\in\mathbb{N}\colon (B-x)-m\in q\}$ so let $m\in(\mathbb{N}x!)\cap D_1$. Then $D_{g(m)}\cap\mathbb{N}m!\subseteq(B-x)-m$ since $g(m+x)=g(m))$ so $(B-x)-m\in q$.

Next we claim:

$$\text{If } p\in cl(FS(\langle n!\rangle_{n=1}^{\infty}))\cap\bigcap_{n=1}^{\infty}cl(\mathbb{N}n)\text{ and } r\in\beta\mathbb{N},\text{ then } g(r\cdot p)=r. \tag{3}$$

To see this it suffices to show that for all $n\in\mathbb{N}$, $g(n\cdot p)=n$, so let $n\in\mathbb{N}$ be given. Let $B=\{\sum_{t\in F}t!$: F is a finite nonempty subset of $\mathbb{N}$ and $\min F\geqq n\}$. Then $B\in p$ so $n\cdot B\in n\cdot p$ and $g[n\cdot B]=\{n\}$. Now by Lemma 1.1 pick $p\in E\cap clFS(\langle n!\rangle_{n=1}^{\infty})$ and let $r\in\beta\mathbb{N}\backslash\mathbb{N}$. Now for each $x\in\mathbb{N}$, $x\cdot p\in E$ so $r\cdot p\in\Gamma$. Let $s=p+r\cdot p$. We show that $s\notin clI$. Suppose instead that $s\in clI$. Observe that $A\in s$. Indeed $FS(\langle n!\rangle_{n=1}^{\infty})\subseteq\{x\in\mathbb{N}$: $A-x\in r\cdot p\}$. (Given $\sum_{n\in F}n!$ one sees that $\mathbb{N}\backslash\{1\}\subseteq\{k$: $(A-\sum_{n\in F}n!)/k\in p\}$ by noting that $\{\sum_{n\in G}n!$: $\min G>\max F$ and $\min G\geqq k\}\subseteq(A-\sum_{nt\in F}n!)/k$.)

We claim that $s\in cl\bigcup_{k=1}^{\infty}(I\cap g^{-1}[\{k\}])$. To see this, let $B\in s$. Since $s\in clI$, we have $cl(A\cap B)\cap I\neq\varnothing$ so we may pick $l\in\mathbb{N}$ and $q_1,q_2,\ldots,q_l\in E$ with $q_1+q_2+\cdots+q_l\in cl(A\cap B)$. We may presume $l\geqq 2$. Now by (2) we have $g(ql)\in\mathbb{N}$. Let $k=g(q_l)$. By (1), $g(q_1+q_2+\cdots+q_l)=k$ so $clB\cap(I\cap g^{-1}[\{k\}])\neq\varnothing$.

Now also $s\in cl(\mathbb{N}+r\cdot p)$ so $cl(\mathbb{N}+r\cdot p)\cap cl\bigcup_{k=1}^{\infty}cl(I\cap g^{-1}[\{k\}])\neq\varnothing$ so by Lemma 1.3 either one has some $n\in\mathbb{N}$ with $n+r\cdot p\in cl\bigcup_{k=1}^{\infty}cl(I\cap g^{-1}[\{k\}])\subseteq\bigcap_{m=1}^{\infty}cl(\mathbb{N}m))$ or one has some $q\in\beta\mathbb{N}$ and some $k\in\mathbb{N}$ with $q+r\cdot p\in cl(I\cap g^{-1}[\{k\}])\subseteq g^{-1}[\{k\}]$. The first possibility would imply that $n\in\bigcap_{m=1}^{\infty}cl(\mathbb{N}m)$. The second would imply that $g(q+r\cdot p)=k$ while by (1) and (3) $g(q+r\cdot p)=g(r\cdot p)=r\notin\mathbb{N}$. □

3. The inclusions are proper

We show in this section that the objects mentioned in Theorem 2.5 are all distinct (except that we have been unable to determine whether $M=T\cap\bigcap_{n=2}^{\infty}S_n$). We proceed from the left in the inclusion diagram from the introduction.

Theorem 3.1. *$I\backslash\Gamma\neq\varnothing$ and $\Gamma\backslash I\neq\varnothing$.*

Proof. That $I\backslash\Gamma\neq\varnothing$ follows from Theorem 2.11. That $\Gamma\backslash I\neq\varnothing$ follows from Theorem 2.12 since $E+I\subseteq I$. □

In the following theorem (and the rest of this section) the inclusions hold by Theorem 2.5 (or are completely trivial). We concentrate on establishing the inequalities.

Theorem 3.2. *$\Gamma\subsetneq clI$, $I\subsetneq clI$, and $clI\subsetneq S_I$.*

Proof. That $\Gamma\neq clI$ follows from the fact from Theorem 3.1 that $I\backslash\Gamma\neq\varnothing$. The remaining two conclusions follow from the fact (Theorem 2.12) that clI is not a semigroup. □

We produce in the following lemma another closed subsemigroup of $\beta\mathbb{N}$ containing the idempotents. It was not included in those discussed in Section 2 because its definition is less natural than those defined there. When we write $\sum_{t\in F}a_t\cdot t!$, we shall assume F is finite and nonempty and each $a_t\in\{1,2,\ldots,t\}$.

Lemma 3.3. *Let $B=\{\sum_{t\in F}a_t\cdot t!$: (1) F is a finite nonempty subset of $\mathbb{N}$; (2) for each $t\in F$, $a_t\in\{1,2,\ldots,t\}$; (3) there exists $t\in F$ such that $a_t>1$; and (4) whenever n, $t\in F$ with*

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$t<n$ either $a_t=a_n=1$ or $a_t>a_n\}$. Then $(\bigcap_{n=1}^{\infty} cl\mathbb{N}n)\backslash clB$ is a closed subsemigroup of $(\beta\mathbb{N}, +)$ containing the idempotents.

Proof. To see that it is a semigroup, let $p,q\in(\bigcap_{n=1}^{\infty} cl\mathbb{N}n)\backslash clB$. Then $p+q\in\bigcap_{n=1}^{\infty} cl\mathbb{N}n$ so we only need to show that $\mathbb{N}\backslash B\in p+q$. To this end we let $x\in\mathbb{N}\backslash B$ and show that $(\mathbb{N}\backslash B)-x\in q$. Write $x=\sum_{t\in F}a_t\cdot t!$ and let $m=\max F+1$. We show that $\mathbb{N}m!\subseteq(\mathbb{N}\backslash B)-x$, so let $y\in\mathbb{N}m!$ and write $y=\sum_{t\in G}b_t\cdot t!$ and note that $\min G\geqq m$.

Now $x\notin B$. Assume first that for all $t\in F$, $a_t=1$. If for all $t\in G$, $b_t=1$, we have $y+x\notin B$ so assume for some $n\in G$, $b_n>1$. Pick any $t\in F$. Then $t<n$ and $a_t=1<b_n$ so again $y+x\notin B$. Now assume we have $t<n$ in F with $a_t\leqq a_n$ and it is not the case that $a_t=a_n=1$. Then directly we have $y+x$ fails to satisfy (4) of the definition so $y+x\notin B$.

Now let $p\in E$. Then $p\in\bigcap_{n=1}^{\infty} cl\mathbb{N}n$ so we show that $\mathbb{N}\backslash B\in p$. Suppose instead that $B\in p$ and let $D=\{\sum_{t\in F}t!$: F is a finite nonempty subset of $\mathbb{N}\}$. Then $D\subseteq\mathbb{N}\backslash B$ so if $D\in p$ we are done. Assume $D\notin p$.

Assume that for some $k\geqq 2$, $\{\sum_{t\in F}a_t\cdot t!$: $\min F\geqq k$ and $\{a_t\colon t\in F\}\subseteq\{1,2,\ldots,k\}\}\in p$. Since $p=p+p+\cdots+p$ (k times) and $p\in\bigcap_{n=1}^{\infty} cl\mathbb{N}n$ we have that $\{\sum_{t\in F}a_t\cdot t!\colon |F|\geqq k\}\in p$. Let $E=B\cap\{\sum_{t\in F}a_t\cdot t!$: $\min F\geqq k$ and $|F|\geqq k$ and $\{a_t\colon t\in F\}\subseteq\{1,2,\ldots,k\}\}$. Then $E\in p$ so pick $x\in E$ such that $E-x\in p$. Write $x=\sum_{t\in F}a_t\cdot t!$ and let $m=\max F+1$. Pick $y\in\mathbb{N}m!\cap(E-x)$, and write $y=\sum_{t\in G}b_t\cdot t!$. Since $x\in B$ and $|F|\geqq k$ and each $a_t\leqq k$, there is some $t\in F$ with $a_t=1$. Since $y\in E$, $y\in B$ so $y\notin D$ so there is some $n\in G$ with $b_n>1$. But then $t<n$ and $a_t<b_n$ so $y+x\notin B$ so $y+x\notin E$ a contradiction.

Thus it must be the case that for all $k\in\mathbb{N}$, $E_k=\{\sum_{t\in F}a_t\cdot t!$: $\{a_t\colon t\in F\}\backslash\{1,2,\ldots,k\}\neq\varnothing\}\in p$.Since $B\in p$, pick x such that $B-x\in p$ and write $x=\sum_{t\in F}a_t\cdot t!$. Let $k=\max\{a_t\colon t\in F\}$ and let $m=\max F+1$. Pick $y\in\mathbb{N}m!\cap E_k\cap(B-x)$ and write $y=\sum_{t\in G}b_t\cdot t!$. Pick $n\in G$ such that $b_n>k$ and pick any $t\in F$. Then $t<n$ and $b_n>a_t$ so $y+x\notin B$, a contradiction. □

Theorem 3.4. $S_I\subsetneq M$.

Proof. Let B be as in Lemma 3.3 and let $H=\{\sum_{t\in F}a_t\cdot t!$: whenever n, $t\in F$ with $t<n$ one had $a_t>a_n\}$. Observe that given any $n\in\mathbb{N}$ there exists $\langle x_t\rangle_{t=1}^{n}$ with $FS(\langle x_t\rangle_{t=1}^{n})\subseteq H$. (For example let $x_t=(n+1-t)\cdot(n+t)!$.) Thus by Theorem 2.7 we may pick $p\in clH\cap\bigcap_{n=2}^{\infty}S_n$. By Lemma 1.1 pick $q=q+q\in\bigcap_{m=1}^{\infty}cl(FS(\langle t!\rangle_{t=m}^{\infty}))$. We claim that $p+q\in M\cap clB$ (so that by Lemma 3.3, $p+q\in M\backslash S_I$).

To see that $p+q\in M$, let $A\in p+q$ and let $n\in\mathbb{N}$ be given. Since $\{x\in\mathbb{N}\colon A-x\in q\}\in p$ and $p\in S_n$, pick $\langle x_t\rangle_{t=1}^{n}$ with $FS(\langle x_t\rangle_{t=1}^{n})\subseteq\{x\in\mathbb{N}\colon A-x\in q\}$. Let $D=\bigcap\{A-z\colon z\in FS(\langle x_t\rangle_{t=1}^{n})\}$. Since $D\in q+q$ pick $\langle y_t\rangle_{t=1}^{\infty}$ with $FS(\langle y_t\rangle_{t=1}^{\infty})\subseteq D$. Then $FS(\langle x_t\rangle_{t=1}^{n})+FS(\langle y_t\rangle_{t=1}^{\infty})\subseteq A$.

To see that $B\in p+q$ we show that $H\subseteq\{x\in\mathbb{N}\colon B-x\in q\}$. So let $x\in H$ and write $x=\sum_{t\in F}a_t\cdot t!$. Let $m=\max F+1$. Then $FS(\langle t!\rangle_{t=m}^{\infty})\subseteq B-x$ so $B-x\in q$. □

As we have remarked, we do not know whether $M=T\cap\bigcap_{n=2}^{\infty}S_n$. It is trivial that $T\backslash\bigcap_{n=2}^{\infty}S_n\neq\varnothing$, indeed that $T\backslash S_2\neq\varnothing$. In fact by Theorem 2.5 $S_2\subseteq C$ and trivially $C\subseteq\bigcap_{n=1}^{\infty}cl(\mathbb{N}n)$ while, given any idempotent p, we have by Theorem 2.4 that $1+p\in cl(\mathbb{N}2+1)\cap T$. This suggests replacing T by $T\cap\bigcap_{n=1}^{\infty}cl(\mathbb{N}n)$.

Theorem 3.5. $\bigcap_{n=2}^{\infty} S_n \setminus T \neq \emptyset$, $(T \cap \bigcap_{n=1}^{\infty} cl(\mathbb{N}n)) \setminus S_2 \neq \emptyset$, and $T \cap \bigcap_{n=2}^{\infty} S_n \subsetneq \bigcap_{n=2}^{\infty} S_n$.

Proof. For the first statement, let $A = \bigcup_{n=1}^{\infty} FS(\langle 2^{2^n+i}\rangle_{i=1}^{n})$. By Theorem 2.7, $(clA) \cap \bigcap_{n=2}^{\infty} S_n \neq \emptyset$. It is easy to see however that one cannot get any $t \in \mathbb{N}$ and any sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $t + FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$ (since all elements of A have binary expansions with support restricted to a small segment of $\mathbb{N}$). Thus $(clA) \cap T = \emptyset$.

Now let $B = \bigcup_{k=4}^{\infty} (2 \cdot (k!) + FS(\langle n! \rangle_{n=k+1}^{\infty}))$, so that B consists of all numbers whose rightmost nonzero factorial digit is a 2, occurring at position 4 or above and all other nonzero digits are 1. Then there do not exist $x, y \in B$ with $x + y \in B$. (Given $x, y \in B$ either the rightmost digit of $x + y$ is 4 or there are two digits in the expansion of $x + y$ which are greater than 1.) Thus $(clB) \cap S_2 = \emptyset$.

Now pick by Lemma 1.1 $p = p + p$ with $p \in \bigcap_{m=1}^{\infty} cl(FS(\langle n! \rangle_{n=m}^{\infty}))$ and pick $q \in \beta\mathbb{N} \setminus \mathbb{N}$ with $\{2 \cdot (k!) \colon k \in \mathbb{N}\} \in q$. Then $p, q \in \bigcap_{m=1}^{\infty} cl\mathbb{N}n)$ so $q + p \in \bigcap_{n}^{\infty} = 1\ cl(\mathbb{N}n)$. By Theorem 2.4, $q + p \in T$. Since $\{2 \cdot (k!) \colon k \in \mathbb{N}$ and $k \geqq 4\} \subseteq \{x \in \mathbb{N} \colon B - x \in p\}$, one has $q + p \in clB$.

The last conclusion of the theorem follows from the first. □

The following result is a special case of Theorem 3.9, but its proof is much simpler so we present it separately.

Theorem 3.6. $S_3 \subsetneq S_2$.

Proof. Let $A = \{2^{2m} - 2^{2n} \colon m, n \in \mathbb{N}$ and $m > n\}$. It is easy to see that one cannot get any $x_1, x_2, x_3 \in A$ with $\{x_1 + x_2, x_1 + x_3, x_2 + x_3\} \subseteq A$. Thus $(clA) \cap S_3 = \emptyset$. To see that $(clA) \cap S_2 \neq \emptyset$ we use Theorem 2.6. So let $\mathscr{F}$ be a finite partition of A. For each $F \in \mathscr{F}$, let $B(F) = \{\{n, m\} \colon n, m \in \mathbb{N}$ and $m > n$ and $2^{2m} - 2^{2n} \in F\}$. By Ramsey's Theorem [**8**, p. 7] pick $F \in \mathscr{F}$ and $n < m < r$ in $\mathbb{N}$ with $\{\{n, m\}, \{n, r\}, m, r\}\} \subseteq B(F)$. Let $x_1 = 2^{2m} - 2^{2n}$ and $x_2 = 2^{2r} - 2^{2m}$. Then $x_1 + x_2 = 2^{2r} - 2^{2n}$ so $\{x_1, x_2, x_1 + x_2\} \subseteq F$. □

For our proof of Theorem 3.9 we need the following result. Given a sequence $\langle F_t \rangle_{t=1}^{n}$ of sets we write $FU(\langle F_t \rangle_{t=1}^{n}) = \{\bigcup_{t \in G} F_t \colon G$ is a (finite) nonempty subset of $\{1, 2, \ldots, n\}\}$.

Theorem 3.7. (Nešetřil and Rödl). *Let $r, n \in \mathbb{N}$. There is a finite set $\mathscr{S}$ of finite nonempty sets such that:*

(a) whenever $\mathscr{S} = \bigcup_{i=1}^{r} \mathscr{B}_i$, there exist $i \in \{1, 2, \ldots, r\}$ and pairwise disjoint $F_1, F_2, \ldots, F_n$ in $\mathscr{S}$ with $FU(\langle F_t \rangle_{t=1}^{n}) \subseteq \mathscr{B}_i$ and

(b) there do not exist pairwise disjoint $F_1, F_2, \ldots, F_{n+1}$ in $\mathscr{S}$ with $FU(\langle F_t \rangle_{t=1}^{n+1} \subseteq \mathscr{S}$.

Proof. [**18**, Theorem 1.1]. (Or see [**7**].) (The fact that $\mathscr{S}$ and the members of $\mathscr{S}$ are finite is not stated, but follows from the proof.) □

The following corollary is not stated in [**7**] or [**18**], and we feel it is interesting in its own right.

Corollary 3.8. *Let $n \in \mathbb{N} \setminus \{1\}$. There is a set $A \subseteq \mathbb{N}$ such that*

(a) *whenever* $\mathcal{F}$ *is a finite partition of* A *there exist* $B\in\mathcal{F}$ *and* $\langle x_t\rangle_{t=1}^n$ *in* $\mathbb{N}$ *with* $FS(\langle x_t\rangle_{t=1}^n)\subseteq B$ *and*

(b) *there does not exist* $\langle x_t\rangle_{t=1}^{n+1}$ *with* $FS(\langle x_t\rangle_{t=1}^{n+1})\subseteq A$.

Proof. Pick by Theorem 3.7 a sequence $\langle \mathcal{S}_r\rangle_{r=1}^\infty$ such that

(i) for each $r\in\mathbb{N}$, $\mathcal{S}_r$ is a finite set of finite nonempty subsets of $\mathbb{N}$ and $\max(\bigcup\mathcal{S}_r)<\min(\bigcup\mathcal{S}_{r+1})$;

(ii) for each $r\in\mathbb{N}$, whenever $\mathcal{S}_r=\bigcup_{i=1}^r\mathcal{B}_i$ there exist $i\in\{1,2,\dots,r\}$ and pairwise disjoint $F_1,F_2,\dots,F_n$ in $\mathcal{S}_r$ with $FU(\langle F_t\rangle_{t=1}^n)\subseteq\mathcal{B}_i$ and

(iii) for each $r\in\mathbb{N}$, there do not exist pairwise disjoint $F_1,F_2,\dots,F_{n+1}$ with $FU(\langle F_t\rangle_{t=1}^{n+1})\subseteq\mathcal{S}_r$.

Let $\mathcal{S}=\bigcup_{r=1}^\infty\mathcal{S}_r$. Then

(iv) whenever $\mathcal{F}$ is a finite partition of $\mathcal{S}$, there exist $\mathcal{B}\in\mathcal{F}$ and pairwise disjoint $F_1,F_2,\dots,F_n$ in $\mathcal{S}$ with $FU(\langle F_t\rangle_{t=1}^n)\subseteq\mathcal{B}$, and

(v) there do not exist pairwise disjoint $F_1,F_2,\dots,F_{n+1}$ in $\mathcal{S}$ with $FU(\langle F_t\rangle_{t=1}^{n+1})\subseteq\mathcal{S}$.

Indeed, (iv) is immediate since if $r=|F|$ one has $\mathcal{S}_r\subseteq\mathcal{S}$. To verify (v), suppose we have pairwise disjoint $F_1,F_2,\dots,F_{n+1}$ in $\mathcal{S}$ with $FU(\langle F_t\rangle_{t=1}^{n+1})\subseteq\mathcal{S}$. Observe that, given any $r\in\mathbb{N}$ and any $G\in\mathcal{S}$, $G\in\mathcal{S}_r$ if and only if $\min(\bigcup\mathcal{S}_r)\leqq\min G$ and $\max G\leqq\max(\bigcup\mathcal{S}_r)$. Pick $r\in\mathbb{N}$ with $F_1\in\mathcal{S}_r$. If any $F_t\notin\mathcal{S}_r$ we have by the above observation that $F_1\cup F_t\notin\mathcal{S}$.

Thus each $F_t\in\mathcal{S}_r$ so, again using the observation, $FU(\langle F_t\rangle_{t=1}^{n+1})\subseteq\mathcal{S}_r$, contradicting (iii).

Now let $A=\{\sum_{t\in F}3^t:F\in\mathcal{S}\}$. Given a finite partition $\mathcal{F}$ of A and $B\in\mathcal{F}$, let $\mathcal{G}(B)=\{F\in\mathcal{S}\colon \sum_{t\in F}3^t\in B\}$. Then $\{\mathcal{G}(B)\colon B\in\mathcal{F}\}$ is a finite partition of $\mathcal{S}$ so by (iv), pick $B\in\mathcal{F}$ and pairwise disjoint $F_1,F_2,\dots,F_n$ in $\mathcal{S}$ with $FU(\langle F_t\rangle_{t=1}^n)\subseteq\mathcal{G}(B)$. For $t\in\{1,2,\dots,n\}$, let $x_t=\sum_{i\in F_t}3^i$. Then $FS(\langle x_t\rangle_{t=1}^n)\subseteq B$, so (a) holds.

To verify (b), suppose we have $x_1,x_2,\dots,x_{n+1}$ in $\mathbb{N}$ with $FS(\langle x_t\rangle_{t=1}^{n+1})\subseteq A$. For each $t\in\{1,2,\dots,n+1\}$, pick F_t such that $x_t=\sum_{i\in F_t}3^i$. We claim that the sets $F_1,F_2,\dots,F_{n+1}$ are pairwise disjoint (so that $FU(\langle F_t\rangle_{t=1}^{n+1})\subseteq\mathcal{S}$, contradicting (v)). Suppose instead we have $t\neq s$ with $F_t\cap F_s\neq\varnothing$. Then $x_t+x_s=\sum_{i\in F_t\Delta F_s}3^i+\sum_{i\in F_t\cap F_s}2\cdot 3^i$. But $x_t+x_s\in A$ so for some G, $x_t+x_s=\sum_{i\in G}3^i$, contradicting the uniqueness of ternary expansions. □

Theorem 3.9. *Let* $n\in\mathbb{N}\backslash\{1\}$. *Then* $S_{n+1}\subsetneqq S_n$.

Proof. Pick A as guaranteed by Corollary 3.8. By (b), $(clA)\cap S_{n+1}=\varnothing$ while by (a) and Theorem 2.7, $(clA)\cap S_n\neq\varnothing$. □

Now we need to show that $C\neq S_2$. We will utilize $\beta\mathbb{Z}$. We brush aside the distinction between ultrafilters on $\mathbb{Z}$ with $\mathbb{N}$ as a member and ultrafilters on $\mathbb{N}$, and thus pretend that $\beta\mathbb{N}\subseteq\beta\mathbb{Z}$. Given $p\in\beta\mathbb{N}$ we let $-p=\{-A\colon A\in p\}$ and note that $-p\in\beta\mathbb{Z}$. (But be cautioned that unless $p\in\mathbb{N}$, $-p+p\neq 0$; in fact $\beta\mathbb{N}\backslash\mathbb{N}$ is a left ideal of $\beta\mathbb{Z}$ so if $p\in\beta\mathbb{N}\backslash\mathbb{N}$ then also $-p+p\in\beta\mathbb{N}\backslash\mathbb{N}$.)

Lemma 3.10. *Let* ϕ *be a homomorphism from* $\beta\mathbb{Z}$ *to the circle group* $\mathbb{T}$ *and let* $p\in\beta\mathbb{N}$. *Then* $\phi(-p)=-\phi(p)$.

Proof. Note that the function f: $\beta\mathbb{N}\to\beta\mathbb{Z}$ defined by $f(p)=-p$ is continuous. For all $n\in\mathbb{N}$, $\phi(-n)=-\phi(n)$ (since $\phi|_{\mathbb{Z}}$ is a group homomorphism). Thus $\phi\circ f$ and $-\phi$ are continuous functions agreeing on $\mathbb{N}$, hence on $\beta\mathbb{N}$. □

Lemma 3.11. *Let $\langle x_n\rangle_{n=1}^{\infty}$ be any increasing sequence in $\mathbb{N}$ and let A and B be infinite subsets of $\mathbb{N}$. Let $D=\{x_n+x_m-x_r-x_s\colon n>m+3>m>r+3>r>s+3$ and n, $s\in A$ and m, $r\in B\}$ and let p, $q\in\beta\mathbb{N}\backslash\mathbb{N}$ with $\{x_n\colon n\in A\}\in p$ and $\{x_n\colon n\in B\}\in q$. Then $D\in -p+-q+q+p$ and $-p+-q+q+p\in C$. In particular D is a rational approximation set.*

Proof. To see that $-p+-q+q+p\in C$ it suffices (as is well known and explained in the introduction to [1]) to let ϕ be a homomorphism from $\beta\mathbb{N}$ to $\mathbb{T}$ and show that $\phi(-p+-q+q+p)=[0]$. To this end let such ϕ be given. Define τ: $\mathbb{Z}\to\mathbb{T}$ by $\tau(0)=[0]$, and $\tau(n)=\phi(n)$ and $\tau(-n)=-\phi(n)$ for $n\in\mathbb{N}$. Then the continuous extension τ^{β} of τ to $\beta\mathbb{Z}$ is a homomorphism and τ^{β} agrees with ϕ on $\beta\mathbb{N}$. Thus, using Lemma 3.10, we have $\phi(-p+-q+q+p)=\tau(-p+-q+q+p)=\tau(-p)+\tau(-q)+\tau(q)+\tau(p)=-\tau(p)-\tau(q)+\tau(q)+\tau(p)=[0]$.

It is completely routine to verify that $D\in -p+-q+q+p$. The "in particular" conclusion follows from Lemma 2.9. □

Lemma 3.12. *Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for each $n\in\mathbb{N}$, $x_{n+1}\geqq 2x_n$. Let A and B be disjoint infinite subsets of $\mathbb{N}$ such that for some i, $j\in\{0,1,2\}$ $A\subseteq\mathbb{N}3+i$ and $B\subseteq\mathbb{N}3+j$. Let $D=\{x_n+x_m-x_r-x_s\colon n>m+3>m>r+3>r>s+3$ and n, $s\in A$ and m, $r\in B\}$. There do not exist a, $b\in D$ with $a+b\in D$.*

Proof. Suppose we have a, $b\in D$ with $a+b\in D$ and pick $n_1>m_1+3>m_1>r_1+3>r_1>s_1+3$, $n_2>m_2+3>m_2>r_2+3>r_2>s_2+3$, and $n_3>m_3+3>m_3>r_3+3>r_3>s_3+3$ such that $a=x_{n_1}+x_{m_1}-x_{r_1}-x_{s_1}$, $b=x_{n_2}+x_{m_2}-x_{r_2}-x_{s_2}$, and $a+b=x_{n_3}+x_{m_3}-x_{r_3}-x_{s_3}$ and $\{n_1,n_2,n_3,s_1,s_2,s_3\}\subseteq A$ and $\{m_1,m_2,m_3,r_1,r_2,r_3\}\subseteq B$. Then we have

$$x_{n_1}+x_{m_1}+x_{n_2}+x_{m_2}+x_{r_3}+x_{s_3}=x_{n_3}+x_{m_3}+x_{r_1}+x_{s_1}+x_{r_2}+x_{s_2}. \tag{*}$$

We may assume without loss of generality that $n_1\geqq n_2$. We claim first that $n_1=n_3$. Suppose $n_1<n_3$. Then since n_1, $n_3\in\mathbb{N}3+i$, the left hand side of (*) is at most $x_{n_3-3}+x_{n_3-6}+x_{n_3-3}+x_{n_3-6}+x_{n_3-6}+x_{n_3-9}\leqq x_{n_3-2}+x_{n_3-5}+x_{n_3-6}+x_{n_3-9}<x_{n_3}$, a contradiction. (Observe that for each n, $x_{n+1}>\sum_{t=1}^{n}x_t$.) Similarly if we had $n_3<n_1$ we would have that the right hand side of (*) is at most $x_{n_1-3}+x_{n_1-6}+x_{n_1-6}+x_{n_1-9}+x_{n_1-6}+x_{n_1-9}\leqq x_{n_1-3}+x_{n_1-5}+x_{n_1-8}+x_{n_1-6}<x_{n_1}$. Thus $n_1=n_3$ so we have

$$x_{m_1}+x_{n_2}+x_{m_2}+x_{r_3}+x_{s_3}=x_{m_3}+x_{r_1}+x_{s_1}+x_{r_2}+x_{s_2} \tag{**}$$

Now $n_2\in A$ and $m_1\in B$ so $n_2\neq m_1$. We claim that $n_2<m_1$ so suppose instead that $n_2>m_1$. If $m_3<n_2$ we have (since $n_2>m_1>r_1+3$) that the right hand side of (**) is at most $x_{n_2-1}+x_{n_2-4}+x_{n_2-7}+x_{n_2-6}+x_{n_2-9}<x_{n_2}$, a contradiction. If $m_3>n_2(>m_1)$ we have that the left hand side of (**) is at most $x_{m_3-3}+x_{m_3-1}+x_{m_3-4}+x_{m_3-3}+x_{m_3-6}\leqq x_{m_3-1}+x_{m_3-2}+x_{m_3-4}+x_{m_3-6}<x_{m_3}$, a contradiction. Thus $n_2<m_1$ as claimed.

Now we claim $m_3 = m_1$. Suppose first that $m_3 < m_1$. Then the right hand side of (**) is at most $x_{m_1-3} + x_{m_1-3} + x_{m_1-6} + x_{m_1-7} + x_{m_1-10} < x_{m_1}$, a contradiction. Similarly if $m_1 < m_3$ one has the left hand side of (**) is at most $x_{m_3-3} + x_{m_3-4} + x_{m_3-7} + x_{m_3-3} x_{m_3-6} < x_{m_3}$, a contradiction. Thus $m = m_1$ so we have

$$x_{n_2} + x_{m_2} + x_{r_3} + x_{s_3} = x_{r_1} + x_{s_1} + x_{r_2} + x_{s_2}. \tag{***}$$

Now we claim that $n_2 < r_1$. Suppose not. Then since $n_2 \in A$ and $r_1 \in B$ we have $n_2 > r_1$ so the right hand side of (***) is at most $x_{n_2-1} + x_{n_2-4} + x_{n_2-6} + x_{n_2-9} < x_{n_2}$, a contradiction. Thus $n_2 < \mathrm{r}_1$ as claimed.

Next we claim $r_3 = r_1$. If $r_3 < r_1$ we have the left hand side of (***) is at most $x_{r_1-1} + x_{r_1-4} + x_{r_1-3} + x_{r_1-6} < x_{r_1}$, a contradiction. If $r_3 > r_1 (> n_2)$ we have the right hand side of (***) is at most $x_{r_3-3} + x_{r_3-6} + x_{r_3-10} + x_{r_3-13} < x_{r_3}$. Thus $r_3 = r_1$ so we have

$$x_{n_2} + x_{m_2} + x_{s_3} = x_{s_1} + x_{r_2} + x_{s_2}. \tag{****}$$

Continuing in this fashion we see that if $n_2 = s_3$ then also $n_2 = s_1$ so that $x_{m_2} + x_{s_3} = x_{r_2} + x_{s_2}$ and hence that $m_2 = r_2$ which is a contradiction.

Thus one must have $n_2 \neq s_3$, and hence that $|\{n_2, m_2, s_3\}| = 3$. Now if $s_1 = s_2$ one has $s_1 = s_2 < r_2 < n_2$ so the right hand side of (****) is at most $x_{n_2-9} + x_{n_2-6} + x_{n_2-9} < x_{n_2}$, a contradiction. Thus $s_1 \neq s_2$ so $|\{s_1, r_2, s_2\}| = 3$. Since $x_{n+1} > \sum_{t=1}^{n} x_t$ for each n, expressions in $FS(\langle x_t \rangle_{t=1}^{\infty})$ are unique. Thus from (****) we have $\{n_2, m_2, s_3\} = \{s_1, r_2, s_2\}$ so that $\{m_2\} = \{n_2, m_2, s_3\} \cap B = \{s_1, r_2, s_2\} \cap B = \{r_2\}$ while $r_2 < m_2$. This contradiction completes the proof. □

Theorem 3.13. *Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for each n, $x_{n+1} \geqq 2x_n$. Let A and B be disjoint infinite subsets of $\mathbb{N}$ and let $p, q \in \beta\mathbb{N} \backslash \mathbb{N}$ such that $\{x_n : n \in A\} \in p$ and $\{x_n : n \in B\} \in q$. Then $-p + -q + q + p \in C \backslash S_2$.*

Proof. Pick $i, j \in \{0, 1, 2\}$ such that $\mathbb{N}3 + i \in p$ and $\mathbb{N}3 + j \in q$. Let $A' = A \cap (\mathbb{N}3 + i)$ and $B' = B \cap (\mathbb{N}3 + j)$. Let $D = \{x_n + x_m - x_r - x_s : n > m + 3 > m > r + 3 > r > s + 3$ and $n, s \in A'$ and $m, r \in B'\}$. By Lemma 3.11, $D \in -p + -q + q + p \in p$ and $-p + -q + q + p \in C$. By Lemma 3.12 $-p + -q + q + p \notin S_2$. □

It is natural to ask whether in lieu of $-p + -q + q + p$ above one might be able to get by with $-p + p$ for some suitable p. We conclude this section by showing that this is not possible.

Theorem 3.14. *Let $p \in \beta\mathbb{N} \backslash \mathbb{N}$. Then $-p + p \in S_2$.*

Proof. Let $A \in -p + p$. Then $\{x \in \mathbb{Z} : A - x \in p\} \in -p$ so $B = \{x \in \mathbb{N} : A + x \in p\} \in p$. Pick $x_1 \in B$, pick $x_2 \in B \cap (A + x_1)$, pick $x_3 \in (A + x_1) \cap (A + x_2)$. Let $y = x_2 - x_1$ and let $z = x_3 - x_2$. Then $y, z \in A$ and $y + z = x_3 - x_1 \in A$. □

4. Connections with other structures

The interaction of the operations $+$ and $\cdot$ on $\beta\mathbb{N}$ has been a very useful

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combinatorial tool. (See [3] for an example where this interaction is utilized several times in succession.)

Recall that, given p and q in $\beta\mathbb{N}$ and $A\subseteq\mathbb{N}$, $A\in p\cdot q$ if and only if $\{x\in\mathbb{N}\colon A/x\in q\}\in p$ where $A/x=\{y\in\mathbb{N}\colon y\cdot x\in A\}$.

It is not generally true that for $n\in\mathbb{N}$ and $p\in\beta\mathbb{N}$ one has $n\cdot p=p+p+\cdots+p$ (n-times). (For example one sees easily that if $n\neq 1$ then $n\cdot p\neq p$ while if $p=p+p$, then $p=p+p+\cdots+p$ (n-times).) On the other hand we do have the following lemma. Recall that, given $p\in\beta\mathbb{N}$ and a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in a topological space X, one has $p\text{-}\lim_{n\in\mathbb{N}} x_n=y$ if and only if for each neighbourhood U of y, $\{n\in\mathbb{N}\colon x_n\in U\}\in p$.

Lemma 4.1. *Let $(G,+)$ be a compact topological group, let $\phi\colon\beta\mathbb{N}\to G$ be a continuous homomorphism, let $p\in\beta\mathbb{N}$, and let $n\in\mathbb{N}$. Then $\phi(n\cdot p)=n\cdot\phi(p)$, where $n\cdot\phi(p)=\phi(p)+\cdots+\phi(p)$ (n-times).*

Proof. Recall that the function λ_n defined by $\lambda_n(p)=n\cdot p$ is continuous since $n\in\mathbb{N}$. Recall further that by the joint continuity of addition in G, we have $n\cdot p\text{-}\lim_{m\in\mathbb{N}}\phi(m)=p\text{-}\lim_{m\in\mathbb{N}}n\cdot\phi(m)$. Thus we have $\phi(n\cdot p)=\phi(n\cdot p\text{-}\lim_{m\in\mathbb{N}}m)=p\text{-}\lim_{m\in\mathbb{N}}\phi(n\cdot m)=p\text{-}\lim_{m\in\mathbb{N}}n\cdot\phi(m)=n\cdot p\text{-}\lim_{m\in\mathbb{N}}\phi(m)=n\cdot\phi(p\text{-}\lim_{m\in\mathbb{N}}m)=n\cdot\phi(p)$. □

Theorem 4.2. *C is a two sided ideal of $(\beta\mathbb{N},\cdot)$.*

Proof. Let G be a compact topological group with identity 0 and let $\phi\colon\beta\mathbb{N}\to G$ be a homomorphism. Let $p\in C$ and let $q\in\beta\mathbb{N}$. Pick nets $\langle x_\eta\rangle_{\eta\in D}$ and $\langle y_\tau\rangle_{\tau\in E}$ in $\mathbb{N}$ converging to p and q respectively.

Then $\phi(q\cdot p)=\phi((\lim_{\tau\in E}y_\tau)\cdot p)=\lim_{\tau\in E}\phi(y_\tau\cdot p)=\lim_{\tau\in E}(y_\tau\cdot\phi(p))=\lim_{\tau\in E}(y_\tau\cdot 0)=0$.

Now let $z=\phi(q)$ and define $\tau\colon\mathbb{N}\to G$ by $\tau(n)=n\cdot z$. Then the continuous extension $\tau^\beta\colon\beta\mathbb{N}\to G$ is a homomorphism. Thus $\phi(p\cdot q)=\phi((\lim_{\eta\in D}x_\eta)\cdot q)=\lim_{\eta\in D}\phi(x_\eta\cdot q)=\lim_{\eta\in D}(x_\eta\cdot\phi(q))=\lim_{\eta\in D}\tau(x_\eta)=\lim_{\eta\in D}\tau^\beta(x_\eta)=\tau^\beta(\lim_{\eta\in D}x_\eta)=\tau^\beta(p)=0$. □

Theorem 4.3. *For each $n\in\mathbb{N}\backslash\{1\}$, S_n is a two sided ideal of $(\beta\mathbb{N},\cdot)$.*

Proof. Let $n\in\mathbb{N}$, let $p\in S_n$ and let $q\in\beta\mathbb{N}$.

To see that $p\cdot q\in S_n$, let $A\in q\cdot p$ and pick $y\in\mathbb{N}$ such that $A/y\in p$. Pick $\langle x_t\rangle_{t=1}^{n}$ with $FS(\langle x_t\rangle_{t=1}^{n})\subseteq A/y$. Then $FS(\langle y\cdot x_t\rangle_{t=1}^{n})\subseteq A$.

To see that $p\cdot q\in S_n$, let $A\in p\cdot q$ and pick $\langle x_t\rangle_{t=1}^{n}$ such that $FS(\langle x_t\rangle_{t=1}^{n})\subseteq\{y\in\mathbb{N}\colon A/z\in q)$. Pick $y\in\bigcap\{A/z\colon z\in FS(\langle x_t\rangle_{t=1}^{n})\}$. Then $FS(\langle y\cdot x_t\rangle_{t=1}^{n})\subseteq A$. □

In the process of our study of the semigroup C, we were led to the following result (and its fortuitous corollary). By a divisible sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ we simply mean an increasing sequence with the property that each x_n divides x_{n+1}.

Recall that we are representing the circle group $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$. By $\mathbb{T}^{\mathbb{T}}$ we mean the set of all functions from $\mathbb{T}$ to $\mathbb{T}$ with the product toplogy (="topology of pointwise convergence").

Theorem 4.4. *Define $h\colon\mathbb{N}\to\mathbb{T}^{\mathbb{T}}$ by $h(n)(\alpha)=n\cdot\alpha$ and let h^β be the continuous extension of h to $\beta\mathbb{N}$. Let $\langle x_n\rangle_{n=1}^{\infty}$ be any divisible sequence in $\mathbb{N}$. Then h^β is one-to-one on $cl\{x_n\colon n\in\mathbb{N}\}$.*

Proof. Let p and q be distinct elements of $cl\{x_n: n\in\mathbb{N}\}$. Pick disjoint A and B contained in $\mathbb{N}$ such that $\{x_n: n\in A\}\in p$ and $\{x_n: n\in B\}\in q$. Since $\{x_n: n\in\mathbb{N}\}=\bigcup_{i=0}^{2}\{x_n: n\equiv i(\text{mod}\,3)\}$ we may presume we have some $i\in\{0,1,2\}$ such that for all n, $m\in A$, $n\equiv m(\text{mod}\,3)$. As a consequence, if n, $m\in A$ and $n<m$ then $n+3\leqq m$ so $x_m\geqq x_{n+3}\geqq 8\cdot x_n$. Now let $t=\sum_{n\in A}\lfloor x_{n+1}/(2x_n)\rfloor/x_{n+1}$, where $\lfloor\,\rfloor$ denotes the greatest integer function. (Since $\langle x_n\rangle_{n=1}^{\infty}$ is a divisible sequence we have each $x_n\geqq 2^{n-1}$so $\lfloor x_{n+1}/(2x_n)\rfloor/x_{n+1}\leqq 1/(2x_n)\leqq 1/2^n$ so the series defining t converges (and $0<t<1$). As before write $[t]=t+\mathbb{Z}$. We show that $h^{\beta}(p)([t])\neq h^{\beta}(q)([t])$.

Let $D=\{[s]: 1/3\leqq s\leqq 4/7\}$ and let $E=\{[s]: 0\leqq s\leqq 9/28\}$. Then D and E are disjoint closed subsets of T. We show that if $n\in A$ then $h(x_n)([t])\in D$ and if $n\in B$ then $h(x_n)([t])\in E$. As a consequence we will have that $h^{\beta}(p)([t])\in D$ and $h^{\beta}(q)([t])\in E$.

To this end we first observe that given any $n\in\mathbb{N}$, $\sum\{(\lfloor x_{k+1}/(2x_k)\rfloor/x_{k+1})\cdot x_n: k\in A$ and $k\geqq n+3\}\leqq 1/14$. Indeed, given the first $k\in A$ with $k\geqq n+3$ one has $(\lfloor x_{k+1}/(2x_k)\rfloor/x_{k+1})\cdot x_n\leqq x_n/(2x_k)\leqq 1/16$. Given k, $m\in A$ with $m>k>n+3$, one has $x_m\geqq x_{k+3}\geqq 8\cdot x_k$. Consequently $\sum\{(\lfloor x_{k+1}/(2x_k)\rfloor/x_{k+1})\cdot x_n$: $k\in A$ and $k\geqq n+3\}\leqq (1/2)\sum_{k=1}^{\infty}1/8^k=1/14$.

Now let $n\in A$. Then $h(x_n)([t])=x_n\cdot[t]=[x_n\cdot t]$. Now $x_n\cdot t=\sum\{(\lfloor x_{k+1}/(2x_k)\rfloor/x_{k+1})\cdot x_n$: $k\in A$ and $k<n\}+(\lfloor x_{n+1}/(2x_n)\rfloor/x_{n+1})\cdot x_n+\sum\{(\lfloor x_{k+1}/(2x_k)\rfloor/x_{k+1})\cdot x_n$: $k\in A$ and $k\geqq n+3\}$. The first of these sums is some integer l and the last of these is at most 1/14. Now consider the middle term. We have $(\lfloor x_{n+1}/(2x_n)\rfloor/x_{n+1})\cdot x_n\leqq 1/2$ and equality holds if x_{n+1}/x_n is even. If x_{n+1}/x_n is odd we have $x_{n+1}\geqq 3x_n$ so $(\lfloor x_{n+1}/(2x_n)\rfloor/x_{n+1})\cdot x_n=(x_{n+1}/(2x_n)-1/2)\cdot x_n/x_{n+1}=1/2-1/2\cdot(x_n/x_{n+1})\geqq 1/2-1/6=1/3$. Thus $l+1/3\leqq x_n\cdot t\leqq l+1/2+1/14$ so $[x_n\cdot t]\in D$ as required.

Finally let $n\in B$. Then $x_n\cdot t=\sum\{(\lfloor x_{k+1}/(2x_k)\rfloor/x_{k+1})\cdot x_n$:$k\in A$ and $k<n\}+\sum\{(\lfloor x_{k+1}/(2x_k)\rfloor/x_{k+1})\cdot x_n$: $k\in A$ and $n<k<n+3\}+\sum\{(\lfloor x_{k+1}/(2x_k)\rfloor/x_{k+1})\cdot x_n$: $k\in A$ and $k\geqq n+3\}$. Again the first sum is some integer l and the last is at most 1/14. The middle sum has at most one term which is at most 1/4. Thus $l\leqq x_n\cdot t\leqq l+1/4+1/14$ so $[x_n\cdot t]\in E$ as required. □

We obtain as a corollary the following result communicated to us by Kenneth Berg. For extensions of this result see **[2]**. Recall that, given f: $\mathbb{T}\to\mathbb{T}$, the enveloping semigroup of f is the closure in $\mathbb{T}^{\mathbb{T}}$ of $\{f^n: n\in\mathbb{N}\}$.

Corollary 4.5. *Define f: $\mathbb{T}\to\mathbb{T}$ by $f(\alpha)=2\cdot\alpha$. Then the enveloping semigroup of f can be identified with $\beta\mathbb{N}$.*

Proof. Note that $f^n(\alpha)=2^n\cdot\alpha$ so if h is defined as in Theorem 4.4, one has for each $n\in\mathbb{N}$, $h(2^n)=f^n$. Thus the enveloping semigroup of f is $h[cl\{2^n: n\in\mathbb{N}\}]$. Since h is one-to-one on this closure, it is a homeomorphism on $cl\{2^n: n\in\mathbb{N}\}$. □

It was shown in **[16]** that if p is a right cancellable element of $\beta\mathbb{N}$, then every element of $cl\{p,p+p,p+p+p,\dots\}$ is right cancellable. As a consequence, any such semigroup has a closure which misses the set of idempotents. We show next that one can get semigroups in $\beta\mathbb{N}$ whose closure is reasonably far removed from the idempotents. (In particular the closure cannot be a semigroup.)

Theorem 4.6. *Let $\langle x_n\rangle_{n=1}^{\infty}$ be any divisible sequence in $\mathbb{N}$ and let $p\in(cl\{x_n\colon n\in\mathbb{N}\})\backslash\mathbb{N}$. Then $cl\{p,p+p,p+p+p,\dots\})\cap T=\varnothing$.*

Proof. We may presume $x_1=1$. (If $x_1>1$, let $y_1=1$ and $y_{n+1}=x_n$ for $n\in\mathbb{N}$. Then $(cl\{y_n\colon n\in\mathbb{N}\})\backslash\mathbb{N}=(cl\{x_n\colon n\in\mathbb{N}\})\backslash\mathbb{N}$.) For each $n\in\mathbb{N}$ let $a_n=x_{n+1}/x_n$. Then each $m\in\mathbb{N}$ has a unique expression of the form $\sum_{t\in F}b_t\cdot x_t$ where for each $t\in F$, $b_t\in\{1,2,\dots,a_t-1\}$. Further x_n divides m if and only if $\min F\geqq n$. Given $m\in\mathbb{N}$, define $c(m)=|F|$ where $m=\sum_{t\in F}b_t\cdot x_t$ as above. Let $c^{\beta}\colon \beta\mathbb{N}\to\beta\mathbb{N}$ be the continuous extension of c. Since c is constantly equal to 1 on $\{x_n\colon n\in\mathbb{N}\}$ we have $c^{\beta}(p)=1$.

Let $X=(\bigcap_{n=1}^{\infty}cl(\mathbb{N}x_n))\cap(\bigcap_{n=1}^{\infty}cl\{m\in\mathbb{N}\colon c(m)>n\})$. We observe that the idempotents are all in X. We have $C\subseteq\bigcap_{n=1}^{\infty}cl(\mathbb{N}x_n)$. To see that the idempotents are contained in $\bigcap_{n=1}^{\infty}cl\{m\in\mathbb{N}\colon c(m)>n\}$, let $e=e+e$ and suppose that for some n, $\{m\in\mathbb{N}\colon c(m)\leqq n\}\in e$. Then, since e is an ultrafilter one has in fact that for some n, $\{m\in\mathbb{N}\colon c(m)=n\}\in e$. Let $A=\{m\in\mathbb{N}\colon c(m)=n\}$ and pick $m\in A$ such that $A-m\in e$. Pick t such that $x_t>m$ and pick $k\in\mathbb{N}x_t\cap(A-m)$. Then $c(k+m)=c(k)+c(m)>n$ so $k+m\notin A$, a contradiction.

Now suppose $(cl\{p,p+p,p+p+p,\dots\})\cap T\neq\varnothing$. By Theorem 2.4, $T=cl\bigcup\{\mathbb{N}+e\colon e\in\beta\mathbb{N}$ and $e+e=e\}$, so $T\subseteq cl(\bigcup_{n=1}^{\infty}n+X)$. Thus $cl\{p,p+p,p+p+p,\dots\}\cap cl(\bigcup_{n=1}^{\infty}n+X)\neq\varnothing$ so by Lemma 1.3 either $cl\{p,p+p,p+p+p,\dots\}\cap(\bigcup_{n=1}^{\infty}n+X)\neq\varnothing$ or $\{p,p+p,p+p+p,\dots\}\cap cl(\bigcup_{n=1}^{\infty}n+X)\neq\varnothing$. But $cl\{p,p+p,p+p+p,\dots\}\subseteq\bigcap_{n=1}^{\infty}cl(\mathbb{N}x_n)$ and $\bigcap_{n=1}^{\infty}cl(\mathbb{N}x_n)\cap(\bigcup_{n=1}^{\infty}n+X)=\varnothing$. Thus we have some $q\in\{p,p+p,p+p+p,\dots\}\cap cl(\bigcup_{n=1}^{\infty}n+X)$. Now $q=p+p+\cdots+p$ (m-times) so $c^{\beta}(q)=m$. Let $A=\{y\in\mathbb{N}\colon c(y)=m\}$. Then $A\in q$ so $clA\cap(\bigcup_{n=1}^{\infty}n+X)\neq\varnothing$, so pick $n\in\mathbb{N}$ with $clA\cap(n+X)\neq\varnothing$ and pick $r\in clA\cap(n+X)$. Pick $k\in\mathbb{N}$ such that $x_k>n$. Now $r-n\in X\subseteq cl(\mathbb{N}x_k)\cap cl\{y\in\mathbb{N}\colon c(y)>m\})$ so $\mathbb{N}x_k\cap\{y\in\mathbb{N}\colon c(y)>m\}\cap(A-n)\neq\varnothing$. Pick $y\in\mathbb{N}x_k\cap\{y\in\mathbb{N}\colon c(y)>m\}\cap(A-n)$. Since $y\in\mathbb{N}x_k$ and $x_k>n$ we have $c(y+n)=c(y)+c(n)>m$ so $y+n\notin A$, a contradiction. □

On the other hand, we see that no semigroup can get too far removed from the idempotents.

Theorem 4.7. *Let S be any subsemigroup of $\beta\mathbb{N}$. Then $(clS)\cap\bigcap_{n=2}^{\infty}S_n\neq\varnothing$.*

Proof. Pick any $p\in S$. Define $\phi\colon\mathbb{N}\to\beta\mathbb{N}$ by $\phi(n)=p+p+\cdots+p$(n times) and let ϕ^{β} be the continuous extension to $\beta\mathbb{N}$. Note that $\phi^{\beta}\colon\beta\mathbb{N}\to\beta\mathbb{N}$ is a homomorphism. Pick any $q\in\bigcap_{n=2}^{\infty}S_n$. Then $\phi^{\beta}(q)\in clS$. We claim that $\phi^{\beta}(q)\in\bigcap_{n=2}^{\infty}S_n$.

We show first that for any $A\in\phi^{\beta}(q)$ and any $n\in\mathbb{N}\backslash\{1\}$, there exist $r_1,r_2,\dots,r_n$ in clA that commute with each other with $FS(\langle r_t\rangle_{t=1}^{n})\subseteq clA$. (The fact that $r_1,r_2,\dots,r_n$ commute with each other is not really relevant except that we do not need to spell out the order of the sums in $FS(\langle r_t\rangle_{t=1}^{n})$.) To see this let $A\in\phi^{\beta}(q)$ and pick $B\in q$ such that $\phi^{\beta}[clB]\subseteq clA$. Now let $n\in\mathbb{N}\backslash\{1\}$ and (since $q\in S_n$) pick $x_1,x_2,\dots,x_n$ in B with $FS(\langle x_t\rangle_{t=1}^{n})\subseteq B$. For each $t\in\{1,2,\dots,n\}$, let $r_t=\phi(x_t)$.

To complete the proof we show by induction on $n\in\mathbb{N}$ that given $A\subseteq\mathbb{N}$, if there exist commuting $r_1,r_2,\dots,r_n$ with $FS(\langle r_t\rangle_{t=1}^{n})\subseteq clA$, then there exist $x_1,x_2,\dots,x_n$ with $FS(\langle x_t\rangle_{t=1}^{n})\subseteq A$. The case $n=1$ is trivial, so let $n\in\mathbb{N}$ and assume the statement is true for n and let $r_1,r_2,\dots,r_{n+1}$ be commuting elements of clA with $FS(\langle r_t\rangle_{t=1}^{n+1})\subseteq clA$. Let

$D=\{x\in\mathbb{N}: A-x\in r_{n+1}\}$. Now given any nonempty $F\subseteq\{1,2,\ldots,n\}$ we have $A\in\sum_{t\in F}r_t+r_{n+1}$ so $D\in\sum_{t\in F}r_t$. That is $FS(\langle r_t\rangle_{t=1}^n)\subseteq clD$. Since also $FS(\langle r_t\rangle_{t=1}^n)\subseteq clA$ we have $FS(\langle r_t\rangle_{t=1}^n)\subseteq cl(A\cap D)$ so by the induction hypothesis choose $\langle x_t\rangle_{t=1}^n$ with $FS(\langle x_t\rangle_{t=1}^n)\subseteq A\cap D$. Now $A\in r_{n+1}$ and for each nonempty $F\subseteq\{1,2,\ldots,n\}$, $A-\sum_{t\in F}x_t\in r_{n+1}$ so pick $x_{n+1}\in A\cap\bigcap\{A-\sum_{t\in F}x_t\colon \varnothing\neq F\subseteq\{1,2,\ldots,n\}\}$. Then $FS(\langle x_t\rangle_{t=1}^{n+1})\subseteq A$. □

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