CONSTRUCTION OF TRANSVERSE FIELDS

H. PUTZ

In this paper we give local conditions for a rectilinear embedding of a non-bounded combinatorial manifold, M^n, in Euclidean space, which are sufficient to prove that M^n has a transverse field (see 1.1 and 1.2, definitions).

In a sequel to this paper (**6**), we will show how with this transverse field we can construct a normal microbundle for the embedded manifold M^n.

Our object in this research was only to obtain an existence theorem for normal microbundles. However, the method of proof via the construction of a transverse field yields as corollaries by Cairns (**1**), Whitehead (**9**), or Tao (**8**), results on smoothing. Earlier smoothing results achieved by the construction of transverse fields in the special case of (global) codimension 1 were obtained by Noguchi (**5**), and Tao (**7**; **8**).

After the research for this paper was completed, a paper of Davis (**2**) came to our attention. His methods are quite different from ours and there is little overlap in the two papers. We will mention his results in the text (see 1.13).

In § 1, we state our local conditions and construct the transverse field, modulo the crucial fact that certain sets, $P[s]$, (see 1.1, definition) are contractible. We also mention the smoothing corollaries which follow from this result. Included is a conjecture, which if true, would give stronger results on smoothings. In § 2, we develop the coordinate structure of $G_{n,p}'$, the space of oriented p-planes in Euclidean $n + p$ space, R^{n+p}, which we then use in § 3 to prove that the sets $P[s]$ are contractible.

0. Conventions. Throughout this paper, unless specifically stated otherwise, M^n will denote a non-bounded combinatorial manifold of dimension n, rectilinearly embedded as a locally finite subcomplex of R^{n+p}. Furthermore, all spaces involved will be assumed to be embedded in some Euclidean space. This is in order to make use of the theorems of (**4**), to which we refer the reader for background definitions, etc.

By a simplex, s, we will mean an "open" simplex, and $\bar{s}$ will denote its closure, that is, s, together with its faces. We will denote by $\mathrm{St}(s, M)$, the open star of s in M, and by $\overline{\mathrm{St}}(s, M)$ the closed star.

When we speak of planes in R^{n+p}, we will mean those that pass through the origin of R^{n+p}, unless specifically stated otherwise.

1. Existence of a transverse field.

Received March 27, 1968 and in revised form, July 5, 1968.

1.1. *Definition.* Let s denote a simplex (open) of M. $P[s]$ will denote the subset (possibly empty) in $G_{n,p}$ (the Grassmannian manifold of all p-planes in R^{n+p}) of p-planes, P, such that, if H is the orthogonal n-plane to P in R^{n+p}, then the orthogonal projection $q: R^{n+p} \to H$, restricted to $\overline{\mathrm{St}}(s, M)$, is a homeomorphism, carrying $\mathrm{St}(s, M)$ onto an open set in H. If P belongs to $P[s]$, then P is said to be *transversal* to M at m, where m is any point of s; see (**9**) for an equivalent definition.

1.2. *Definition.* A continuous map $g: M \to G_{n,p}$ is a *transverse field* if the set $g(s)$ is contained in the set $P[s]$, for every simplex s of M.

1.3. *Definition.* The embedding of M^n in R^{n+p} is *locally normal* if for each vertex, m of M, the set $P[m]$ is non-empty.

1.4. *Remark.* If the simplex t is a face of the simplex s, then since $\overline{\mathrm{St}}(s, M)$ is contained in $\overline{\mathrm{St}}(t, M)$, we have the fact that the set $P[t]$ is contained in the set $P[s]$. Thus, for a locally normal embedding, the sets $P[s]$ are non-empty.

1.5. *Definition.* The rectilinear embedding of M^n in R^{n+p} is of *local codimension* k, if for each vertex m, of M, there is an $(n + k)$-plane K (depending on m), such that $\mathrm{St}(m, M)$ is contained in some translation of K.*

1.6. *Remark.* If the embedding of M^n in R^{n+p} is of local codimension k and s is a simplex of M passing through the origin of R^{n+p}, then $\mathrm{St}(s, M)$ is contained in a plane of dimension $n + k$. Let t and r be two n-simplices in $\mathrm{St}(s, M)$. Then, if L and H are the two n-planes determined by t and r, respectively, the intersection of these two planes obviously has dimension greater than or equal to $n - k$.

1.7. THEOREM. *If the embedding of M^n in R^{n+p} is locally normal, and of local codimension* 1, *then for any simplex, s, of M, the subset $P[s]$ of $G_{n,p}$ is open and contractible.*

Proof. See 3.9.

The principal result of this paper is the following.

1.8. THEOREM. *For a locally normal embedding of M^n in R^{n+p} of local codimension* 1, *M^n has a transverse field.*

Proof. We must construct a continuous map $h: M \to G_{n,p}$, such that for each simplex, s, of M, $h(s)$ is contained in $P[s]$. We do this by induction. For any vertex, m of M, let $h(m)$ be any p-plane in $P[m]$. Assume that h has been defined on the $k - 1$ skeleton of M, and let s be a k-simplex. Then, for t a face of s, $h(t)$ is defined and is contained in $P[t]$, which is contained in $P[s]$; see 1.4, remark. Thus, h maps the boundary of $\bar{s}$ into $P[s]$. By the theorem of

*This suggestive terminology of local codimension is due to Davis (**2**).

1.7, $P[s]$ is contractible, thus we may extend h to a continuous map of $\bar{s}$ into $P[s]$.

1.9. COROLLARY. *There exists a piecewise smooth map* $g: M \to G_{n,p}$, *such that* $g(s)$ *is contained in* $P[s]$ *for each simplex* s *of* M.

Proof. Let h be the map of the previous theorem. Thus, $h(\bar{s})$ is contained in $P[s]$. From the fact that $P[s]$ is open in $G_{n,p}$ (1.7, theorem) and the fact that $h(\bar{s})$ is compact, we see that there exists a strictly positive continuous function δ mapping M to the positive reals, such that, if

$$g: M \to G_{n,p} \quad \text{and} \quad ||g(x) - h(x)|| < \delta(x)$$

(where $||\ ||$ denotes a metric on $G_{n,p}$), then $g(s)$ is contained in $P[s]$.

Now let V denote the second regular neighbourhood of M in R^{n+p}, and let $r: V \to M$ denote the retraction of this open set onto M. Thus, we have a continuous map $hr: V \to G_{n,p}$, defined on the smooth manifold V. Let $g: V \to G_{n,p}$ be a smooth map, such that its restriction to M is pointwise within $\delta(x)$ of the map h. (Such a map exists by the proof of (**4**, Theorem 4.2).) If we let the restriction map also be denoted by g, we have the proof of the corollary.

Since M has a transverse field, it has a compatible differentiable structure according to Cairns (**1**), or Whitehead (**9**). Thus, as a corollary to the theorem of 1.8 we have the following result.

1.10. THEOREM. *Let* M^n *be a non-bounded combinatorial manifold. If there exists a locally normal embedding of local codimension* 1 *in some Euclidean space, then* M^n *has a differentiable structure compatible with its given combinatorial structure.*

1.11. *Smoothing corollaries*. Of our two hypotheses (local normality and local codimension 1) in the theorem of 1.8, clearly the first is necessary. However, for results on smoothing, the position of M^n in Euclidean space is immaterial, and it may be possible, following Noguchi (**5**), to avoid using this hypothesis as follows.

CONJECTURE. *Let* M^n *be embedded in some Euclidean space with local codimension* 1. *Then, if the Schoenflies conjecture holds for dimension* $\leq n$ *it is possible to shift the embedding of* M^n *slightly so that the new embedding will still have local codimension* 1, *and also be locally normal.*

We do not pursue this conjecture here, but remark that if it were true, then by what we have already done, the following would immediately follow.

THEOREM. *If the Schoenflies conjecture holds in dimension* $\leq n$, *and* M^n *is embedded in some Euclidean space with local codimension* 1, *then* M^n *has a differentiable structure compatible with its given combinatorial structure.*

We remark that Tao (**8**) has proven the previous theorem via the method of transverse fields in the case when M^n is immersed (piecewise linear local homeomorphism) in R^{n+1}. Following Tao (**8**), the truth of the conjecture stated for an immersion would yield the following theorem which would be the most general smoothing result obtained via transverse fields.

THEOREM. *If the Schoenflies conjecture holds in dimension* $\leqq n$ *and* M^n *is immersed in some Euclidean space with local codimension* 1, *then* M^n *has a differentiable structure compatible with its given combinatorial structure.*

Finally we mention that without relying on the above conjecture for immersions we have the following result.

1.12. THEOREM. *Let* M^n *be a non-bounded combinatorial manifold. If there exists an immersion of* M^n *in some Euclidean space which is locally normal and of local codimension* 1, *then* M^n *has a differentiable structure compatible with its given combinatorial structure.*

Proof. The theorem of 1.8 holds in the case of an immersion, since its proof rests on the theorem of 1.7, which is a local statement. Our theorem then follows from propositions of Tao (**8**, Propositions 1, 2, 3, 4), where the immersion version of the theorem of 1.8 is the key hypothesis.

1.13. *Results of Davis.* Davis in (**2**), among other results, obtained necessary and sufficient conditions that a compact non-bounded combinatorial manifold be embeddable with local codimension 1. He also showed that the Klein bottle is embeddable with local codimension 1. Thus, local codimension 1 is distinct from global codimension 1. (That is, one might think that the embeddability of M^n with local codimension 1 might imply that M^n might be actually embeddable in R^{n+1}.) This indicates that our theorem in 1.10 is a real generalization of the work of Noguchi (**5**) and Tao (**7**).

2. The coordinate structure of the space $G_{n,p}'$ of oriented p-planes in Euclidean $n + p$ space.

2.1. *Construction.* Denote by $G_{n,p}'$ the set of p-planes in R^{n+p} together with a choice of orientation for each plane. We construct coordinate neighbourhoods for $G_{n,p}'$ as follows. We fix once and for all, throughout the remainder of this paper, an orientation of R^{n+p}. That is, pick a basis of R^{n+p}, and it is to be understood that, for any other basis of R^{n+p} mentioned, the matrix associated with the change of basis is to have a positive determinant. Now let P be a p-plane in R^{n+p}, and let $R_1, \ldots, R_p$ be a basis of P, which we complete to a basis of R^{p+n}. Consider another basis $S_1, \ldots, S_p$ of P, and complete it to a basis of R^{n+p}. Then the basis $R_1, \ldots, R_p$ of P is *equivalent* to the basis $S_1, \ldots, S_p$ of P if the associated $p \times p$ matrix has a positive determinant. Thus, if (R_i) denotes the equivalence class of the basis $R_1, \ldots, R_p$, of P, the points of $G_{n,p}'$ can be denoted $(P, (R_i))$. Let $y_1, \ldots, y_{n+p}$ be a basis of R^{n+p}, and let P

denote the p-plane spanned by $y_1, \ldots, y_p$. We shall construct a coordinate neighbourhood of the point $(P, (y_i))$ in $G_{n,p}'$. Throughout this paper we identify points, a, of the Euclidean space R^{pn}, with the $p \times (n + p)$ matrices $A = (I; a)$, where I denotes the $p \times p$ identity matrix.

We define a map f: $R^{pn} \to G_{n,p}'$ by $f(a) = (P(a), (R_i(a)))$, where $P(a)$ is the p-plane spanned by the vectors $R_i(a) = \sum a_{ij} y_j$, $i = 1, \ldots, p$, $j = 1, \ldots, p + n$. Thus $f(0) = (P, (y_i))$. Associated with the mapping f we define a map f': $R^{pn} \to G_{n,p}'$ by $f'(a) = (P(a), (R_i'(a)))$, where $P(a)$ is the p-plane spanned by the vectors $R_i'(a)$, and where $R_1'(a) = -R_1(a)$, and $R_i'(a) = R_i(a)$, $i = 2, \ldots, p$. Thus, $f'(a)$ is the oriented p-plane, oppositely oriented to the oriented p-plane $f(a)$. One easily sees that these coordinate neighbourhoods define a differentiable structure for $G_{n,p}'$, and that $G_{n,p}'$ is a double covering of $G_{n,p}$ under the covering map which sends $(P, (R_i))$ to the p-plane P.

The following theorem is the key result in our proof that the sets $P[s]$ are contractible.

2.2. THEOREM. *Let $y_1, \ldots, y_{p+n}$ and $x_1, \ldots, x_{p+n}$ be two bases of R^{n+p}, such that $y_j = x_j$, $j = p + 2, \ldots, p + n$. Let f and g denote the associated coordinate functions, as defined in the previous construction. Let U denote the intersection in $G_{n,p}'$ of $f(R^{pn})$ and $g(R^{pn})$. Then $f^{-1}(U)$ is a convex set in R^{pn}.*

Proof. Suppose that $(P(a), (R_i(a))) = (P(b), (S_k(b)))$, where $R_i(a) = \sum a_{ij} y_j$, $S_k(b) = \sum b_{kr} x_r$, $i, k = 1, \ldots, p$, $j, r = 1, \ldots, p + n$. Now $R_i(a) = \sum d_{ik}(a) S_k(b)$, where the $p \times p$ matrix $D(a) = (d_{ik}(a))$ has a positive determinant. Now $y_j = \sum c_{jr} x_r$, where the $(n + p) \times (n + p)$ matrix $C = (c_{jr})$ has a positive determinant. Note that by the choice of bases y_j and x_r, that $c_{jr} = \delta_{jr}$ (the Kronecker delta), for $j = p + 2, \ldots, p + n$. In matrix terms, we have then $AC = D(a)B$, where $D(a)$ has positive determinant.

Thus, we have reduced the proof to showing that the solutions, A, in R^{pn} (recall that we identify points in R^{pn} with $p \times (p + n)$ matrices, whose first p columns form the $p \times p$ identity matrix I) of the matrix equation $AC = D(a)B$ forms a convex set (where C is fixed, the first p columns of A and B form the $p \times p$ identity matrix, and the determinant of $D(a)$ is positive, but $D(a)$ is otherwise unrestricted.)

Let us assume that A in R^{pn} is a solution to $AC = DB$. We will derive a necessary condition for A. We will then show that for any A satisfying this condition, there exist matrices D and B such that $AC = DB$. Finally, we shall show that the set of points A in R^{pn} satisfying this necessary and sufficient condition is a convex set in R^{pn}.

Necessity. Suppose that $AC = DB$ and let $G = D^{-1}$. Then since $GAC = B$, the first p-columns of GAC equal the $p \times p$ identity matrix I. Consider the first $p + 1$ entries of the ith row of GA. This then is the row vector (with

https://doi.org/10.4153/CJM-1969-125-6 Published online by Cambridge University Press

$p+1$ entries) $(G_i; G_i \cdot A^{p+1})$, where A^{p+1} is the $(p+1)$st column of A, and G_i is the ith row of G. By the particular choice of the bases x_r and y_j, we have for the matrix C of the change of basis, $c_{jr} = 0$ for $j = p+2, \ldots, p+n$ and $r = 1, \ldots, p$. Hence, in determining the first p columns of GAC, we need only consider the first $p+1$ rows of the jth column of C, $j = 1, \ldots, p$. We write this column vector (with $p+1$ entries) as $(F^j; c_{p+1,j})$, where F^j is the column vector consisting of the first p entries of the jth column vector of C. Thus $GAC = B$ implies that $G_i \cdot F^j + (G_i \cdot A^{p+1})c_{p+1,j} = \delta_{ij}$. That is, if Z is the $p \times p$ matrix whose jth column vector is $F^j + c_{p+1,j}A^{p+1}$, then $GZ = I$, and therefore $Z = D$. Thus a necessary condition for A to be a solution of the equation $AC = DB$ is that the determinant of the $p \times p$ matrix

$$Z = (c_{ij} + a_{i,p+1}c_{p+1,j}),$$

$i, j = 1, \ldots, p$, be positive. Note that this condition only concerns the $(p+1)$st column of A.

Sufficiency. Suppose that the point A in R^{pn} is such that the matrix $Z = (c_{ij} + a_{i,p+1}c_{p+1,j})$, $i, j = 1, \ldots, p$, has a positive determinant. Let $G = Z^{-1}$. Then, if B is defined to be the matrix GAC, we see, by retracing our steps above, that the first p columns of B form the identity matrix. Thus, $AC = DB$, where $D = G^{-1} = Z$, and the determinant of D is positive. Thus the condition on A is sufficient.

Convexity. We shall show that the set of points A in R^{pn} satisfying the condition that the determinant of the matrix $Z = (c_{ij} + a_{i,p+1}c_{p+1,j})$ be positive, is a convex set. First we prove the following result.

SUBLEMMA. *Let us consider an arbitrary square $p \times p$ matrix of the form $E = (c_{ij} + s_i t_j)$. Then the determinant of E is equal to $\sum z_k s_k + c$, $k = 1, \ldots, p$, where z_k is a function only of the terms c_{ij} and t_r, and not of the terms s_q, and $c = \det(c_{ij})$.*

Proof. $\det E = \sum (\text{sign } \pi)(c_{1\pi(1)} + s_1 t_{\pi(1)}) \ldots (c_{p\pi(p)} + s_p t_{\pi(p)})$, where the sum is taken over all permutations π of p symbols. A typical term in this expansion of the determinant of E is

$$(\text{sign } \pi)\, c_{j_1\pi(j_1)} \ldots c_{j_r\pi(j_r)} t_{j_{r+1}} s_{\pi(j_{r+1})} \ldots t_{j_p} s_{\pi(j_p)},$$

where $j_1 < \ldots < j_r$, and $j_{r+1} < \ldots < j_p$. We claim that if $p - r \geqq 2$, then any such term is cancelled. For let π' denote the permutation defined by $\pi'(j_k) = \pi(j_k)$, for $k \neq r+1, p$, and $\pi'(j_{r+1}) = \pi(j_p)$, $\pi'(j_p) = \pi(j_{r+1})$. Since $\text{sign } \pi' = -\text{sign } \pi$, we see that the term in the expansion involving the permutation π' cancels the term involving the permutation π, when $p - r \geqq 2$, which proves the sublemma.

Applying the sublemma to our matrix Z, we see that the necessary and sufficient condition on the point A in R^{pn} is that $\sum z_k a_{k,p+1} + c > 0$, $k = 1, \ldots, p$. The set of points A in R^{pn} satisfying this condition are all

https://doi.org/10.4153/CJM-1969-125-6 Published online by Cambridge University Press

points lying on one side of the $(pn - 1)$-dimensional hyperplane in R^{pn} determined by $\sum z_k a_{k,p+1} + c = 0$. This set is convex, and the theorem of 2.2 is proved.

The remainder of this section consists of rather tedious constructions and lemmas, concerning mainly various choices of bases in R^{p+n}. These are of no intrinsic interest, but are necessary preliminaries, so that we may, in § 3, prove that the sets $P[s]$ are contractible. Our use of the space of oriented p-planes, rather than the space of p-planes, is a technical device to facilitate this proof.

Let $y_1, \ldots, y_{p+n}$ denote a basis of R^{p+n}, and let $q\colon R^{p+n} \to R^p$ denote the projection defined by $q(y_i) = y_i$, $i = 1, \ldots, p$, $q(y_j) = 0$, $j = p + 1, \ldots, p + n$. Denote by H the n-plane spanned by the vectors y_j, $j = p + 1, \ldots, p + n$, and let P denote a p-plane in R^{p+n}.

2.3. LEMMA. *The map $q\colon P \to R^p$ is non-singular if and only if the p-plane P can be spanned by vectors $R_1, \ldots, R_p$, such that $q(R_i) = y_i$, $i = 1, \ldots, p$.*

2.4. LEMMA. *The intersection of the p-plane P with the n-plane H is the zero vector if and only if $q\colon P \to R^p$ is non-singular.*

2.5. COROLLARY. *Let $f, f'\colon R^{pn} \to G_{n,p}'$ denote the coordinate maps constructed in 2.1 with respect to the basis, $y_1, \ldots, y_{p+n}$; thus $f(a) = (P(a), (R_i(a)))$ and $f'(a) = (P(a), (R_i'(a)))$. Then the intersection of P and H is the zero vector if and only if $P = P(a)$, for some point a in R^{pn}.*

The proofs of the results in 2.3–2.5 above are obvious, and will be left to the reader.

2.6. LEMMA. *Let H be an n-plane in R^{n+p} with a given orientation, and let $y_{p+1}, \ldots, y_{p+n}$ be a basis for this oriented n-plane. Let us complete this to a basis $y_1, \ldots, y_{p+n}$ of R^{p+n}. Denote by $z_{p+1}, \ldots, z_{p+n}$ another basis of the oriented n-plane H (thus the matrix of the transformation taking z_j to y_j,*

$$j = p + 1, \ldots, p + n,$$

has positive determinant), and complete it to a basis $z_1, \ldots, z_{p+n}$ of R^{p+n}. If $f, g\colon R^{pn} \to G_{n,p}'$ denote the respective coordinate maps defined in 2.1, then $f(R^{pn}) = g(R^{pn})$, and the homeomorphism $f^{-1}g\colon R^{pn} \to R^{pn}$ maps convex sets to convex sets.

Proof. As in the proof of the theorem of 2.2, we are led to consider the matrix equation $AC = DB$, where C is the matrix of the transformation of R^{p+n} which takes the vector y_j to the vector x_j. However, now the matrix C is simpler than it was in 2.1, since

$$C = \begin{pmatrix} X & Y \\ O & Z \end{pmatrix},$$

where X is a $p \times p$ matrix of positive determinant, and O dnoetes the null

https://doi.org/10.4153/CJM-1969-125-6 Published online by Cambridge University Press

$n \times p$ matrix. Thus, $D = X$, a constant matrix, and $B = D^{-1}AC$. Hence, $f(R^{pn}) = g(R^{pn})$. The fact that $g^{-1}f(A) = D^{-1}AC = B$, shows that $g^{-1}f$ maps convex sets to convex sets.

We now prove some lemmas concerning $\overline{\mathrm{St}}(s, M)$. In order to avoid the notational inconvenience involved with translated planes, we will assume, without loss of generality, that the simplex s passes through the origin of R^{p+n}. Denote by K and L the n-planes determined by two adjacent n-simplices r and t in $\mathrm{St}(s, M)$. Let P be a p-plane whose intersection with K and whose intersection with L is the zero vector; denote by H the n-plane orthogonal to P, and by $q: R^{p+n} \to H$ the orthogonal projection. Let W be the $(p + n - 1)$-plane determined by P and the intersection of $\bar{r}$ and $\bar{t}$.

2.7. LEMMA. *The map* $q: \bar{r} \cup \bar{t} \to H$ *is one-to-one if and only if* W *separates* r *and* t.

The proof is easy, and is left to the reader.

2.8. *Definition.* We now choose an orientation of $\overline{\mathrm{St}}(s, M)$. ($\overline{\mathrm{St}}(s, M)$ is orientable, whether M is, or not.) Let $s_1, \ldots, s_v$ denote the n-simplices of $\overline{\mathrm{St}}(s, M)$, with s_i adjacent to s_{i+1}. Let H_i denote the n-plane determined by the n-simplex s_i, and choose an orientation of H_i determined by the orientation s_i received from the orientation of $\overline{\mathrm{St}}(s, M)$. Let $y_{p+1}{}^i, \ldots, y_{p+n}{}^i$ be a basis of the oriented plane H_i, and complete this to a basis $y_1{}^i, \ldots, y_{p+n}{}^i$, of R^{p+n}, such that $y_j{}^i$ is perpendicular to H_i, for $j = 1, \ldots, p$. Let $f_i: R^{pn} \to G_{n,p}{}'$ denote the coordinate map associated with the basis $y_j{}^i$, $i = 1, \ldots, v$, defined in 2.1.

We define $U_i(s)$ to be the set $f_i(R^{pn})$ in $G_{n,p}{}'$.

2.9. LEMMA. *Let* P *denote a* p*-plane,* H *the perpendicular* n*-plane, and* $q: R^{p+n} \to H$*, the orthogonal projection. Then, the map* q *is one-to-one on the union* $\bar{s}_i \cup \bar{s}_{i+1}$ *of adjacent oriented simplices, if and only if there exist points* a_i *and* a_{i+1} *in* R^{pn} *such that* $f_i(a_i) = f_{i+1}(a_{i+1}) = P'$*, an oriented* p*-plane, with* $\pi(P') = P$*, where* $\pi: G_{n,p}{}' \to G_{n,p}$ *is the covering map defined in* 2.1.

Proof. Let W denote the $(p + n - 1)$-plane determined by P and the intersection of $\bar{s}_i$ and $\bar{s}_{i+1}$. By the lemma of 2.7, it is sufficient to prove that $f_i(a_i) = f_{i+1}(a_{i+1}) = P'$ if and only if W separates s_i and s_{i+1}. To show this, we choose new bases of the oriented planes, H_i and H_{i+1}, as follows. For notational convenience, let $s_i = s_1$, $s_{i+1} = s_2$, and let $\bar{t}$ denote the intersection of $\bar{s}_1$ and $\bar{s}_2$. The orientation of s_1 induces an orientation of t, and hence of the $(n - 1)$-plane determined by t. Let $z_{p+2}{}^1, \ldots, z_{p+n}{}^1$ be a basis of this oriented $(n - 1)$-plane, and complete this to a basis of the oriented n-plane H_1, by the addition of a vector $z_{p+1}{}^1$. Similarly, the n-simplex s_2 induces a basis $z_{p+1}{}^2, \ldots, z_{p+n}{}^2$ of the oriented n-plane H_2, where $z_{p+2}{}^2, \ldots, z_{p+n}{}^2$ span the same $(n - 1)$-plane as $z_{p+2}{}^1, \ldots, z_{p+n}{}^1$. However, since the orientation of the simplex t induced by the orientation of s_2 is opposite to the one induced by s_1, the matrix of the transformation taking the vector z_j^1 to the vector z_j^2, $j = p + 2, \ldots, p + n$, has a negative determinant. Now choose vectors $z_1^1, \ldots, z_p^1$ such that they span the p-plane P, and such that $z_1^1, \ldots, z_{p+n}^1$ is a basis of R^{p+n}. Similarly, we obtain a basis $z_1^2, \ldots, z_{p+n}^2$ of R^{p+n}, where the vectors $z_1^2, \ldots, z_p^2$ span P. Let $g_1, g_2: R^{pn} \to G_{n,p}'$ be the coordinate maps associated with the bases z^1 and z^2, respectively. Thus,

$$g_1(0) = (P, (z_1^1, \ldots, z_p^1)) \quad \text{and} \quad g_2(0) = (P, (z_1^2, \ldots, z_p^2)).$$

Now by the lemma of 2.6, $f_1(R^{pn}) = g_1(R^{pn})$, and $f_2(R^{pn}) = g_2(R^{pn})$. Hence, to prove the lemma, it is sufficient to prove that $g_1(0) = g_2(0)$ if and only if the $(p + n - 1)$-plane W separates the simplices s_1 and s_2.

First, suppose that $g_1(0) = g_2(0)$. Then, the transformation which maps z_j^1 to z_j^2, $j = 1, \ldots, p$, has positive determinant. Now the transformation which takes z_j^1 to z_j^2, $j = p + 2, \ldots, p + n$, has negative determinant. Therefore, since the transformation of R^{p+n} taking z_j^1 to z_j^2, $j = 1, \ldots, p + n$, has positive determinant, we see that $g_1(0) = g_2(0)$ implies that $z_{p+1}^1 = cz_{p+1}^2 + \sum c_k z_k^2$, $k \neq p + 1$, with $c < 0$. Thus, W clearly separates s_1 and s_2.

Conversely, if W separates s_1 and s_2, then $z_{p+1}^1 = cz_{p+1}^2 + \sum c_k z_k^2$, $k \neq p + 1$, with $c < 0$. In order that the transformation which takes z_j^1 to z_j^2, $j = 1, \ldots, p + n$, have positive determinant, it is necessary that the transformation which takes z_j^1 to z_j^2, $j = 1, \ldots, p$, have positive determinant. Therefore, $g_1(0) = g_2(0)$, which completes the proof of the lemma.

Finally, we consider the following situation. Recall that H_i denotes the n-plane determined by the n-simplex s_i in $\mathrm{St}(s, M)$, $i = 2, \ldots, v$. (We are assuming, without loss of generality, that the simplex s passes through the origin of R^{p+n}.) Since the rectilinear embedding of M in R^{p+n} is of local co-dimension 1, we have by the remark of 1.6 that the intersection of H_i and H_1 is a plane of dimension greater than or equal to $n - 1$. If the dimension is n, the following lemma is trivial, hence let us assume that the dimension is $n - 1$.

Let $z_{p+2}^1 = z_{p+2}^i, \ldots, z_{p+n}^1 = z_{p+n}^i$, be a basis of this $(n - 1)$-plane. Complete this to a basis $z_{p+1}^1, \ldots, z_{p+n}^1$ of the oriented n-plane H_1, and also complete it to a basis $z_{p+1}^i, \ldots, z_{p+n}^i$ of the oriented n-plane H_i. Now complete both to bases z^1 and z^i of R^{pn}.

Let $g_1, g_i: R^{pn} \to G_{n,p}'$ denote the coordinate maps associated to the bases z^1 and z^i, respectively. By the lemma of 2.6, $f_1(R^{pn}) = g_1(R^{pn})$ and $f_i(R^{pn}) = g_i(R^{pn})$.

2.10. Lemma. *Let V_i denote the intersection in $G_{n,p}'$ of $f_1(R^{pn})$ and $f_i(R^{pn})$. Then $K_i = f_1^{-1}(V_i)$ is a convex set in R^{pn}.*

Proof. By the preceding discussion, the set V_i is the intersection of $g_1(R^{pn})$ and $g_i(R^{pn})$. Let $K_i' = g_1^{-1}(V_i)$. By the theorem of 2.2, K_i' is a convex set in R^{pn}. Now $K_i = f_1^{-1}(V_i) = f_1^{-1}g_1g_1^{-1}(V_i) = f_1^{-1}g_1(K_i')$. By the lemma of 2.6, the fact that K_i' is convex implies that K_i is convex.

3. The set $P[s]$ is open and contractible. The proof that $P[s]$ is open is direct. In order to prove that $P[s]$ is contractible, we prove that it equals another set, $Q[s]$, which is more easily seen to be contractible. The purpose of introducing the space $G_{n,p}'$ of oriented p-planes in R^{n+p}, in § 2, was to enable us to define this set $Q[s]$, which we now proceed to do. Throughout this section we will assume, without loss of generality, that the simplex s passes through the origin of R^{p+n}.

3.1. *Definition of* $Q[s]$. Let $s_1, \ldots, s_v$ be the oriented n-simplices of the oriented complex $\overline{\mathrm{St}}(s, M)$, with s_i adjacent to s_{i+1}. Furthermore, denote by $U_i(s) = f_i(R^{pn})$ the set in $G_{n,p}'$ which was defined in 2.8. We define $Q'[s]$ to be the intersection in $G_{n,p}'$ of the subsets $U_1(s), \ldots, U_v(s)$. Then $Q[s]$ is defined by setting $Q[s] = \pi(Q'[s])$, where π: $G_{n,p}' \to G_{n,p}$ is the covering map defined in 2.1. We note that $Q[s]$ is homeomorphic to $Q'[s]$, since the restriction of the covering map π to any subset $U_i(s)$ is a homeomorphism.

3.2. LEMMA. *The set $P[s]$ is contained in $Q[s]$.*

Proof. Let the p-plane P belong to $P[s]$. By the lemma of 2.9, there exist points a_1 and a_2 in R^{pn} such that $f_1(a_1) = f_2(a_2) = P'$, and $\pi(P') = P$. Applying the lemma of 2.9 again, we see that there exist points a_2' and a_3 such that $f_2(a_2') = f_3(a_3) = P''$, and $\pi(P'') = P$. However, πf_2: $R^{pn} \to G_{n,p}$ is a homeomorphism; therefore, $P'' = P'$, and $a_2' = a_2$. Continuing in this manner, we have points $a_1, \ldots, a_v$ in R^{pn} such that $f_1(a_1) = \ldots = f_v(a_v) = P'$, and $\pi(P') = P$. Thus, P' belongs to $Q'[s]$, and therefore P belongs to $Q[s]$. We note in addition that, by the remark of 1.4, this implies that the set $Q[s]$ is non-empty.

3.3. THEOREM. *The set $Q[s]$ is open and contractible in $G_{n,p}$.*

Proof. The set $Q[s]$ is obviously open in $G_{n,p}$. Since $Q[s]$ is homeomorphic to $Q'[s]$, it is sufficient to show that $Q'[s]$ is contractible. Now

$$Q'[s] = U_1(s) \cap \ldots \cap U_v(s) = \bigcap (U_1(s) \cap U_i(s)),$$

$i = 2, \ldots, v$. By the lemma of 2.10,

$$f_1^{-1}(Q'[s]) = \bigcap f_1^{-1}(U_1(s) \cap U_i(s)) = \bigcap K_i,$$

$i = 2, \ldots, v$, where K_i is a convex set. Thus, $f_1^{-1}(Q'[s])$ is convex, and therefore contractible. Since f_1: $R^{pn} \to G_{n,p}'$ is a homeomorphism, we have then that the set $Q'[s]$ is contractible, which proves the theorem.

3.4. LEMMA. *Let the p-plane P belong to the set $Q[s]$, let H denote the orthogonal n-plane, and let q: $\overline{\mathrm{St}}(s, M) \to H$ denote the orthogonal projection. Then the map q is one-to-one on $\bar{t}$, for any simplex t in* $\mathrm{St}(s, M)$.

Proof. It is obviously sufficient to prove the lemma for an n-simplex s_i. Let H_i denote the n-plane determined by the n-simplex s_i. If the map q is not one-to-one on $\bar{s}_i$, then the intersection of P and H_i has dimension greater than or equal to one. By the corollary of 2.5, this contradicts the fact that $P = \pi(P')$, where P' belongs to $f_i(R^{pn})$.

3.5. Lemma. *Let H be an n-plane in R^{p+n}, q: $\mathrm{St}(s, M) \to H$ the orthogonal projection map. If the map q is one-to-one, then q: $\overline{\mathrm{St}}(s, M) \to H$ is a homeomorphism, mapping* $\mathrm{St}(s, M)$ *onto an open set in H.*

Proof. Suppose that there exist points x and y in $\overline{\mathrm{St}}(s, M) - \mathrm{St}(s, M)$ such that $q(x) = q(y)$. This implies that the image under the map q of the rays x to m and y to m coincide for any point m in s. However, these rays, with the exception of the points x and y, lie in $\mathrm{St}(s, M)$. Hence, we have a contradiction to the fact that the map q is one-to-one on $\mathrm{St}(s, M)$. Thus, q: $\overline{\mathrm{St}}(s, M) \to H$ is a homeomorphism. Now since $P[s]$ is non-empty by the remark of 1.4, we know that there exists a homeomorphism of $\mathrm{St}(s, M)$ with an open set in Euclidean n-space. Hence, by the theorem on the invariance of domain (see **3**, pp. 95, 96, Theorem VI 9), the image under the map q of $\mathrm{St}(s, M)$ is open in H.

3.6. *Remark.* Let H be an n-plane in R^{p+n}, and let P denote the orthogonal p-plane. The condition that the orthogonal projection q: $\overline{\mathrm{St}}(s, M) \to H$ is one-to-one (which, by the previous lemma, ensures that the map q carries $\mathrm{St}(s, M)$ homeomorphically onto an open set in H) is easily seen to be equivalent to the condition that for any two distinct points x, y in $\overline{\mathrm{St}}(s, M)$, the vector $x - y$ does not belong to the p-plane P.

3.7. Proposition. *The set $P[s]$ is open in $G_{n,p}$.*

Proof. Let the p-plane P belong to the set $P[s]$. By the preceding remark, we have to find a neighbourhood U of P in $G_{n,p}$ such that if P_1 belongs to U, then $x - y$ does not belong to P_1, for every two distinct points x, y in $\overline{\mathrm{St}}(s, M)$. First we prove the following result.

Sublemma. *Given any two distinct points x and y in $\overline{\mathrm{St}}(s, M)$; then there exist points x' and y' such that $x' - y'$ is parallel to $x - y$, and such that x' belongs to $\bar{t}_j$, y' belongs to $\bar{t}_k$, where t_j and t_k are two simplices of $\overline{\mathrm{St}}(s, M)$ whose closures do not intersect.*

Proof. For the sake of completeness, we include the proof, which is taken from (**9**, p. 197). Let x belong to the simplex t_1 and let y belong to the simplex t_2. We use induction on the number, $\dim t_1 + \dim t_2$. If the sum of the dimensions is zero, the lemma is trivially true. Now, let us assume that $\bar{t}_1$ and $\bar{t}_2$ have a non-empty intersection. (If the intersection is empty, the lemma is trivial.) Let m be any point of this intersection. For the real number r greater than or equal to one, set $x(r) = m + r(x - m)$, and $y(r) = m + r(y - m)$. Thus, $x(r) - y(r)$ is parallel to $x - y$. Let r_1 be the maximum of the numbers r such that $x(r)$ belongs to $\bar{t}_1$, and $y(r)$ belongs to $\bar{t}_2$. Then, either $x(r_1)$ belongs to the boundary of $\bar{t}_1$, or $y(r_1)$ belongs to the boundary of $\bar{t}_2$. Thus, we have:

$x(r_1)$ belongs to t_3, $y(r_1)$ belongs to t_4, $x(r_1) - y(r_1)$ is parallel to $x - y$, and $\dim t_3 + \dim t_4$ is smaller than $\dim t_1 + \dim t_2$. Hence, the lemma is established by induction.

Now let L denote the subcomplex of the cell complex $\overline{\mathrm{St}}(s, M) \times \overline{\mathrm{St}}(s, M)$, consisting of cells $\bar{t}_j \times \bar{t}_k$, such that the intersection of $\bar{t}_j$ and $\bar{t}_k$ is empty. Let h be a mapping of L into the sphere, S^{p+n-1}, defined by

$$h(x, y) = (x - y)/||x - y||.$$

Then, by the remark of 3.6, the p-plane P belongs to the set $P[s]$ if and only if the intersection of $h(L)$ and P is empty. Since L is compact, $h(L)$ is closed in the sphere S^{p+n-1}. Thus, we can choose a neighbourhood U of P in $G_{n,p}$ such that, if P_1 belongs to U, then the intersection of $h(L)$ and P_1 is empty. Thus, P_1 belongs to $P[s]$, which proves the proposition.

3.8. THEOREM. *The set* $Q[s] - P[s]$ *is open in* $G_{n,p}$.

Proof. Let the p-plane P belong to the set $Q[s] - P[s]$ in $G_{n,p}$. Denote by H the n-plane orthogonal to P, and let q: $\mathrm{St}(s, M) \to H$ denote the orthogonal projection map. The fact that the p-plane P does not belong to $P[s]$ implies that there are points x_1, y_1 in $\overline{\mathrm{St}}(s, M)$ such that $q(x_1) = q(y_1)$. By the lemma of 3.5, we may, without loss of generality, assume that x_1 and y_1 both lie in $\mathrm{St}(s, M)$.

SUBLEMMA. *There exist points x and y in* $\mathrm{St}(s, M)$ *such that* $q(x) = q(y)$, *and the maps* q: $\mathrm{St}(x, M) \to H$, q: $\mathrm{St}(y, M) \to H$ *are one-to-one.*

Proof. If (x_1, y_1) is not such a pair, let us assume that q: $\mathrm{St}(x_1, M) \to H$ is not one-to-one. Thus, we have points x_2 and y_2 in $\mathrm{St}(x_1, M)$, such that $q(x_2) = q(y_2)$. Again, if (x_2, y_2) is not the desired pair of points, assume that q: $\mathrm{St}(x_2, M) \to H$ is not one-to-one. Let x_1 belong to the simplex t_1 of $\mathrm{St}(s, M)$. Then, the dimension of t_1 is less than n. (For, if the dimension of t_1 is n, then $\mathrm{St}(x_1, M) = t_1$, and by the lemma of 3.4, the map q is one-to-one on t_1, which is a contradiction.) Now the point x_2 belongs to a simplex t_2, which has t_1 as a face. Again, we see that the dimension of t_2 is less than n. The simplex t_2 is not the simplex t_1. (For, if it were, then $\mathrm{St}(x_1, M) = \mathrm{St}(x_2, M)$. This, together with the fact that $q(x_2) = q(y_2)$, would imply that on the simplex t containing y_2 which has t_1 as a face, the map q: $\bar{t} \to H$ is not one-to-one. This would contradict the lemma of 3.4.) Hence we have that the dimension of t_1 is strictly less than the dimension of t_2 which is strictly less than n. This process, therefore, must eventually end with a pair of points (x, y) in $\mathrm{St}(s, M)$, such that the maps, q: $\mathrm{St}(x, M) \to H$, q: $\mathrm{St}(y, M) \to H$, are both one-to-one, and such that $q(x) = q(y)$, which is the statement of the lemma.

From this lemma, and the lemma of 3.5, the images under the map q of $\mathrm{St}(x, M)$ and $\mathrm{St}(y, M)$ are open sets in H which intersect. Thus, there exist two n-simplices, s_k and s_r of $\mathrm{St}(s, M)$, containing points z_k and z_r, respectively, such that $q(z_k) = q(z_r)$. Hence, the vector $z_k - z_r$ lies in the p-plane P. Thus, $s_k - s_r$ is a neighbourhood of $z_k - z_r$ in the plane $H_k + H_r$. Recall that H_k and H_r are the n-planes determined by the n-simplices s_k and s_r, respectively. The dimension of $H_k + H_r$ is $n + 1$. (This is because the embedding of M in R^{n+p} is of local codimension 1, so that by the remark of 1.6, the intersection of H_k and H_r has dimension greater than or equal to $n - 1$. This dimension cannot be equal to n, since then, the sum of H_k and H_r would equal either, say H_k, then, the fact that the p-plane P belongs to $Q[s]$ would imply by the corollary of 2.5, that the intersection of P and $H_k + H_r$ is zero. This contradicts the fact that this intersection contains the vector $z_k - z_r$. Actually, we only require in the remainder of this proof that $H_k + H_r$ have dimension greater than n, so that our hypothesis that the embedding of M in R^{p+n} be of local codimension 1 is not crucial here. We leave it to the reader to verify this.)

Now choose a basis $e_1, \ldots, e_{p+n}$ of R^{p+n} such that: $e_p = z_k - z_r$, the vectors $e_1, \ldots, e_p$ span the p-plane P, and the vectors $e_p, \ldots, e_{p+n}$ span $H_k + H_r$. (We note that $P + H_k + H_r = R^{p+n}$, because of the fact that P belongs to $Q[s]$ implies, by the corollary of 2.5, that $P + H_k = R^{p+n}$.)

Let $f\colon R^{pn} \to G_{n,p}$ be defined by $f(a) = P(a)$, where $P(a)$ is the p-plane spanned by the vectors $R_i(a) = \sum a_{ij}e_j$, $i = 1, \ldots, p$, $j = 1, \ldots, p + n$. (Recall that we identify points a in R^{pn} with $p \times (p + n)$ matrices $A = (a_{ij})$, whose first p-columns form the $p \times p$ identity matrix.) Now $R_p(a) = e_p + \sum a_{pj}e_j$, $j = p + 1, \ldots, p + n$, and therefore $R_p(a)$ belongs to $H_k + H_r$. Choose the points a in R^{pn} sufficiently near the origin to ensure that $R_p(a)$ will lie in the neighbourhood $s_k - s_r$ of $e_p = z_k - z_r$ in $H_k + H_r$, and also close enough to ensure that the p-plane $P(a)$ lies in the open set $Q[s]$. Thus $P(a)$ contains a vector $z_k' - z_r'$, where z_k' belongs to the simplex s_k, and z_r' belongs to the simplex s_r. Hence, by the remark of 3.6, $P(a)$ belongs to the set $Q[s] - P[s]$, which proves the theorem of 3.8.

3.9. *Proof of the theorem of* 1.7. By the proposition of 3.7, the set $P[s]$ is open, and it is a subset of $Q[s]$ by the lemma of 3.2. It is non-empty by the remark of 1.4. By the theorem of 3.8, $Q[s] - P[s]$ is open in $Q[s]$. Since, by the theorem of 3.3, $Q[s]$ is connected, it follows that $P[s] = Q[s]$. Thus, by the theorem of 3.3, $P[s]$ is contractible and open in $G_{n,p}$.

Added in proof.

3.10. THEOREM. *Let M^n be a non-bounded combinatorial manifold which has a locally normal embedding of local codimension* 1 *in R^{n+p}. Let $g, h\colon M \to G_{n,p}$ be two transverse fields and let $M(g)$ and $M(h)$ denote the resulting differential structures on M which are compatible with its given combinatorial structure. Then $M(g)$ is diffeomorphic to $M(h)$.*

Proof. Since the sets $P[s]$ are contractible, it is easy to see that the two transverse fields g and h are *transversely homotopic*. That is, there exists a homotopy $F: M \times I \to G_{n,p}$ such that $F_0 = g$, $F_1 = h$, and $F_t: M \to G_{n,p}$ is a transverse field. The theorem then follows from (**9**, Theorem 1.11).

References

1. S. S. Cairns, *Homeomorphisms between topological manifolds and analytic manifolds*, Ann. of Math. (2) *41* (1940), 796–808.

2. H. Davis, *Manifolds with local codimension one*, Thesis, University of Illinois, 1965 (to appear).

3. W. Hurewicz and H. Wallman, *Dimension theory* (Princeton Mathematical Series, Vol. 4, Princeton Univ. Press, Princeton, 1941).

4. J. R. Munkres, *Elementary differential topology* (Princeton Univ. Press, Princeton, N.J., 1963).

5. H. Noguchi, *The smoothing of combinatorial n-manifolds in* $(n + 1)$ *space*, Ann. of Math. (2) *72* (1960), 201–215.

6. H. Putz, *Transverse field implies normal microbundle*, Proc. Amer. Math. Soc. *23* (1969), 232–236.

7. J. Tao, *Some properties of* $(n - 1)$*-manifolds in the Euclidean n-space*, Osaka Math. J. *10* (1958), 137–146.

8. ——— *On the smoothing of a combinatorial n-manifold immersed in the Euclidean* $(n + 1)$*-space*, Osaka Math. J. *13* (1961), 229–249.

9. J. H. C. Whitehead, *Manifolds with transverse fields in Euclidean space*, Ann. of Math. (2) *73* (1961), 154–211.

Temple University,
Philadelphia, Pennsylvania