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On the Dual König Property of the Order-interval Hypergraph of Two Classes of N-free Posets

Isma Bouchemakh and Kaci Fatma

Abstract. Let P be a finite N-free poset. We consider the hypergraph $\mathcal{H}(P)$ whose vertices are the elements of P and whose edges are the maximal intervals of P. We study the dual König property of $\mathcal{H}(P)$ in two subclasses of N-free class.

1 Introduction

Let $(P, \leq)$ be a finite partially ordered set (briefly *poset* P). A subset of P is called a *chain* (resp. *antichain*) if every two elements in P are comparable (resp. incomparable). The number of elements in a chain is the *length* of the chain. The *height* of an element $x \in P$, denoted by *h(x)*, is the length of a longest chain in P having x as its maximum element. The *height* of a poset P, denoted *h(P)*, is the length of a longest chain in P. The *i-level* or *height-i-set* of P, denoted by N_i, is the set of all elements of P that have height i.

Let p and q be two elements of P. We say *q covers p*, and we denote $p \prec q$, if $p \prec v \leq q$ implies $v = q$. Furthermore we denote by Max P (resp. Min P) the set of all maximal (resp. minimal) elements of P. A subset I of P of the form $I = \{v \in P, p \leq v \leq q\}$ (denoted $[p, q]$) is called an *interval*. It is maximal if p (resp. q) is a minimal (resp. maximal) element of P. Denote by $\mathcal{I}(P)$ the family of maximal intervals of P. The hypergraph $\mathcal{H}(P)$=(P,$\mathcal{I}(P)$) whose vertices are the elements of P and whose edges are the maximal intervals of P is said to be the *order-interval hypergraph of P*.

A subset A (resp. T) of P is called *independent* (resp. a *point cover* or *transversal set*) if every edge of $\mathcal{H}$ contains at most one point of A (resp. at least one point of T). A subset $\mathcal{M}$ (resp. $\mathcal{R}$) of $\mathcal{I}$ is called a *matching* (resp. an *edge cover*) if every point of P is contained in at most one member of $\mathcal{M}$ (resp. at least one member of member of $\mathcal{R}$). Let

$$\begin{aligned}
\alpha(\mathcal{H}) &= \max\{|A| : A \text{ is independent}\},\\
\tau(\mathcal{H}) &= \min\{|T| : T \text{ is a point cover}\},\\
\nu(\mathcal{H}) &= \max\{|\mathcal{M}| : \mathcal{M} \text{ is a matching}\},\\
\rho(\mathcal{H}) &= \min\{|\mathcal{R}| : \mathcal{R} \text{ is an edge cover}\}.
\end{aligned}$$

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These numbers are called the *independence number*, the *point covering number*, the *matching number*, and the *edge covering number* of $\mathcal{H}(P)$, respectively. It is easy to see that $\nu(\mathcal{H}) \leq \tau(\mathcal{H})$ and $\alpha(\mathcal{H}) \leq \rho(\mathcal{H})$. We say that $\mathcal{H}$ has the *König property* if $\nu(\mathcal{M}) = \tau(\mathcal{M})$ and *dual König property* if $\nu(\mathcal{H}^*) = \tau(\mathcal{H}^*)$, *i.e.*, $\alpha(\mathcal{H}) = \rho(\mathcal{H})$, since $\alpha(\mathcal{H}) = \nu(\mathcal{H}^*)$ and $\rho(\mathcal{H}) = \tau(\mathcal{H}^*)$. This class of hypergraphs has been studied intensively in the past, and we find interesting results from an algorithmic point of view as well as min-max relations [2–8].

Let $P_1 = (E_1, \leq_1)$ and $P_2 = (E_2, \leq_2)$ be two posets such that E_1 and E_2 are disjoint. The *disjoint sum* $P_1 + P_2$ of P_1 and P_2 is the poset defined on $E_1 \cup E_2$ such that $x \leq y$ in $P_1 + P_2$ if and only if ($x, y \in P_1$ and $x \leq_1 y$) or ($x, y \in P_2$ and $x \leq_2 y$). The *linear sum* $P_1 \oplus P_2$ of P_1 and P_2 is the poset defined on $E_1 \cup E_2$ such that $x \leq y$ in $P_1 \oplus P_2$ if and only if ($x, y \in P_1$ and $x \leq_1 y$) or ($x, y \in P_2$ and $x \leq_2 y$) or ($x \in P_1$ and $y \in P_2$).

Let $A \subseteq \operatorname{Max} P_1$ and $B \subseteq \operatorname{Min} P_2$ with A and B are not empty. The *quasi-series composition* of P_1 and P_2 denoted $P = (P_1, A) * (P_2, B)$ is the poset $P = (E_1 \cup E_2, \leq)$ such that $x \leq y$ if ($x, y \in E_1$ and $x \leq_1 y$) or ($x, y \in E_2$ and $x \leq_2 y$) or ($x \in E_1$, $y \in E_2$), and there exist $\alpha \in A$, $\beta \in B$ such that $x \leq_1 \alpha$ and $\beta \leq_2 y$.

2 N-free Poset

A poset P is said to be *series-parallel*, if it can be constructed from singletons P_0 (P_0 is the poset having only one element) using only the two operations disjoint sum and linear sum. It may be characterized by the fact that it does not contain the poset N as an induced subposet [12, 13]. P is called N-*free* if and only if its Hasse diagram does not contain four vertices v_1, v_2, v_3, v_4, where $v_1 < v_2$, $v_2 > v_3$ and $v_3 < v_4$, and v_1 and v_4, v_1 and v_3, v_2 and v_4, are incomparable. The class of N-free posets contains the class of series-parallel posets. Habib and Jegou [10] defined the *Quasi- Series-Parallel* (QSP) class of posets, as the smallest class of posets that contains P_0 and closed under quasi-series composition and linear sum. They proved that a poset is N-free if and only if it is a QSP poset. The following theorem gives many other characterizations of N-free posets (see [9–11]).

Theorem 2.1 *The four following properties are equivalent:*

(i) *P is QSP.*
(ii) *P is an N-free poset.*
(iii) *P is a C.A.C. (Chain-Antichain Complete) order* i.e., *every maximal chain intersects each maximal antichains.*
(iv) *The Hasse diagram of P is a line-digraph.*
(v) *For every two elements $p, q \in P$, if $N(p) \cap N(q) \neq \emptyset$, then $N(p) = N(q)$, where $N(p)$ denoted the set of all elements of P that cover p in P.*

It is known that the order-interval hypergraph $\mathcal{H}(P)$ has the König and dual König properties for the class of series-parallel posets [3]. In [4], it was proved that $\mathcal{H}(P)$ has again the dual König property for the class of a posets that contains the series-parallel posets and whose members have comparability graphs that are distance-hereditary graphs or generalizations of them. If P is an N-free poset, the König property is not satisfied in general; see [4]. The poset of Figure 1 is an example where $\nu(\mathcal{H}(P)) = 1$,

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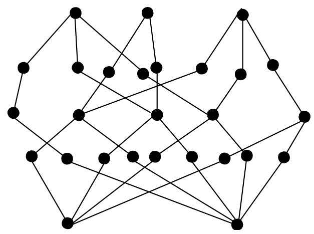

Figure 1: $\nu(\mathcal{H}(P)) = 1$ and $\tau(\mathcal{H}(P)) = 2$

$\tau(\mathcal{H}(P)) = 2$. In this paper, we consider two classes of N-free posets and prove that the dual König property of the order-interval hypergraph of these classes of posets are satisfied.

2.1 Blocks in an N-free Poset

There is a useful representation of an N-free poset, namely the *block* (see [1]). If P is an N-free poset with levels $N_1, \dots, N_r$, a block of P is maximal complete bipartite graph in the Hasse diagram of P. More precisely, a block of P is a pair (A_i, B_i), where $A_i, B_i \subset P$ such that A_i is the set of all lower covers of every $x \in B_i$ and B_i is the set of all upper covers of every $y \in A_i$. By convention, $(\varnothing, \operatorname{Min} P)$ and $(\operatorname{Max} P, \varnothing)$ are blocks

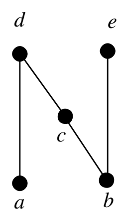

Figure 2: P is N-free with blocks $(\varnothing, \{a, b\}), (\{b\}, \{c, e\}), (\{a, c\}, \{d\})$ and $(\{d, e\}, \varnothing)$.

In this paper, we say that (A_i, B_i) and (A_j, B_j) are *adjacent* if there exists at least one vertex of $A_i \cup B_i$ in the same interval in P with at least one vertex of $A_j \cup B_j$. For example, the blocks $(\{b\}, \{c, e\})$ and $(\{a, c\}, \{d\})$ of poset of Figure 2 are adjacent.

2.2 N-free Poset of Type 1

Definition 2.2 Let P be a connected poset with levels $N_1, N_2, \dots, N_r$. We say that P is of *Type 1* if there exists an integer n such that the induced subposet $P_{n,n+1}$ formed from the consecutive levels $N_n \cup N_{n+1}$ is of the form $N_n \oplus N_{n+1}$.

For the class of posets of Type 1, we give the following result.

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Theorem 2.3 *Let P be a poset of Type 1. Then $\mathcal{H}(P)$ has the dual König property, and we have $\alpha(\mathcal{H}(P)) = \rho(\mathcal{H}(P)) = \mathrm{Max}\{|\mathrm{Max}\,P|, |\mathrm{Min}\,P|\}$.*

Proof We set $\mathrm{Min}\,P = \{a_1, a_2, \ldots, a_k\}$ and $\mathrm{Max}\,P = \{b_1, b_2, \ldots, b_l\}$. Consider the family of edges $\mathcal{I}$ of $\mathcal{H}(P)$ such that

$$\mathcal{I} = \begin{cases} \{[a_j, b_j], j = 1, \ldots, k\} \cup \{[a_k, b_j], j = k+1, \ldots, l\} & \text{if } k \le l, \\ \{[a_j, b_j], j = 1, \ldots, l\} \cup \{[a_j, b_l], j = l+1, \ldots, k\} & \text{if } k > l. \end{cases}$$

It is not difficult to see that $\mathcal{I}$ is an edge-covering family of $\mathcal{H}(P)$ of cardinality equal to $\mathrm{Max}\{|\mathrm{Max}\,P|, |\mathrm{Min}\,P|\}$. Hence, $\alpha(\mathcal{H}(P)) = \rho(\mathcal{H}(P)) = \mathrm{Max}\{|\mathrm{Max}\,P|, |\mathrm{Min}\,P|\}$ ■

In particular, the order-interval hypergraph of the N-free poset of Type 1 has the dual König property.

3 N-free Poset of Type 2

Definitions

(a) Let P be a connected N-free poset with levels $N_1, N_2, \ldots, N_r$. We say that P is a poset of *Type* 2 if there exists an integer n such that N_n is the first level where the induced subposet $P_{n,r}$ is disconnected of the form $P_{n,r} = P_1 + P_2 + \cdots + P_l$, and for all $i \in L = \{1, \ldots, l\}$, P_i is connected poset of Type 1.

(b) We say that the subposet P_i is *linked* with the subposet P_j by a vertex z of N_1, if we can obtain intervals of the form $[z, x]$ and $[z, y]$ for each $x \in \mathrm{Max}\,P_i$ and $y \in \mathrm{Max}\,P_j$, and we say z links P_i with P_j.

(c) We say that P_i is linked with P_j by the subset R of N_1 if for every element z of R, z links P_i with P_j.

Example 3.1 The poset P in Figure 3 is N-free of Type 2; it is easy to see that N_2 is the first level where $P_{2,3} = P_1 + P_2$ is disconnected poset where P_1 and P_2 are posets of Type 1. On the other hand, Q is an N-free poset but not of Type 2.

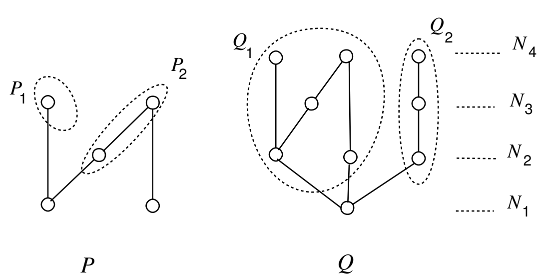

Figure 3:

In order to prove the dual König property of $\mathcal{H}(P)$, where P is N-free of Type 2, let us introduce the following notation.

Notation

(a) For every subposet P_k, we denote by R_k the subset of N_1, where every element of R_k is comparable with all elements of $\operatorname{Max} P_k$, and R_k does not link P_k with any other poset $P_s, s \in L$. The set R_k can be empty.

(b) For every subposet P_k, we denote by R'_{ik}, $i \in I_k = \{1, 2, \dots, |N_1|\}$, the subset of N_1 that links P_k with the same family of poset $\{P_s\}_{s\in L}$. We can obtain $R'_{ik} = R'_{jl}$ for $i \neq j$ and $k \neq l$.

Observation 3.2 *The family $\{R'_{ik}\}_{k\in L, i\in I_k}$ is pairwise disjoint.*

See Figure 4 for an illustration of the class of N-free posets of Type 2.

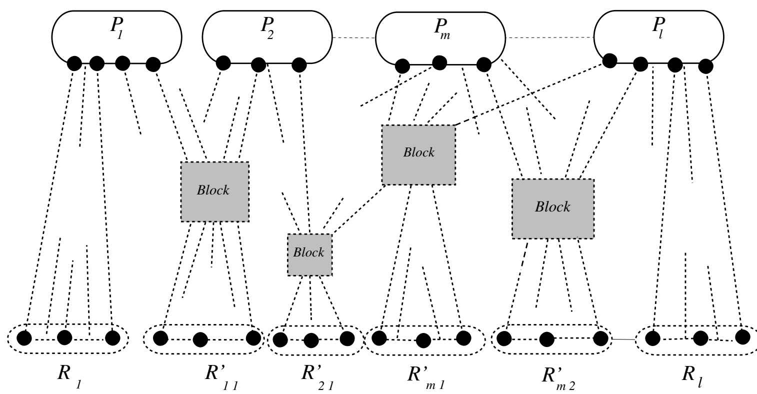

Figure 4: Illustration of an N-free poset of Type 2

3.1 Maximal Stable Sets of $\mathcal{H}(P)$

In our poset, it is clear that for a linked subposet family $F_k = \{P_l\}_{l\in L}$, we can obtain blocks (A_i, B_i) in the level N_{n-j}, for $j \in \{0, 1, \dots, n-1\}$, *i.e.*, B_i intersects N_{n-j}, and every element x of A_i, x links a subfamily F_s of F_k, we say (A_i, B_i) links F_s. Such blocks must exist in N_n since P is N-free poset of Type 2.

We note the following observation.

Observation 3.3 *For every block (A_i, B_i) that links F_s, B_i has the following partition: $B_i = \bigcup_{t\in T} B_{i,t}$, where $\forall x \in B_{i,t}$, x is comparable with a vertex of* Min P_t, *where $P_t \in F_s$, and $|F_s| = |T|$*

Let us now give two algorithms to find maximal stable sets of an N-free poset of Type 2; the second algorithm can be applied only after the first.

Maximal Stable-set 1 Algorithm

INPUT: An N-free poset P of Type 2. $F_1, F_2, \dots, F_m$ all linked subposet families of P.

(a) For each k, from $k = 1$ to m.

(b) For each j, from $j = 0$ to $n-1$, in N_{n-j} we determine $C_{k,j}$ by taking for every block (A_i, B_i) that links a subfamily of F_k, one vertex from each $B_{i,t}$ such that:

 (i) if there exists a family $\{B_{i,t}\}_i$ from block family that are adjacent pairwise, we take only one vertex from only one set of $\{B_{i,t}\}_i$;

 (ii) we delete every vertex which is in the same interval with a vertex of $C_{k,t}$, $t < j$.

(c) Put $C_k = \bigcup_{j=0}^{n-1} C_{k,j}$.

(d) Output $\mathcal{C} = (\bigcup_{k=1}^{m} C_k) \cup (\bigcup_{l \in L} R_l)$. End

Theorem 3.4 *The set $\mathcal{C}$ is a maximal stable set of $\mathcal{H}(P)$.*

Proof $\mathcal{C}$ is a stable set by construction of every C_k. It remains the maximality of $\mathcal{C}$. We say that an interval I crosses a block (A_i, B_i) if I intersects B_i. Let us show that for every interval I of P, I contains one vertex of $\mathcal{C}$, and this means that for every $x \in P$, $\mathcal{C} \cup \{x\}$ will not be a stable set.

In the case where I does not cross any block, the minimal vertex of I will be in R_l.

Now, in the case where I crosses a block (A_i, B_i), let y be a commun vertex of B_i and I. If $y \in \mathcal{C}$, then I intersects $\mathcal{C}$. Otherwise, $y \notin \mathcal{C}$, which means that y is in the same interval J with an element y' of $\mathcal{C}$. Consequently, I and J will have minimal vertices in R'_{pq} and maximal vertices in Max P_l; this gives $y' \in I$. ∎

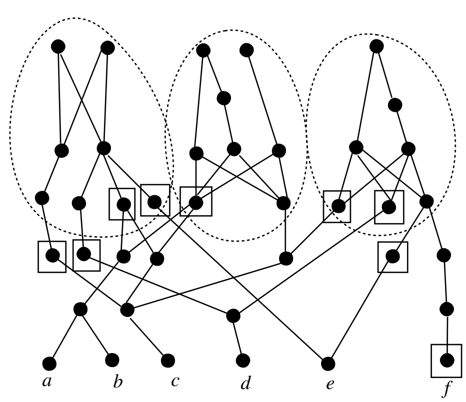

Figure 5: An N-free poset P of Type 2. Applying the Maximal Stable-Set 1 algorithm on P; the framed vertices form a maximal stable set of $\mathcal{H}(P)$.

Example 3.5 The poset of Figure 5 is N-free of Type 2, where P_1, P_2, and P_3 are the supbosets surrounded from left to right. We have $R'_{11} = R'_{12} = \{a, b\}$, $R'_{21} = R'_{22} =$

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$R'_{13} = \{c\}$, $R'_{31} = R'_{23} = \{d\}$, $R'_{41} = R'_{33} = \{e\}$, and $R_3 = \{f\}$. The framed vertices form the maximal stable set $\mathcal{C}$ of $\mathcal{H}(P)$ obtained by the Maximal Stable-set 1 algorithm.

We will need the following definition.

Definition 3.6 In $\mathcal{H}(P)$, for every vertex $x \in P$, a *stable adjacent* M_x to x is the set of all vertices y such that x and y are in the same interval of P, where M_x is stable. M_x can be equal to $\{x\}$. We say M_D is a stable adjacent to the set D of P if it is a maximal stable subset of the union of all M_x, $x \in D$, where $|M_D| \geq |D|$.

We can write $\bigcup_{k=1}^{m} C_k = D_1 \cup D_2 \cup \cdots \cup D_m$ for the stable set obtained from the Maximal Stable-set 1 algorithm, where D_i are subsets of blocks of P. We determine a new maximal stable set $\mathcal{C}'$ from $\mathcal{C}$ as follows.

Maximal Stable-set 2 Algorithm

INPUT: An N-free poset P of Type 2, and maximal stable set $\mathcal{C}$.
OUTPUT: A new maximal stable set $\mathcal{C}'$.

1. $\mathcal{C}' := \mathcal{C}$.
2. For each i, from $i = 1$ to m.
3. We determine M_{D_i} the stable adjacent to D_i such that $\mathcal{C} - (\bigcup_{t=1}^{t=i} D_t) \cup (\bigcup_{t=1}^{t=i} M_{D_t})$ is stable.
4. We take $\mathcal{C}' := \mathcal{C} - (\bigcup_{t=1}^{t=i} D_t) \cup (\bigcup_{t=1}^{t=i} M_{D_t})$.
5. Stop.

By construction of $\mathcal{C}'$, we deduce the following result.

Proposition 3.7 *The set $\mathcal{C}'$ is a maximal stable set of $\mathcal{H}(P)$.*

We denote by C'_k the set of all vertices obtained from every $x_i \in C_k$ using the Maximal Stable-set 2 algorithm.

As a consequence of the previous algorithms, we make the following observation.

Observation 3.8 *Consider the subposet family F_k linked by R'_{pq}.*

(i) *The set R'_{pq} has the following partition: $R'_{pq} = \bigcup_s R'_{pq,s}$, where for every s, $R'_{pq,s}$ is a stable adjacent to A_s a subset of C'_k.*

(ii) *It will be possible to obtain that the family $\{A_s\}_s$ is pairwise disjoint.*

Proof To prove the second observation, we suppose that x is a common vertex of A_s and $A_{s'}$. Let I (resp. J) an interval containing x with minimal element $c_j \in R'_{pq,s}$ (resp. $c_{j'} \in R'_{p'q',s'}$). In I (resp. J) there exists a vertex z (resp. z') that is incomparable with every vertex of $R'_{p'q',s'}$ (resp. $R'_{pq,s}$) (we take as an example, the vertex z (resp. z') such that $c_j < z$ (resp. $c_{j'} < z'$)). Otherwise, we will obtain $R'_{pq,s} = R'_{p'q',s'}$ since P is N-free. In this case, we can reconstruct $\mathcal{C}$ by starting with z and z' respectively to obtain two new disjoint sets. ∎

In the remainder of this paper, we suppose that $\mathcal{C}'$ verifies Observation 3.8(ii).

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Example 3.9 The poset of Figure 6 is N-free of Type 2, where $\mathcal{C} = \{a, b\}$. Applying the Maximal Stable-set 2 algorithm, we obtain two different maximal stable sets: $\mathcal{C}'_1$ is the framed vertex set and $\mathcal{C}'_2$ is the surrounded vertex set. We remark that $\mathcal{C}'_2$ verifies Observation 3.8(ii), while $\mathcal{C}'_1$ does not.

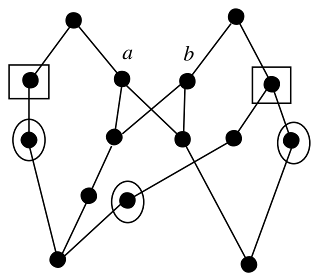

Figure 6: Two different maximal stable sets of $\mathcal{H}(P)$ by applying the Maximal Stable-set 2 algorithm.

3.2 Edge Covering Family of $\mathcal{H}(P)$

In this section, we will present an algorithm to construct an edge covering family of $\mathcal{H}(P)$ where P is an N-free of Type 2.

We set

$$\operatorname{Max} P_l = \{b_1^l, b_2^l, \ldots, b^l_{|\operatorname{Max} P_l|}\}, \qquad R_l = \{a_1, a_2, \ldots, a_{|R_l|}\},$$
$$R'_{pq,s} = \{c_1, c_2, \ldots, c_{|R'_{pq,s}|}\}, \qquad \bigcup_{i \in I_l} R'_{il} = \{c'_1, c'_2, \ldots, c'_{m_l}\}.$$

Theorem 3.10 *If for every $k \in L$ we have*

$$|\operatorname{Max} P_k| \geq |R_k| + \sum_{i \in I_k} |R'_{ik}|, \tag{3.1}$$

then $\mathcal{H}(P)$ has the dual König property and $\alpha(\mathcal{H}(P)) = \rho(\mathcal{H}(P)) = |\operatorname{Max} P|$.

Proof For every P_k, we consider the edge family:

$$\mathcal{I}_k = \Big\{[a_i, b_i], i = 1, \ldots, |R_k|\Big\} \cup \Big\{[c'_{j-|R_k|}, b_j], j = |R_k| + 1, \ldots, |R_k| + m_k\Big\}$$
$$\cup \Big\{[c'_{m_k}, b_s], s = m_k + |R_k| + 1, \ldots, |\operatorname{Max} P_k|\Big\}.$$

The union of all $\mathcal{I}_k$, $k \in L$ is an edge covering family of $\mathcal{H}(P)$ with cardinality equal to $|\operatorname{Max} P|$ and as $\operatorname{Max} P$ is a stable set of $\mathcal{H}(P)$ then $\alpha(\mathcal{H}(P)) = \rho(\mathcal{H}(P)) = |\operatorname{Max} P|$. ∎

We remark that by applying the Maximal Stable-set 2 algorithm to P, we can obtain different maximal stable sets of $\mathcal{H}(P)$, and this depends on the choice of M_{D_i}. In the next algorithm we need to characterize the set $\mathcal{C}'$ as follows.

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$\mathcal{C}'$ is determined such that for every subposet family F_k that contains subposets P_l verifying (3.1), we determine M_{D_i} different to D_i, and if $x \in D_i$ is incomparable with all vertices of $\operatorname{Max} P_l$, then M_x will be too. For other subposet families, M_{D_i} does not contain a vertex of $\operatorname{Max} P_m$, where R_m is not empty.

Edge-Cover Algorithm

INPUT: An N-free poset P of Type 2 and the maximal stable set $\mathcal{C}'$.

OUTPUT: An edge covering family $\mathcal{I}(\mathcal{H}(P))$.

Step 1 For every R_l, where P_l does not verify (3.1), we construct the edge family E_l with $|R_l|$ intervals as follows:

1.1 If $|R_l| \le |\operatorname{Max} P_l|$: $E_l = \{[a_j, b_j^l], j = 1, 2, \dots, |R_l|\}$.

1.2 Otherwise: $E_l = \{[a_j, b_j^l], j = 1, 2, \dots, |\operatorname{Max} .P_l|\} \cup \{[a_t, b_{|\operatorname{Max} P_l|}], t = |\operatorname{Max} P_l| + 1, \dots, |R_l|\}$.

Step 2 For every P_l, where P_l verifies (3.1), we construct the edge family J_l as follows:

$$J_l = \Big\{ [a_i, b_i], i = 1, \dots, |R_l| \Big\} \cup \Big\{ [c'_{j-|R_l|}, b_j], j = |R_l| + 1, \dots, |R_l| + m_l \Big\} \\ \cup \Big\{ [c'_{m_l}, b_s], s = m_l + |R_l| + 1, \dots, |\operatorname{Max} P_l| \Big\}.$$

We obtain $|\operatorname{Max} P_l|$ intervals.

Step 3 First, determine all linked subposet families $F_1, F_2, \dots, F_m$. Then apply this step to $F_k = \{P_l\}_{l \in S_k}$, which is linked by R'_{pq} for $k = 1$ to $k = m$.

In this step, we use the vertices b_t^l of $\operatorname{Max} P_l$, $P_l \in F_k$, which are not used in Step 1 or in the application of this step to F_t, where $t < k$; otherwise, we use vertices already used.

Let A'_s be the set A_s deleting all vertices comparable with $\operatorname{Max} P_m$, where P_m verifies (3.1), and $F'_k = \{P_l\}_{l \in S'_k}$ be the family F_k deleting all subposets verifying (3.1) . For every $R'_{pq,s}$ we construct the edge family I_s as follows:

3.1 If $|A'_s| \le |R'_{pq,s}|$, then $I_s = \Big\{ [c_j, b_t^l], j = 1, 2, \dots, |A'_s| \text{ and } l \in S'_k \Big\}$. We obtain $|A'_s|$ intervals.

3.2 If $|A'_s| > |R'_{pq,s}|$, then

$$I_s = \Big\{ [c_j, b_t^l], j = 1, 2, \dots, |R'_{pq,s}| \text{ and } l \in S''_k \subset S'_k \Big\} \cup \big\{ [c_1, b_t^l], l \in (S'_k - S''_k) \big\}.$$

We obtain $|A'_s|$ intervals.

Step 4 It remains some minimal vertices c_j that are not used in Steps 1 and 3 such that $c_j \in R'_{pq,s}$ and R'_{pq} does not link any subposet verifying (3.1). In this step, we construct J_{c_j} the interval containing c_j and b_t^l a maximal vertex that is not already used, otherwise, J_{c_j} is any interval containing c_j.

Step 5 We take $\mathcal{I}(\mathcal{H}(P))$ to be the set of all intervals obtained from Step 1 to Step 4.

END

Theorem 3.11 *The Edge-Cover algorithm applied to an N-free poset P of Type 2, yields an edge-covering family of $\mathcal{H}(P)$.*

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Proof We can assert that every z of P that is a minimal element, comparable with a vertex of R_m or comparable with a vertex of $\operatorname{Max} P_l$, where P_l verifies (3.1) is covered by $\mathcal{I}(\mathcal{H}(P))$.

Moreover, if $z > x$, where $x \in A'_s$, then z would be covered by the interval of $\mathcal{I}(\mathcal{H}(P))$ that intersects A'_s.

In other cases, suppose that there exists z of P that is not covered by $\mathcal{I}(\mathcal{H}(P))$. We distinguish two cases.

Case 1. If z is a maximal of P_l and no interval obtained from Step 3 or Step 4 covers z, then P_l necessarily would verify (3.1). This contradicts the construction of intervals in these steps.

Case 2. Let $J \notin \mathcal{I}(\mathcal{H}(P))$ containing z and x, where $x \in A'_s$ and $x \nless z$. Let I be the interval of $\mathcal{I}(\mathcal{H}(P))$ containing x. The only form of I and J is that they will have maximal elements in $\operatorname{Max} P_l$ and two different minimal elements in $R'_{pq,s}$. Then z is not covered by I, then for every couple (t, t') of (I, J), where $t \le x$ and $t' \le z$, we will have $t \nleq t'$. We suppose that such a couple exists.

If t and t' are not in the same interval and $A'_s \cup \{t, t'\} - \{x\}$ is stable, then x can be replaced by t and t' in $\mathcal{C}'$, and this contradicts the construction of $\mathcal{C}'$. Otherwise, we can reconstruct A'_s starting by z. In this case, $R'_{pq,s}$ will be partitioned into at least two subsets, and by applying the Edge-Cover algorithm; z will be covered by the new family. ∎

As a consequence of Theorem 3.11, we have the following corollary.

Corollary 3.12 *If in the Edge-Cover algorithm, for every vertex x of* $\operatorname{Max} P$ *(resp.* $\operatorname{Min} P$*), x is taken only once in the construction of $\mathcal{I}(\mathcal{H}(P))$, then P will have the dual König property.*

Proof In this case, we will have $|\mathcal{I}(\mathcal{H}(P))| = |\operatorname{Max} P|$ (resp. $|\operatorname{Min} P|$), and as $\operatorname{Max} P$ and $\operatorname{Min} P$ are stable sets of $\mathcal{H}(P)$, therefore

$$\alpha(\mathcal{H}(P)) = \rho(\mathcal{H}(P)) = |\operatorname{Max} P| \quad (\text{resp., } \alpha(\mathcal{H}(P)) = \rho(\mathcal{H}(P)) = |\operatorname{Min} P|).$$

∎

Theorem 3.13 *Let P be an N-free poset of type 2. Then $\mathcal{H}(P)$ has the dual König property.*

Proof The main idea of the proof is to use $\mathcal{I}(\mathcal{H}(P))$ obtained from the Edge-Cover algorithm for constructing a stable set $\mathcal{C}(\mathcal{H})$ of $\mathcal{H}(P)$ with the same size as $\mathcal{I}(\mathcal{H}(P))$.

Let B_1 (resp. B_2) be the union of all R_l (resp. $\operatorname{Max} P_k$), where P_l (resp. P_k) does not verify (resp. verifies) (3.1).

From Step 1 (resp. Step 2) of the Edge-Cover algorithm, B_1 (resp. B_2) is a stable set with the cardinality equal to the cardinality of the union of all E_l (resp. J_l). It becomes clear that $B_1 \cup B_2$ is stable set.

The union of all I_s of Step 3.1 can be partitioned into 2 subsets. The first denoted by D_1, which is the union of all I_s, where $R'_{pq,s}$ does not link subposets verifying (3.1), and the second is denoted by D_2. Let $B_{3,1}$ be the union of all $R'_{pq,s}$, where R'_{pq} does not link subposets verifying (3.1) and $|R'_{pq,s}| > |A_s|$. $B_{3,1}$ is a stable set with cardinality equal to $|D_1|$ plus the cardinality of the union of all J_{c_j} of Step 4.

We denote by $B_{3,2}$ the union of all A'_s such that $|A'_s| > |R'_{pq,s}|$ or $|A'_s| \le |R'_{pq,s}|$, where R'_{pq} links subposets verifying (3.1). From Observation 3.8(ii), we deduce that there is no commun vertex x of A_s and $A_{s'}$ that is covered by two different intervals of $\mathcal{I}(\mathcal{H}(P))$. Consequently, $|B_{3,2}|$ is equal to $|D_2|$ plus the cardinality of the union of all I_s of Step 3.2. Consider the set $\mathcal{C}(\mathcal{H}) = B_1 \cup B_2 \cup B_{3,1} \cup B_{3,2}$.

Hence, it is not difficult to see that $\mathcal{C}(\mathcal{H})$ is a stable set with size $|\mathcal{I}(\mathcal{H}(P))|$. ■

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L'IFORCE Laboratory, University of Sciences and Technology Houari Boumediene, Faculty of Mathematics, B.P. 32 El-Alia, Bab-Ezzouar, 16111, Algiers, Algeria
e-mail: ibouchemakh@usthb.dz

L'IFORCE Laboratory, Mohamed Khider University of Biskra,, Department of Mathematics, 07000, Algeria
e-mail: kaci_fatma2000@yahoo.fr