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On Hecke L-series associated with cubic characters

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Abstract

In this work we study the moments of central values of Hecke L-functions associated with cubic characters, and establish quantitative non-vanishing result for the L-values.

1. Introduction and statement of the main result

Let K be a fixed number field with the discriminant D_K and the ring of integers O_K, and denote by $L(s,\chi)$ the L-series associated with primitive finite-order characters $\chi \bmod q$, $(q, D_K) = 1$, where $q \in O_K$. The L-values of such functions at the center of the critical strip, $L(1/2,\chi)$, encode intrinsic arithmetic information about the field K, and they are intensively studied from various perspectives. A theorem of Rohrlich [Roh89] asserts that there are infinitely many characters χ such that $L(1/2,\chi) \neq 0$. In light of some applications, we also need to know more precise information that the characters χ can be chosen with pre-assigned order $n \geqslant 2$. For $n = 2$, this has been established by Goldfeld, Hoffstein and Patterson [GHP82]. In general this question fits well in the theory of automorphic forms on the metaplectic group. In this work, we prove a quantitative non-vanishing theorem in the case $n = 3$, by establishing moments of these central values for L-functions associated to cubic characters.

We will assume here $K = \mathbb{Q}(\zeta_3)$ $(= \mathbb{Q}(\sqrt{-3}))$, where $\zeta_n = e^{2\pi i/n}$ is the nth root of unity. Our method however works for any number field K containing ζ_3 after generalizing the large sieve inequality in [Hea00]. It is well known that the imaginary quadratic field K has class number 1, and in the ring of integers $O_K = \mathbb{Z}[\zeta_3]$ every ideal coprime to 3 has unique generator $\equiv 1 \pmod 3$. For $1 \neq c \in O_K$ which is square-free and $\equiv 1 \pmod 9$ (which we assume throughout the paper), let $(\frac{\cdot}{c})_3$ be the cubic residue symbol. Since $\chi_c = (\frac{\cdot}{c})_3$ is trivial on units, it can be regarded as a primitive character of the ray class group $h_{(c)} = I_{(c)}/P_{(c)}$, where, as usual, $I_{(c)} = \{\mathcal{A} \in I,\ (\mathcal{A},(c)) = 1\}$, $P_{(c)} = \{(a) \in P,\ a \equiv 1 \ (\mathrm{mod}(c))\}$, with I and P standing for the group of fractional ideals in K and the subgroup of principal ideals respectively.

The Hecke L-function associated with χ_c is defined by

$$L(s,\chi_c) = \sum_{0 \neq \mathcal{A} \subset O_K} \chi_c(\mathcal{A})(N\mathcal{A})^{-s}$$

for $\Re(s) > 1$, where $\mathcal{A}$ runs over all non-zero integral ideals in K, and $N\mathcal{A}$ is the norm of $\mathcal{A}$. As shown by Hecke, $L(s,\chi_c)$ admits analytic continuation to the whole s-plane as an entire function, and satisfies the functional equation

$$\Lambda(s,\chi_c) = W(\chi_c)(N(c))^{-1/2}\Lambda(1-s,\overline{\chi_c}),$$

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where $W(\chi_c)$ is the Gauss sum of χ_c,

$$W(\chi_c) = \sum_{a \in O_K/(c)} \chi_c(a) e\left(\mathrm{Tr}\left(\frac{a}{\delta c}\right)\right),$$

(δ) is the different of K,

$$\Lambda(s, \chi_c) = (|D_K| N(c))^{s/2} (2\pi)^{-s} \Gamma(s) L(s, \chi_c),$$

and D_K $(= -3)$ is the discriminant of K. In fact $L(s, \chi_c)$ coincides with the L-function associated with the newform $f(z) \in S_1(\Gamma_0(N), \chi)$ (see [Iwa97], for instance), where $N = |D_K| N(c)$, $\chi(n) = (D_K/n)\chi_c((n))$ and

$$f(z) = \sum_{\mathcal{A} \subset O_K} \chi_c(\mathcal{A}) e(zN(\mathcal{A})).$$

Here, as usual, $(\frac{D_K}{\cdot})$ is the Kronecker symbol of K, and $e(z) = e^{2\pi i z}$. If we define

$$G_c = \sum_{a \in O_K/(c)} \chi_c(a) e\left(\mathrm{Tr}\left(\frac{a}{c}\right)\right),$$

then $W(\chi_c) = \chi_c(\delta) G_c$.

The function $L(1/2, \chi_c)$ can be represented by finite Dirichlet series. There are two important analytic ingredients entering here. One is the asymptotic moments of a cubic Gauss sum, which have been studied by Patterson ([Pat77], see also [KP84] and [LP02]), using the metaplectic Eisenstein series and cubic reciprocity law. We will make use of this to control and handle the dual sum appearing in the approximate functional equation. This is the new feature in that the cubic Gauss sum in the ϵ-factor of the functional equation cannot be evaluated according to congruence classes. The other crucial tool, which is also employed in the recent works of Perelli and Pomykala [PP97] and Soundararajan [Sou00] in the case of quadratic twist, is the analogue of the large sieve inequality established by Heath-Brown [Hea00] for the cubic characters, to bound effectively the non-diagonal contributions

$$\sum_{N(m) \leqslant M}^{*} \left| \sum_{N(n) \leqslant N}^{*} c_n \left(\frac{n}{m}\right)_3 \right|^2 \ll_\epsilon (M + N + (MN)^{2/3})(MN)^\epsilon \sum_{N(n) \leqslant N}^{*} |c_n|^2,$$

for any $\epsilon > 0$, where $\sum^{*}$ denotes summation over square-free elements of $\mathbb{Z}[\zeta_3]$ congruent to 1 (mod 3).

The goal of this paper is to establish the following theorem.

THEOREM. *For $y \to \infty$ and any $\epsilon > 0$, we have*

$$\sum_{c \equiv 1 \,(\mathrm{mod}\, 9)}^{*} L(1/2, \chi_c) \exp(-N(c)/y) = Ay + O_\epsilon(y^{21/22+\epsilon})$$

and

$$\sum_{c \equiv 1 \,(\mathrm{mod}\, 9)}^{*} |L(1/2, \chi_c)|^2 \exp(-N(c)/y) \ll_\epsilon y^{1+\epsilon},$$

where A is the constant defined in § 3.

COROLLARY. *For $y \to \infty$, we have*

$$\#\{c \in \mathbb{Z}[\zeta_3],\ c \equiv 1 \ (\mathrm{mod}\, 3),\ N(c) \leqslant y,\ L(1/2, (*/c)_3) \neq 0\} \gg_\epsilon y^{1-\epsilon},$$

for any $\epsilon > 0$.

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We remark that, using a different approach, i.e. the analytic properties of the Mellin transform of the induced maximal parabolic Eisenstein series on the cubic cover of $GL(3)$, Farmer, Hoffstein and Lieman [FHL99, Main Theorem] have obtained a more general asymptotic formula for the first moment of the *modified* L-values. One different feature, if any, seems that our moments are over *square-free* integers.

2. Approximate functional equation

Let $x > 1$. By evaluating the integral

$$\frac{1}{2\pi i}\int_{(2)}(2\pi)^{-(s+1/2)}\Gamma(s+1/2)L(s+1/2)\frac{x^s}{s}\,ds$$

in two ways, we derive the following expression for $L(1/2,\chi_c)$:

$$\begin{aligned}L(1/2,\chi_c) = &\sum_{0\neq\mathcal{A}\subset O_K}\chi_c(\mathcal{A})(N(\mathcal{A}))^{-1/2}\Gamma(1/2,2\pi N(\mathcal{A})/x) + W(\chi_c)(N(c))^{-1/2}\\ &\times\sum_{0\neq\mathcal{A}\subset O_K}\overline{\chi_c(\mathcal{A})}(N(\mathcal{A}))^{-1/2}\Gamma(1/2,2\pi N(\mathcal{A})x/(|D_K|N(c))),\end{aligned}\tag{1}$$

where

$$\Gamma(1/2,\xi) = \frac{1}{2\pi i}\int_{(2)}\frac{\Gamma(s+1/2)}{\Gamma(1/2)}\frac{\xi^{-s}}{s}\,ds = \frac{1}{\Gamma(1/2)}\int_{\xi}^{\infty}t^{-1/2}e^{-t}\,dt$$

is the (normalized) incomplete Γ-function.

From Eisenstein's supplement theorem to the cubic reciprocity law (see [IR90], for instance), we have

$$\chi_c(1-\zeta_3) = 1.$$

Note that $1-\zeta_3 = \sqrt{-3}\,\zeta_3^2$. Since any integral non-zero ideal $\mathcal{A}$ in $\mathbb{Z}[\zeta_3]$ has unique generator $(1-\zeta_3)^r a$, where $r\in\mathbb{Z}$, $a\in\mathbb{Z}[\zeta_3]$, $r\geqslant 0$, $a\equiv 1 \pmod 3$, it follows from the cubic reciprocity law (see [IR90]) that $\chi_c(\mathcal{A}) = \chi_a(c)$.

In the following sections, we study separately, by making use of (1), the moments (with $y\to\infty$)

$$\sideset{}{^*}\sum_{c\equiv 1 \,(\mathrm{mod}\, 9)} L(1/2,\chi_c)\exp(-N(c)/y),$$

and

$$\sideset{}{^*}\sum_{c\equiv 1 \,(\mathrm{mod}\, 9)} |L(1/2,\chi_c)|^2\exp(-N(c)/y).$$

3. The main term of the first moment

We have, by the above discussion (choosing $x = y^b$, where $0<b<1$ will be specified later),

$$\begin{aligned}\Sigma_1 &= \sideset{}{^*}\sum_{c\equiv 1 \,(\mathrm{mod}\, 9)}\sum_{0\neq\mathcal{A}\subset O_K}\chi_c(\mathcal{A})(N(\mathcal{A}))^{-1/2}\Gamma(1/2,2\pi N(\mathcal{A})/x)\exp(-N(c)/y)\\ &= \sideset{}{^*}\sum_{c\equiv 1 \,(\mathrm{mod}\, 9)}\sum_{r\geqslant 0,\ a\equiv 1 \,(\mathrm{mod}\, 3)}\frac{\chi_a(c)}{3^{r/2}N(a)^{1/2}}\Gamma(1/2,2\pi 3^r N(a)/x)\exp(-N(c)/y)\\ &= \sum_{r,\,a}\Bigg(\sideset{}{^*}\sum_{c\equiv 1 \,(\mathrm{mod}\, 9)}\chi_a(c)\exp(-N(c)/y)\Bigg)\frac{\Gamma(1/2,2\pi 3^r N(a)/x)}{3^{r/2}N(a)^{1/2}}.\end{aligned}\tag{2}$$

For a, a cube, the inner sum above is

$$\begin{aligned}\sum^{*}_{c\equiv 1\,(\mathrm{mod}\,9),\,(a,c)=1} & \exp(-N(c)/y)\\ &= \frac{1}{2\pi i}\int_{(2)}\Gamma(s)y^s\biggl(\sum^{*}_{c\equiv 1\,(\mathrm{mod}\,9),\,(a,c)=1}\frac{1}{N(c)^s}\biggr)\,ds\\ &= \frac{1}{\#h_{(9)}}\sum_{\chi\,(\mathrm{mod}\,9)}\frac{1}{2\pi i}\int_{(2)}\Gamma(s)y^s\biggl(\sum_{\mathcal{A}\neq 0,\,(\mathcal{A},a)=1}\chi(\mathcal{A})|\mu(\mathcal{A})|(N(\mathcal{A}))^{-s}\biggr)\,ds,\end{aligned}$$

where χ runs over all ray class characters mod 9, and $\mu(\cdot)$ is the Möbius function. Moving the line of integration to $\Re(s)=1/2+\epsilon$, we see that the above sum equals asymptotically $C_a y + O_\epsilon(y^{1/2+\epsilon}|a|^\epsilon)$, where

$$C_a = \frac{\mathrm{res}_{s=1}\zeta_K(s)}{\#h_{(9)}\zeta_K(2)}\prod_{\mathcal{P}|(9a)}(1+N(\mathcal{P})^{-1})^{-1},$$

$\zeta_K(s)$ being the Dedekind zeta function of K.

Thus the contribution from cubes a in (2) is exactly

$$Ay + O(yx^{-1/4}+y^{1/2+\epsilon}), \tag{3}$$

where

$$A = \frac{3+\sqrt{3}}{2}\,\frac{\mathrm{res}_{s=1}\zeta_K(s)}{\#h_{(9)}\zeta_K(2)}\sum_{(\mathcal{A},3)=1}\frac{1}{N(\mathcal{A})^{3/2}}\prod_{\mathcal{P}|(9)\mathcal{A}}(1+N(\mathcal{P})^{-1})^{-1}.$$

4. The remainder terms of the first moment

For non-cube a, χ_a is non-trivial and we have the analogue of the Polya–Vinogradov inequality (see Lemma 2 of [HP79]): for any $\epsilon>0$,

$$\sum_{c\equiv 1\,(\mathrm{mod}\,3)}\chi_a(c)\exp(-N(c)/y) \ll_\epsilon N(a)^{1/2+\epsilon}. \tag{4}$$

Note that we can assume $N(c)\ll y^{1+\epsilon}$ and $N(a)\ll x^{1+\epsilon}$ in view of the exponential decay of the test functions. We infer that

$$\begin{aligned}\sum^{*}_{c\equiv 1\,(\mathrm{mod}\,9)} & \chi_a(c)\exp(-N(c)/y)\\ &= \sum_{c\equiv 1\,(\mathrm{mod}\,9)}\chi_a(c)\exp(-N(c)/y)\sum_{d^2|c,\,d\equiv 1\,(\mathrm{mod}\,3)}\mu(d)\\ &= \sum_{d\equiv 1\,(\mathrm{mod}\,3),\,N(d)\leqslant B}\mu(d)\chi_a(d^2)\sum_{c\equiv \overline{d}^2\,(\mathrm{mod}\,9)}\chi_a(c)\exp(-N(d^2c)/y)\\ &\quad+\sum_{b\equiv 1\,(\mathrm{mod}\,3)}\chi_a(b^2)\biggl(\sum_{d|b,\,N(d)>B,\,d\equiv 1\,(\mathrm{mod}\,3)}\mu(d)\biggr)\sum^{*}_{c\equiv \overline{b}^2\,(\mathrm{mod}\,9)}\chi_a(c)\exp(-N(b^2c)/y)\\ &= R+S,\end{aligned}$$

say. Here $B>y^{1/2}x^{-1/2}$ (and hence $x\geqslant y/N(b)^2$), and it will be chosen optimally. Using the ray class characters (mod 9) to detect the congruence condition $c\equiv\overline{d}^2$ (mod 9), and applying (4), we see that the contribution of R to Σ_1 is at most xBy^ϵ. To deal with S, we appeal to the large sieve type inequality for cubic characters [Hea00]: for any $\epsilon > 0$,

$$\sum_{N(m)\leqslant M}^{*} \bigg| \sum_{N(n)\leqslant N}^{*} c_n \left(\frac{n}{m}\right)_3 \bigg|^2 \ll_\epsilon (M + N + (MN)^{2/3})(MN)^\epsilon \sum_{N(n)\leqslant N}^{*} |c_n|^2. \tag{5}$$

We extract square divisors of a by writing $a = a_1 a_2^2$, where $a_1, a_2 \equiv 1 \pmod 3$ and a_1 is square-free. We deduce, by means of (5) and Cauchy's inequality, that the contribution of S to Σ_1 is at most

$$x^{1/2} y^{1/2+\epsilon} + \frac{y^{5/6+\epsilon} x^{1/3}}{B^{2/3}}.$$

Choosing $B = y^{1/2} x^{-2/5}$, we obtain

$$\Sigma_1 = Ay + O(yx^{-1/4} + y^{1/2+\epsilon} x^{3/5}). \tag{6}$$

Next we need to bound

$$\begin{aligned} \Sigma_2 = &\sum_{c\equiv 1 \,(\mathrm{mod}\, 9)}^{*} W(\chi_c)(N(c))^{-1/2} \exp(-N(c)/y) \\ &\times \sum_{0\neq \mathcal{A}\subset O_K} \overline{\chi_c(\mathcal{A})}(N(\mathcal{A}))^{-1/2} \Gamma(1/2, 2\pi N(\mathcal{A}) x/(|D_K| N(c))). \end{aligned}$$

In view of the equalities $W(\chi_c) = G_c$ and $G_c^3 = \mu(c) c^2 \overline{c}$, we may drop the above restriction '$*$'. Denote $\widetilde{G_c} = G_c N(c)^{-1/2}$. Thus, we have

$$\Sigma_2 = \sum_{r\geqslant 0,\, a\equiv 1 \,(\mathrm{mod}\, 3)} \frac{1}{3^{r/2} N(a)^{1/2}} \sum_{c\equiv 1 \,(\mathrm{mod}\, 9)} \widetilde{G_c}\overline{\chi_a(c)} \exp(-N(c)/y) \Gamma(1/2, 2\pi 3^r N(a) x/(|D_K| N(c))).$$

We need the following crucial bound (see (1) of [Pat81]):

$$\sum_{N(c)\leqslant X,\, c\equiv 1 \,(\mathrm{mod}\, 9)} \widetilde{G_c}\overline{\chi_a(c)} \ll_\epsilon X^{5/6+\epsilon}. \tag{7}$$

By partial summation and applying (7), we obtain

$$\Sigma_2 \ll_\epsilon y^{5/6+\epsilon} (y/x)^{1/2}. \tag{8}$$

From (6) and (8), we conclude that

$$\begin{aligned} \sum_{c\equiv 1 \,(\mathrm{mod}\, 9)}^{*} L(1/2, \chi_c) \exp(-N(c)/y) &= Ay + O_\epsilon(yx^{-1/4} + y^{1/2+\epsilon} x^{3/5} + y^{5/6+\epsilon}(y/x)^{1/2}) \\ &= Ay + O_\epsilon(y^{1/2+\epsilon} x^{3/5} + y^{5/6+\epsilon}(y/x)^{1/2}) \\ &= Ay + O_\epsilon(y^{21/22+\epsilon}), \end{aligned} \tag{9}$$

on taking $x = y^{25/33}$.

5. The second moment

We apply (5) to the approximate functional equation (1) (with $x = (|D_K| N(c))^{1/2}$) and obtain the bound for the square moment of $L(1/2, \chi_c)$:

$$\sum_{c\equiv 1 \,(\mathrm{mod}\, 9)}^{*} |L(1/2, \chi_c)|^2 \exp(-N(c)/y) \ll_\epsilon y^{1+\epsilon}. \tag{10}$$

From (9) and (10) the Theorem follows.

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6. Proof of the Corollary

By Cauchy's inequality, we infer from (9) and (10) that

$$\begin{aligned} y &\ll \sideset{}{^*}\sum_{c\equiv 1\,(\mathrm{mod}\,9)} L(1/2,\chi_c)\exp(-N(c)/y) \\ &\ll \Bigg(\sideset{}{^*}\sum_{c\equiv 1\,(\mathrm{mod}\,9),\ L(1/2,\chi_c)\neq 0} \exp(-N(c)/y)\Bigg)^{1/2} \\ &\quad\times \Bigg(\sideset{}{^*}\sum_{c\equiv 1\,(\mathrm{mod}\,9)} |L(1/2,\chi_c)|^2\exp(-N(c)/y)\Bigg)^{1/2} \\ &\ll_\epsilon y^{1/2+\epsilon}\#\{c\equiv 1\ (\mathrm{mod}\,9),\ N(c)\leqslant y^{1+\epsilon},\ \mu(c)\neq 0,\ L(1/2,\chi_c)\neq 0\}^{1/2} + O(1). \end{aligned} \tag{11}$$

Now the Corollary follows immediately from (11), by changing $y^{1+\epsilon}$ to y.

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