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ON COUNTING TYPES OF SYMMETRIES IN FINITE UNITARY REFLECTION GROUPS

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Let K be a field of characteristic zero. Let V be an n-dimensional vector space over K. A linear automorphism of V is said to be *of type* i if it leaves fixed a subspace of dimension i. A reflection is a linear automorphism of type $n - 1$ which has finite order. A finite reflection group is a finite group of linear automorphisms which is generated by reflections. These groups are especially interesting because the full group of symmetries of a regular polytope is always a finite reflection group. There is also a strong connection between these groups and Lie groups.

Shephard and Todd [**2**] have discovered and verified and L. Solomon [**3**] has given a general proof of the following counting principle: Let G be a finite reflection group. Let g_i denote the number of elements in G of type i; then the polynomial

$$g_n x^n + g_{n-1}x^{n-1} + \ldots + g_0$$

always factors into the form

$$(x + m_1)(x + m_2) \ldots (x + m_n),$$

where $m_1, \ldots, m_n$ are positive integers such that $m_1 + 1, \ldots, m_n + 1$ are the degrees of a minimal generating set for the homogeneous polynomial invariants of G. From now on let $d_k = m_k + 1$. The $m_1, \ldots, m_n$ are called the *exponents* of the group. See Coxeter [**1**, pp. 149–150] for an historical discussion of this principle.

In this paper we extend the above result to a counting principle on the eigenvalues of the elements of a finite reflection group. We shall prove the following theorem:

THEOREM. *Let G be a finite reflection group, let p be a positive integer, and let u be a primitive* pth *root of unity. If g_i is the number of elements in G for which the eigenvalue u occurs with multiplicity i, then the polynomial*

$$g_n x^n + g_{n-1}x^{n-1} + \ldots + g_0$$

factors into the form

$$c(x + m_{l_1})(x + m_{l_2}) \ldots (x + m_{l_r}),$$

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The referee remarks that this theorem was proved by Ian G. Macdonald in a seminar at the Institute for Advanced Study, Princeton, in 1968. Macdonald's proof, along the same lines, has not been published.

where $m_{l_1}, \ldots, m_{l_r}$ *are the exponents of* G *for which* $p|d_k$ *and* c *is the product of the remaining* d_k.

Note: If $u = 1$, then the above statement reduces to the original counting principle.

Proof. Let $w_1(g), w_2(g), \ldots, w_n(g)$ be the eigenvalues of $g \in G$. According to Solomon [**3**], we can write

$$\frac{1}{|G|}\sum_{g\in G}\frac{\sigma_{n,k}(w_1(g),\ldots,w_n(g))}{(1-w_1(g)t)\ldots(1-w_n(g)t)} = \frac{\sigma_{n,k}(t^{m_1},\ldots,t^{m_n})}{(1-t^{d_1})\ldots(1-t^{d_n})}$$

for $k = 0, \ldots, n$, where $\sigma_{n,k}$ is the kth elementary symmetric function in n variables.

A computation shows that

$$\frac{1}{|G|}\sum_{g\in G}\frac{\sigma_{n,k}(1-w_1(g)t,\ldots,1-w_n(g)t)}{(1-w_1(g)t)\ldots(1-w_n(g)t)} = \frac{\sigma_{n,k}(1-t^{d_1},\ldots,1-t^{d_n})}{(1-t^{d_1})\ldots(1-t^{d_n})}$$

By expanding and canceling within each term, we get:

$$\frac{1}{|G|}\sum_{g\in G}\sigma_{n,n-k}\left(\frac{1}{1-w_1(g)t},\ldots,\frac{1}{1-w_n(g)t}\right) = \sigma_{n,n-k}\left(\frac{1}{1-t^{d_1}},\ldots,\frac{1}{1-t^{d_n}}\right).$$

Thus the average over the group of any elementary symmetric function in the $1/(1-w_i(g)t)$ is the same elementary symmetric function in the $1/(1-t^{d_i})$.

Using these elementary symmetric functions as coefficients of a polynomial in X gives us:

$$\frac{1}{|G|}\sum_{k=0}^{n}\sum_{g\in G}\sigma_{n,k}\left(\frac{1}{1-w_1(g)t},\ldots,\frac{1}{1-w_n(g)t}\right)X^k = \sum_{k=0}^{n}\sigma_{n,k}\left(\frac{1}{1-t^{d_1}},\ldots,\frac{1}{1-t^{d_n}}\right)X^k,$$

which factors into:

$$\frac{1}{|G|}\sum_{g\in G}\left(\frac{X}{1-w_1(g)t}+1\right)\ldots\left(\frac{X}{1-w_n(g)t}+1\right) = \left(\frac{X}{1-t^{d_1}}+1\right)\ldots\left(\frac{X}{1-t^{d_n}}+1\right).$$

If, in the above expression, we let $X = (1-ut)Y$, set $t = u^{-1}$, then on the left each $X/(1-w_i(g)t)+1$ yields $Y+1$ if $u = w_i(g)$ and 1 if not. Thus for each $g \in G$, the product $(X/(1-w_1(g)t)+1)\ldots(X/(1-w_n(g)t)+1)$ yields $(Y+1)^i$, where i is the multiplicity of the eigenvalue u in g. Thus on the left we get:

$$\frac{1}{|G|}\sum_{i=0}^{n} g_i(Y+1)^i.$$

On the right, for each $k = 1, \ldots, n$, we have

$$\frac{1}{1 - t^{d_k}} = \frac{1}{d_k} \sum_{j=0}^{m_k} \frac{1}{1 - \eta_j t},$$

where $\eta_0, \ldots, \eta_{m_k}$ are the d_k-th roots of unity. Hence the factor $X(1 - t^{d_k}) + 1$ yields $(Y/d_k) + 1$ if $u \in \{\eta_0, \ldots, \eta_{m_k}\}$, and 1 otherwise.

Since u is a primitive pth root of unity, $u \in \{\eta_0, \ldots, \eta_{m_k}\}$ if and only if $p|d_k$. Thus on the right we get $((Y/d_{l_1}) + 1) \ldots ((Y/d_{l_r}) + 1)$, where $m_{l_1}, \ldots, m_{l_r}$ are the exponents of G such that $p|d_k$. Equating the two sides yields:

$$\sum_{i=0}^{n} g_i(\gamma + 1)^i = \frac{|G|}{(d_{l_1}) \ldots (d_{l_r})} (Y + d_{l_1}) \ldots (Y + d_{l_r}).$$

Now it follows from the original result that $|G| = d_1 \ldots d_n$. Setting $x = Y + 1$ gives:

$$\sum_{i=0}^{n} g_i x^i = c(x + m_{l_1}) \ldots (x + m_{l_r}),$$

where c is the product of the remaining d_k's.

References

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3. L. Solomon, *Invariants of finite reflection groups*, Nagoya Math. J. *22* (1963), 57–64.

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