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GORENSTEIN MODULES AND GORENSTEIN MODEL STRUCTURES

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Abstract. Given a complete hereditary cotorsion pair $(\mathcal{X}, \mathcal{Y})$, we introduce the concept of $(\mathcal{X}, \mathcal{X} \cap \mathcal{Y})$-Gorenstein projective modules and study its stability properties. As applications, we first get two model structures related to Gorenstein flat modules over a right coherent ring. Secondly, for any non-negative integer n, we construct a cofibrantly generated model structure on Mod(R) in which the class of fibrant objects are the modules of Gorenstein injective dimension $\leq n$ over a left Noetherian ring R. Similarly, if R is a left coherent ring in which all flat left R-modules have finite projective dimension, then there is a cofibrantly generated model structure on Mod(R) such that the cofibrant objects are the modules of Gorenstein projective dimension $\leq n$. These structures have their analogous in the category of chain complexes.

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1. Introduction. Enochs and Jenda and coauthors introduced the so-called Gorenstein injective, Gorenstein projective and Gorenstein flat modules and developed Gorenstein homological algebra. In **[23]**, Hovey showed that when R is a Gorenstein ring, the category of R-modules has two Quillen equivalent model structures which are related to Gorenstein projective modules and Gorenstein injective modules. Their homotopy categories are what we call the stable module category of the ring R. There is also a third Quillen equivalent model structure called the flat model structure, as was shown in **[17]**. In **[16]**, Gillespie showed that when R is a Ding-Chen ring, the category of R-modules has a model structure in which the cofibrant objects are the Gorenstein flat modules.

In **[25]**, Marco and Pérez constructed new complete cotorsion pairs in the categories of modules and chain complexes over a Gorenstein ring R. If n is a positive integer, it is shown that the class of modules with Gorenstein projective (or Gorenstein flat) dimension $\leq n$ forms the left half of a complete cotorsion pair. Analogous results also hold for chain complexes over R. In any Gorenstein category, the class of objects with Gorenstein injective dimension $\leq n$ is the right half of a complete cotorsion pair. As applications, these cotorsion pairs induce new model structures. In the recent work of **[6]**, it is shown that the Gorenstein injective cotorsion pair $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ is a complete cotorsion pair cogenerated by a set whenever R is Noetherian. On the other hand, it is shown that the Gorenstein projective cotorsion pair $(\mathcal{GP}, \mathcal{GP}^{\perp})$ is a complete cotorsion pair cogenerated by a set whenever R is a coherent ring in which all flat modules have finite projective dimension. So, the category of R-modules has a model structure in which the fibrant objects are the Gorenstein injective modules when R is a left Noetherian ring. It is a generalisation of the model structure introduced in [**23**] for Gorenstein rings. On the other hand, for the (left) coherent rings mentioned above, the category of R-modules has a model structure in which the cofibrant objects are the Gorenstein projective modules also generalizing one from [**23**].

Recently, Sather-Wagstaff et al. [**29**] studied modules of the form $\mathrm{im}(G_0 \to G^0)$ for some exact sequence of Gorenstein projective R-modules

$$\mathbf{G} = \cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

such that the two complexes $\mathrm{Hom}_R(\mathbf{G}, H)$ and $\mathrm{Hom}_R(H, \mathbf{G})$ are exact for each Gorenstein projective R-module H, and they proved that these modules are nothing but Gorenstein projective [**29**, Theorem A]. In [**34**], the authors studied the stability of Gorenstein flat modules. In order to develop Tate and complete cohomology theories in the pure derived category of flat modules, the authors of [**1**] introduced the definition of **F**-Gorenstein flat R-modules (see Example 3.2).

The point of this paper is to describe how the homotopy theory on modules (or complexes) over a Gorenstein ring generalises to a homotopy theory on modules (or complexes) over a more general ring.

The paper is structured as follows: In Section 2, we collect definitions and results that we need throughout the paper.

In Section 3, we first introduce the concept of $(\mathcal{X}, \mathcal{X} \cap \mathcal{Y})$-Gorenstein projective modules for a complete hereditary cotorsion pair $(\mathcal{X}, \mathcal{Y})$. We denote by $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ the class of $(\mathcal{X}, \mathcal{X} \cap \mathcal{Y})$-Gorenstein projective R-modules. If we set $\mathcal{GP}^2_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})} := \mathcal{GP}_{(\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})},\mathcal{X}\cap\mathcal{Y})}$, then it is obvious that there is a containment $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})} \subseteq \mathcal{GP}^2_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$. Moreover, we get the following

THEOREM A. *If $(\mathcal{X}, \mathcal{Y})$ is a complete hereditary cotorsion pair in* $\mathrm{Mod}(R)$*, then* $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})} = \mathcal{GP}^2_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$.

In Section 4, we first get a weak AB-context and a model structure which is related to $(\mathcal{X}, \mathcal{X} \cap \mathcal{Y})$-Gorenstein projective modules (see Theorem 4.2). Second, as applications of Theorem A, we get two model structures which are related to Gorenstein flat modules over a right coherent ring (see Theorem 4.5). Let $\mathcal{P}_n$(resp. $\mathcal{F}_n$, $\mathcal{I}_n$) be the class of left R-modules with projective (resp. flat , injective) dimension $\leq n$ and $\mathcal{GP}_n$ (resp. $\mathcal{GF}_n$, $\mathcal{GI}_n$) be the class of left R-modules with Gorenstein projective (resp. flat , injective) dimension $\leq n$ for any non-negative integer n. Then, we get the following.

THEOREM B. *Let R be a ring and n a non-negative integer. Then, the following hold:*

(1) *If R is left Noetherian, then there is a cofibrantly generated model structure on* $\mathrm{Mod}(R)$ *in which the cofibrant objects are the R-modules in ${}^{\perp_1}\mathcal{I}_n$, the fibrant objects are the R-modules in $\mathcal{GI}_n$ and the trivial objects are the R-modules in the class*

$$\{N | \exists \text{ an exact sequence } 0 \to C \to F \to N \to 0 \text{ with } F \in {}^{\perp_1}\mathcal{GI}_n,\ C \in \mathcal{I}_n\}.$$

(2) *If R is a left coherent ring in which all flat left R-modules have finite projective dimension, then there is a cofibrantly generated model structure on* $\mathrm{Mod}(R)$ *in which the cofibrant objects are the R-modules in $\mathcal{GP}_n$, the fibrant objects are the R-modules in $\mathcal{P}_n^{\perp}$ and the trivial objects are the R-modules in the class*

$$\{N | \exists \text{ an exact sequence } 0 \to C \to F \to N \to 0 \text{ with } F \in \mathcal{P}_n,\ C \in \mathcal{GP}_n^{\perp}\}.$$

(3) *If R is right coherent, there is a cofibrantly generated model structure on* Mod(R) *in which the cofibrant objects are the R-modules in $\mathcal{GF}_n$, the fibrant objects are the R-modules in $\mathcal{F}_n^{\perp}$ and the trivial objects are the R-modules in the class*

$$\{N|\exists \textit{ an exact sequence } 0 \to C \to F \to N \to 0 \textit{ with } F \in \mathcal{F}_n, C \in \mathcal{GF}_n^{\perp}\}.$$

In Section 5, we study the Gorenstein model structures on Ch(R). As applications of Theorem B, we get some similar model structures on Ch(R).

2. Preliminaries. We start this section by collecting all the background material that will be necessary in the sequel. For unexplained concepts and notations, we refer the reader to [**12**, **14**, **20**, **27**, **35**].

NOTATION 2.1. Throughout this paper, R is an associative ring. Mod(R) is the class of left R-modules and Ch(R) is the class of complexes of left R-modules. In what follows "R-modules" and "complexes" mean "left R-modules" and "complexes of left R-modules", respectively. We use the term "subcategory" to mean a "full and additive subcategory that is closed under isomorphisms". Let $\mathcal{P}$, $\mathcal{F}$ and $\mathcal{I}$ be the classes of projective, flat and injective R-modules, respectively. For an R-module M, M^+ denotes the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

2.1. Cotorsion pairs. Let $\mathcal{D}$ be an abelian category with enough projective and injective objects and $\mathcal{X}$ a subcategory of $\mathcal{D}$. Any complex $\mathbf{L}$ in $\mathcal{D}$ is $\mathrm{Hom}_{\mathcal{D}}(-, \mathcal{X})$-*exact* if the sequence $\mathrm{Hom}_{\mathcal{D}}(\mathbf{L}, X)$ is exact for each $X \in \mathcal{X}$. For an object $M \in \mathcal{D}$, we write $M \in {}^{\perp}\mathcal{X}$ (resp. $M \in {}^{\perp_1}\mathcal{X}$) if $\mathrm{Ext}_{\mathcal{D}}^{\geqslant 1}(M, X) = 0$ (resp. $\mathrm{Ext}_{\mathcal{D}}^{1}(M, X) = 0$) for each $X \in \mathcal{X}$. Dually, we can define $M \in \mathcal{X}^{\perp}$ and $M \in \mathcal{X}^{\perp_1}$. Recall that a subcategory $\mathcal{X}$ is resolving if it contains all projective objects such that for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{D}$ with $Z \in \mathcal{X}$, it follows that $X \in \mathcal{X}$ if and only if $Y \in \mathcal{X}$. Dually, we have the notion of co-resolving subcategory.

2.2.1. Following Enochs [**12**], Hovey [**24**] and Salce [**28**], a *cotorsion pair* is a pair of classes $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{D}$ such that $\mathcal{X}^{\perp_1} = \mathcal{Y}$ and ${}^{\perp_1}\mathcal{Y} = \mathcal{X}$. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be *hereditary* [**17**] if $\mathrm{Ext}_{\mathcal{D}}^{i}(X, Y) = 0$ for all $i \geqslant 1$ and all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, or equivalently, $\mathcal{X}$ is resolving($\mathcal{Y}$ is co-resolving).

2.2.2. Let M be an object in $\mathcal{D}$. A monomorphism $\phi : M \to B$ with $B \in \mathcal{X}$ is said to be a *special $\mathcal{X}$-preenvelope* of M if $\mathrm{coker}(\phi) \in {}^{\perp_1}\mathcal{X}$. Dually, we have the definition of a special $\mathcal{X}$-precover. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is called *complete* if every object M of $\mathcal{D}$ has a special $\mathcal{Y}$-preenvelope and a special $\mathcal{X}$-precover. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be *cogenerated* by a set $\mathcal{S} \subseteq \mathcal{D}$ if $\mathcal{Y} = \mathcal{S}^{\perp}$. Every cotorsion pair in Mod(R) or Ch(R) cogenerated by a set is complete. Perhaps the most useful complete hereditary cotorsion pair is the flat cotorsion pair $(\mathcal{F}, \mathcal{C})$. Here, $\mathcal{C}$ is the collection of all modules C such that $C \in \mathcal{F}^{\perp_1}$. Such modules are called cotorsion modules.

2.2. Model structures. Recall that a model category is a bicomplete category $\mathcal{D}$ equipped with three classes of morphisms called cofibrations, fibrations and weak equivalences, satisfying certain conditions. We leave the details of the definition to the reader (for example, see [**23**]). Let $\mathcal{D}$ be a bicomplete abelian category with enough projective and injective objects, equipped with a model structure. An object $X \in \mathcal{D}$

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is *cofibrant* if $0 \longrightarrow X$ is a cofibration; *fibrant* if $X \longrightarrow 0$ is a fibration; and *trivial* if $0 \longrightarrow X$ is a weak equivalence. A model category $\mathcal{D}$ is said to be *abelian* if it satisfies the following conditions: (1) f is a (trivial) cofibration if, and only if, f is a monomorphism with (trivial) cofibrant cokernel. (2) f is a (trivial) fibration if, and only if, f is an epimorphism with (trivial) fibrant kernel. Given two cotorsion pairs $(\mathcal{C}, \mathcal{F}')$ and $(\mathcal{C}', \mathcal{F})$ in an abelian category, we shall say that they are *compatible* if there exists a class of objects $\mathcal{W}$ such that $\mathcal{C}' = \mathcal{C} \cap \mathcal{W}$ and $\mathcal{F}' = \mathcal{F} \cap \mathcal{W}$. Let $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ be two complete compatible cotorsion pairs in a bicomplete abelian category $\mathcal{D}$ with enough projective and injective objects, where the class $\mathcal{W}$ is thick. Then, there exists a unique abelian model structure on $\mathcal{D}$ such that $\mathcal{C}$ is the class of cofibrant objects, $\mathcal{F}$ is the class of fibrant objects and $\mathcal{W}$ is the class of trivial objects [**23**, Theorem 2.2].

2.3. Complexes. A (cochain) complex

$$\cdots \longrightarrow X^{-1} \xrightarrow{\delta^{-1}} X^0 \xrightarrow{\delta^0} X^1 \longrightarrow \cdots$$

of R-modules will be denoted by (X, δ_X) or simply X. The nth boundary (resp., cycle, homology) module of X is defined as $\mathrm{im}\delta^{n-1}$ (resp., $\ker\delta^n$, $\ker\delta^n/\mathrm{im}\delta^{n-1}$) and it is denoted by $B^n(X)$ (resp., $Z^n(X)$, $H^n(X)$). Let M be an R-module. We denote by $\overline{M}$ the complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow M \xrightarrow{1_M} M \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

except in the degrees -1 and 0, where we put M.

2.4. Gorenstein modules and complexes. An R-module X is Gorenstein projective if there exists an exact sequence

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

of projective modules which is $\mathrm{Hom}_R(-, \mathcal{P})$-exact and such that $X = \mathrm{Ker}(P^0 \longrightarrow P^1)$. Dually, X is Gorenstein injective if there exists an exact sequence

$$\cdots \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

of injective modules which is $\mathrm{Hom}_R(\mathcal{I}, -)$-exact and such that $X = \mathrm{Ker}(I^0 \longrightarrow I^1)$. An R-module E is said to be Gorenstein flat if there exists an exact sequence of flat modules

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

such that $E = \mathrm{Ker}(F^0 \longrightarrow F^1)$ and the sequence

$$\cdots \longrightarrow I \otimes_R F_1 \longrightarrow I \otimes_R F_0 \longrightarrow I \otimes_R F^0 \longrightarrow I \otimes_R F^1 \longrightarrow \cdots$$

is exact, for every injective module I. We denote by $\mathcal{GP}$, $\mathcal{GI}$ and $\mathcal{GF}$ the classes of Gorenstein projective, Gorenstein injective and Gorenstein flat modules, respectively.

It is known that a complex $X \in \mathrm{Ch}(R)$ is Gorenstein projective (injective) [**33**] if and only if every X^i is Gorenstein projective (injective). If R is right coherent, then a complex $X \in \mathrm{Ch}(R)$ is Gorenstein flat [**33**] if and only if every X^i is Gorenstein flat.

2.5. Resolutions and dimension. Let $\mathcal{D}$ be an abelian category with enough projective and injective objects, $\mathcal{X}$ a subcategory of $\mathcal{D}$ and $M \in \mathcal{D}$. An $\mathcal{X}$-*resolution* of M is an exact sequence

$$X^+ = \cdots \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0$$

with $X_i \in \mathcal{X}$ for all $i \geq 0$. The $\mathcal{X}$-*projective dimension* of M is defined as

$$\mathcal{X}\text{-pd}_R(M) = \inf\{\sup\{n \geqslant 0 \mid X_n \neq 0\} \mid X^+ \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

An $\mathcal{X}$-*co-resolution* and the $\mathcal{X}$-*injective dimension* of M are defined dually. Define

$$res\widehat{X} = \{M \in \mathcal{D} | \mathcal{X}\text{-pd}_R(M) < \infty\}.$$

The Gorenstein projective, injective and flat dimensions of an R-module M are usually denoted by $Gpd_R(M)$, $Gid_R(M)$ and $Gfd_R(M)$, respectively. We use $pd_R(M)$, $fd_R(M)$ and $id_R(M)$ to denote, respectively, projective, flat and injective dimension of M. In [**4**], it is proved that for a ring R,

$$\sup\{Gpd_R(M) | M \text{ is a left } R\text{-module}\} = \sup\{Gid_R(M) | M \text{ is a left } R\text{-module}\}.$$

The common value of the terms of this equality is called, the left Gorenstein global dimension of R, and denoted by $l.Ggldim(R)$. Also, the left Gorenstein weak global dimension of a ring R,

$$l.wGgdim(R) = \sup\{Gfd_R(M) | M \text{ is a left } R\text{-module}\},$$

is investigated. If $X \in \text{Ch}(R)$, then the Gorenstein projective (injective) dimension of X [**32**, Proposition 4.8] is

$$Gpd_R(X) = \sup\{Gpd_R(X^i) | i \in \mathbb{Z}\} (Gid_R(X) = \sup\{Gid_R(X^i) | i \in \mathbb{Z}\}).$$

If R is right coherent, then the Gorenstein flat dimension of X [**34**, Theorems 3.11, 3.13] is

$$Gfd_R(X) = \sup\{Gfd_R(X^i) | i \in \mathbb{Z}\}.$$

3. Stability of *$(\mathcal{X}, \mathcal{X} \cap \mathcal{Y})$*-Gorenstein projective modules. Throughout this paper, $(\mathcal{X}, \mathcal{Y})$ denotes a complete hereditary cotorsion pair in $\text{Mod}(R)$. We begin with the following

DEFINITION 3.1. [**26**] Let $\mathcal{X}$ and $\mathcal{Y}$ be two classes of R-modules such that $\mathcal{P} \subseteq \mathcal{X}$. An R-module M is called *$(\mathcal{X}, \mathcal{Y})$-Gorenstein projective* if there exists an $\text{Hom}_R(-, \mathcal{Y})$-exact acyclic complex

$$\cdots \to A_1 \to A_0 \to A^0 \to A^1 \to \cdots$$

in $\text{Mod}(R)$ with each A_i, $A^i \in \mathcal{X}$ such that $M \cong \text{im}(A_0 \to A^0)$. Dually, one can define the *$(\mathcal{X}, \mathcal{Y})$-Gorenstein injective* module.

We denote by $\mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}(\mathcal{GI}_{(\mathcal{X}\cap\mathcal{Y}, \mathcal{Y})})$ the class of *$(\mathcal{X}, \mathcal{X} \cap \mathcal{Y})$*-Gorenstein projective (*$(\mathcal{X} \cap \mathcal{Y}, \mathcal{Y})$*-Gorenstein injective) R-modules.

EXAMPLE 3.2.

(1) $(\mathcal{P}, \mathcal{P} \cap \text{Mod}(R))$-Gorenstein projective modules are exactly the Gorenstein projective modules [**12**].

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(2) $(\mathcal{F}, \mathcal{F} \cap \mathcal{F}^{\perp_1})$-Gorenstein projective modules are exactly the **F**-Gorenstein flat modules [**1**].

By the above definition and [**30**, Lemma 2.9(a)],we get the following.

LEMMA 3.3. *Let M be an R-module and $(\mathcal{X}, \mathcal{Y})$ be a complete hereditary cotorsion pair in* $\mathrm{Mod}(R)$*. Then, the following conditions are equivalent.*

(1) *M is an $(\mathcal{X}, \mathcal{X} \cap \mathcal{Y})$-Gorenstein projective projective R-module;*
(2) *$M \in {}^{\perp}(\mathcal{X} \cap \mathcal{Y})$ and there exists a $\mathrm{Hom}_R(-, \mathcal{X} \cap \mathcal{Y})$-exact acyclic sequence in* $\mathrm{Mod}(R)$

$$0 \to M \to A^0 \to A^1 \to \cdots$$

with each $A^i \in \mathcal{X}$.

LEMMA 3.4. *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of R-modules and $(\mathcal{X}, \mathcal{Y})$ be a complete hereditary cotorsion pair in* $\mathrm{Mod}(R)$*. If $X \in \mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}$ and $Z \in \mathcal{X}$, then $Y \in \mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}$.*

Proof. Since $X \in \mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}$, then there exists an exact sequence $0 \to X \to A^0 \to L^0 \to 0$ of R-modules with $A^0 \in \mathcal{X}$ and $L^0 \in \mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}$. By Snake Lemma, we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A^0 \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & Z \arrow[r] & 0 \\
 & L^0 \arrow[r, equal] \arrow[d] & L^0 \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

Note that $U \in \mathcal{X}$ since $A^0 \in \mathcal{X}$ and $Z \in \mathcal{X}$. Then, the exact sequence $0 \to Y \to U \to L^0 \to 0$ of R-modules is $\mathrm{Hom}_R(-, \mathcal{X} \cap \mathcal{Y})$-exact. So $Y \in \mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}$ by Lemma 3.3. □

LEMMA 3.5. *Let $(\mathcal{X}, \mathcal{Y})$ be a complete hereditary cotorsion pair in* $\mathrm{Mod}(R)$*. If $M \in \mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}$, then there exists a $\mathrm{Hom}_R(-, \mathcal{X} \cap \mathcal{Y})$-exact acyclic complex in* $\mathrm{Mod}(R)$

$$0 \to M \to A^0 \to A^1 \to \cdots$$

with each $A^i \in \mathcal{X} \cap \mathcal{Y}$.

Proof. Note that $M \in \mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}$. Then, there exists an exact sequence $0 \to M \to F^0 \to L^1 \to 0$ of R-modules with $F^0 \in \mathcal{X}$ and $L^1 \in \mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}$. Since $(\mathcal{X}, \mathcal{Y})$ is a complete hereditary cotorsion pair, there exists an exact sequence $0 \to F^0 \to A^0 \to K^1 \to 0$ of R-modules with $A^0 \in \mathcal{X} \cap \mathcal{Y}$ and $K^1 \in \mathcal{X}$. By Snake Lemma, we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & F^0 \arrow[r] \arrow[d] & L^1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r] & A^0 \arrow[r] \arrow[d] & T^1 \arrow[r] \arrow[d] & 0 \\ & & K^1 \arrow[r, equal] \arrow[d] & K^1 \arrow[d] & \\ & & 0 & 0. & \end{tikzcd}$$

Note that $L^1 \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ and $K^1 \in \mathcal{X}$, it follows that $T^1 \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ by Lemma 3.4. So $T^1 \in {}^{\perp}(\mathcal{X} \cap \mathcal{Y})$ and the exact sequence $0 \to M \to A^0 \to T^1 \to 0$ of R-modules is $\mathrm{Hom}_R(-, \mathcal{X} \cap \mathcal{Y})$-exact. We proceed in this manner to get a $\mathrm{Hom}_R(-, \mathcal{X} \cap \mathcal{Y})$-exact acyclic complex

$$0 \to M \to A^0 \to A^1 \to \cdots$$

in $\mathrm{Mod}(R)$ with each $A^i \in \mathcal{X} \cap \mathcal{Y}$. □

Proposition 3.6. *If $(\mathcal{X}, \mathcal{Y})$ is a complete hereditary cotorsion pair in* $\mathrm{Mod}(R)$*, then $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})} = \mathcal{GP}$ if and only if $\mathcal{P} = \mathcal{X} \cap \mathcal{Y}$.*

Proof. If $\mathcal{P} = \mathcal{X} \cap \mathcal{Y}$, then, by Lemma 3.5 and Lemma 3.3, we get that $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})} = \mathcal{GP}$. Assume that $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})} = \mathcal{GP}$. We first prove the containment $\mathcal{P} \subseteq \mathcal{X} \cap \mathcal{Y}$. Let $X \in \mathcal{P} \subseteq \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$. Then, there exists by Lemma 3.5 an exact sequence $0 \to X \to B \to L \to 0$ of R-modules with $B \in \mathcal{X} \cap \mathcal{Y}$ and $L \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$. However, $L \in \mathcal{G}(\mathcal{P})$ by assumption, so the exact sequence above splits and $X \in \mathcal{X} \cap \mathcal{Y}$. For the reverse containment, let $X \in \mathcal{X} \cap \mathcal{Y}$. Note that $X \in \mathcal{GP}$ by assumption. Then, there exists an exact sequence $0 \to X \to P \to K \to 0$ of R-modules with $P \in \mathcal{P}$ and $K \in \mathcal{GP}$. Since $K \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ by assumption, the exact sequence above splits by Lemma 3.3. So $X \in \mathcal{P}$. This completes the proof. □

Let $\mathcal{X}$ be a class of R-modules. We call $\mathcal{X}$ *projectively resolving* [**22**] if $\mathcal{P} \subseteq \mathcal{X}$ and for every exact sequence $0 \to X' \to X \to X'' \to 0$ of R-modules with $X'' \in \mathcal{X}$ the conditions $X' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent.

Proposition 3.7. *If $(\mathcal{X}, \mathcal{Y})$ is a complete hereditary cotorsion pair in* $\mathrm{Mod}(R)$*, then the class $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ is projectively resolving, closed under arbitrary direct sums and direct summands.*

Proof. By definitions, we easily check that $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ is closed under arbitrary direct sums, since $\mathcal{X} = {}^{\perp_1}\mathcal{Y}$ To prove that $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ is projectively resolving, we consider any exact sequence $0 \to X \to Y \to Z \to 0$ of R-modules with $Z \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$. First assume that $X \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$. By Lemma 3.5, there exist $\mathrm{Hom}_R(-, \mathcal{X} \cap \mathcal{Y})$-exact acyclic complexes:

$$0 \to X \to A^0 \to A^1 \to \cdots \quad \text{and} \quad 0 \to Z \to C^0 \to C^1 \to \cdots$$

in Mod(R) with all A^i, $C^i \in \mathcal{X} \cap \mathcal{Y}$. Dualising the proof of [**12**, Lemma 8.2.1], we can construct a $\mathrm{Hom}_R(-, \mathcal{X} \cap \mathcal{Y})$ -exact acyclic complex

$$0 \to Y \to B^0 \to B^1 \to \cdots$$

in Mod(R) with all $B^i \in \mathcal{X} \cap \mathcal{Y}$. By assumption, $X \in {}^{\perp}(\mathcal{X} \cap \mathcal{Y})$ and $Z \in {}^{\perp}(\mathcal{X} \cap \mathcal{Y})$. Thus, $Y \in {}^{\perp}(\mathcal{X} \cap \mathcal{Y})$ by [**30**, Lemma 2.7(a)], and hence $Y \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ by Lemma 3.3.

Next, we assume that $Y \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$. Then, there exists a short exact sequence $0 \to Y \to F^0 \to L^0 \to 0$ of R-modules with $F^0 \in \mathcal{X}$ and $L^0 \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$. By Snake Lemma, we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] & F^0 \arrow[r] \arrow[d] & T^1 \arrow[r] \arrow[d] & 0 \\
 & & L^0 \arrow[r, equal] \arrow[d] & L^0 \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

Since $L^0 \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ and $Z \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$, we get that $T^1 \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ by the proof above. Thus, there exists a $\mathrm{Hom}_R(-, \mathcal{X} \cap \mathcal{Y})$-exact acyclic complex

$$0 \to T^1 \to F^1 \to F^2 \to \cdots$$

in Mod(R) with all $F^i \in \mathcal{X}$. Assembling the sequence above and $0 \to X \to F^0 \to T^1 \to 0$, we get a $\mathrm{Hom}_R(-, \mathcal{X} \cap \mathcal{Y})$-exact acyclic complex

$$0 \to X \to F^0 \to F^1 \to F^2 \to \cdots$$

in Mod(R) with all $F^i \in \mathcal{X}$, as desired.

Finally, we have to show that the class $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ is closed under direct summands. Since $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ is projectively resolving and closed under arbitrary direct sums, $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ is closed under direct summands by [**22**, Proposition 1.4]. $\square$

In what follows, we shall consider the following class of objects in $M \in$ Mod(R), for any pair $(\mathcal{X}, \mathcal{Y})$ of classes of R-modules

$$\mathcal{GP}^2_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})} := \mathcal{GP}_{(\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})},\mathcal{X}\cap\mathcal{Y})}.$$

The following is the main result of this section.

THEOREM 3.8. *If $(\mathcal{X}, \mathcal{Y})$ is a complete hereditary cotorsion pair in* Mod(R)*, then*

$$\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})} = \mathcal{GP}^2_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}.$$

Proof. The containment $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})} \subseteq \mathcal{GP}^2_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ is clear. For the reverse containment, let $M \in \mathcal{GP}^2_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$. Then, there exists a $\mathrm{Hom}_R(-, \mathcal{X} \cap \mathcal{Y})$-exact acyclic complex

$$\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

in Mod(R) with all G_i and G^i in $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ and $M \cong \mathrm{im}(G_0 \to G^0)$. By [**30**, Lemma 2.9(a)], we get that $M \in {}^{\perp}(\mathcal{F} \cap \mathcal{C})$. Put $L^i = \mathrm{im}(G^i \to G^{i+1})$ for any $i \geqslant 0$. One easily checks that $L^i \in {}^{\perp}(\mathcal{X} \cap \mathcal{Y})$ for each $i \geqslant 0$. Since $G^0 \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$, there exists a $\mathrm{Hom}_R(-, \mathcal{X} \cap \mathcal{Y})$-exact acyclic complex $0 \to G^0 \to F^0 \to K^1 \to 0$ of R-modules with $F^0 \in \mathcal{X}$ and $K^1 \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$. Consider the following pushout diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & G^0 \arrow[r] \arrow[d] & L^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & F^0 \arrow[r] \arrow[d] & T^0 \arrow[r] \arrow[d] & 0 \\
 & & K^1 \arrow[r, equal] \arrow[d] & K^1 \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

Since $K^1 \in {}^{\perp}(\mathcal{X} \cap \mathcal{Y})$ and $L^0 \in {}^{\perp}(\mathcal{X} \cap \mathcal{Y})$, we get that $T^0 \in {}^{\perp}(\mathcal{X} \cap \mathcal{Y})$. By Snake Lemma, we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & L^0 \arrow[r] \arrow[d] & G^1 \arrow[r] \arrow[d] & L^1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T^0 \arrow[r] \arrow[d] & U^1 \arrow[r] \arrow[d] & L^1 \arrow[r] & 0 \\
 & K^1 \arrow[r, equal] \arrow[d] & K^1 \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

Since $G^1, K^1 \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$, by Proposition 3.7, we get that $U^1 \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$. Then, there is an acyclic complex

$$0 \to T^0 \to U^1 \to G^2 \to G^3 \to \cdots$$

in Mod(R) which is $\mathrm{Hom}_R(-, \mathcal{X} \cap \mathcal{Y})$-exact with $U^1 \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ and $G^i \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ for $i \geqslant 2$. We proceed in this manner to get a $\mathrm{Hom}_R(-, \mathcal{X} \cap \mathcal{Y})$-exact acyclic complex

$$0 \to M \to F^0 \to F^1 \to F^2 \to \cdots$$

in Mod(R) with all $F^i \in \mathcal{X}$. So $M \in \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ by Lemma 3.3. This completes the proof. $\square$

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4. Weak AB-Contexts and Gorenstein model structures on Mod(R)**.** In this section, we get a weak AB-Context induced by $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ and some new Gorenstein model structures.

LEMMA 4.1. *Let* $(\mathcal{X}, \mathcal{Y})$ *be a complete hereditary cotorsion pair in* Mod(R)*,* M *an* R*-module and* $n \geq 0$*. Then,*

(1) $\mathcal{X}\cap\mathcal{Y} = res\widehat{\mathcal{X}\cap\mathcal{Y}} \cap \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ *and* $\mathcal{X}\cap\mathcal{Y}$ *is an injective cogenerator for* $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$.
(2) $\mathcal{X}\cap\mathcal{Y}-pd(M)\leq n$ *if and only if* $M\in\mathcal{Y}$ *and* $\mathcal{X}-pd(M)\leq n$*, and hence,* $res\widehat{\mathcal{X}\cap\mathcal{Y}} = \mathcal{Y}\cap res\widehat{\mathcal{X}}$.
(3) $res\widehat{\mathcal{X}\cap\mathcal{Y}}$ *is closed under cokernels of monomorphisms, extensions and summands and* $res\widehat{\mathcal{X}\cap\mathcal{Y}} \subseteq res\widehat{\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}}$.
(4) $\mathcal{X} = res\widehat{\mathcal{X}}\cap\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$.
(5) $\mathcal{X}\cap\mathcal{Y} = \mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}\cap\mathcal{GP}^{\perp_1}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$.

Proof.

(1) The containment $\mathcal{X}\cap\mathcal{Y}\subseteq res\widehat{\mathcal{X}\cap\mathcal{Y}}\cap\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ is straightforward. For the reverse containment, let $M\in res\widehat{\mathcal{X}\cap\mathcal{Y}}\cap\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$, then there exists an exact sequence $0\to K\to X\to M$ with $X\in\mathcal{X}\cap\mathcal{Y}$ and $\mathcal{X}\cap\mathcal{Y}-pd(K)<\infty$. We have $\mathrm{Ext}^1_R(M,K)=0$ by Lemma 3.3 and [**30**, Lemma 2.8], so this sequence splits. Hence, M is a summand of X. But $\mathcal{X}\cap\mathcal{Y}$ is closed under summands, so $M\in\mathcal{X}\cap\mathcal{Y}$.
By Lemma 3.5, we know that for any $M\in\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$, M has an injective $\mathcal{X}\cap\mathcal{Y}$-preenvelope $f : M\to X$ with coker(f) $\in\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$. Moreover, we get $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}\perp\mathcal{X}\cap\mathcal{Y}$ by Lemma 3.3 which implies that $\mathcal{X}\cap\mathcal{Y}$ is an injective cogenerator for $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$.

(2) If $\mathcal{X}\cap\mathcal{Y}-pd(M)\leq n$, then there exists an exact sequence

$$0\to X_n\to\cdots\to X_1\to X_0\to M\to 0$$

with every $X_i\in\mathcal{X}\cap\mathcal{Y}$, and hence $M\in\mathcal{Y}$ and $\mathcal{X}-pd(M)\leq n$ since $\mathcal{Y}$ is a co-resolving subcategory. Conversely, Let $M\in\mathcal{Y}$ and $\mathcal{X}-pd(M)\leq n$, we get an exact sequence

$$0\to Y_n\to\cdots\to Y_1\to Y_0\to M\to 0$$

with every $Y_i\in\mathcal{X}$ such that each $L_i=\mathrm{im}(Y_i\to Y_{i-1})\in\mathcal{Y}$ which implies that each $Y_i\in\mathcal{X}\cap\mathcal{Y}$. And hence, $\mathcal{X}\cap\mathcal{Y}-pd(M)\leq n$.

(3) By (2) and [**36**, Proposition 3.5], we know that $res\widehat{\mathcal{X}\cap\mathcal{Y}}$ is closed under cokernels of monomorphisms and extensions since $\mathcal{Y}$ is co-resolving. Now, it is sufficient to show that $res\widehat{\mathcal{X}}$ is closed under summands. Assume that $M=X\oplus Y\in res\widehat{\mathcal{X}}$, then, by [**36**, Proposition 2.3], there exists a non-negative integer n such that $\Omega^n(M)\in\mathcal{X}={}^{\perp_1}\mathcal{Y}$, it follows that $M\in{}^{\perp_{n+1}}\mathcal{Y}$, and hence $X\in{}^{\perp_{n+1}}\mathcal{Y}$. So we get that $\Omega^n(X)\in\mathcal{X}={}^{\perp_1}\mathcal{Y}$ which implies $X\in res\widehat{\mathcal{X}}$ since $(\mathcal{X},\mathcal{Y})$ is a complete hereditary cotorsion pair.

(4) First, the containment $\mathcal{X}\subseteq res\widehat{\mathcal{X}}\cap\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ is clear. For the reverse containment, let $M\in res\widehat{\mathcal{X}}\cap\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$ and $\mathcal{X}-pd(M)=n$. If $n=0$, $M\in\mathcal{X}$ is clear. By induction, it is enough to show that $M\in\mathcal{X}$ when $\mathcal{X}-pd(M)=1$. In this case, there exists an exact sequence $0\to K\to X\to$

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$M \to 0$ with $X, K \in \mathcal{X}$. Since $(\mathcal{X}, \mathcal{Y})$ is a complete hereditary cotorsion pair, there exists an exists an exact sequence $0 \to K \to G \to Y \to 0$ with $G \in \mathcal{X} \cap \mathcal{Y}$, $Y \in \mathcal{X}$. By pushout, we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & K \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & G \arrow[r] \arrow[d] & U^1 \arrow[r] \arrow[d] & M \arrow[r] & 0 \\ & Y \arrow[r, equal] \arrow[d] & Y \arrow[d] & & \\ & 0 & 0. & & \end{tikzcd}$$

Since $X, Y \in \mathcal{X}$, we get that $U^1 \in \mathcal{X}$. Note that $G \in \mathcal{X} \cap \mathcal{Y}$, $M \in \mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}$, hence the middle row in the above diagram splits. So $M \in \mathcal{X}$.

(5) By Lemma 3.3, we first get the containment $\mathcal{X} \cap \mathcal{Y} \subseteq \mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})} \cap \mathcal{GP}^{\perp_1}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}$. For the reverse containment, let $M \in \mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})} \cap \mathcal{GP}^{\perp_1}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}$. Then, by Lemma 3.5, there exists an exact sequence $0 \to M \to X \to N \to 0$ with $X \in \mathcal{X} \cap \mathcal{Y}$, $N \in \mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}$, the above exact sequence splits since $\mathrm{Ext}^1_R(N, M) = 0$. So $M \in \mathcal{X} \cap \mathcal{Y}$. □

THEOREM 4.2. *Let $(\mathcal{X}, \mathcal{Y})$ be a complete hereditary cotorsion pair in* $\mathrm{Mod}(R)$. *Then,*

(1) *The triple $(\mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}, res\widehat{\mathcal{X}\cap\mathcal{Y}}, \mathcal{X} \cap \mathcal{Y})$ is a weak AB-context.*
(2) *If $res\widehat{\mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}} = \mathrm{Mod}(R)$, that is, $(\mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}, res\widehat{\mathcal{X}\cap\mathcal{Y}}, \mathcal{X} \cap \mathcal{Y})$ is an AB-context, then there is a model structure on $\mathrm{Mod}(R)$ in which the cofibrant objects are the $(\mathcal{X}, \mathcal{Y})$-Gorenstein projective R-modules, the fibrant objects are the R-modules in $\mathcal{Y}$ and the trivial objects are the R-modules with finite $\mathcal{X}$-projective dimension.*

Proof.

(1) It follows from Lemma 4.1 and Proposition 3.7.
(2) It suffices to show that $(\mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})} \cap res\widehat{\mathcal{X}}, \mathcal{Y})$ and $(\mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}, \mathcal{Y} \cap res\widehat{\mathcal{X}})$ are complete cotorsion pairs. By Lemma 4.1(4), we know that $(\mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})} \cap res\widehat{\mathcal{X}}, \mathcal{Y}) = (\mathcal{X}, \mathcal{Y})$ is complete. By Lemma 3.3, Lemma 4.1 and [**30**, Lemma 2.8], we know that $\mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})} \perp (res\widehat{\mathcal{X}\cap\mathcal{Y}} = \mathcal{Y} \cap res\widehat{\mathcal{X}})$. Since $res\widehat{\mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}} = \mathrm{Mod}(R)$ by hypothesis, we have $\mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})} = {}^{\perp_1} res\widehat{\mathcal{X}\cap\mathcal{Y}}$ and $res\widehat{\mathcal{X}\cap\mathcal{Y}} = \mathcal{GP}^{\perp_1}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}$ by [**21**, Theorem 1.12.10 4,5]. Thus, $(\mathcal{GP}_{(\mathcal{X}, \mathcal{X}\cap\mathcal{Y})}, \mathcal{Y} \cap res\widehat{\mathcal{X}})$ is a complete cotorsion pair by [**21**, Theorem 1.12.10 3]. □

The following lemma is useful. We recall that $\mathcal{F}$ is the class of flat R-modules and $\mathcal{C} := \mathcal{F}^{\perp_1}$ is the class of cotorsion R-modules.

LEMMA 4.3. *Let M be an R-module and R be right coherent. Then, the following conditions are equivalent:*

(1) *M is an $(\mathcal{F}, \mathcal{F} \cap \mathcal{C})$-Gorenstein projective;*
(2) *M is Gorenstein flat.*
(3) *M is an $(\mathcal{GF}, \mathcal{F} \cap \mathcal{C})$-Gorenstein projective.*

Proof.
(1) $\Rightarrow$ (2) By (1), there is an exact sequence

$$\mathbf{F} : \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of flat R-modules with $M \cong \mathrm{im}(F_0 \to F^0)$ such that $\mathrm{Hom}_R(\mathbf{F}, X)$ is exact for each $X \in \mathcal{F} \cap \mathcal{C}$. Let I be an injective right R-module. Then, $(I \otimes_R \mathbf{F})^+ \cong \mathrm{Hom}_R(\mathbf{F}, I^+)$. Since I is an injective right R-module, $I^+ \in \mathcal{F} \cap \mathcal{C}$. Thus, $\mathrm{Hom}_R(\mathbf{F}, I^+)$ is exact by hypothesis, and hence $I \otimes_R \mathbf{F}$ is exact. So M is Gorenstein flat.

(2) $\Rightarrow$ (1) By (2), there is an exact sequence

$$\mathbf{F} : \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of flat R-modules with $M \cong \mathrm{im}(F_0 \to F^0)$ such that $I \otimes_R \mathbf{F}$ is exact for every injective right R-module I. Let $X \in \mathcal{F} \cap \mathcal{C}$. Then, we have a pure exact sequence $0 \to X \to X^{++} \to X^{++}/X \to 0$ of R-modules by [**35**, Proposition 2.3.5]. Note that $X \in \mathcal{F}$ implies $X^{++} \in \mathcal{F}$. Thus, $X^{++}/X \in \mathcal{F}$, and hence the above pure exact sequence splits. Note that $\mathrm{Hom}_R(\mathbf{F}, X^{++}) \cong (X^+ \otimes_R \mathbf{F})^+$. Since $X^+ \otimes_R \mathbf{F}$ is exact by hypothesis, the complex $\mathrm{Hom}_R(\mathbf{F}, X^{++})$ is exact. So $\mathrm{Hom}_R(\mathbf{F}, X)$ is exact.

(1) $\Leftrightarrow$ (3) It follows from Theorem 3.8 and the equivalence (1) $\Leftrightarrow$ (2), since $(\mathcal{F}, \mathcal{C})$ is a complete hereditary cotorsion pair. $\square$

A ring R is called an *n-FC ring* [**9**, **10**] if it is both left and right coherent and FP-id$({}_RR)$ = FP-id$(R_R) = n$. We say R is a *Ding-Chen ring* if it is an n-FC ring for some $n \geq 0$. If R is a *Ding-Chen ring*, then a right (left) R-module N has finite flat dimension if and only if it has finite FP-injective dimension. Let $\mathcal{GF}_n = \{M \in \mathrm{Mod}(R) | \mathcal{G}f\, d_R(M) \leq n\}$ and $\mathcal{F}_n = \{M \in \mathrm{Mod}(R) | f\, d_R(M) \leq n\}$ for any $n \geq 0$. By [**31**, Lemma1.5], we know that $(\mathcal{F}_n, \mathcal{F}_n^{\perp_1})$ is a hereditary cotorsion pair cogenerated by a set. The following result extends [**25**, Theorem 5.4] which was proved for Gorenstein rings.

LEMMA 4.4. *Let R be a right coherent ring. Then, for any non-negative integer n, $(\mathcal{GF}_n\, \mathcal{GF}_n^{\perp_1})$ is a hereditary cotorsion pair cogenerated by a set.*

Proof. It follows from [**13**, Theorems 2.11, 2.12] and [**8**, Theorem 2.2, Corollary 2.3, Corollary 2.7]. $\square$

The following result extends [**16**, Theorem 4.10], [**19**, Theorem 3.12] and [**18**, Theorem 3.3]

THEOREM 4.5. *Let R be a right coherent ring and n a non-negative integer. Then*

(1) *There is an injective model structure on* $\mathrm{Mod}(R)$ *in which any R-module is cofibrant, the fibrant objects are the Gorenstein cotorsion R-modules and the trivially cofibrant objects are the Gorenstein flat R-modules.*
(2) *There is a cofibrantly generated model structure on* $\mathrm{Mod}(R)$ *in which the cofibrant objects are the R-modules in $\mathcal{GF}_n$, the fibrant objects are the R-modules in $\mathcal{F}_n^{\perp}$ and*

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the trivial objects are the R-modules in the class

$$\{N | \exists \text{ an exact sequence } 0 \to C \to F \to N \to 0 \text{ with } F \in \mathcal{F}_n, C \in \mathcal{GF}_n^{\perp}\}.$$

Proof.

(1) Notes that $(\mathcal{GF}, \mathcal{GF}^{\perp})$ is a complete hereditary cotorsion pair, and moreover, by Lemma 4.1(5) and 4.3, we have $\mathcal{F} \cap \mathcal{C} = \mathcal{GF} \cap \mathcal{GF}^{\perp}$. Since R is right coherent, $\mathcal{F}$ is covariantly finite by [**12**, Proposition 6.5.1]. Thus, $\mathcal{F} \cap \mathcal{C} = \mathcal{GF} \cap \mathcal{GF}^{\perp}$ is covariantly finite by [**2**, Corollary 3.11, p. 93]. Then, the result follows from [**2**, Theorem 4.7, p. 150].

(2) By [**23**, Lemma 6.7, Corollary 6.8], Lemma 4.4 and [**15**, Theorem 1.1 and Theorem 1.2], it is sufficient to show that $\mathcal{F}_n \cap \mathcal{F}_n^{\perp} = \mathcal{GF}_n \cap \mathcal{GF}_n^{\perp}$. Let $X \in \mathcal{GF}_n^{\perp} \cap \mathcal{GF}_n$, then there exists an exact sequence $0 \to X \to T \to N \to 0$, where N is Gorenstein flat and $T \in \mathcal{F}_n$ by [**7**, Lemma 2.19]. Therefore, this exact sequence splits and so $X \in \mathcal{F}_n \cap \mathcal{F}_n^{\perp}$. Then, we get $\mathcal{F}_n^{\perp} \cap \mathcal{F}_n \supseteq \mathcal{GF}_n \cap \mathcal{GF}_n^{\perp}$. Conversely, let $X \in \mathcal{F}_n \cap \mathcal{F}_n^{\perp}$, $Y \in \mathcal{GF}_n$, then we get an exact sequence $0 \to Y \to T \to N \to 0$, where N is Gorenstein flat and $T \in \mathcal{F}_n$ by [**7**, Lemma 2.19]. On the other hand, we have an exact sequence $\mathrm{Ext}_R^1(T, X) \to \mathrm{Ext}_R^1(Y, X) \to \mathrm{Ext}_R^2(N, X)$, where $\mathrm{Ext}_R^1(T, X) = 0$ since $T \in \mathcal{F}_n$. Since $X \in \mathcal{F}_n \cap \mathcal{F}_n^{\perp}$, we have that X is a cotorsion module and thus there exists an exact sequence

$$0 \to F_n \to F_{n-1} \to F_{n-2} \to \cdots \to F_1 \to F_0 \to X \to 0$$

with every F_i flat cotorsion. On the other hand, By Lemma 4.1 (5) and Lemma 4.3, we know that $\mathcal{F} \cap \mathcal{C} = \mathcal{GF} \cap \mathcal{GF}^{\perp}$. Therefore, $X \in \mathcal{GF}^{\perp}$ and $\mathrm{Ext}_R^2(N, X) = 0$, this implies $\mathrm{Ext}_R^1(Y, X) = 0$. Therefore, $X \in \mathcal{GF}_n \cap \mathcal{GF}_n^{\perp}$ and $\mathcal{F}_n^{\perp} \cap \mathcal{F}_n \subseteq \mathcal{GF}_n \cap \mathcal{GF}_n^{\perp}$. This completes the proof. $\square$

By Theorem 4.2 and Lemma 4.3, we get the following

COROLLARY 4.6. *Let R be a ring. Then, the following hold:*

(1) *If R has finite left Gorenstein global dimension, then there is a model structure on* Mod(R) *in which the cofibrant objects are Gorenstein projective R-modules, the fibrant objects are the R-modules and the trivial objects are the R-modules with finite projective dimension.*

(2) *If R has finite left Gorenstein global dimension, then there is a model structure on* Mod(R) *in which the cofibrant objects are Gorenstein projective R-modules, the fibrant objects are the R-modules with finite projective dimension and the trivial objects are the R-modules.*

(3) *If R is right coherent with finite left Gorenstein weak global dimension, then there is a model structure on* Mod(R) *in which the cofibrant objects are Gorenstein flat R-modules, the fibrant objects are the cotorsion R-modules and the trivial objects are the R-modules with finite flat dimension.*

(4) *If R is right coherent with finite left Gorenstein weak global dimension, then there is a model structure on* Mod(R) *in which the cofibrant objects are Gorenstein flat R-modules, the fibrant objects are the Gorenstein cotorsion R-modules and the trivial objects are the R-modules.*

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Proof.

(1) It follows from Theorem 4.2 (2), by using the cotorsion pair $(\mathcal{P}, \mathrm{Mod}(R))$.
(2) We first set $(\mathcal{X}, \mathcal{Y})= (\mathcal{P}, \mathrm{Mod}(R))$. Let R be of finite left Gorenstein global dimension. Then, by the proof of Theorem 4.2, we get that $(\mathcal{GP}, \mathcal{GP}^{\perp_1})$ is a complete hereditary cotorsion pair with $\mathcal{GP} \cap \mathcal{GP}^{\perp_1} = \mathcal{P}$ and $\mathcal{GP}^{\perp_1}$ is the class of R-modules with finite projective dimension. Now consider $(\mathcal{X}, \mathcal{Y})= (\mathcal{GP}, \mathcal{GP}^{\perp_1})$. Then, the item (2) follows from Proposition 3.6 and Theorem 4.2.
(3) It follows from Theorem 4.2 and Lemma 4.3.
(4) Set $(\mathcal{X}, \mathcal{Y})= (\mathcal{GF}, \mathcal{GF}^{\perp})$, then it follows from Theorem 4.2 and Lemma 4.3. □

A left R-module M is said to be of type FP_∞ [**6**] if M has a projective resolution by finitely generated projectives. A left R-module N is called *level* [**6**] if $\mathrm{Tor}^1_R(M, N) = 0$ for all right R-modules M of type FP_∞. Let $\mathcal{GP}_n = \{M \in \mathrm{Mod}(R) | Gpd_R(M) \leq n\}$ and $\mathcal{P}_n = \{M \in \mathrm{Mod}(R) | pd_R(M) \leq n\}$ for any $n \geq 0$. If every level R-module has finite projective dimension, then, by [**6**, Theorem 8.5, Lemma 8.6, Proposition 8.10], $(\mathcal{GP}, \mathcal{GP}^{\perp})$ is a hereditary cotorsion pair cogenerated by a set and hence, by [**8**, Theorem 2.2, Corollary 2.3, Corollary 2.7] we have that $(\mathcal{GP}_n, \mathcal{GP}_n^{\perp})$ is a hereditary cotorsion pair cogenerated by a set. The following result extends [**25**, Theorem 4.1] and [**23**, Theorem 8.6]

THEOREM 4.7. *Let R be a ring which every level left R-module has finite projective dimension and n a non-negative integer. Then, there is a cofibrantly generated model structure on* $\mathrm{Mod}(R)$ *in which the cofibrant objects are the R-modules in $\mathcal{GP}_n$, the fibrant objects are the R-modules in $\mathcal{P}_n^{\perp}$ and the trivial objects are the R-modules in the class* $\{N | \exists$ *an exact sequence* $0 \to C \to F \to N \to 0$ *with* $F \in \mathcal{P}_n, C \in \mathcal{GP}_n^{\perp}\}$.

Proof. By [**23**, Lemma 6.7, Corollary 6.8] and [**15**, Theorem 1.1], it is sufficient to show that $\mathcal{P}_n^{\perp} \cap \mathcal{P}_n = \mathcal{GP}_n \cap \mathcal{GP}_n^{\perp}$. Let $X \in \mathcal{GP}_n^{\perp} \cap \mathcal{GP}_n$, then there exists an exact sequence $0 \to X \to T \to N \to 0$, where N is Gorenstein projective and $T \in \mathcal{P}_n$ by [**7**, Lemma 2.17]. Then, the above exact sequence splits, and so $X \in \mathcal{P}_n \cap \mathcal{P}_n^{\perp}$. Thus, $\mathcal{P}_n^{\perp} \cap \mathcal{P}_n \supseteq \mathcal{GP}_n \cap \mathcal{GP}_n^{\perp}$.

Conversely, let $X \in \mathcal{P}_n \cap \mathcal{P}_n^{\perp}$, $Y \in \mathcal{GP}_n$, then we get an exact sequence $0 \to Y \to T \to N \to 0$, where N is Gorenstein projective and $T \in \mathcal{P}_n$ by [**7**, Lemma 2.17]. On the other hand, we have an exact sequence $\mathrm{Ext}^1_R(T, X) \to \mathrm{Ext}^1_R(Y, X) \to \mathrm{Ext}^2_R(N, X)$, where $\mathrm{Ext}^1_R(T, X) = 0$ since $T \in \mathcal{P}_n$, and $\mathrm{Ext}^2_R(N, X) = 0$ by [**22**, Theorem 2.20]. So $X \in \mathcal{GP}_n^{\perp}$ and $\mathcal{P}_n^{\perp} \cap \mathcal{P}_n = \mathcal{GP}_n \cap \mathcal{GP}_n^{\perp}$. This completes the proof. □

Let $(\mathcal{X}, \mathcal{Y})$ be a complete hereditary cotorsion pair in $\mathrm{Mod}(R)$ and $\mathcal{B}_n = \{M | \mathrm{Ext}^{n+1}_R(X, M) = 0$ for any $X \in \mathcal{X}\}$. Then, it is easy to see that $X \in \mathcal{B}_n$ if and only if there exists an exact sequence

$$0 \to X \to X_0 \to X_1 \to \cdots \to X_{n-1} \to X_n \to 0$$

with each $X_i \in \mathcal{X}$.

LEMMA 4.8. *Let $(\mathcal{X}, \mathcal{Y})$ be a hereditary cotorsion pair in* $\mathrm{Mod}(R)$ *cogenerated by a set $\mathcal{S}$, $\mathcal{B}_n = \{M | \mathrm{Ext}^{n+1}_R(X, M) = 0$ for any $X \in \mathcal{X}\}$. Then, $({}^{\perp_1}\mathcal{B}_n, \mathcal{B}_n)$ is a hereditary cotorsion pair in* $\mathrm{Mod}(R)$ *cogenerated by a set.*

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Proof. Note that $\mathcal{B}_n = \Omega^n(\mathcal{X})^{\perp_1}$, it follows that $({}^{\perp_1}\mathcal{B}_n, \mathcal{B}_n)$ is a cotorsion pair. Since $(\mathcal{X}, \mathcal{Y})$ is a complete hereditary cotorsion pair, it is easy to see that $\mathcal{B}_n$ is coresolving, so $({}^{\perp_1}\mathcal{B}_n, \mathcal{B}_n)$ is hereditary. Obviously, $\mathcal{B}_n \subseteq \Omega^n(\mathcal{S})^{\perp_1}$ since $\mathcal{S} \subseteq \mathcal{X}$. For any $X \in \Omega^n(\mathcal{S})^{\perp_1}$, we have $\mathrm{Ext}^1_R(Y, X) = 0$ for every $Y \in \Omega^n(\mathcal{S})$, and so $\mathrm{Ext}^{n+1}_R(Z, X) = 0$ for any $Z \in \mathcal{S}$. Thus, $\Omega^{-n}(X) \in \mathcal{S}^{\perp_1} = \mathcal{Y}$ and $X \in \mathcal{B}_n$. Therefore, $\mathcal{B}_n = \Omega^n(\mathcal{S})^{\perp_1}$. For any $X \in \mathcal{S}$, choose an nth syzygy A_X, then $\{A_X | X \in \mathcal{S}\}^{\perp_1} = \mathcal{B}_n$. □

If R is a left Noetherian ring and n is a non-negative integer, then, by [**6**, Lemma 5.6, Proposition 5.10], $({}^{\perp_1}\mathcal{GI}, \mathcal{GI})$ is a hereditary cotorsion pair cogenerated by a set. Let $\mathcal{GI}_n = \{M \in \mathrm{Mod}(R) | Gid_R(M) \leq n\}$ and $\mathcal{I}_n = \{M \in \mathrm{Mod}(R) | id_R(M) \leq n\}$ for any $n \geq 0$. Then, by Lemma 4.8, we have that $({}^{\perp_1}\mathcal{GI}_n, \mathcal{GI}_n)$ is a hereditary cotorsion pair cogenerated by a set. The following result extends [**25**, Theorem 4.4] and [**23**, Theorem 8.6]

THEOREM 4.9. *Let R be a left Noetherian ring and n a non-negative integer. Then, there is a cofibrantly generated model structure on* $\mathrm{Mod}(R)$ *in which the cofibrant objects are the R-modules in ${}^{\perp_1}\mathcal{I}_n$, the fibrant objects are the R-modules in $\mathcal{GI}_n$ and the trivial objects are the R-modules in the class $\{N | \exists$ an exact sequence $0 \to C \to F \to N \to 0$ with $F \in {}^{\perp_1}\mathcal{GI}_n, C \in \mathcal{I}_n\}$.*

Proof. By [**23**, Lemma 6.7, Corollary 6.8] and [**15**, Theorem 1.1], it is sufficient to show that ${}^{\perp_1}\mathcal{I}_n \cap \mathcal{I}_n = {}^{\perp_1}\mathcal{GI}_n \cap \mathcal{GI}_n$. Let $X \in {}^{\perp_1}\mathcal{GI}_n \cap \mathcal{GI}_n$, then there exists an exact sequence $0 \to N \to T \to X \to 0$, where N is Gorenstein injective and $T \in \mathcal{I}_n$ by [**7**, Lemma 2.18]. Then,the above exact sequence splits, and so $X \in {}^{\perp_1}\mathcal{I}_n \cap \mathcal{I}_n$. Thus, ${}^{\perp_1}\mathcal{GI}_n \cap \mathcal{GI}_n \subseteq {}^{\perp_1}\mathcal{I}_n \cap \mathcal{I}_n$.

Conversely, let $X \in {}^{\perp_1}\mathcal{I}_n \cap \mathcal{I}_n$, $Y \in \mathcal{GI}_n$, then we get an exact sequence $0 \to N \to T \to Y \to 0$, where N is Gorenstein injective and $T \in \mathcal{I}_n$ by [**7**, Lemma 2.18]. On the other hand, we have an exact sequence $\mathrm{Ext}^1_R(X, T) \to \mathrm{Ext}^1_R(X, Y) \to \mathrm{Ext}^2_R(X, N)$, where $\mathrm{Ext}^1_R(X, T) = 0$ since $T \in \mathcal{I}_n$, and $\mathrm{Ext}^2_R(X, N) = 0$ by [**22**, Theorem 2.22]. So $X \in {}^{\perp_1}\mathcal{GI}_n$ and ${}^{\perp_1}\mathcal{I}_n \cap \mathcal{I}_n = {}^{\perp_1}\mathcal{GI}_n \cap \mathcal{GI}_n$. This completes the proof. □

5. Gorenstein model structures on Ch(R).

In this section, we use the methods of Hovey in [**23**, Theorem 2.2] to provide some model structures on the category of complexes. Let R be a ring with unit and the $\mathbb{Z}$-graded ring $A = R[x]/(x^2)$, which has a direct sum decomposition $A = \cdots \oplus 0 \oplus (x) \oplus R \oplus 0 \oplus \cdots$, where the scalars $r \in R$ are the elements of degree 0, and the elements in the ideal (x) form the terms of degree -1. It is not hard to see that the categories $\mathrm{Mod}(A)$ and $\mathrm{Ch}(R)$ are isomorphic. The isomorphism of categories between $\mathrm{Mod}(A)$ and $\mathrm{Ch}(R)$ automatically preserves injectives and projectives and takes flat A-modules to flat complexes [**19**, **25**]. Therefore, the isomorphism preserves Gorenstein injectives (resp. projectives, flats). These facts shall allow us to prove the following.

LEMMA 5.1. *Let R be a ring and $A = R[x]/(x^2)$. Then, the following statements hold:*

(1) *R has finite left Gorenstein global dimension if and only if A is so;*
(2) *R is right coherent if and only if A is so;*
(3) *Let R be right coherent. Then, R has finite left Gorenstein weak global dimension if and only if A is so;*
(4) *R is a Ding-Chen ring if and only if A is so;*
(5) *R is a left Noetherian if and only if A is so;*

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(6) *R is a ring in which every level left R-module has finite projective dimension if and only if A is so.*

Proof.

(1) Note that the isomorphism of categories between $\mathrm{Mod}(A)$ and $\mathrm{Ch}(R)$ automatically preserves injectives and projectives. Then, the item (1) follows from [**32**, Proposition 4.8] and [**11**, Theorem 2.28].
(2) It follows from the fact that the isomorphism of categories between $\mathrm{Mod}(A)$ and $\mathrm{Ch}(R)$ preserves products and takes flat A-modules to flat complexes.
(3) It follows from (2) and [**34**, Theorems 3.11, 3.13].
(4) It follows from (3) and [**3**, Theorems 2.8, 2.16].
(5) It follows from the fact that the isomorphism of categories between $\mathrm{Mod}(A)$ and $\mathrm{Ch}(R)$ preserves co-products and takes injective A-modules to injective complexes.
(6) Let R be a ring in which every level left R-module has finite projective dimension. Then, there exists a positive integer n such that every level left R-module has finite projective dimension $\leq n$ since the class of level left R-modules is closed under direct sums. Let X be a level A-module, then as an R-complex, it is exact and every $Z^i(X)$ is a level R-module by [**5**, Proposition 4.6]. Let

$$0 \to K \to P_{n-1} \to \cdots \to P_0 \to X \to 0$$

be an exact sequence of complexes with each P_i projective, then K and

$$0 \to Z^i(K) \to Z^i(P_{n-1}) \to \cdots \to Z^i(P_0) \to Z^i(X) \to 0$$

are exact for each $i \in Z$, and hence $Z^i(K)$ is projective, this implies that K is projective. Conversely, let K be a level left R-module, then $X = \overline{K}$ is a level left A-module and its projective dimension is finite. And hence K has finite projective dimension. □

By Lemma 5.1 and Corollary 4.6, we get the following.

COROLLARY 5.2. *Let R be a ring. Then, the following statements hold.*

(1) *If R has finite left Gorenstein global dimension, then there is a model structure on $Ch(R)$ in which the cofibrations are the monomorphisms with Gorenstein projective cokernel, the fibrations are the epimorphisms and the trivial objects are the exact complexes with cycles of finite projective dimension.*
(2) *If R has finite left Gorenstein global dimension, then there is a model structure on $Ch(R)$ in which the cofibrations are the monomorphisms with Gorenstein projective cokernel, the fibrations are the epimorphisms with kernel in $\{X \in Ch(R) | X$ is exact with cycles of finite projective dimension$\}$ and the trivial objects are the complexes in $Ch(R)$.*
(3) *If R is right coherent with finite left Gorenstein weak global dimension, then there is a model structure on $Ch(R)$ in which the cofibrations are the monomorphisms with Gorenstein flat cokernels, the fibrations are the epimorphisms with cotorsion kernels and the trivial objects are the exact complexes with cycles of finite flat dimension.*

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(4) *If R is right coherent with finite left Gorenstein weak global dimension, then there is a model structure on $Ch(R)$ in which the cofibrations are the monomorphisms with Gorenstein flat cokernels, the fibrations are the epimorphisms with Gorenstein cotorsion kernels and the trivial objects are the complexes in $Ch(R)$.*

Let $\widehat{\mathcal{P}_n}$(resp. $\widehat{\mathcal{F}_n}$, $\widehat{\mathcal{I}_n}$) be the class of complexes in $\mathrm{Ch}(R)$ with projective (resp. flat , injective) dimension $\leq n$ and $\widehat{\mathcal{GP}_n}$(resp. $\widehat{\mathcal{GF}_n}$, $\widehat{\mathcal{GI}_n}$) be the class of complexes in $\mathrm{Ch}(R)$ with Gorenstein projective (resp. flat , injective) dimension $\leq n$ for any non-negative integer n. By Theorems 4.5, 4.7, 4.9 and Lemma 5.1 and the isomorphic relations between $\mathrm{Mod}(A)$ and $\mathrm{Ch}(R)$, we get the following.

Theorem 5.3. *Let R be a right coherent ring. Then, there is an injective model structure on* $\mathrm{Ch}(R)$ *in which any complex is cofibrant, the fibrant objects are the complexes in $\widehat{\mathcal{GF}_0}^{\perp_1}$ and the trivially cofibrant objects are the Gorenstein flat complexes.*

Theorem 5.4. *Let R be a ring and n a non-negative integer. Then, the following statements hold.*

(1) *If R is left Noetherian, then there is a cofibrantly generated model structure on* $\mathrm{Ch}(R)$ *in which the cofibrant objects are the complexes in ${}^{\perp_1}\widehat{\mathcal{I}_n}$, the fibrant objects are the complexes in $\widehat{\mathcal{GI}_n}$ and the trivial objects are the complexes in the class*

$$\{N \in \mathrm{Ch}(R) | \exists \text{ an exact sequence } 0 \to C \to F \to N \to 0 \text{ with } F \in^{\perp_1} \widehat{\mathcal{GI}_n},\ C \in \widehat{\mathcal{I}_n}\}.$$

(2) *If R is a left coherent ring in which all flat left R-modules have finite projective dimension, then there is a cofibrantly generated model structure on* $\mathrm{Ch}(R)$ *in which the cofibrant objects are the complexes in $\widehat{\mathcal{GP}_n}$, the fibrant objects are the complexes in $\widehat{\mathcal{P}_n}^{\perp}$ and the trivial objects are the complexes in the class*

$$\{N \in \mathrm{Ch}(R) | \exists \text{ an exact sequence } 0 \to C \to F \to N \to 0 \text{ with } F \in \widehat{\mathcal{P}_n},\ C \in \widehat{\mathcal{GP}_n}^{\perp}\}.$$

(3) *If R is a right coherent, there is a cofibrantly generated model structure on* $\mathrm{Ch}(R)$ *in which the cofibrant objects are the complexes in $\widehat{\mathcal{GF}_n}$, the fibrant objects are the complexes in $\widehat{\mathcal{F}_n}^{\perp}$ and the trivial objects are the complexes in the class*

$$\{N \in \mathrm{Ch}(R) | \exists \text{ an exact sequence } 0 \to C \to F \to N \to 0 \text{ with } \\ F \in \widehat{\mathcal{F}_n},\ C \in \widehat{\mathcal{GF}_n}^{\perp}\}.$$

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