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AN ASYMPTOTIC APPROACH TO CENTRALLY PLANNED PORTFOLIO SELECTION

ZONGXIA LIANG,* *Tsinghua University*
YANG LIU ,** *The Chinese University of Hong Kong, Shenzhen*

Abstract

We formulate a centrally planned portfolio selection problem with the investor and the manager having S-shaped utilities under a recently popular first-loss contract. We solve for the closed-form optimal portfolio, which shows that a first-loss contract can sometimes behave like an option contract. We propose an asymptotic approach to investigate the portfolio. This approach can be adopted to illustrate economic insights, including the fact that the portfolio under a convex contract becomes more conservative when the market state is better. Furthermore, we discover a means of Pareto improvement by simultaneously considering the investor's utility and increasing the manager's incentive rate. This is achieved by establishing the collection of Pareto points of a single contract, proving that it is a strictly decreasing and strictly concave frontier, and comparing the Pareto frontiers of different contracts. These results may be helpful for the illustration of risk choices and the design of Pareto-optimal contracts.

Keywords: Utility theory; first-loss contract; asymptotic analysis; Pareto improvement

2020 Mathematics Subject Classification: Primary 91B16
Secondary 91G10

1. Introduction

In the continuous-time model of asset management, an investor authorizes a portfolio manager to allocate the total wealth process $X = \{X_t : 0 \le t \le T\}$. There is a contract $\Theta(X_T)$ sharing profit and loss between the two parties based on the performance of the wealth at the terminal evaluation time $T > 0$. Such a model involves different economic parties (the investor, the manager, etc.) and various relationships (conflict, win–win, etc.) between them.

As indicated in [32], many effects may prevent the manager from simply maximizing his/her own utility, including reputation consideration, capital raising, etc. As a result, the utility of the other party (the investor) is a major concern for the manager. From the perspective of welfare, it is ideal if both parties benefit and the total welfare is fully utilized. In this regard, Pareto-optimality, which means that the utility of one party cannot be improved without reducing the utilities of the other parties, is widely applied to study the complex relationship between the principal and the agent; see, e.g., [13]. Here we are interested in the following questions: (a)

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* Postal address: Department of Mathematical Sciences, Tsinghua University, 30 Shuangqing Road, Haidian Dist, Beijing 100084, China. Email address: liangzongxia@mail.tsinghua.edu.cn
** Postal address: Division of Mathematics, School of Science and Engineering, The Chinese University of Hong Kong, Shenzhen, 2001 Longxiang Blvd, Longgang Dist, Shenzhen, Guangdong 518172, China. Email address: yangliu16@cuhk.edu.cn (corresponding author)

What is the Pareto-optimal portfolio? (b) What is the difference between the optimal portfolio oriented to the investor and the one oriented to the manager? (c) Is there any contract that yields a Pareto improvement (strictly improves the utilities of both parties) over other contracts?

In this paper, we study Pareto portfolio selection and compare different contracts from the perspective of a central planner. The central planner can be interpreted in various ways: as Pareto efficiency in welfare economics, a regulatory institution in finance, etc. A centrally planned perspective is sometimes a starting step for the principal–agent problem. The paper [21] uses the centrally planned perspective to solve for the optimal portfolio delegation when the parties have CRRA utilities. The paper [12] equivalently studies utility optimization with bargaining power in the principal–agent problem; see also [31]. The centrally planned problem can also be applied to solve a constrained utility maximization problem in the hedge fund context. In the fundraising stage, the investor pays great attention to how his/her utility is realized. To increase attraction, the manager typically incorporates a participation constraint into the investment decision model, requiring the investor to achieve at least a lower-bound reservation utility; see [32] and [22].

Generally, we present the classic formulation of multi-objective programming (cf. [27], [34], and [11]) and the definition of Pareto-optimality.

Definition 1. *(Multi-objective programming and Pareto-optimality.)* A multi-objective problem is given by

$$\sup \mathbf{f}(X) \equiv (f_1(X), \ldots, f_m(X)), \qquad \text{such that } X \in \mathcal{D},$$

where $\mathcal{D}$ is the decision domain and $\mathbf{f} = (f_1, \ldots, f_m) : \mathcal{D} \to \mathbb{R}^m$ is a multidimensional function with $m \geq 2$. A feasible solution $X^* \in \mathcal{D}$ to the above problem is called (Pareto-) optimal if and only if there exists no other solution $X \in \mathcal{D}$ such that $f_i(X^*) \leq f_i(X)$ for all $i = 1, \ldots, m$ and $f_j(X^*) < f_j(X)$ for at least one index $j \in \{1, \ldots, m\}$.

As explained, such a problem aims to deal with multiple objectives in potential conflict, and a solution is Pareto-optimal if there is no way to improve one objective without worsening at least one of the other objectives. This definition implies that there may be many solutions satisfying Pareto-optimality.

In our context, the centrally planned portfolio selection problem is formulated in the sense of two-objective programming:

$$\sup_{\pi \in \mathcal{V}[0,T]} \left(\mathbb{E}\left[\widehat{U}_1\left(\Theta\left(X_T^\pi\right)\right)\right], \mathbb{E}\left[\widehat{U}_0\left(X_T^\pi - \Theta\left(X_T^\pi\right) - x_0\right)\right]\right), \tag{1}$$

where $\mathcal{V}[0, T]$ is the set consisting of all portfolio allocation processes $\pi = \{\pi_t : 0 \leq t \leq T\}$, X_T^π is the corresponding terminal wealth, and $X_0 = x_0 \in \mathbb{R}$ is the initial wealth. Both objectives are expected utility maximization, where $\widehat{U}_1(\cdot)$ and $\widehat{U}_0(\cdot)$ are utility functions of the profit and loss (P&L) of the manager and the investor, respectively. Here $\Theta(\cdot)$ is the function of the terminal fund wealth X_T^π representing the manager's P&L under a certain contract. The residue wealth $X_T^\pi - \Theta\left(X_T^\pi\right) - x_0$ is the investor's P&L.

As in Definition 1, a portfolio $\pi^* \in \mathcal{V}[0, T]$ with terminal wealth $X_T^{\pi^*}$ is called Pareto-optimal to Problem (1) if there exists no portfolio $\pi \in \mathcal{V}[0, T]$ with terminal wealth X_T^π such that $\mathbb{E}\left[\widehat{U}_1\left(\Theta\left(X_T^{\pi^*}\right)\right)\right] \leq \mathbb{E}\left[\widehat{U}_1\left(\Theta\left(X_T^\pi\right)\right)\right]$ and $\mathbb{E}\left[\widehat{U}_0\left(X_T^{\pi^*} - \Theta\left(X_T^{\pi^*}\right) - x_0\right)\right] \leq \mathbb{E}\left[\widehat{U}_0\left(X_T^\pi - \Theta\left(X_T^\pi\right) - x_0\right)\right]$, where at least one of the inequalities holds strictly. That is, a Pareto-optimal portfolio is one for which there exists no strictly better portfolio with respect to the two different objectives. Generally, there are many Pareto-optimal portfolios for Problem (1). The

Pareto point is a two-dimensional vector of the two parties' expected utilities under one optimal portfolio. The collection of all Pareto points forms a set in the two-dimensional plane:

$$\mathrm{PF} \triangleq \left\{\left(\mathbb{E}\left[\widehat{U}_1\left(\Theta\left(X_T^{\pi^*}\right)\right)\right], \mathbb{E}\left[\widehat{U}_0\left(X_T^{\pi^*} - \Theta\left(X_T^{\pi^*}\right) - x_0\right)\right]\right)\right\}_{\pi^* \in \mathcal{O}[0,T]} \subset \mathbb{R}^2, \tag{2}$$

where $\mathcal{O}[0, T]$ is the set consisting of all optimal portfolios π^* to Problem (1). In particular, the model solved here is based on S-shaped utilities $\widehat{U}_0$ and $\widehat{U}_1$ (cf. [33]) and a recently popular first-loss contract Θ (cf. [16]). The details of the model setting will be discussed in Section 2.

Our paper contributes in two ways. First, we provide a new asymptotic approach to finding the optimal portfolio. For the centrally planned problem (1), we solve out the closed-form Pareto-optimal portfolio π^* in Theorem 1. We divide π^* into two cases (the first-loss case and the option case), which shows that a first-loss contract can sometimes behave like an option contract. Further, we decompose π^* into three terms ($\pi^* = \pi^{(1)} + \pi^{(2)} + \pi^{(3)}$ with Merton term $\pi^{(1)}$, *aggressive term* $\pi^{(2)}$, and *conservative term* $\pi^{(3)}$), and then propose an asymptotic analysis with respect to the market state. The asymptotic approach includes a monotonic analysis, a dynamic analysis, and a terminal-time analysis. It can be adopted to illustrate various economic insights, including the fact that the portfolio under a convex contract becomes more conservative under a better market state (cf. [9]) and that the wealth process is always above the liquidation boundary (cf. [5]).

The classification result for the optimal wealth can be also seen as a sequel to [22]. The reference points in the composite S-shaped utilities of the manager and the investor are the same (both x_0 in (7)) in our model, while the reference points are set to be different in [22]. In addition, the closed-form optimal portfolio is lacking in [22]. By contrast, the present paper focuses on the implementation of the asymptotic approach. The relevant discussion about the optimal portfolio in Theorem 2 is the main contribution.

Second, we find a novel contract providing a Pareto improvement. We first investigate the pattern of the collection (2). In Theorem 3 it is proved to be a strictly decreasing and strictly concave frontier, and hence it is referred to as a *Pareto frontier*. Next, we use Pareto frontiers to compare different contracts and find that among first-loss contracts with long evaluation time, the investor benefits from the contract with smaller incentive rate and smaller managerial ownership proportion. When the evaluation time is short, we discover a means of Pareto improvement by simultaneously adding the investor's utility into the manager's investment objective and increasing the manager's incentive rate. The improvement is shown to hold in the sense of both expected utility and certainty equivalent.

Finally, we compare our setting and results with those in some related literature. Problem (1) has some connection to the classical risk-sharing problem in insurance. A seminal work, [7], proposed and studied this problem in the reinsurance market. Recent advances on the topic include [28], [1], [6], [8], and [2]. The goal of the risk-sharing problem is to establish the Pareto-optimal solution (i.e., equilibrium solution) for all agents, given a fixed total risk. The criteria of the agents include various kinds of risk measures. In contrast, the total wealth X_T^{π} in Problem (1) is not fixed, and can be optimized according to the objective. As a result, our problem does not aim to share the risk, but to maximize the expected utility of all the agents.

In addition, the paper [15] investigates the weighted utility optimization problem for the participating endowment contract and suggests the existence of a Pareto improvement through parametric sensitivity analysis. In this paper, we not only claim the existence, but also specify

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the means of obtaining a Pareto improvement: by simultaneously considering the investor's utility and increasing the manager's incentive rate. Actually, we use a different method to achieve this: we establish a Pareto frontier for a single contract, prove certain theoretical properties of it (Theorem 3), and then numerically compare the Pareto frontiers of different contracts (Figures 10–11). We hope that this result will be helpful for contract design, where the aim is to improve the utilities of both parties.

The remainder of the paper is organized as follows. In Section 2, we describe the model setting of Problem (1). In Section 3, we solve the problem and derive the closed-form solution in Theorem 1. In Section 4, we propose the approach of asymptotic analysis in Theorem 2. In Section 5, we establish the Pareto frontier in Theorem 3 and find the contract providing a Pareto improvement. The conclusions are in the last section. The appendices state the standard procedure for solving the general non-concave utility maximization problem and give the proofs of all lemmas and theorems.

2. Model setting

In this section, we describe the model setting for our centrally planned portfolio selection problem (1). The key features include a complete market with a liquidation boundary, S-shaped utilities, and a first-loss contract.

2.1. Financial market

The financial market is composed of one risk-free asset and one risky asset, where the former has a risk-free return rate r and no volatility, while the latter has a higher expected return rate $\mu > r$ and a positive volatility $\sigma > 0$. Denote by $\theta \triangleq \frac{\mu - r}{\sigma} > 0$ the market price of risk. The augmented filtered complete probability space $\big(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P}\big)$ in the financial market is generated by a Brownian motion $W = \{W_t : 0 \le t \le T\}$ on $(\Omega, \mathcal{F}_T, \mathbb{P})$; i.e., the equipped filtration $\mathcal{F} = \{\mathcal{F}_t\}_{0\le t\le T}$ is denoted by $\mathcal{F}_t = \sigma\{W_u : 0 \le u \le t\}$ and satisfies the usual conditions. The interest rate is a constant $r > 0$, and the price of the risk-free asset $R = \{R_t : 0 \le t \le T\}$ is $\mathrm{d}R_t = rR_t\mathrm{d}t$. The price of the risky asset $S = \{S_t : 0 \le t \le T\}$ is a geometric Brownian motion satisfying $\mathrm{d}S_t = S_t\big(\mu\mathrm{d}t + \sigma\mathrm{d}W_t\big) \equiv S_t\big(r\mathrm{d}t + \sigma(\mathrm{d}W_t + \theta\mathrm{d}t)\big)$. As a result, the financial market is complete, and we denote the state price density process $\{\xi_t : 0 \le t \le T\}$ by $\xi_t \triangleq \exp\big(-\big(r + \frac{1}{2}\theta^2\big)t - \theta W_t\big)$. The wealth process $X = \{X_t : 0 \le t \le T\}$ is uniquely determined by a portfolio process $\pi = \{\pi_t : 0 \le t \le T\}$ and an initial wealth x_0:

$$\mathrm{d}X_t = rX_t\mathrm{d}t + \pi_t\sigma(\mathrm{d}W_t + \theta\mathrm{d}t). \tag{3}$$

For any time $t \in [0, T]$, π_t represents the amount of wealth invested in the risky asset at time t. A portfolio π belongs to an admissible portfolio set $\mathcal{V}[0, T]$ if and only if

(i) π is an $\{\mathcal{F}_t\}_{0\le t\le T}$-progressively measurable $\mathbb{R}$-valued process and $\int_0^T |\pi_t|^2\mathrm{d}t < \infty$ almost surely, and

(ii) X is not lower than the liquidation boundary, i.e., $X_t \ge bx_0e^{-r(T-t)}$ almost surely for all $t \in [0, T]$.

The first condition guarantees the existence and finiteness of the wealth process X. The second condition comes from the so-called liquidation boundary bx_0, which ensures that the wealth at time t is always above the liquidation level $bx_0e^{-r(T-t)}$, where $b \in [0, 1)$. The liquidation boundary is also considered in the studies of [17] and [16].

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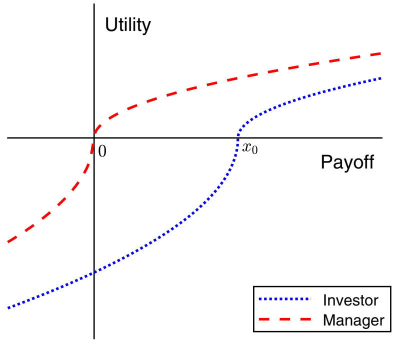

FIGURE 1. S-shaped utilities: $\widehat{U}_1$ (manager), $\widehat{U}_0(\cdot - x_0)$ (investor).

2.2. S-shaped utilities and first-loss contracts

In our model, we assume that the investor ($i = 0$) and the manager ($i = 1$) have S-shaped utilities:

$$\widehat{U}_i(x) = \begin{cases} x^{p_i}, & x \geq 0, \\ -\lambda_i(-x)^{q_i}, & x < 0. \end{cases} \tag{4}$$

The functions $\widehat{U}_0(\cdot)$ and $\widehat{U}_1(\cdot)$ reflect respectively the investor's and the manager's preferences, which are S-shaped utilities as in cumulative prospect theory. They are illustrated in Figure 1. Note that in Figure 1 we plot $\widehat{U}_0(\cdot - x_0)$ instead of $\widehat{U}_0(\cdot)$ to make the curves clearer. For $i = 0, 1$, $p_i \in (0, 1)$ and $q_i \in (0, 1)$ respectively measure the degrees of risk-aversion and risk-seeking, and $\lambda_i > 1$ measures the degree of loss-aversion. The investor and the manager compare their P&L with a reference point 0.

The S-shaped utility is related to cumulative prospect theory (cf. [33]), where the individual displays loss-aversion below a certain reference level and risk-aversion above that level. The theoretical results in this area from behavioral economics are supported by empirical evidence (cf. [3]) and have been successfully applied in many fields of decision-making, including in the contexts of hedge funds (cf. [20]) and insurance (cf. [23]). Although the investor is not a decision-maker for the investment as the manager is, the investor also sets a profit goal as his/her own reference level in the S-shaped utility in order to choose an appropriate contract.

Next we consider a recently popular first-loss contract (cf. [16]). The mechanism of the contract is as follows. There is a guarantee benchmark and a first-loss capital. The guarantee benchmark is the initial fund wealth, and the first-loss capital is provided by the manager in the total fund wealth. When the fund wealth is just below the guarantee benchmark, the manager has to lose first and the investor loses nothing until the first-loss capital is all lost. As a reward, a higher incentive rate for good performance is shared with the manager. The reference level in the S-shaped utilities and the guarantee benchmark in the first-loss contract naturally fit each other. This is why we use S-shaped utilities and the first-loss contract in our model; this is also reflected in [4] and [20], which study S-shaped utilities, and in [16], which studies both of them.

Specifically, the manager's P&L is given by a function $\Theta(X_T)$ of the terminal wealth X_T, i.e.,

$$\Theta(x) = \begin{cases} (\omega + \alpha(1-\omega))(x - x_0), & x \geq x_0, \\ x - x_0, & (1-\omega)x_0 \leq x < x_0, \\ -\omega x_0, & bx_0 \leq x < (1-\omega)x_0. \end{cases} \tag{5}$$

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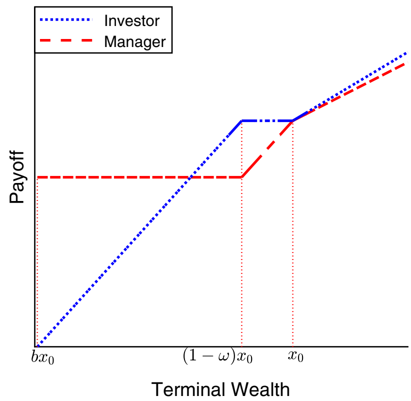

FIGURE 2. P&L: $\Theta(X_T)$ (manager), $X_T - \Theta(X_T) - x_0$ (investor).

The investor's P&L is given by $X_T - \Theta(X_T) - x_0$. A first-loss contract has two parameters: $\omega \in [0, 1]$ is the proportion of managerial ownership in the fund wealth, and $\alpha \in [0, 1]$ is the incentive rate for good performance above the benchmark x_0. In a first-loss contract, the manager provides the first-loss capital ωx_0. The investor does not suffer a loss until the terminal wealth is under $(1-\omega)x_0$. Practically, the manager is incapable of covering the whole loss of funds for the investor, and thus the liquidation boundary is (much) lower than the minimum guarantee (i.e., $0 < bx_0 < (1-\omega)x_0$). Figure 2 gives a numerical demonstration of the above details.

We do not want too many contract parameters involved. Thus, we consider a specific first-loss contract and fix it in the optimization problem. Nevertheless, our insights regarding the centrally planned problem (1) and the following results can be applied in other models.

3. Optimal solutions

In this section, we investigate our centrally planned portfolio selection problem (1) and obtain closed-form solutions. Above all, we state a weighted utility optimization problem:

$$\sup_{\pi \in \mathcal{V}[0,T]} \mathbb{E}\big[U_\gamma\big(X_T^\pi\big)\big], \tag{6}$$

where the objective function is a weighted utility of the terminal payoffs of the investor and the manager,

$$U_\gamma\big(X_T^\pi\big) = (1-\gamma)\widehat{U}_0\big(X_T^\pi - \Theta\big(X_T^\pi\big) - x_0\big) + \gamma \widehat{U}_1\big(\Theta\big(X_T^\pi\big)\big),$$

with $\gamma \in [0, 1]$ interpreted as the weight of the manager's utility. By composition, one can see that the reference points of the actual utility functions of the investor, $\widehat{U}_0\big(\cdot - \Theta(\cdot) - x_0\big)$, and the manager, $\widehat{U}_1\big(\Theta(\cdot)\big)$, are both x_0. Analytically, we have

$$U_\gamma\big(X_T^\pi\big) = \begin{cases} \gamma(\omega + \alpha(1-\omega))^{p_1}\big(X_T^\pi - x_0\big)^{p_1} & \\ \quad + (1-\gamma)((1-\alpha)(1-\omega))^{p_0}\big(X_T^\pi - x_0\big)^{p_0}, & X_T^\pi \geq x_0, \\ -\gamma\lambda_1\big(x_0 - X_T^\pi\big)^{q_1}, & (1-\omega)x_0 < X_T^\pi < x_0, \\ -\gamma\lambda_1(\omega x_0)^{q_1} - (1-\gamma)\lambda_0\big((1-\omega)x_0 - X_T^\pi\big)^{q_0}, & X_T^\pi < (1-\omega)x_0. \end{cases} \tag{7}$$

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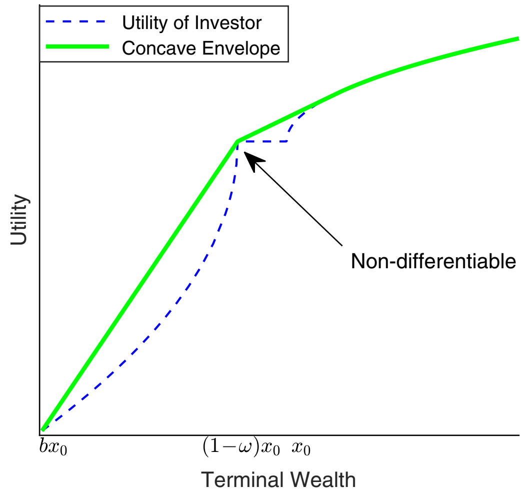

FIGURE 3. Investor's composite S-shaped utility $U_0\left(X_T^\pi\right) \equiv \widehat{U}_0\left(X_T^\pi - \Theta\left(X_T^\pi\right) - x_0\right)$ and its concave envelope.

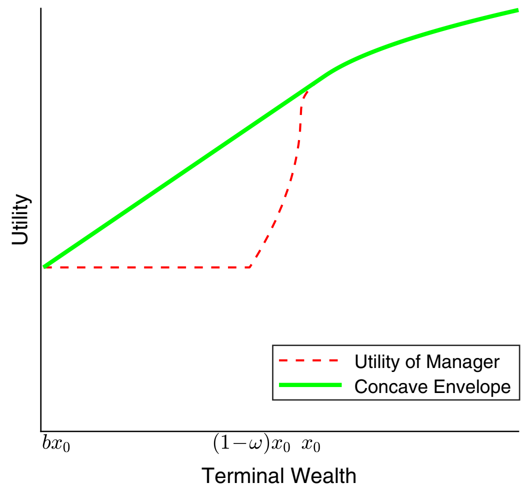

FIGURE 4. Manager's composite S-shaped utility $U_1\left(X_T^\pi\right) \equiv \widehat{U}_1\left(\Theta\left(X_T^\pi\right)\right)$ and its concave envelope.

The cases of $\gamma = 0$ and $\gamma = 1$ correspond to the composite utility functions of the investor and the manager; see Figures 3–4.

We proceed to solve the centrally planned problem (1). Proposition 1 shows that the solution to Problem (1) can be characterized by that to Problem (6). Recall from Section 1 that the optimal solution to Problem (1) is in the Pareto sense as in Definition 1.

Proposition 1. *For any fixed* $\gamma \in [0, 1]$, *the optimal solution to Problem* (6) *is Pareto-optimal to Problem* (1).

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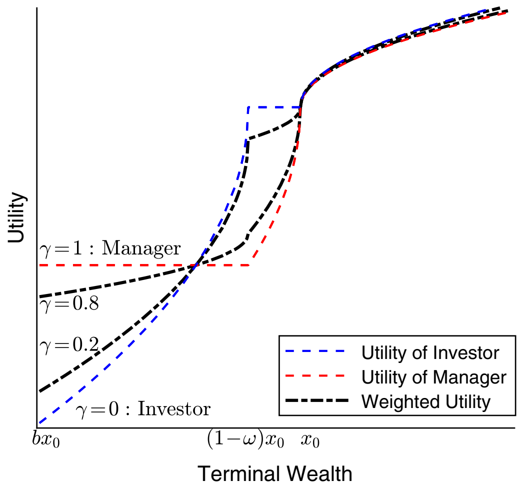

FIGURE 5. Single utilities and weighted utilities.

Problem (6) is the so-called weighting problem, or the weighted sum method in the literature on multi-objective optimization, and contributes to finding the Pareto-optimal solution; see [27] and [11]. However, the equivalence of the weighting problem (6) and the centrally planned problem (1) does not hold in general for non-concave utilities. Indeed, the converse statement in Proposition 1 requires some conditions on convexity. For example, if $\{X_T^\pi : \pi \in \mathcal{V}[0, T]\}$ is a convex set and U_0 and U_1 are strictly concave functions, we can prove that a Pareto optimum for Problem (1) is an optimal solution to some weighting problem (6). Technically, these conditions guarantee the existence of some $\gamma \in [0, 1]$ via the hyperplane separation theorem. We refer to Theorems 14–16 in [11] and Theorem 2.1 in [8] for more details.

Next, one applies the standard procedure for non-concave utility optimization, which is provided in Appendix A, to solve Problem (6), as the utility U_γ is non-concave in our model. The key step is to solve the corresponding utility maximization problem

$$\sup_{X_T \in D} \mathbb{E}\big[U_\gamma^{**}(X_T)\big], \tag{8}$$

where D is defined as the set consisting of all terminal wealth values generated by all portfolios; the expression for D is given in Appendix A. We use concavification techniques to establish the concave envelope of U_γ. This is defined as the smallest concave function dominating U_γ and is denoted by U_γ^{**}; see Figure 6 for an illustration. The concave envelope is also defined by the Legendre transform; see Equation (24) in Appendix A. Lemma 1 discusses the existence of two tangent lines and establishes the concave envelope of U_γ; it is applied to obtain a solution to Problem (1) in Theorem 1.

Lemma 1.

(1) *There exists a unique tangent point* $(1 + c_1)x_0 \in (x_0, +\infty)$ *deduced from* $(bx_0, U_\gamma(bx_0))$ *to* $U_\gamma|_{(x_0,+\infty)}$, *which satisfies*

$$\frac{U_\gamma((1 + c_1)x_0) - U_\gamma(bx_0)}{(1 + c_1)x_0 - bx_0} = U'_\gamma((1 + c_1)x_0). \tag{9}$$

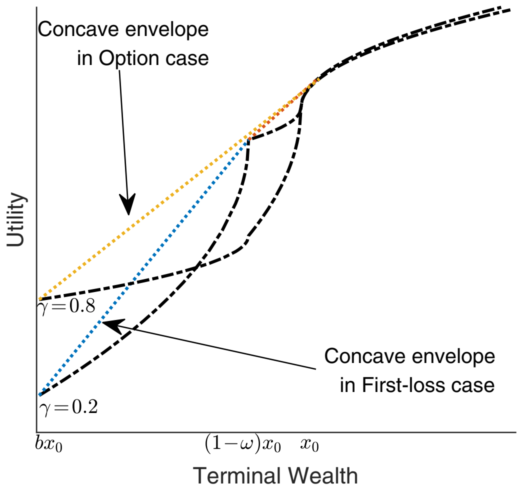

FIGURE 6. Weighted utilities and concave envelopes.

Define κ_1 *as the slope of the tangent line:*

$$\kappa_1 \triangleq U'_\gamma((1+c_1)x_0). \tag{10}$$

(2) *There exists a unique tangent point* $(1+c_2)x_0 \in (x_0, +\infty)$ *deduced from* $((1-\omega)x_0, U_\gamma((1-\omega)x_0))$ *to* $U_\gamma|_{(x_0,+\infty)}$*, which satisfies*

$$\frac{U_\gamma((1+c_2)x_0) - U_\gamma((1-\omega)x_0)}{(1+c_2)x_0 - (1-\omega)x_0} = U'_\gamma((1+c_2)x_0). \tag{11}$$

Define κ_2 *as the slope of the tangent line:*

$$\kappa_2 \triangleq U'_\gamma((1+c_2)x_0). \tag{12}$$

From the procedure for finding the concave envelope of U_γ in Lemma 1, we notice that $(1-\omega)x_0$ is a non-differentiable point for U_γ, as the left and right derivatives for $U_\gamma(x)$ at $x=(1-\omega)x_0$ are not the same. Define κ_3 as the slope of the chord between $(bx_0, U_\gamma(bx_0))$ and $((1-\omega)x_0, U_\gamma((1-\omega)x_0))$:

$$\kappa_3 \triangleq \frac{U_\gamma((1-\omega)x_0) - U_\gamma(bx_0)}{(1-\omega)x_0 - bx_0}. \tag{13}$$

For analytical tractability, we assume that the investor and the manager have the same preference parameters: $p_0 = p_1 = p$, $q_0 = q_1 = q$. Without this assumption, the method of non-concave utility optimization is valid, but the solution is not explicit. As a result, we obtain the closed-form optimal wealth process and the optimal portfolio in Theorem 1.

Theorem 1. *The closed-form optimal wealth process* X^* *and the optimal portfolio* π^* *for Problem* (1) *are given by the following:*

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(1) (First-loss case.) If $\kappa_2 < \kappa_3$, then the optimal terminal wealth X_T^* is

$$X_T^* = \left(b\mathbb{1}_{\left\{\xi_T > \frac{\kappa_3}{\nu^*}\right\}} + (1-\omega)\mathbb{1}_{\left\{\frac{\kappa_2}{\nu^*} < \xi_T \le \frac{\kappa_3}{\nu^*}\right\}} + \left(1 + c_2 \left(\frac{\kappa_2}{\nu^* \xi_T}\right)^{\frac{1}{1-p}}\right) \mathbb{1}_{\left\{\xi_T \le \frac{\kappa_2}{\nu^*}\right\}} \right) x_0, \tag{14}$$

and the optimal wealth process X^* is

$$\begin{aligned} X_t^* &= \xi_t^{-1} \mathbb{E}\left[\xi_T X_T^* | \mathcal{F}_t\right] \\ &= x_0 e^{-r(T-t)} \left(b + (1-\omega-b)\Phi(g_{3,t}) + \omega \Phi(g_{2,t}) + c_2 \frac{\Phi'(g_{2,t})}{\Phi'(d_{2,t})} \Phi(d_{2,t}) \right). \end{aligned} \tag{15}$$

The optimal amount allocated to the risky asset, i.e., the optimal portfolio π^*, is

$$\begin{aligned} \pi_t^* &= \underbrace{\frac{\theta}{(1-p)\sigma} X_t^*}_{\text{Merton term}} + \underbrace{\frac{1}{\sigma\sqrt{T-t}} \Big(\Phi'(g_{3,t}) \cdot \big(1-\omega-b\big) + \Phi'(g_{2,t}) \cdot \big(\omega + c_2\big) \Big) x_0 e^{-r(T-t)}}_{\text{aggressive term}} \\ &\quad + \underbrace{\frac{-\theta}{(1-p)\sigma} \Big(\big(1-\Phi(g_{3,t})\big) \cdot b + \big(\Phi(g_{3,t}) - \Phi(g_{2,t})\big) \cdot (1-\omega) + \Phi(g_{2,t}) \cdot 1 \Big) x_0 e^{-r(T-t)}}_{\text{conservative term}} \\ &\triangleq \pi_t^{(1)} + \pi_t^{(2)} + \pi_t^{(3)}. \end{aligned} \tag{16}$$

(2) (Option case.) If $\kappa_2 \ge \kappa_3$, then the optimal terminal wealth X_T^* is

$$X_T^* = \left(b\mathbb{1}_{\left\{\xi_T > \frac{\kappa_1}{\nu^*}\right\}} + \left(1 + c_1 \left(\frac{\kappa_1}{\nu^* \xi_T}\right)^{\frac{1}{1-p}}\right) \mathbb{1}_{\left\{\xi_T \le \frac{\kappa_1}{\nu^*}\right\}} \right) x_0, \tag{17}$$

and the optimal wealth process X^* is

$$\begin{aligned} X_t^* &= \xi_t^{-1} \mathbb{E}\left[\xi_T X_T^* | \mathcal{F}_t\right] \\ &= x_0 e^{-r(T-t)} \left(b + (1-b)\Phi(g_{1,t}) + c_1 \frac{\Phi'(g_{1,t})}{\Phi'(d_{1,t})} \Phi(d_{1,t}) \right). \end{aligned} \tag{18}$$

The optimal amount allocated to the risky asset, i.e., the optimal portfolio π^*, is

$$\begin{aligned} \pi_t^* &= \underbrace{\frac{\theta}{(1-p)\sigma} X_t^*}_{\text{Merton term}} + \underbrace{\frac{\Phi'(g_{1,t})}{\sigma\sqrt{T-t}} (1 + c_1 - b) x_0 e^{-r(T-t)}}_{\text{aggressive term}} \\ &\quad + \underbrace{\frac{-\theta}{(1-p)\sigma} \Big(\big(1-\Phi(g_{1,t})\big) \cdot b + \Phi(g_{1,t}) \cdot 1 \Big) x_0 e^{-r(T-t)}}_{\text{conservative term}} \\ &\triangleq \pi_t^{(1)} + \pi_t^{(2)} + \pi_t^{(3)}, \end{aligned} \tag{19}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function; the parameter ν^* (a Lagrangian multiplier) is given by $\mathbb{E}[\xi_T X_T^*] = x_0$; κ_1, κ_2 and κ_3 are respectively defined in

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Equations (10), (12), and (13); and $g_{i,t}$ $(i=1, 2, 3)$ *and* $d_{i,t}$ $(i=1, 2)$ *are functions of the slope* κ_i*:*

$$g_{i,t} = \frac{\log\left(\frac{\kappa_i}{\nu^* \xi_t}\right) + \left(r - \frac{\theta^2}{2}\right)(T-t)}{\theta\sqrt{T-t}}, \qquad d_{i,t} = g_{i,t} + \frac{\theta\sqrt{T-t}}{1-p}. \tag{20}$$

Theorem 1 shows that a first-loss contract can sometimes behave like an option contract. There are two cases of the optimal terminal wealth, which we call respectively the option case and the first-loss case. In the option case, the optimal terminal outcome is either liquidation or profit for both parties. Thus, it behaves like an option: either out of the money or in the money. This is why it is called the option case. In the first-loss case, the optimal terminal wealth shows the typical feature of the first-loss contract, which has a third outcome: the manager loses all the first-loss capital ωx_0 while the investor loses nothing. For the manager, the latter outcome is as bad as liquidation. However, for the investor, it is a better outcome than liquidation.

We numerically plot some Pareto-optimal portfolios in Figure 7, where U_0 belongs to the first-loss case while $U_{0.5}$ and U_1 belong to the option case. The option case ($U_{0.5}$ and U_1) has a one-peak–two-valley pattern for the optimal portfolio, which originates from one linear part in the corresponding concave envelope. Some of the literature refers to a peak–valley pattern in the option case (cf. [17]). This reflects that aggressive risk-taking behavior happens when the fund wealth is smaller than the benchmark x_0 and the time to evaluation is limited.

Significantly, the first-loss case (U_0) has a two-peak–three-valley pattern for the optimal portfolio, which originates from two linear parts in the corresponding concave envelope. Compared to the option case, the first-loss case is less risk-seeking at the first-loss residue capital $(1-\omega)x_0$. An intuitive explanation is that the optimal terminal wealth has the third outcome of staying at the first-loss residue capital $(1-\omega)x_0$, which is favorable to the investor. When the investor is relatively more weighted in the Pareto problem (1), the concave envelope of U_γ consists of $\big((1-\omega)x_0,\, U_\gamma((1-\omega)x_0)\big)$ and the optimal wealth is the first-loss case, which is shown in Figure 6.

We emphasize that even if the assumptions $p_0 = p_1$ and $q_0 = q_1$ do not hold, the pattern of the optimal portfolio is similar to Figure 7 if the concave envelope of the objective function is similar. This is because the shape of the optimal portfolio totally depends on the shape of the concave envelope of the objective. From the expression for the objective U_γ, the concave envelope is still categorized into two cases: two linear sections or one linear section. The former possibility leads to the option case for the terminal wealth, while the latter leads to the first-loss case. Furthermore, the option case leads to a one-peak–two-valley pattern for the portfolio, while the first-loss case leads to a two-peak–three-valley pattern.

4. Asymptotic analysis approach

4.1. Three-term decomposition of the optimal portfolio

To further illustrate the patterns in Figure 7 based on Theorem 1, we analyze the structure of the optimal portfolio, which is significantly different from that in the classical theory (cf. [25, 26]). To be precise, the optimal portfolio is decomposed into three terms in Theorem 1, and the second and third terms reflect the novelty of the optimal portfolio solved out in our model.

The first term, $\pi_t^{(1)} = \frac{\theta}{(1-p)\sigma} X_t^*$, is the *Merton term*, the constant-percentage risky investment based on the CRRA utility. In [25, 26], the optimal percentage allocated to the risky asset is a constant $\frac{\theta}{(1-p)\sigma}$.

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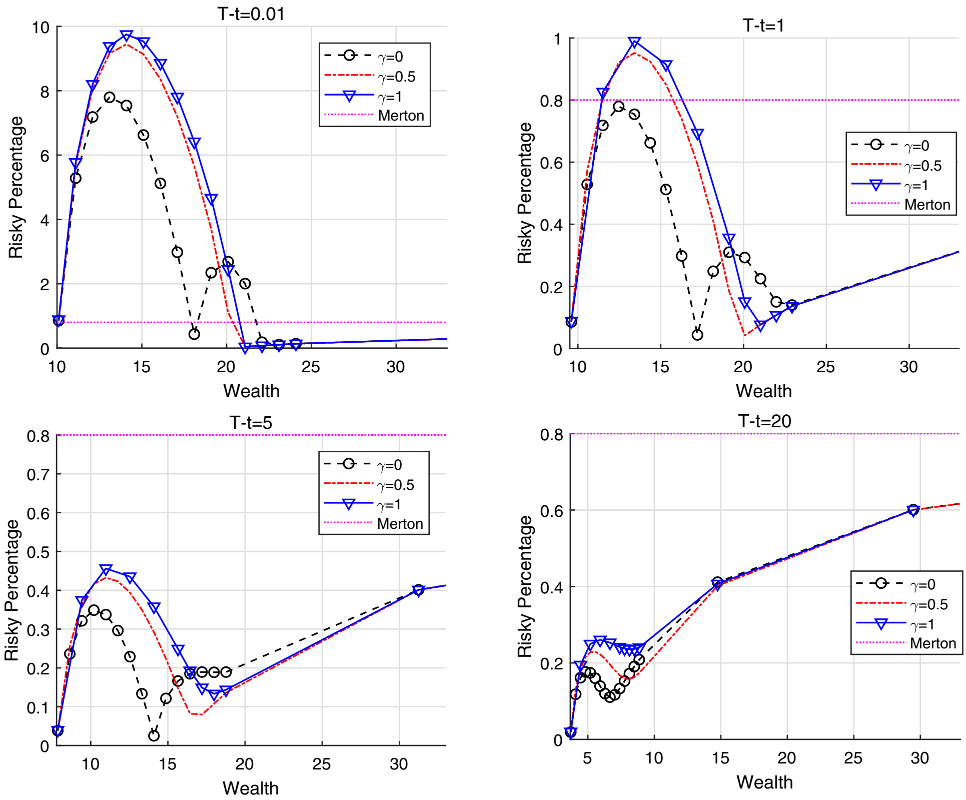

FIGURE 7. The optimal percentage allocated to the risky asset $\frac{\pi_t^*}{X_t^*}$ with respect to the wealth X_t^*. The market parameters are $r = 0.05$, $\sigma = 0.3$, $\theta = 0.12$. The utility parameters are $p = 0.5$, $q = 0.6$, $\lambda = 2.25$. The guarantee benchmark (initial wealth) is $x_0 = 20$, the liquidation boundary is $bx_0 = 10$, and the first-loss residue capital is $(1-\omega)x_0 = 18$. The incentive rate is $\alpha = 0.4$. The Merton line is $\frac{\theta}{(1-p)\sigma} = 0.8$.

The second term, $\pi_t^{(2)}$, is called the *aggressive term*. It increases the allocation on the risky asset, and it arises from the non-concavity of the weighted utility. We observe that $\pi_t^{(2)}$ is related to the linear intervals of the corresponding concave envelope. Quantitatively, the aggressive term is a sum of the product of each linear interval length, the function of the linear interval slope, and the exponential discount factor.

The third term, $\pi_t^{(3)}$, is called the *conservative term*. It decreases the allocation to the risky asset, and it arises from certain wealth levels related to the concave envelope: the liquidation boundary bx_0, the first-loss residue capital $(1-\omega)x_0$, and the benchmark x_0. Quantitatively, the conservative term is a weighted sum of a certain multiple of these wealth levels. The multiple is the product of the Merton constant $\frac{\theta}{(1-p)\sigma}$ and the exponential discount factor. The weight is the function of the slope of the corresponding linear part's slope; the weights can be interpreted as probabilities, with the sum 1.

4.2. Asymptotic analysis of the optimal portfolio

In this subsection, we conduct an asymptotic analysis of X^* and π^* in Theorem 2 below and discuss the resulting economic insights.

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Theorem 2. *We have the following statements on X^* and π^*:*

(i) (Monotonic analysis.) X_t^* and $\pi_t^{(1)}$ are decreasing with respect to ξ_t, while $\pi_t^{(3)}$ is increasing with respect to ξ_t. In the first-loss case, $\pi_t^{(2)}$ is increasing if

$$\xi_t \in \left(0,\ \frac{\kappa_2}{\nu^*} e^{\left(r-\frac{\theta^2}{2}\right)(T-t)}\right),$$

and $\pi_t^{(2)}$ is decreasing if

$$\xi_t \in \left(\frac{\kappa_3}{\nu^*} e^{\left(r-\frac{\theta^2}{2}\right)(T-t)},\ +\infty\right).$$

In the option case, $\pi_t^{(2)}$ is increasing if

$$\xi_t \in \left(0,\ \frac{\kappa_1}{\nu^*} e^{\left(r-\frac{\theta^2}{2}\right)(T-t)}\right)$$

and decreasing otherwise.

(ii) (Dynamic analysis.) As $\xi_t \to 0$, we have

$$X_t^* \to +\infty, \quad \pi_t^* \to +\infty, \quad \frac{\pi_t^*}{X_t^*} \to \frac{\theta}{(1-p)\sigma}, \quad \pi_t^{(2)} \to 0, \quad \frac{\pi_t^{(3)}}{-\frac{\theta}{(1-p)\sigma} x_0 e^{-r(T-t)}} \to 1.$$

(iii) (Dynamic analysis.) As $\xi_t \to +\infty$, we have

$$\frac{X_t^*}{bx_0 e^{-r(T-t)}} \to 1, \quad \pi_t^* \to 0, \quad \frac{\pi_t^*}{X_t^*} \to 0, \quad \pi_t^{(2)} \to 0, \quad \frac{\pi_t^{(3)}}{-\frac{b\theta}{(1-p)\sigma} x_0 e^{-r(T-t)}} \to 1.$$

We list the above results of (ii) and (iii) as follows:

TABLE 1. Dynamic analysis.

	X_t^*	π_t^*	$\frac{\pi_t^*}{X_t^*}$	$\pi_t^{(2)}$	$\pi_t^{(3)}$
$\xi_t \to 0$	$+\infty$	$+\infty$	$\frac{\theta}{(1-p)\sigma}$	0	$-\frac{\theta}{(1-p)\sigma} x_0 e^{-r(T-t)}$
$\xi_t \to +\infty$	$bx_0 e^{-r(T-t)}$	0	0	0	$-\frac{\theta}{(1-p)\sigma} bx_0 e^{-r(T-t)}$

(iv) *(Terminal-time analysis.) In the option case, we have the following:*

TABLE 2. Option case: terminal-time analysis.

	X_T^*	π_T^*	$\frac{\pi_T^*}{X_T^*}$	$\pi_T^{(2)}$	$\pi_T^{(3)}$
$\xi_T < \frac{\kappa_1}{\nu^*}$	$\left(1 + c_1\left(\frac{\kappa_1}{\nu^* \xi_T}\right)^{\frac{1}{1-p}}\right) x_0$	$\frac{\theta}{(1-p)\sigma}\left(\frac{\kappa_1}{\nu^* \xi_T}\right)^{\frac{1}{1-p}} c_1 x_0$	$\frac{\theta}{(1-p)\sigma} \frac{X_T^* - x_0}{X_T^*}$	0	$-\frac{\theta}{(1-p)\sigma} x_0$
$\xi_T = \frac{\kappa_1}{\nu^*}$	$\{bx_0, (1+c_1)x_0\}$	$+\infty$	$+\infty$	$+\infty$	$-\frac{\theta}{(1-p)\sigma} \frac{b+1}{2} x_0$
$\xi_T > \frac{\kappa_1}{\nu^*}$	bx_0	0	0	0	$-\frac{\theta}{(1-p)\sigma} bx_0$

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(v) *(Terminal-time analysis.) In the first-loss case, we have the following:*

TABLE 3. First-loss case: terminal-time analysis.

	X_T^*	π_T^*	$\frac{\pi_T^*}{X_T^*}$	$\pi_T^{(2)}$	$\pi_T^{(3)}$
$\xi_T < \frac{\kappa_2}{\nu^*}$	$\left(1+c_2\left(\frac{\kappa_2}{\nu^*\xi_T}\right)^{\frac{1}{1-p}}\right)x_0$	$\frac{\theta}{(1-p)\sigma}\left(\frac{\kappa_2}{\nu^*\xi_T}\right)^{\frac{1}{1-p}}c_2x_0$	$\frac{\theta}{(1-p)\sigma}\frac{X_T^*-x_0}{X_T^*}$	0	$-\frac{\theta}{(1-p)\sigma}x_0$
$\xi_T = \frac{\kappa_2}{\nu^*}$	$\{(1-\omega)x_0, (1+c_2)x_0\}$	$+\infty$	$+\infty$	$+\infty$	$-\frac{\theta}{(1-p)\sigma}\frac{1+1-\omega}{2}x_0$
$\frac{\kappa_2}{\nu^*} < \xi_T < \frac{\kappa_3}{\nu^*}$	$(1-\omega)x_0$	0	0	0	$-\frac{\theta}{(1-p)\sigma}(1-\omega)x_0$
$\xi_T = \frac{\kappa_3}{\nu^*}$	$\{bx_0, (1-\omega)x_0\}$	$+\infty$	$+\infty$	$+\infty$	$-\frac{\theta}{(1-p)\sigma}\frac{1-\omega+b}{2}x_0$
$\xi_T > \frac{\kappa_3}{\nu^*}$	bx_0	0	0	0	$-\frac{\theta}{(1-p)\sigma}bx_0$

A basic observation for our analysis is that ξ_t, the state price process at time t, can be interpreted as an indicator showing whether the market is in a good state (small ξ_t) or a bad state (large ξ_t).

Theorem 2(i) provides a monotonic analysis and shows that the wealth process X_t^* is decreasing with respect to ξ_t. This statement additionally explains why a small ξ_t means a good market state (leading to a large wealth X_t^*) and a large ξ_t means a bad market state (leading to a small wealth X_t^*). More importantly, our three-term decomposition provides an approach for the monotonic analysis, because the monotonicity of π_t^* with respect to ξ_t is unknown in general. While the first term $\pi_t^{(1)}$, as a multiple of X_t^*, is decreasing in ξ_t, the conservative term $\pi_t^{(3)}$ is increasing in ξ_t. The combined effect of the three terms is that the optimal strategy π_t^* is not monotone in ξ_t. Yet we can use the decomposition to show the monotonicity of the separate terms. If the market improves (ξ_t decreases), the first term $\pi_t^{(1)}$ contributes to an increase on the risky investment, although the third term $\pi_t^{(3)}$ decreases. If the market gets worse (ξ_t increases), the first term $\pi_t^{(1)}$ contributes to a decrease on the risky investment, although the third term $\pi_t^{(3)}$ increases. Two further scenarios ($\xi_t \to 0$ and $\xi_t \to +\infty$) are respectively analyzed in Theorem 2(ii)–(iii), where $\pi_t^{(1)}$ is a dominating term. In the middle range of ξ_t, the monotonicity of π_t^* is highly affected by the aggressive term $\pi_t^{(2)}$ and the conservative term $\pi_t^{(3)}$.

Theorem 2(ii) provides a dynamic analysis and shows that when the market is in a good state ($\xi_t \to 0$), the wealth is well accumulated and the optimal percentage allocated to the risky asset is asymptotically the Merton constant $\frac{\theta}{(1-p)\sigma}$. The aggressive term $\pi_t^{(2)}$ tends to zero, showing that there is no need to take on more risk in the good state. The conservative term $\pi_t^{(3)}$ tends to $\frac{\theta}{(1-p)\sigma}x_0$, the Merton constant multiplied by the initial wealth x_0. This shows that in the good state, the weighted sum $\pi_t^{(3)}$ is tracking one of the wealth levels, the benchmark x_0.

Theorem 2(iii) shows that when the market is in a bad state ($\xi_t \to +\infty$), the investment performs poorly and tends to be liquidated, and the optimal strategy is to invest everything in the risk-free asset. The aggressive term $\pi_t^{(2)}$ tends to zero, showing that it is impossible to take on any risk in the asymptotic liquidation state. The conservative term $\pi_t^{(3)}$ tends to $\frac{\theta}{(1-p)\sigma}bx_0$, the Merton constant multiplied by the liquidation boundary bx_0. This shows that in the bad

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state, the weighted sum $\pi_t^{(3)}$ is tracking one of the wealth levels, the liquidation boundary bx_0. As a result, the amount allocated to the risky asset tends to 0 when the optimal wealth tends to be liquidated, and the wealth is never below the liquidation level $bx_0e^{-r(T-t)}$ (which coincides with [5, 14, 16]). In addition, (ii) and (iii) show that if the wealth X_t^* increases, the higher wealth level which $\pi_t^{(3)}$ is tracking will increase, and thus the conservative term $\pi_t^{(3)}$ will increase.

Theorem 2(iv) provides a terminal-time analysis and shows the asymptotic behavior as one approaches the terminal evaluation time T in the option case. We compare the state price ξ_T with a threshold value $\frac{\kappa_1}{\nu^*}$. When the market is in a good state ($\xi_T < \frac{\kappa_1}{\nu^*}$), the optimal final wealth is higher than the benchmark x_0 and results in a profit. The term $\pi_t^{(2)}$ tends to zero, and $\pi_t^{(3)}$ tends to $-\frac{\theta}{(1-p)\sigma}x_0$, which is coincident with (ii). However, the optimal percentage $\frac{\theta}{(1-p)\sigma}\frac{X_T^*-x_0}{X_T^*}$ is smaller than the Merton constant $\frac{\theta}{(1-p)\sigma}$, implying that the optimal investment becomes more conservative when the time is limited and the market is good (which coincides with [9]). The bad state is coincident with (iii).

Interestingly, when the market is in a specific state ($\xi_T = \frac{\kappa_1}{\nu^*}$), the optimal portfolio π^* tends to infinity while the optimal wealth process X^* tends to liquidation, implying that the best strategy is to gamble as much as possible, but that the resulting outcome is either liquidation or profit for both parties. In this case, $\pi_t^{(2)}$ and $\pi_t^{(3)}$ behave strangely. The aggressive term tends to infinity, reflecting the strategy of gambling for a profit, while the conservative term tends to $\frac{\theta}{(1-p)\sigma}\frac{b+1}{2}x_0$, the exact midpoint between the conservative terms in (ii) and (iii). Theorem 2(v) shows similar asymptotic behavior as one approaches the terminal evaluation time T in the first-loss case.

To sum up, because of $\pi^{(2)}$, the portfolio gambles on the risky asset whenever the utility is not strictly concave, resulting in two peaks in the first-loss case and one peak in the option case; because of $\pi^{(3)}$, the portfolio becomes conservative at some wealth levels, resulting in three valleys in the first-loss case and two valleys in the option case. The term $\pi^{(3)}$ further explains why the portfolio becomes more conservative in a better market state (cf. [9]) and the total wealth is always above the liquidation boundary bx_0 (cf. [5]).

5. Pareto improvement

5.1. Pareto frontier

In this subsection, we establish the collection of Pareto points (2) under the optimal portfolio π^* given in Theorem 1. Based on Proposition 1, we can equivalently establish $\text{PF} \equiv \big\{\big(\mathbb{E}\big[U_1\big(\hat{X}_\gamma\big)\big], \mathbb{E}\big[U_0\big(\hat{X}_\gamma\big)\big]\big) : \gamma \in [0, 1]\big\} \subset \mathbb{R}^2$, where, with a little abuse of notation, $\hat{X}_\gamma \triangleq X_T^{\pi^{*,\gamma}}$ denotes the terminal wealth of the optimal portfolio $\pi^{*,\gamma}$ in Problem (6) with weight $\gamma \in [0, 1]$. The point $\big(\mathbb{E}\big[U_1\big(\hat{X}_\gamma\big)\big], \mathbb{E}\big[U_0\big(\hat{X}_\gamma\big)\big]\big)$ is a manager–investor expected utility (EU) pair.

Our first result is Theorem 3 below, showing that the collection (2) is a decreasing and concave frontier, which is referred to as the *Pareto frontier* (PF). Above all, we need to introduce the manager's single utility maximization problem,

$$\sup_{X \in D} \mathbb{E}\big[\widehat{U}_1\big(\Theta(X_T)\big)\big], \tag{21}$$

and the investor's single utility maximization problem,

$$\sup_{X \in D} \mathbb{E}\big[\widehat{U}_0\big(X_T - \Theta(X_T) - x_0\big)\big]. \tag{22}$$

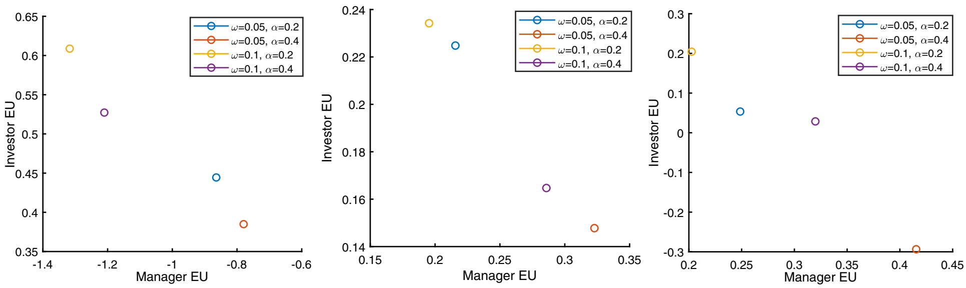

FIGURE 8. Fixed-weight contract comparison: each circle point corresponds to a manager–investor EU pair under a contract with (ω, α). For all panels, $T = 0.01$; left panel, $\gamma = 0$; middle panel, $\gamma = 0.5$; right panel, $\gamma = 1$.

Similarly to Theorem 1, we show in Proposition 2 that there exist (distribution-wise unique) solutions to Problems (21) and (22).

Proposition 2. *There exist a (distribution-wise unique) optimum $\hat{X}_1$ to Problem (21) and a (distribution-wise unique) optimum $\hat{X}_0$ to Problem (22).*

We now characterize the PF, in Theorem 3.

Theorem 3. *We have the following:*

1. *(Strictly decreasing.) If $\left(\mathbb{E}[U_1(\hat{X}_0)], \mathbb{E}[U_0(\hat{X}_0)]\right) \neq \left(\mathbb{E}[U_1(\hat{X}_1)], \mathbb{E}[U_0(\hat{X}_1)]\right)$, then $\mathbb{E}\left[U_1\left(\hat{X}_\gamma\right)\right]$ is strictly increasing with respect to $\gamma \in [0, 1]$, and $\mathbb{E}\left[U_0\left(\hat{X}_\gamma\right)\right]$ is strictly decreasing with respect to $\gamma \in [0, 1]$. Moreover, the collection (2) is a strictly decreasing frontier.*
2. *(Strictly concave.) For any Pareto point $\left(\mathbb{E}\left[U_1\left(\hat{X}_\gamma\right)\right], \mathbb{E}\left[U_0\left(\hat{X}_\gamma\right)\right]\right)$, there exists an affine function with slope $-\frac{\gamma}{1-\gamma}$ passing through the point and dominating the collection (2). Moreover, if the collection (2) is continuous, then it is a strictly concave frontier with pointwise sub-differential $-\frac{\gamma}{1-\gamma}$.*

Theorem 3 states that the collection (2) is a strictly decreasing and strictly concave frontier if it is not reduced to a single point. A decreasing PF shows that the total welfare is shared between the investor and the manager. Nevertheless, in some multi-objective problems (e.g., when U_1 is a multiple of U_0), if Problem (21) and Problem (22) have the same solution, the PF will reduce to a single point (i.e., $\left(\mathbb{E}\left[U_1\left(\hat{X}_0\right)\right], \mathbb{E}\left[U_0\left(\hat{X}_0\right)\right]\right) = \left(\mathbb{E}\left[U_1\left(\hat{X}_1\right)\right], \mathbb{E}\left[U_0\left(\hat{X}_1\right)\right]\right)$).

5.2. Comparison of fixed-weight contracts

We first compare different contracts with the weight γ fixed ($\gamma = 0, 0.5, 1$); see Figures 8–9. Each contract contains two parameters (ω, α). Recall in Equation (5) that ω is the manager's proportion in the total asset and α is the incentive rate for the wealth above the benchmark x_0.

We find that there is no Pareto improvement, in terms of the utilities of both the investor and the manager, between any two fixed-weight contracts. In any subfigure, for any two contracts, there is no contract outperforming the other one in both objectives, no matter whether the evaluation time is long or short. This motivates us to incorporate the weight parameter γ as one of the contract parameters in order to seek possible Pareto improvement.

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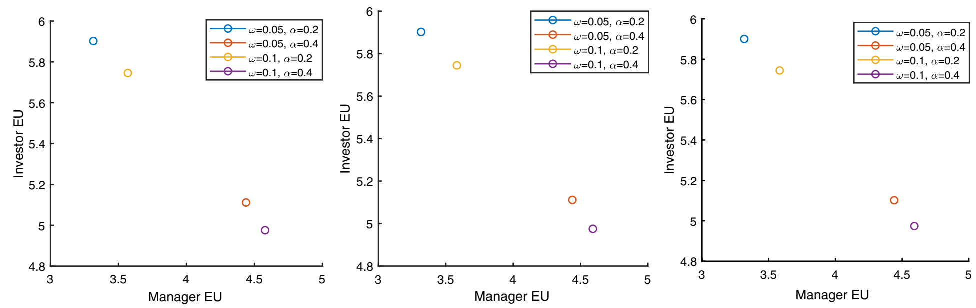

FIGURE 9. Fixed-weight contract comparison: each circle point corresponds to a manager–investor EU pair under a contract with (ω, α). For all panels, $T = 20$; left panel, $\gamma = 0$; middle panel, $\gamma = 0.5$; right panel, $\gamma = 1$.

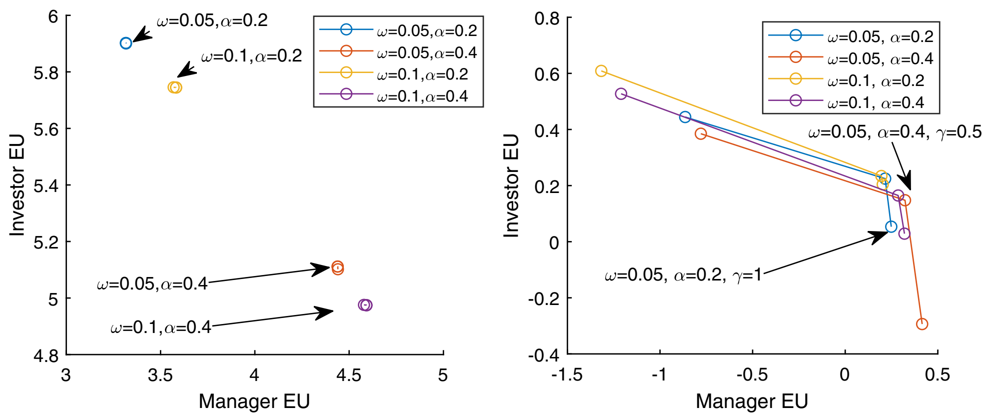

FIGURE 10. PFs of different contracts (left panel, $T = 20$; right panel, $T = 0.01$). Each circle point corresponds to a manager–investor EU pair (a Pareto point) under a contract with (ω, α, γ). A PF is plotted by a three-point curve, consisting of three Pareto points, for $\gamma = 0, 0.5, 1$, under a specific contract setting (ω, α). In the right panel, each three-point curve represents a PF which is decreasing and concave. In the left panel, each three-point curve is almost reduced to a single point.

In addition, when the evaluation time is long, we can see that the EUs of the investor (and of the manager) are similar for different weight parameters. The investor gets a higher EU for a smaller incentive rate α and a smaller managerial ownership proportion ω. This finding will be revisited below.

5.3. A contract providing a Pareto improvement

In this subsection, we focus on PFs of different contracts. We propose a novel contract, simultaneously changing the weight and the parameter to improve the utilities of both parties, as illustrated in Figure 10, which depicts several PFs under different sets of contract parameters (ω, α).

In the left panel of Figure 10, when the evaluation time is long ($T = 20$), the Pareto-optimal portfolios are quite similar and do not involve gambling (see the fourth subfigure in Figure 7). As a result, the PF of each contract is almost reduced to a single point, as if the Pareto points

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of each PF are clustered around the single point. Meanwhile, the PFs of different contracts are separated. Graphically, a new frontier of these PFs appears and is again decreasing, which shows that the total welfare is shared among the different contracts when the evaluation time is long. We find that among first-loss contracts with long evaluation time, the investor benefits from contracts with smaller incentive rate α or smaller managerial ownership proportion ω.

Significantly, we can find a contract providing a Pareto improvement from the right panel of Figure 10. When the evaluation time is short ($T = 0.01$), the performance of Pareto-optimal portfolios under each contract is much different (see the first subplot in Figure 7). As a result, the PFs of different contracts are clustered. If we treat the objective weight γ as a parameter of a contract, there exists some contract dominating another contract with a different weight. For instance, the contract with $\omega = 0.05$, $\alpha = 0.4$ and objective weight $\gamma = 0.5$ dominates another contract with $\omega = 0.05$, $\alpha = 0.2$ and objective weight $\gamma = 1$ on the PF, which indicates that one contract has been found to be strictly better than others.

Therefore, if we simultaneously consider the investor's utility in the investment objective and increase the incentive rate, we can obtain a Pareto improvement on some other contract ($\omega = 0.05$, $\alpha = 0.2$, $\gamma = 1$) in the traditional maximization of the single utility of the manager. This may be beneficial for investors and also helpful in enabling fund managers to design competitive contracts.

5.4. Certainty equivalent

To enlarge the scope of our results above, we investigate the certainty equivalent in addition to the expected utility. Denote by $\widehat{U}_i^{-1}$ the inverse function of $\widehat{U}_i$, $i = 0, 1$. For $\gamma \in [0, 1]$, we define the P&L certainty equivalents of the manager and the investor under the optimal wealth $\hat{X}_\gamma$ (note that $\hat{X}_\gamma \equiv X_T^{\pi^*,\gamma}$) as follows:

$$\mathrm{CE}_1(\hat{X}_\gamma) \triangleq \widehat{U}_1^{-1}\left(\mathbb{E}\big[\widehat{U}_1(\Theta(\hat{X}_\gamma))\big]\right), \qquad \mathrm{CE}_0 \triangleq \widehat{U}_0^{-1}\left(\mathbb{E}\left[\widehat{U}_0(\hat{X}_\gamma - \Theta(\hat{X}_\gamma) - x_0)\right]\right).$$

In our context, the certainty equivalent means the guaranteed amount of cash allocated to the corresponding agent, who would consider this to be the same amount as the optimal terminal wealth. That is,

$$\widehat{U}_1(\mathrm{CE}_1(\hat{X}_\gamma)) = \mathbb{E}\big[\widehat{U}_1(\Theta(\hat{X}_\gamma))\big], \qquad \widehat{U}_0(\mathrm{CE}_0(\hat{X}_\gamma)) = \mathbb{E}\big[\widehat{U}_0(\hat{X}_\gamma - \Theta(\hat{X}_\gamma) - x_0)\big].$$

Indeed, as $\widehat{U}_i$, $i = 0, 1$, are increasing, this is equivalent to maximizing the manager's (or the investor's) expected utility and certainty equivalent. The advantage of using the certainty equivalent is that we can compare the numerical values for the manager and the investor on the same scale. We define the Pareto frontier of the certainty equivalents (PFCE) as follows:

$$\mathrm{PFCE} = \left\{\left(\mathrm{CE}_1(\hat{X}_\gamma), \mathrm{CE}_0(\hat{X}_\gamma)\right) : \gamma \in [0, 1]\right\} \subset \mathbb{R}^2.$$

Figure 11 shows the PFCEs under the different contract settings. The curves in Figure 11 show a similar pattern to those in Figure 10, and the behaviors in terms of CE and EU are similar.

From the left panel of Figure 11, we find that if the evaluation time is long (i.e., T is large), the investor has a higher certainty equivalent if the contract has a smaller incentive rate α or a smaller managerial ownership proportion ω. In the right panel of Figure 11, if the evaluation time is short (i.e., T is small), we can find a similar means of Pareto improvement on CE. If we simultaneously consider the investor's utility in the investment objective and increase the incentive rate ($\omega = 0.05$, $\alpha = 0.4$, $\gamma = 0.5$), we can obtain a Pareto improvement over some

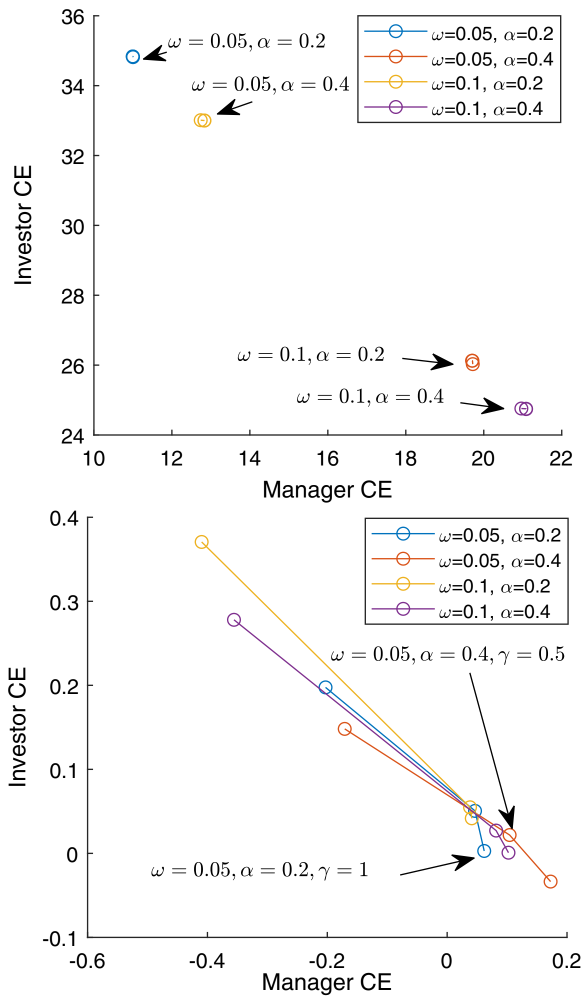

FIGURE 11. PFCEs of different contracts (Left panel: $T = 20$, Right panel: $T = 0.01$). Each circle points means an Manager-Investor CE pair (a Pareto point) under a contract with (ω, α, γ). A PFCE is plotted by a three-point curve, consisting of three Pareto points of $\gamma = 0, 0.5, 1$ under a specific contract setting (ω, α). In the right panel, each three-point curve represents a PF, which is decreasing and concave. In the left panel, each three-point curve is almost reduced to a single point.

other contract $(\omega = 0.05,\ \alpha = 0.2,\ \gamma = 1)$ in the traditional problem of maximization of the single utility of the manager. This indicates that the Pareto improvement results hold in wide generality, as the numerical values for the manager and the investor are compared on the same scale of CE.

6. Conclusion

We formulate the centrally planned portfolio selection problem under S-shaped utilities and a recently popular first-loss contract. The optimal portfolio shows that a first-loss contract can sometimes behave like an option contract. We propose an asymptotic analysis approach and

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decompose the optimal portfolio into three terms (the Merton term $\pi^{(1)}$, the aggressive term $\pi^{(2)}$, and the conservative term $\pi^{(3)}$). From our asymptotic analysis, we find that because of $\pi^{(2)}$, the portfolio involves gambling on the risky asset whenever the utility is not strictly concave, resulting in two peaks in the first-loss case and one peak in the option case. Furthermore, we find that among first-loss contracts with long evaluation time, the investor benefits from contracts with smaller incentive rate and smaller managerial ownership proportion. In addition, when the evaluation time is short, the contract involving consideration of the investor's utility and increasing the incentive rate leads to a Pareto improvement.

The approach of asymptotic analysis can be clearly applied to more general models to illustrate economic phenomena. The structure of the optimal portfolio also holds for other non-concave utility optimization problems. The Pareto improvement is shown qualitatively in our paper. We leave for future research the question of how to address the Pareto improvement in a quantitative way, together with the above questions.

Appendix A. Standard procedure for non-concave utility maximization

In this section, based on the martingale and duality methods (cf. [19]) and concavification techniques (cf. [9], [20], [16]), we state the standard procedure for non-concave utility maximization.

The first step is to transform the optimization problem for controlled allocation processes into that for terminal wealth variables, i.e., to transform Problem (6) to Problem (23). Based on the martingale method, if the optimal terminal wealth X_T^* is obtained, the optimal allocation $\left\{\pi_t^* : 0 \le t \le T\right\}$ can be duplicated through a martingale representation. The terminal wealth optimization problem is

$$\sup_{X_T \in D} \mathbb{E}[U_\gamma(X_T)], \tag{23}$$

where

$$D \triangleq \{X_T : X_T \in \mathcal{F}_T,\ \ \mathbb{E}[\xi_T X_T] = x_0;\ X_T \ge b x_0,\ \text{ a.s.}\}.$$

The second step of the martingale method is to solve out the optimal terminal wealth X_T^* of Problem (23) by duality methods and concavification techniques, because the utility U_γ is non-concave. By the theory of convex analysis (cf. [30]), the concave envelope of U_γ is given by its biconjugate function U_γ^{**}:

$$\begin{cases} U_\gamma^*(y) \triangleq \sup\limits_{x \in \mathrm{dom}\, U_\gamma} (U_\gamma(x) - yx), \\ U_\gamma^{**}(x) \triangleq \inf\limits_{y \in \mathrm{dom}\, U_\gamma^*} (U_\gamma^*(y) + xy). \end{cases} \tag{24}$$

Applying Lagrange duality methods to Problem (23), we find that the optimal terminal wealth is given by

$$X_T^* = \mathbb{X}(\nu^* \xi_T) = \arg \sup_{x \in \mathrm{dom} U} \left(U_\gamma(x) - \nu^* \xi_T x\right) = \left(\left(U_\gamma^{**}\right)'\right)^{-1}\left(\nu^* \xi_T\right).$$

The uniqueness of X^* holds because U_γ^{**} is concave and ξ_T has no atom.

The last step is to use the martingale representation to duplicate the optimal terminal wealth in the financial market. As $\{\xi_t X_t^* : 0 \le t \le T\}$ is a martingale, using the expression for X_T^* we obtain that

$$\begin{aligned} X_t^* &= \xi_t^{-1} \mathbb{E}\left[\xi_T \mathbb{X}\left(\nu^* \xi_T\right) | \mathcal{F}_t\right] \\ &= \mathbb{E}\left[Z_{t,T} \mathbb{X}\left(\nu^* \xi_t Z_{t,T}\right) | \mathcal{F}_t\right], \end{aligned} \tag{25}$$

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where $Z_{t,T} \triangleq \frac{\xi_T}{\xi_t} = \exp\left[-\left(r + \frac{\theta^2}{2}\right)(T-t) - \theta\left(W_T - W_t\right)\right]$. Because $Z_{t,T}$ is independent of $\mathcal{F}_t$, X_t^* is a function of ξ_t, i.e.,

$$X_t^* = f(t, \xi_t),$$

where

$$f(t, \xi) = \mathbb{E}\left[Z_{t,T}\mathbb{X}\left(\nu^* \xi Z_{t,T}\right)|\mathcal{F}_t\right]. \tag{26}$$

Applying Itô's lemma to $\left\{X_t^*, 0 \le t \le T\right\}$ and comparing to Equation (3), we derive the optimal allocation:

$$\pi_t^* = \sigma^{-1}\left(-\theta\xi_t \frac{\partial f}{\partial \xi}(t, \xi_t)\right), \tag{27}$$

where f is defined by Equation (26).

Appendix B. Proofs of lemmas and theorems

B.1. Proof of Proposition 1

For any optimum $\hat{X}$ to Problem (6), it suffices to prove that there exists no admissible terminal wealth $X \in D$ such that $\mathbb{E}\left[U_0\left(\hat{X}\right)\right] \le \mathbb{E}[U_0(X)]$ and $\mathbb{E}\left[U_1\left(\hat{X}\right)\right] \le \mathbb{E}[U_1(X)]$, and at least one of the inequalities holds strictly.

We prove this statement by contradiction. Suppose that there exists some $X \in D$ satisfying $\mathbb{E}\left[U_0\left(\hat{X}\right)\right] \le \mathbb{E}[U_0(X)]$ and $\mathbb{E}\left[U_1\left(\hat{X}\right)\right] < \mathbb{E}[U_1(X)]$. We have

$$\mathbb{E}\left[U_\gamma\left(\hat{X}\right)\right] \equiv (1-\gamma)\mathbb{E}\left[U_0\left(\hat{X}\right)\right] + \gamma\mathbb{E}\left[U_1\left(\hat{X}\right)\right] < (1-\gamma)\mathbb{E}[U_0(X)] + \gamma\mathbb{E}[U_1(X)] \equiv \mathbb{E}[U_\gamma(X)],$$

for any $\gamma \in (0, 1]$. Then the optimality of $\hat{X}$ to Problem (6) implies that $\hat{X}$ has to be the optimum to Problem (6) with $\gamma = 0$ (i.e., Problem (22)). On the other hand, $\mathbb{E}\left[U_0\left(\hat{X}\right)\right] \le \mathbb{E}[U_0(X)]$ implies that X is also an optimum to Problem (22), so the inequality becomes an equality, $\mathbb{E}\left[U_0\left(\hat{X}\right)\right] = \mathbb{E}[U_0(X)]$. By the uniqueness of the optimal solution, we have $X \overset{d}{=} \hat{X}$, contradicting the hypothesis that $\mathbb{E}\left[U_1\left(\hat{X}\right)\right] < \mathbb{E}[U_1(X)]$.

B.2. Proof of Lemma 1

(1) We now solve for the tangent point (denoted by $(1+c_1)x_0$) deduced from the point $(bx_0, U_\gamma(bx_0))$ to the curve $U_\gamma|_{[x_0,+\infty)}$.

We define $F_1(x) \triangleq U_\gamma(x) - U_\gamma(bx_0) - U'_\gamma(x)(x - bx_0)$, $x \in [x_0, +\infty)$. As $U'_\gamma(x_0+) = +\infty$ and $U'_\gamma(+\infty) = 0$, we obtain that $F'_1|_{(x_0,+\infty)}(x) > 0$, $F_1(x_0+) = -\infty$, and $F_1(+\infty) = +\infty$. By the intermediate value property of the continuous function F_1, there exists a unique $(1+c_1)x_0 \in [x_0, +\infty)$ $(c_1 > 0)$ satisfying $F_1((1+c_1)x_0) = 0$, i.e., such that (9) holds. In addition, the slope of the tangent line is $\kappa_1 \triangleq U'_\gamma((1+c_1)x_0)$.

(2) We now solve for the tangent point (denoted by $(1+c_2)x_0$) deduced from the point $((1-\omega)x_0, U_\gamma((1-\omega)x_0))$ to the curve $U_\gamma|_{[x_0,+\infty)}$.

We define $F_2(x) \triangleq U_\gamma(x) - U_\gamma((1-\omega)x_0) - U'_\gamma(x)(x - (1-\omega)x_0)$, $x \in [x_0, +\infty)$. As $U'_\gamma(x_0+) = +\infty$ and $U'_\gamma(+\infty) = 0$, we obtain that $F'_2|_{(x_0,+\infty)}(x) > 0$, $F_2(x_0+) = -\infty$, and $F_2(+\infty) = +\infty$. By the intermediate value property of the continuous function F_2, there exists a unique $(1+c_2)x_0 \in [x_0, +\infty)$ $(c_2 > 0)$ satisfying $F_2((1+c_2)x_0) = 0$, i.e., such that (11) holds. In addition, the slope of the tangent line is $\kappa_2 \triangleq U'_\gamma((1+c_2)x_0)$.

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B.3. Proof of Theorem 1

We use the standard procedure in Appendix A to solve Problem (6). We classify two cases of the concave envelope of U_γ by comparing κ_2 with κ_3.

(1) (First-loss case.) If $\kappa_2 < \kappa_3$, then the concave envelope of U_γ is given by

$$U_\gamma^{**}(x) = \begin{cases} U_\gamma(bx_0) + \kappa_3(x - bx_0), & bx_0 \le x < (1-\omega)x_0, \\ U_\gamma((1-\omega)x_0) + \kappa_2(x-(1-\omega)x_0), & (1-\omega)x_0 \le x < (1+c_2)x_0, \\ U_\gamma(x), & x \ge (1+c_2)x_0. \end{cases}$$

In this case, we obtain that

$$\mathbb{X}(y) \triangleq \arg\sup_{x \ge bx_0} [U_\gamma(x) - yx] = \begin{cases} \left(1 + c_2\left(\frac{y}{\kappa_2}\right)^{\frac{1}{p-1}}\right) x_0, & 0 < y \le \kappa_2, \\ (1-\omega)x_0, & \kappa_2 < y \le \kappa_3, \\ bx_0, & y > \kappa_3. \end{cases}$$

The optimal terminal wealth is given by

$$\begin{aligned} X_T^* &= \mathbb{X}(\nu^*\xi_T) \\ &= \left(b1_{\{\xi_T > \frac{\kappa_3}{\nu^*}\}} + (1-\omega)1_{\{\frac{\kappa_2}{\nu^*} < \xi_T \le \frac{\kappa_3}{\nu^*}\}} + \left(1 + c_2\left(\frac{\nu^*\xi_T}{\kappa_2}\right)^{\frac{1}{p-1}}\right) 1_{\left\{\xi_T \le \frac{\kappa_2}{\nu^*}\right\}} \right) x_0. \end{aligned}$$

Using the martingale representation equations (25) and (27), we have the optimal wealth process X^* given by

$$\begin{aligned} X_t^* &= \xi_t^{-1}\mathbb{E}[\xi_T X_T^* | \mathcal{F}_t] \\ &= e^{-r(T-t)}x_0 \left[b + (1-\omega-b)\Phi(g_{3,t}) + \omega\Phi(g_{2,t}) + c_2 \frac{\Phi'(g_{2,t})}{\Phi'(d_{2,t})}\Phi(d_{2,t}) \right], \end{aligned}$$

and the optimal risky asset allocation

$$\begin{aligned} \frac{\pi_t^*}{X_t^*} = \Bigg(\frac{1}{1-p} + \frac{x_0}{e^{r(T-t)}X_t^*} \Bigg[&\frac{1-\omega-b}{\theta\sqrt{T-t}}\Phi'(g_{3,t}) + \frac{\omega + c_2}{\theta\sqrt{T-t}}\Phi'(g_{2,t}) \\ &- \frac{1}{1-p}(b + (1-\omega-b)\Phi(g_{3,t}) + \omega\Phi(g_{2,t})) \Bigg] \Bigg) \frac{\theta}{\sigma}, \end{aligned}$$

where $g_{2,t}$, $g_{3,t}$, and $d_{2,t}$ are given by Equation (20), and $\Phi(\cdot)$ is the standard normal cumulative distribution function.

(2) (Option case.) If $\kappa_2 \ge \kappa_3$, then the concave envelope of U_γ is given by

$$U_\gamma^{**}(x) = \begin{cases} U_\gamma(bx_0) + \kappa_1(x - bx_0), & bx_0 \le x < (1+c_1)x_0, \\ U_\gamma(x), & x \ge (1+c_1)x_0. \end{cases}$$

In this case, we obtain that

$$\mathbb{X}(y) \triangleq \arg\sup_{x \ge bx_0} [U_\gamma(x) - yx] = \begin{cases} \left(1 + c_1\left(\frac{y}{\kappa_1}\right)^{\frac{1}{p-1}}\right) x_0, & 0 < y \le \kappa_1, \\ bx_0, & y > \kappa_1. \end{cases}$$

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The optimal terminal wealth is given by

$$X_T^* = \mathbb{X}(\nu^*\xi_T)$$
$$= \left(b\mathbb{1}_{\left\{\xi_T > \frac{\kappa_1}{\nu^*}\right\}} + \left(1 + c_1\left(\frac{\nu^*\xi_T}{\kappa_1}\right)^{\frac{1}{p-1}}\right)\mathbb{1}_{\{\xi_T \le \frac{\kappa_1}{\nu^*}\}}\right)x_0.$$

Using the martingale representation equations (25) and (27), we have the optimal wealth process

$$X_t^* = \xi_t^{-1}\mathbb{E}[\xi_T X_T^* | \mathcal{F}_t]$$
$$= e^{-r(T-t)}x_0\left[b + (1-b)\Phi(g_{1,t}) + c_1\frac{\Phi'(g_{1,t})}{\Phi'(d_{1,t})}\Phi(d_{1,t})\right],$$

and the optimal risky asset allocation

$$\frac{\pi_t^*}{X_t^*} = \left(\frac{1}{1-p} + \frac{x_0}{e^{r(T-t)}X_t^*}\left[\frac{c_1+1-b}{\theta\sqrt{T-t}}\Phi'(g_{1,t}) - \frac{1}{1-p}(b+(1-b)\Phi(g_{1,t}))\right]\right)\frac{\theta}{\sigma},$$

where $g_{1,t}$ and $d_{1,t}$ are given by Equation (20), and $\Phi(\cdot)$ is the standard normal cumulative distribution function.

B.4. Proof of Theorem 2

(i) We prove this statement separately in the two cases.
First-loss case. Using (15), we take the derivative of X_t^* with respect to ξ_t and obtain

$$\frac{\partial X_t^*}{\partial \xi_t}$$
$$= x_0 e^{-r(T-t)}\Bigg((1-\omega-b)\Phi'(g_{3,t})\left(-\frac{1}{\xi_t\theta\sqrt{T-t}}\right) + \omega\Phi'(g_{2,t})\left(-\frac{1}{\xi_t\theta\sqrt{T-t}}\right)$$
$$+ c_2\Phi'(d_{2,t})\frac{\Phi'(g_{2,t})}{\Phi'(d_{2,t})}\left(-\frac{1}{\xi_t\theta\sqrt{T-t}}\right)$$
$$+ c_2\Phi(d_{2,t})\frac{\Phi'(g_{2,t})(-g_{2,t})\left(-\frac{1}{\xi_t\theta\sqrt{T-t}}\right)\Phi'(d_{2,t}) - \Phi'(d_{2,t})(-d_{2,t})\left(-\frac{1}{\xi_t\theta\sqrt{T-t}}\right)\Phi'(g_{2,t})}{\left(\Phi'(d_{2,t})\right)^2}\Bigg)$$
$$= \frac{-x_0e^{-r(T-t)}}{\xi_t\theta\sqrt{T-t}}\left((1-\omega-b)\Phi'(g_{3,t}) + (\omega+c_2)\Phi'(g_{2,t}) + c_2\Phi(d_{2,t})\frac{\Phi'(g_{2,t})\theta\sqrt{T-t}}{\Phi'(d_{2,t})(1-p)}\right) < 0.$$

Hence the wealth process X_t^* is decreasing in ξ_t. Using (16), we have

$$\frac{\partial \pi_t^{(1)}}{\partial \xi_t} = \frac{\theta}{(1-p)\sigma}\frac{\partial X_t^*}{\partial \xi_t} < 0,$$

and

$$\frac{\partial \pi_t^{(3)}}{\partial \xi_t} = \frac{x_0e^{-r(T-t)}}{\xi_t(1-p)\sigma\sqrt{T-t}}\left((1-\omega-b)\Phi'(g_{3,t}) + \omega\Phi'(g_{2,t})\right) > 0.$$

Hence $\pi_t^{(1)}$ is decreasing in ξ_t, while $\pi_t^{(3)}$ is increasing in ξ_t. We compute

$$\frac{\partial \pi_t^{(2)}}{\partial \xi_t} = \frac{x_0e^{-r(T-t)}}{\xi_t\sigma\theta(T-t)}\left((1-\omega-b)\Phi'(g_{3,t})g_{3,t} + (\omega+c_2)\Phi'(g_{2,t})g_{2,t}\right),$$

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which implies that the monotonicity of $\pi_t^{(2)}$ depends on the signs of $g_{3,t}$ and $g_{2,t}$. Letting both $g_{3,t} > 0$ and $g_{2,t} > 0$ and using $\kappa_2 < \kappa_3$, we have that $\pi_t^{(2)}$ is increasing if $\xi_t \in \left(0, \frac{\kappa_2}{\nu^*} e^{\left(r-\frac{\theta^2}{2}\right)(T-t)}\right)$. Letting both $g_{3,t} < 0$ and $g_{2,t} < 0$, we have that $\pi_t^{(2)}$ is decreasing if $\xi_t \in \left(\frac{\kappa_3}{\nu^*} e^{\left(r-\frac{\theta^2}{2}\right)(T-t)}, +\infty\right)$.

Option case. Using (18), we take the derivative of X_t^* with respect to ξ_t and obtain

$$\begin{aligned}
&\frac{\partial X_t^*}{\partial \xi_t} \\
&= x_0 e^{-r(T-t)} \left((1-b)\Phi'(g_{1,t}) \left(-\frac{1}{\xi_t \theta \sqrt{T-t}}\right) + c_1 \Phi'(d_{1,t}) \frac{\Phi'(g_{1,t})}{\Phi'(d_{1,t})} \left(-\frac{1}{\xi_t \theta \sqrt{T-t}}\right) \right. \\
&\quad \left. + c_1 \Phi(d_{1,t}) \frac{\Phi'(g_{1,t})(-g_{1,t}) \left(-\frac{1}{\xi_t\theta\sqrt{T-t}}\right) \Phi'(d_{1,t}) - \Phi'(d_{1,t})(-d_{1,t}) \left(-\frac{1}{\xi_t\theta\sqrt{T-t}}\right) \Phi'(g_{1,t})}{\left(\Phi'(d_{1,t})\right)^2} \right) \\
&= \frac{-x_0 e^{-r(T-t)}}{\xi_t \theta \sqrt{T-t}} \left((1+c_1-b)\Phi'(g_{1,t}) + c_1 \Phi(d_{1,t}) \frac{\Phi'(g_{1,t})}{\Phi'(d_{1,t})} \frac{\theta\sqrt{T-t}}{1-p} \right) < 0.
\end{aligned}$$

Hence the wealth process X_t^* is decreasing in ξ_t. Using (19), we have

$$\frac{\partial \pi_t^{(1)}}{\partial \xi_t} = \frac{\theta}{(1-p)\sigma} \frac{\partial X_t^*}{\partial \xi_t} < 0,$$

and

$$\frac{\partial \pi_t^{(3)}}{\partial \xi_t} = \frac{x_0 e^{-r(T-t)}}{\xi_t (1-p)\sigma\sqrt{T-t}} (1-b)\Phi'(g_{1,t}) > 0.$$

Hence $\pi_t^{(1)}$ is decreasing in ξ_t, while $\pi_t^{(2)}$ and $\pi_t^{(3)}$ are increasing in ξ_t. We compute

$$\frac{\partial \pi_t^{(2)}}{\partial \xi_t} = \frac{x_0 e^{-r(T-t)}}{\xi_t \sigma \theta (T-t)} (1+c_1-b)\Phi'(g_{1,t}) g_{1,t},$$

which implies that the monotonicity of $\pi_t^{(2)}$ depends on the sign of $g_{1,t}$. Letting $g_{1,t} > 0$, we have that $\pi_t^{(2)}$ is increasing if $\xi_t \in \left(0, \frac{\kappa_1}{\nu^*} e^{\left(r-\frac{\theta^2}{2}\right)(T-t)}\right)$ and decreasing otherwise.

(ii) The conclusion holds in both the option case and the first-loss case. For the option case, using Equation (20), as $\xi_t \to 0$, we have $g_{1,t} \to +\infty$ and $d_{1,t} \to +\infty$. Based on the expression (18) for X_t^*,

$$X_t^* \to +\infty, \qquad \frac{\Phi'(g_{1,t})}{\Phi'(d_{1,t})} \to +\infty.$$

Thus, we obtain that if $\xi_t \to 0$, then

$$\pi_t^{(2)} \to 0, \qquad \frac{\pi_t^{(3)}}{-\frac{\theta}{(1-p)\sigma} x_0 e^{-r(T-t)}} \to 1, \qquad \frac{\pi_t^*}{X_t^*} \to \frac{\theta}{(1-p)\sigma}.$$

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(iii) The conclusion holds in both the option case and the first-loss case. For the option case, similarly, we can derive that if $\xi_t \to +\infty$, then

$$\frac{X_t^*}{bx_0e^{-r(T-t)}} \to 1, \qquad g_{1,t} \to -\infty.$$

Based on the properties of the standard normal probability density function $\Phi'(\cdot)$, we know that $\Phi'(g_{1,t}) \to 0$, and thus obtain that if $\xi_t \to +\infty$, then

$$\pi_t^{(2)} \to 0, \qquad \frac{\pi_t^{(3)}}{-\frac{b}{1-p}\frac{\theta}{\sigma}x_0e^{-r(T-t)}} \to 1, \qquad \pi_t^* \to 0.$$

(iv) We can verify that the result in (iii) holds when $t \to T$, based on the following facts:

- if $\nu^*\xi_T < \kappa_1$, then

$$\lim_{t\to T} g_{1,t} = +\infty, \qquad \lim_{t\to T} \frac{\Phi'(g_{1,t})}{\theta\sqrt{T-t}} = 0;$$

- if $\nu^*\xi_T = \kappa_1$, then

$$\lim_{t\to T} g_{1,t} = 0, \qquad \lim_{t\to T} \frac{\Phi'(g_{1,t})}{\theta\sqrt{T-t}} = +\infty;$$

- if $\nu^*\xi_T > \kappa_1$, then

$$\lim_{t\to T} g_{1,t} = -\infty, \qquad \lim_{t\to T} \frac{\Phi'(g_{1,t})}{\theta\sqrt{T-t}} = 0.$$

(v) Because the proof is similar to that of (iii), we omit it here.

B.5. Proof of Proposition 2

We use the standard procedure in Appendix A to solve Problem (21) and Problem (22). Similarly to Theorem 1, we obtain respectively a (distribution-wise) unique optimum $\hat{X}_1$ to Problem (21) and a (distribution-wise) unique optimum $\hat{X}_0$ to Problem (22).

B.6. Proof of Theorem 3

1. It suffices to prove the following claim: for any $0 \le \gamma_1 < \gamma_2 \le 1$, the corresponding optima $\hat{X}_{\gamma_1}$ and $\hat{X}_{\gamma_2}$ satisfy either

$$\begin{cases} \mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big] = \mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big], \\ \mathbb{E}\big[U_0\big(\hat{X}_{\gamma_1}\big)\big] = \mathbb{E}\big[U_0\big(\hat{X}_{\gamma_2}\big)\big], \end{cases} \tag{28}$$

or

$$\begin{cases} \mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big] < \mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big], \\ \mathbb{E}\big[U_0\big(\hat{X}_{\gamma_1}\big)\big] > \mathbb{E}\big[U_0\big(\hat{X}_{\gamma_2}\big)\big]. \end{cases} \tag{29}$$

The claim shows that for any $0 \le \gamma_1 < \gamma_2 \le 1$, if $\hat{X}_{\gamma_1}$ and $\hat{X}_{\gamma_2}$ do not represent the same point on the PF, we have $\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big] < \mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big]$ and $\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_1}\big)\big] > \mathbb{E}\big[U_0\big(\hat{X}_{\gamma_2}\big)\big]$. Thus, the strictly decreasing result follows directly.

Now we prove the claim. Because $\hat{X}_{\gamma_1}$ and $\hat{X}_{\gamma_2}$ are correspondingly optimal to the problems with γ_1 and γ_2, we have

$$(1-\gamma_2)\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_2}\big)\big]+\gamma_2\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big]\geq(1-\gamma_2)\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_1}\big)\big]+\gamma_2\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big],$$

$$(1-\gamma_1)\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_1}\big)\big]+\gamma_1\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big]\geq(1-\gamma_1)\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_2}\big)\big]+\gamma_1\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big].$$

Thus,

$$\begin{aligned}&\gamma_2\Big(\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big]-\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big]\Big)\geq(1-\gamma_2)\Big(\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_1}\big)\big]-\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_2}\big)\big]\Big),\\&(1-\gamma_1)\Big(\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_1}\big)\big]-\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_2}\big)\big]\Big)\geq\gamma_1\Big(\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big]-\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big]\Big).\end{aligned}\tag{30}$$

Case 1. If $\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big]-\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big]\leq 0$ holds, we have

$$\begin{aligned}0\geq\gamma_2\Big(\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big]-\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big]\Big)&\geq(1-\gamma_2)\Big(\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_1}\big)\big]-\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_2}\big)\big]\Big)\\&\geq(1-\gamma_1)\Big(\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_1}\big)\big]-\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_2}\big)\big]\Big)\\&\geq\gamma_1\Big(\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big]-\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big]\Big).\end{aligned}$$

Thus,

$$0\geq\gamma_2\Big(\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big]-\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big]\Big)\geq\gamma_1\Big(\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big]-\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big]\Big),$$

which implies that $\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big]-\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big]=0$. By (30), we have

$$\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_1}\big)\big]-\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_2}\big)\big]\geq 0.$$

If $\gamma_2<1$, we have $\big(\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big],\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_1}\big)\big]\big)=\big(\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big],\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_2}\big)\big]\big)$.
If $\gamma_2=1$, we have $\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big]=\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big]$ and $\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_1}\big)\big]\geq\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_2}\big)\big]$, which implies that $\hat{X}_{\gamma_1}$ and $\hat{X}_{\gamma_2}$ both solve Problem (21). Using Proposition 2, we must have $\hat{X}_{\gamma_1}\overset{d}{=}\hat{X}_{\gamma_2}$, and thus $\big(\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big],\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_1}\big)\big]\big)=\big(\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big],\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_2}\big)\big]\big)$.
Thus, the equations (28) hold.

Case 2. If $\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big]-\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big]>0$ holds, we have

$$0\leq\gamma_1\Big(\mathbb{E}\Big[U_1\Big(\hat{X}_{\gamma_2}\Big)\Big]-\mathbb{E}\Big[U_1\Big(\hat{X}_{\gamma_1}\Big)\Big]\Big)\leq(1-\gamma_1)\Big(\mathbb{E}\Big[U_0\Big(\hat{X}_{\gamma_1}\Big)\Big]-\mathbb{E}\Big[U_0\Big(\hat{X}_{\gamma_2}\Big)\Big]\Big).$$

If $\gamma_1>0$, we have $\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big]>\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big]$ and $\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_1}\big)\big]>\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_2}\big)\big]$.
If $\gamma_1=0$, we have $\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big]>\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big]$ and $\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_1}\big)\big]\geq\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_2}\big)\big]$. Suppose $\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_1}\big)\big]=\mathbb{E}\big[U_0\big(\hat{X}_{\gamma_2}\big)\big]$. This implies that $\hat{X}_{\gamma_1}$ and $\hat{X}_{\gamma_2}$ both solve Problem (22). Based on Proposition 2, we must have $\hat{X}_{\gamma_1}\overset{d}{=}\hat{X}_{\gamma_2}$, which contradicts $\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_2}\big)\big]>\mathbb{E}\big[U_1\big(\hat{X}_{\gamma_1}\big)\big]$. Thus, the equations (29) hold. This completes the proof of the claim.

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2. For any Pareto point $(u_1, u_0) \in \text{PF}$, there exists some $\gamma \in [0, 1]$ (which may not be unique) such that $\hat{X}_\gamma$ solves Problem (6) with γ and $\left(\mathbb{E}\left[U_1\left(\hat{X}_\gamma\right)\right], \mathbb{E}\left[U_0\left(\hat{X}_\gamma\right)\right]\right) = (u_1, u_0)$. This implies that for any admissible terminal wealth X_T,

$$(1-\gamma)\mathbb{E}[U_0(X)] + \gamma\mathbb{E}[U_1(X)] \le (1-\gamma)\mathbb{E}\left[U_0\left(\hat{X}_\gamma\right)\right] + \gamma\mathbb{E}\left[U_1\left(\hat{X}_\gamma\right)\right] = (1-\gamma)u_0 + \gamma u_1.$$

Define the affine function

$$L_\gamma(x) \triangleq -\frac{\gamma}{1-\gamma}x + \frac{(1-\gamma)u_0 + \gamma u_1}{1-\gamma}, \qquad x \in \mathbb{R}.$$

Then the PF is dominated by the affine function L_γ, and the point (u_1, u_0) lies on L_γ. Thus, if the PF is continuous, then the PF is strictly concave and $-\frac{\gamma}{1-\gamma}$ is one sub-differential of the Pareto point (u_1, u_0). (When $\gamma = 1$, the sub-differential is $-\infty$.)

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