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Normalization of Closed Ekedahl–Oort Strata

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Abstract. We apply our theory of partial flag spaces developed with W. Goldring to study a group-theoretical generalization of the canonical filtration of a truncated Barsotti–Tate group of level 1. As an application, we determine explicitly the normalization of the Zariski closures of Ekedahl–Oort strata of Shimura varieties of Hodge-type as certain closed coarse strata in the associated partial flag spaces.

Introduction

Let H be a truncated Barsotti–Tate group of level 1 (BT1), over an algebraically closed field k of characteristic p, and let $\sigma\colon k \to k$ denote the map $x \mapsto x^p$. Denote by $D := \mathbf{D}(H)$ its Dieudonné module, which is a finite-dimensional k-vector space D endowed with a σ-linear endomorphism F and a σ^{-1}-linear endomorphism V satisfying $FV = VF = 0$ and $\mathrm{Im}(F) = \mathrm{Ker}(V)$. Oort [Oor01] showed that there exists a flag of D that is stable by V and F^{-1} and is coarsest among all such flags, called the *canonical filtration* of D. After choosing a basis of D, we obtain a filtration of k^n (where $n = \dim_k(D)$ is the height of H). The stabilizer of this flag is a parabolic subgroup $P(H) \subset GL_n$, well-defined up to conjugation. We want to emphasize that this construction attaches a group-theoretical object $P(H)$ to a truncated Barsotti–Tate group of level 1.

The theory of F-zips developed in [MW04, PWZ11, PWZ15] establishes the precise link between BT1's and group theory. Specifically, isomorphism classes of BT1's of height n and dimension d correspond bijectively to E-orbits in GL_n, where E is the zip group (see Section 4.1). The stack of F-zips of type (n,d) can be defined as the quotient stack $F\text{-}\mathsf{Zip}^{n,d} = [E\backslash GL_n]$. Moreover, there is a natural morphism of stacks $BT_1^{n,d} \to F\text{-}\mathsf{Zip}^{n,d}$, where $BT_1^{n,d}$ is the stack of BT1's of height n and dimension d over k. More generally, let G be a connected reductive group over $\mathbf{F}_p$, and $P, Q \subset G$ parabolic subgroups (defined over some finite extension of $\mathbf{F}_p$). Let $L \subset P$ and $M \subset Q$ be Levi subgroups and assume that $\varphi(L) = M$, where $\varphi\colon G \to G$ is the Frobenius homomorphism. One can define the stack of G-zips of type $\mathcal{Z} = (G, P, L, Q, M, \varphi)$ as the quotient stack $G\text{-}\mathsf{Zip}^{\mathcal{Z}} = [E\backslash G]$, where $E \subset P \times Q$ is the zip group (see Section 1.1). For example, if G is the automorphism group of a PEL-datum, then $G\text{-}\mathsf{Zip}^{\mathcal{Z}}(k)$ classifies BT1's over k of type $\mathcal{Z}$ endowed with this additional structure. If W denotes the Weyl group of G, the E-orbits in G are parametrized by a subset ${}^IW \subset W$ (see Section 1.3).

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Denote by $G_w \subset G$ the E-orbit corresponding to $w \in {}^I W$ and put $Z_w := [E \backslash G_w]$ the corresponding zip stratum.

In [GKb], we defined for each parabolic $P_0 \subset P$ the stack of partial zip flags $G\text{-}\mathsf{ZipFlag}^{(\mathcal{Z},P_0)}$ endowed with a natural projection $\pi\colon G\text{-}\mathsf{ZipFlag}^{(\mathcal{Z},P_0)} \to G\text{-}\mathsf{Zip}^{\mathcal{Z}}$ that makes it a P/P_0-bundle over $G\text{-}\mathsf{Zip}^{\mathcal{Z}}$. This defines a tower of stacks above $G\text{-}\mathsf{Zip}^{\mathcal{Z}}$. Moreover, the stack $G\text{-}\mathsf{ZipFlag}^{(\mathcal{Z},P_0)}$ admits two natural stratifications. In general, one is finer than the other, but they coincide when P_0 is a Borel subgroup. The fine strata $Z_{P_0,w}$ are parametrized by $w \in {}^{I_0}W$, where ${}^{I_0}W \subset W$ is a subset containing ${}^I W$ (see Section 2.1). The strata $Z_{P_0,w}$ attached to elements $w \in {}^I W$ are called minimal and satisfy $\pi(Z_{P_0,w}) = Z_w$, and the restriction $\pi\colon Z_{P_0,w} \to Z_w$ is finite étale. When $P_0 = P$, the stack $G\text{-}\mathsf{ZipFlag}^{(\mathcal{Z},P)}$ coincides with $G\text{-}\mathsf{Zip}^{\mathcal{Z}}$ and the fine stratification is the stratification by E-orbits, whereas the coarse stratification is given by $P \times Q$-orbits. In general, we say that a stratum $Z_{P_0,w}$ has coarse closure if it is open in the coarse stratum containing it. If $Z_{P_0,w}$ has coarse closure, its Zariski closure $\overline{Z}_{P_0,w}$ is normal.

In the formalism of G-zips, one can attach to each $w \in {}^I W$ a parabolic subgroup $P_w \subset P$. In the case $G = GL_n$, if H is a BT1 corresponding to $w \in {}^I W$ under the correspondence between BT1's and E-orbits, then P_w is the parabolic $P(H)$ defined above. This is proved in Proposition 4.3.1. Since P_w is canonically attached to w, it is natural to ask what special property is satisfied by the stratum $Z_{P_w,w}$ of $G\text{-}\mathsf{ZipFlag}^{(\mathcal{Z},P_w)}$. Our main theorem answers this question.

Theorem 1 (Th. 3.1.3) *Let $w \in {}^I W$. The following properties hold:*

(i) *$\pi\colon Z_{P_w,w} \to Z_w$ is an isomorphism.*
(ii) *$Z_{P_w,w}$ has coarse closure.*

Furthermore, among all parabolic subgroups P_0 such that ${}^z B \subset P_0 \subset P$, the parabolic P_w is the smallest parabolic satisfying (i) *and the largest one satisfying* (ii).

The Borel subgroup ${}^z B$ of the above theorem is defined in Section 1.2. Note that property (i) is obviously satisfied for $P_0 = P$ and property (ii) is satisfied for $P_0 = {}^z B$ because fine and coarse strata coincide in this case. Hence, the canonical parabolic P_w is the unique intermediate parabolic such that both properties are satisfied. As a consequence, we deduce that the normalization of the Zariski closure $\overline{Z}_w$ is $\widetilde{Z}_w := \operatorname{Spec}(\mathcal{O}(\overline{Z}_{P_w,w}))$ (see Corollary 3.3.2).

Let X be the special fiber of a good reduction Hodge-type Shimura variety, and let G be the attached reductive $\mathbf{F}_p$-group (see Section 3.4). In [Zha], Zhang has constructed a smooth map of stacks $\zeta\colon X \to G\text{-}\mathsf{Zip}^{\mathcal{Z}}$, where $\mathcal{Z}$ is the zip datum attached to X as in [GKb, §6.2]. The Ekedahl–Oort stratification of X is defined as the fibers of ζ. For $w \in {}^I W$, set $X_w := \zeta^{-1}(Z_w)$. For any parabolic ${}^z B \subset P_0 \subset P$, define the partial flag space X_{P_0} as the fiber product

$$\begin{tikzcd} X_{P_0} \arrow[r, "\zeta_{P_0}"] \arrow[d, "\pi"'] & G\text{-}\mathsf{ZipFlag}^{(\mathcal{Z},P_0)} \arrow[d, "\pi_{P_0}"] \\ X \arrow[r, "\zeta"'] & G\text{-}\mathsf{Zip}^{\mathcal{Z}}. \end{tikzcd}$$

For $w \in {}^{I_0}W$, define the fine stratum $X_{P_0,w} := \zeta_{P_0}^{-1}(Z_{P_0,w})$ of X_{P_0}. The space X_{P_0} is a generalization of the flag space considered by Ekedahl and Van der Geer [EvdG09], where they consider flags refining the Hodge filtration of an abelian variety.

Corollary 1 (Cor. 3.4.1) *Let $w \in {}^{I}W$. The normalization of $\overline{X}_w$ is the Stein factorization of the map $\pi: \overline{X}_{P_w,w} \to \overline{X}_w$. It is isomorphic to $\overline{X}_w \times_{G\text{-}\mathsf{Zip}^{\mathcal{Z}}} \widetilde{Z}_w$.*

For Siegel-type Shimura varieties, an analogous result to Corollary 1 was proved by Boxer in [Box15, Thm. 5.3.1] using different methods.

We now give an overview of the paper. In Section 1, we review the theory of G-zips and prove a result on point stabilizers for later use. Section 2 is devoted to the stack of partial G-zips and its stratifications. We define minimal strata and give an explicit form for the restriction of the map π to a minimal stratum (Proposition 2.2.1). In Section 3, we define the notion of canonical parabolic and explain its relevance with respect to the normalization of a closed stratum of $G\text{-}\mathsf{Zip}^{\mathcal{Z}}$. We prove Theorem 3.1.3 after giving criteria for properties (i) and (ii) above. Finally, we explain in Section 4 the correspondence between the classical theory of BT1's and the theory of G-zips, following [PWZ11]. We establish the link between the parabolic P_w and the canonical parabolic of a BT1.

1 Review of G-zips

We will need to review some facts about the stack of G-zips found in [PWZ11] and prove a result on the stabilizer of an element by the group E.

1.1 The Stack $G\text{-}\mathsf{Zip}^{\mathcal{Z}}$

We fix an algebraic closure k of $\mathbf{F}_p$. A zip datum is a tuple $\mathcal{Z} = (G, P, L, Q, M, \varphi)$, where G is a connected reductive group over $\mathbf{F}_p$, $\varphi: G \to G$ is the Frobenius homomorphism, $P, Q \subset G$ are parabolic subgroups of G_k, $L \subset P$ and $M \subset Q$ are Levi subgroups of P and Q, respectively. One imposes the condition $\varphi(L) = M$. One can attach to $\mathcal{Z}$ a zip group E defined by

$$E := \left\{ (p,q) \in P \times Q, \varphi(\overline{p}) = \overline{q} \right\}$$

where $\overline{p} \in L$ and $\overline{q} \in M$ denote the projections of p and q via the isomorphisms $P/R_u(P) \simeq L$ and $Q/R_u(Q) \simeq M$. We let $G \times G$ act on G via $(a,b) \cdot g := agb^{-1}$, and we obtain by restriction an action of E on G. The stack of G-zips is then isomorphic to the quotient stack $G\text{-}\mathsf{Zip}^{\mathcal{Z}} \simeq [E \backslash G]$. When we want to specify the zip datum $\mathcal{Z}$, we sometimes write $E_{\mathcal{Z}}$ for the zip group E.

1.2 Frame

A frame for $\mathcal{Z}$ is a triple (B, T, z), where (B, T) is a Borel pair and $z \in G(k)$ satisfying the following conditions:

(a) $B \subset Q$,
(b) ${}^{z}T \subset L$,

(c) ${}^{z}B \subset P$,
(d) $\varphi({}^{z}B \cap L) = B \cap M$,
(e) $\varphi({}^{z}T) = T$.

We fix throughout a frame (B, T, z), and we define the following:

(1) $\Phi \subset X^*(T)$: the set of T-roots of G.
(2) Φ_+ : the set of positive roots with respect to B.
(3) $\Delta \subset \Phi_+$: the set of positive simple roots.
(4) For $\alpha \in \Phi$, let $s_\alpha \in W$ be the corresponding reflection. Then $(W, \{s_\alpha\}_{\alpha\in\Delta})$ is a Coxeter group, and we denote the length function by $\ell \colon W \to \mathbf{N}$.
(5) For $K \subset \Delta$, Let $W_K \subset W$ be the subgroup generated by the s_α for $\alpha \in K$. Let $w_0 \in W$ be the longest element and $w_{0,K}$ the longest element in W_K.
(6) If $R \subset G$ is a parabolic subgroup containing B and D is the unique Levi subgroup of R containing T, then the type of R (or of D) is the unique subset $K \subset \Delta$ such that $W(D, T) = W_K$. The type of an arbitrary parabolic R is the type of its unique conjugate containing B. Let $I \subset \Delta$ (resp. $J \subset \Delta$) be the type of P (resp. Q).
(7) For $K \subset \Delta$, ${}^{K}W$ (resp. W^K) : the subset of elements $w \in W$ that are minimal in the coset $W_K w$ (resp. wW_K).
(8) For $K, R \subset \Delta$, ${}^{K}W^{R} := {}^{K}W \cap W^{R}$.
(9) For an element $x \in {}^{I}W^{J}$, define $I_x := J \cap {}^{x^{-1}}I$. By [PWZ11, Proposition 2.7], any element $w \in W_I x W_J$ can be uniquely written as

$$w = xw_J, \quad \text{with} \quad w_J \in {}^{I_x}W_J. \tag{1.2.1}$$

For $w \in W$, one has an equivalence:

$$w \in {}^{I}W \iff {}^{z}B \cap L = {}^{zw}B \cap L. \tag{1.2.2}$$

1.3 Stratification

For $w \in W$, choose a representative $\dot{w} \in N_G(T)$ such that $(w_1w_2)^{\cdot} = \dot{w}_1\dot{w}_2$ whenever $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ (this is possible by choosing a Chevalley system; see [ABD+66, Exp. XXIII, §6]). For $h \in G(k)$, denote by $\mathcal{O}_{\mathcal{Z}}(h)$ the E-orbit of h in G and define $\mathfrak{o}_{\mathcal{Z}}(h) := [E\backslash\mathcal{O}_{\mathcal{Z}}(h)]$. By [PWZ11, Theorem 7.5], there is a bijection

$${}^{I}W \to \{E\text{-orbits in } G\}, \quad w \mapsto G_w := \mathcal{O}(z\dot{w}). \tag{1.3.1}$$

Furthermore, for all $w \in {}^{I}W$, one has

$$\dim(G_w) = \ell(w) + \dim(P). \tag{1.3.2}$$

For $w \in {}^{I}W$, we endow the locally closed subset G_w with the reduced structure, and we define the corresponding zip stratum of $G\text{-Zip}^{\mathcal{Z}}$ by $Z_w := [E\backslash G_w]$.

1.4 Point Stabilizers

Definition 1.4.1 ([PWZ11]) Let $w \in {}^{I}W$. There is a largest subgroup M_w of ${}^{w^{-1}z^{-1}}L$ satisfying $\varphi({}^{zw}M_w) = M_w$.

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In [PWZ11, §5.1], this subgroup is denoted by H_w. It is a Levi subgroup of G contained in M. We also define

$$L_w := {}^{zw}M_w \subset L \tag{1.4.1}$$

$$P_w := L_w{}^zB \subset P$$

$$Q_w := M_wB \subset Q \tag{1.4.2}$$

Since $\varphi(L_w) = M_w$, we obtain a zip datum $\mathcal{Z}_w := (G, P_w, L_w, Q_w, M_w, \varphi)$. Note that (B, T, z) is again a frame for $\mathcal{Z}_w$. If an algebraic group G acts on a k-scheme X and $x \in X(k)$, we denote by $\mathrm{Stab}_G(x)$ the scheme-theoretical stabilizer of x. For an algebraic group H, we denote by H_{red} the underlying reduced algebraic group and by H° the identity component of H.

Lemma 1.4.2

(i) *One has* $\mathrm{Stab}_E(z\dot{w})_{\mathrm{red}} = A \ltimes R$, *where* $A \subset L_w \times M_w$ *is the finite group*

$$A := \{(x, \varphi(x)),\ x \in L_w,\ {}^{z\dot{w}}\varphi(x) = x\} \tag{1.4.3}$$

and R *is a unipotent smooth connected normal subgroup.*

(ii) *One has* $\mathrm{Stab}_E(z\dot{w})^\circ \subset {}^zB \times B$.

Proof The first part is Theorem 8.1 in [PWZ11]. To prove (ii), it suffices to show that $\mathrm{Stab}_E(z\dot{w})^\circ \subset {}^zB \times G$, or, equivalently, $\mathrm{Lie}(\mathrm{Stab}_E(z\dot{w})) \subset \mathrm{Lie}({}^zB) \times \mathrm{Lie}(G)$. We follow the proof of Theorem 8.5 of [PWZ11]. An arbitrary tangent vector of E at 1 has the form $(1 + dp, 1 + dv)$ for $dp \in \mathrm{Lie}(P)$ and $dv \in \mathrm{Lie}(V)$. This element stabilizes $z\dot{w}$ if and only if $dp = \mathrm{Ad}_{z\dot{w}}(dv)$. Hence,

$$dp \in \mathrm{Lie}(P) \cap \mathrm{Ad}_{z\dot{w}}(\mathrm{Lie}(V)) = \mathrm{Lie}(P \cap {}^{zw}V).$$

Hence, it suffices to show $P \cap {}^{zw}V \subset {}^zB$. This amounts to $L \cap {}^{zw}V \subset L \cap {}^zB$ and equivalently $M \cap \varphi({}^{zw}V) \subset M \cap \varphi({}^zB) = M \cap B$. This is proved in Proposition 4.12 of [PWZ11]. More precisely, the authors define in construction 4.3 a group V_x (note that the element z is denoted by g there), where $w = xw_J$ is a decomposition as in (1.2.1). One has $V_x = M \cap \varphi({}^{z\dot{x}}V) = M \cap \varphi({}^{zw}V)$ because $w_J \in W_J$, so ${}^{w_J}V = V$. Proposition 4.12 of [PWZ11] shows that $(M \cap B, T, 1)$ is a frame for $\mathcal{Z}_{\dot{x}}$, so, in particular, one has $V_x \subset M \cap B$. This terminates the proof of the lemma. ■

2 The Stack of Partial Zip Flags

We recall in this section some of the results of [GKb].

2.1 Fine and Coarse Flag Strata

For each parabolic subgroup P_0 satisfying ${}^zB \subset P_0 \subset P$, in [GKb, §2] we defined a stack $G\text{-}\mathsf{ZipFlag}^{(\mathcal{Z}, P_0)}$ that parametrizes G-zips of type $\mathcal{Z}$ endowed with a compatible P_0-torsor. There is an isomorphism

$$G\text{-}\mathsf{ZipFlag}^{(\mathcal{Z}, P_0)} \simeq \left[E \backslash (G \times P/P_0) \right],$$

where E acts on $G \times P/P_0$ by $(a,b)\cdot(g, xP_0) := (agb^{-1}, axP_0)$. Furthermore, there is a natural projection map $\pi\colon G\text{-}\mathsf{ZipFlag}^{(\mathcal{Z},P_0)} \to G\text{-}\mathsf{Zip}^{\mathcal{Z}}$ that is a P/P_0-bundle.

Denote by $L_0 \subset P_0$ the Levi subgroup containing ${}^zT$ (note that $L_0 \subset L$). We define a second zip datum $\mathcal{Z}_0 = (G, P_0, L_0, Q_0, M_0, \varphi)$ by setting:

$$M_0 := \varphi(L_0) \subset M \quad \text{and} \quad Q_0 := M_0 B \subset Q.$$

Note that (B,T,z) is again a frame of $\mathcal{Z}_0$. By [GKb, §3.1], there is a natural morphism of stacks

$$\Psi_{P_0}\colon G\text{-}\mathsf{ZipFlag}^{(\mathcal{Z},P_0)} \longrightarrow G\text{-}\mathsf{Zip}^{\mathcal{Z}_0}$$

which is an $\mathbb{A}^r$-bundle for $r = \dim(P/P_0)$ ([GKb, Proposition 3.1.1]). It is induced by the map $G \times P \to G$, $(g,a) \mapsto a^{-1}g\varphi(\overline{a})$. Let I_0 and J_0 denote respectively the types of P_0 and Q_0. For $w \in {}^{I_0}W$, we define the fine flag stratum $Z_{P_0,w}$ of $G\text{-}\mathsf{ZipFlag}^{(\mathcal{Z},P_0)}$ as the locally closed substack

$$Z_{P_0,w} := \Psi_{P_0}^{-1}\big(\mathfrak{o}_{\mathcal{Z}_0}(z\dot{w})\big)$$

endowed with the reduced structure. Explicitly, one has $Z_{P_0,w} = [E\backslash G_{P_0,w}$ where $G_{P_0,w}$ is the algebraic subvariety of $G \times P/P_0$ defined by

$$G_{P_0,w} := \big\{(g, aP_0) \in G \times P/P_0,\ a^{-1}g\varphi(\overline{a}) \in \mathcal{O}_{\mathcal{Z}_0}(z\dot{w})\big\}.$$

Denote by $\mathrm{Brh}^{\mathcal{Z}_0}$ the quotient stack $[P_0\backslash G/Q_0]$, called the Bruhat stack. Since $E_{\mathcal{Z}_0} \subset P_0 \times Q_0$, there is a natural projection morphism $\beta\colon G\text{-}\mathsf{Zip}^{\mathcal{Z}_0} \to \mathrm{Brh}^{\mathcal{Z}_0}$. The composition $\Psi_{P_0} \circ \beta$ gives a smooth map of stacks

$$\psi_{P_0}\colon G\text{-}\mathsf{ZipFlag}^{(\mathcal{Z},P_0)} \longrightarrow \mathrm{Brh}^{\mathcal{Z}_0}.$$

By [Wed14, Lem. 1.4], the set $\{z\dot{x}, x \in {}^{I_0}W^{J_0}\}$ is a set of representatives of the $P_0 \times Q_0$-orbits in G (pay attention to the fact that ${}^zB \subset P$). For $x \in {}^{I_0}W^{J_0}$, write $\mathfrak{b}(x) := [P_0\backslash(P_0 z\dot{x} Q_0)/Q_0]$ (locally closed substack of $\mathrm{Brh}^{\mathcal{Z}_0}$) and define the coarse flag stratum $\mathbf{Z}_{P_0,x}$ as $\mathbf{Z}_{P_0,x} := \psi_{P_0}^{-1}(\mathfrak{b}(x))$ endowed with the reduced structure. Explicitly, one has $\mathbf{Z}_{P_0,x} = [E\backslash \mathbf{G}_{P_0,x}]$ where $\mathbf{G}_{P_0,x}$ is the subvariety of $G \times P/P_0$ defined by

$$\mathbf{G}_{P_0,x} := \big\{(g, aP_0) \in G \times P/P_0,\ ag\varphi(\overline{a})^{-1} \in P_0 z\dot{x} Q_0\big\}.$$

All fine and coarse flag strata are smooth. A coarse stratum is a union of fine strata and the Zariski closure of a coarse flag stratum is normal. In each coarse stratum there is a unique open fine stratum.

Definition 2.1.1 We say that a fine flag stratum Z has coarse closure if it is open in the coarse stratum that contains it, equivalently, if its Zariski closure coincides with the Zariski closure of a coarse flag stratum.

In particular, the Zariski closure $\overline{Z}$ of such a stratum is normal ([GKb, Prop. 2.2.1(1)]).

2.2 Minimal Strata

Recall that we defined in [GKb] a minimal flag stratum as a flag stratum $Z_{P_0,w}$ parametrized by an element $w \in {}^IW$. For a minimal stratum one has $\pi(Z_{P_0,w}) = Z_w$ and the induced morphism $\pi\colon Z_{P_0,w} \to Z_w$ is finite ([GKb, Proposition 3.2.2]). The following proposition shows that it is also étale. For $w \in {}^IW$, denote by $\widetilde{\pi}\colon G_{P_0,w} \to G_w$ the first projection; it is an E-equivariant map.

Proposition 2.2.1 *Let $w \in {}^IW$ and denote by $S := \mathrm{Stab}_E(z\dot{w})$ the stabilizer of $z\dot{w}$ in E and define $S_{P_0} := S \cap (P_0 \times G)$.*

(i) *There is a commutative diagram*

$$\begin{tikzcd} Z_{P_0,w} \arrow[r, "\simeq"] \arrow[d, "\pi"'] & {[1/S_{P_0}]} \arrow[d] \\ Z_w \arrow[r, "\simeq"] & {[1/S]} \end{tikzcd}$$

where the horizontal maps are isomorphisms and the right-hand side vertical map is the natural projection.

(ii) *The map $\pi\colon Z_{P_0,w} \to Z_w$ is finite étale.*

(iii) *The map $\pi\colon Z_{P_0,w} \to Z_w$ is an isomorphism if and only if the inclusion $S \subset P_0 \times G$ holds.*

Proof We first prove (i). There is a natural identification $G_w \simeq [E/S]$, because G_w is the E-orbit of $z\dot{w}$. It follows that $Z_w \simeq [E\backslash E/S] \simeq [1/S]$. Similarly, we claim that the variety $G_{P_0,w}$ consists of a single E-orbit. This was proved in [GKa] Proposition 5.4.5 in the case when P_0 is a Borel subgroup. For a general P_0, we can reduce to the Borel case as follows: By [GKb, Proposition 3.2.2], we have a natural E-equivariant surjective projection map $G_{{}^zB,w} \to G_{P_0,w}$, hence $G_{P_0,w}$ consists of a single E-orbit. We thus can identify $Z_{P_0,w} \simeq [E\backslash E/S']$ where $S' = \mathrm{Stab}_E(z\dot{w}, 1)$. It is clear that $S' = S_{P_0}$, so the result follows.

We now show (ii). By [GKb, Proposition 3.2.2], we know that $\pi\colon Z_{P_0,w} \to Z_w$ is finite. By (i), it is equivalent to show that S/S_{P_0} is an étale scheme. By Lemma 1.4.2 (ii), we have $S^\circ \subset S \cap ({}^zB \times G) \subset S_{P_0}$. Hence, the quotient map $S \to S/S_{P_0}$ factors through a surjective map $\pi_0(S) \to S/S_{P_0}$, which shows that S/S_{P_0} is étale.

Finally, the last assertion follows immediately from (i). ∎

3 The Canonical Parabolic

We fix an element $w \in {}^IW$. Recall that we defined in (1.4.2) a parabolic subgroup $P_w \subset P$.

3.1 Definition

Let P_0 be a parabolic subgroup of G such that ${}^zB \subset P_0 \subset P$.

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Definition 3.1.1 We say that P_0 is a canonical parabolic subgroup for w if the following properties are satisfied:

(i) The map $\pi\colon Z_{P_0,w} \to Z_w$ is an isomorphism.
(ii) The stratum $Z_{P_0,w}$ has coarse closure.

Using the notations of Proposition 2.2.1, property (i) is equivalent to $S \subset P_0 \times G$. The justification of this definition is the following: For $P_0 = P$, condition (i) is obviously satisfied. On the other hand, if $P_0 = {}^zB$, then (ii) is satisfied, because coarse and fine strata coincide. For a given w, a canonical parabolic for w is an intermediate parabolic subgroup P_0 satisfying both conditions. A priori neither the existence nor the uniqueness of such a parabolic is clear.

We give justification for this definition. Let P_0 be a canonical parabolic subgroup for w. We have morphisms

$$\pi\colon \overline{Z}_{P_0,w} \to \overline{Z}_w \quad \text{and} \quad \widetilde{\pi}\colon \overline{G}_{P_0,w} \to \overline{G}_w,$$

which yield isomorphisms $Z_{P_0,w} \simeq Z_w$ and $G_{P_0,w} \simeq G_w$. Since $Z_{P_0,w}$ has coarse closure, the stack (resp. variety) $\overline{Z}_{P_0,w}$ (resp. $\overline{G}_{P_0,w}$) is normal. We deduce the following proposition.

Proposition 3.1.2 *Let P_0 be a canonical parabolic subgroup for $w \in {}^IW$. Write $w = xw_J$ as in (1.2.1). Then the normalization of $\overline{G}_w$ is the Stein factorization of the map $\widetilde{\pi}\colon \overline{G}_{P_0,w} \to \overline{G}_w$. It is isomorphic to* $\operatorname{Spec}(\mathcal{O}(\overline{\mathbf{G}}_{P_0,x}))$, *where*

$$\overline{\mathbf{G}}_{P_0,x} = \overline{G}_{P_0,w} = \left\{ (g, aP_0) \in G \times P/P_0,\ a^{-1} g \varphi(\overline{a}) \in \overline{P_0 z \dot{w} Q_0} \right\}$$

and the first projection induces an isomorphism $G_{P_0,w} \simeq G_w$.

The following theorem is the main result of this paper. Its proof will follow from the results of §3.2 and §3.3.

Theorem 3.1.3 *Let $w \in {}^IW$. The parabolic subgroup P_w is the unique canonical parabolic subgroup for w. More precisely, among all parabolic subgroups ${}^zB \subset P_0 \subset P$, the following hold:*

(i) *P_w is the smallest parabolic P_0 such that $\pi\colon Z_{P_0,w} \to Z_w$ is an isomorphism.*
(ii) *P_w is the largest parabolic P_0 such that $Z_{P_0,w}$ has coarse closure.*

3.2 A Criterion for Condition (i)

Lemma 3.2.1 *Let ${}^zB \subset P_0 \subset P$ be a parabolic subgroup. The following assertions are equivalent:*

(i) *the map $\pi\colon Z_{P_0,w} \to Z_w$ is an isomorphism;*
(ii) *one has $P_w \subset P_0$.*

Proof Using the notation of Proposition 2.2.1, we know that $\pi\colon Z_{P_0,w} \to Z_w$ is an isomorphism if and only if $S_{P_0} := S \cap (P_0 \times G) = S$. By the same proposition, we know that the quotient S/S_{P_0} is a finite affine étale scheme over k. In particular, we have $S \subset P_0 \times G$ if and only if $S_{\mathrm{red}} \subset P_0 \times G$.

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By Lemma 1.4.2, we can write $S_{\mathrm{red}} = A \ltimes R$ with A the finite group given by equation (1.4.3) of Lemma 1.4.2 and R a smooth unipotent connected normal subgroup. Write $R_{P_0} := R \cap (P_0 \times G)$. The inclusion $R \subset S$ induces a closed embedding $R/R_{P_0} \to S/S_{P_0}$. Hence, R/R_{P_0} is a finite, smooth, connected k-scheme, so $R/R_{P_0} = \mathrm{Spec}(k)$, hence $R \subset P_0 \times G$. It follows that $S \subset P_0 \times G$ if and only if $A \subset P_0 \times G$, which is equivalent to $A_1 \subset P_0$, where

$$A_1 := \{x \in L_w,\ {}^{z\dot{w}}\varphi(x) = x\}.$$

By Steinberg's theorem we can write $z\dot{w} = a^{-1}\varphi(a)$ with $a \in G(k)$. Then it is easy to see that the subgroup ${}^{a}L_w$ is defined over $\mathbf{F}_p$, and the inclusion $A_1 \subset P_0$ is equivalent to

$$({}^{a}L_w)(\mathbf{F}_p) \subset {}^{a}P_0. \tag{3.2.1}$$

Note that both ${}^{a}L_w$ and ${}^{a}P_0$ contain the torus ${}^{az}T$, which is defined over $\mathbf{F}_p$. Thus, Lemma 3.2.2 below shows that (3.2.1) is equivalent to ${}^{a}L_w \subset {}^{a}P_0$, hence $L_w \subset P_0$, which is the same as $P_w \subset P_0$. This terminates the proof. ■

Lemma 3.2.2 *Let G be a connected reductive group over $\mathbf{F}_p$. Let L be a Levi $\mathbf{F}_p$-subgroup of G and P be a parabolic subgroup of G_k. Assume that there exists a maximal $\mathbf{F}_p$-torus T contained in $L \cap P$ and that $L(\mathbf{F}_p) \subset P$. Then one has $L \subset P$.*

Proof Define a subgroup of G by

$$H := L \cap \bigcap_{i \in \mathbf{Z}} \sigma^i(P) = \bigcap_{i \in \mathbf{Z}} L \cap \sigma^i(P).$$

It is clear that $H \subset L$, H is defined over $\mathbf{F}_p$, and $L(\mathbf{F}_p) = H(\mathbf{F}_p)$. Furthermore, H is an intersection of parabolic subgroups of L containing T. Hence, it suffices to prove the following claim. Let G be a connected reductive group over $\mathbf{F}_p$, $T \subset G$ a maximal $\mathbf{F}_p$-torus, and $T \subset H \subset G$ an $\mathbf{F}_p$-subgroup, which is an intersection of parabolic subgroups of G_k containing T, and assume that $H(\mathbf{F}_p) = G(\mathbf{F}_p)$. Then one has $H = G$.

We now prove the claim. Using inductively [DM91, Prop. 2.1], one shows that an intersection of parabolic subgroups $P_1, \dots, P_m$ containing T is connected and can be written as a semidirect product

$$\bigcap_{i=1}^{m} P_i = L_0 \ltimes U_0,$$

where L_0 is a Levi subgroup of G containing T and U_0 is a unipotent connected subgroup of G, normalized by L_0. Applying this to H, we can write $H = L_0 \ltimes U_0$. Since H is defined over $\mathbf{F}_p$, so are L_0 and U_0.

By [Car93, Thm. 3.4.1], the highest power of p dividing $|G(\mathbf{F}_p)|$ is p^N, where $N = |\Phi_+|$ is the dimension of any maximal unipotent subgroup of G_k. Since $G(\mathbf{F}_p) = H(\mathbf{F}_p) = L_0(\mathbf{F}_p) \times U_0(\mathbf{F}_p)$, we deduce that for all maximal unipotent subgroup U' in L_0, the subgroup $U' \times U_0$ is unipotent maximal in G. In particular, H contains a Borel subgroup, so H is a parabolic subgroup. Then $H = G$ follows from [ABD+66, XXVI, 5.11] ■

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3.3 A Criterion for Condition (ii)

We examine Definition 3.1.1(ii). Let ${}^zB \subset P_0 \subset P$ be a parabolic subgroup and let L_0, M_0, Q_0, $\mathcal{Z}_0$ as defined in Section 2.1.

Lemma 3.3.1 *Let ${}^zB \subset P_0 \subset P$ be a parabolic subgroup. The following assertions are equivalent:*

(i) *$Z_{P_0,w}$ has coarse closure;*
(ii) *one has ${}^{z\dot{w}}M_0 = L_0$.*

Proof The stratum $Z_{P_0,w}$ has coarse closure if and only if $\mathcal{O}_{E_{\mathcal{Z}_0}}(z\dot{w}) \subset P_0 z\dot{w} Q_0$ is an open embedding, which is equivalent to the equality of their dimensions. Note that (B, T, z) is again a frame of $\mathcal{Z}_0$, so formula (1.3.2) shows that

$$\dim\big(\mathcal{O}_{E_{\mathcal{Z}_0}}(z\dot{w})\big) = \dim(P_0) + \ell(w).$$

On the other hand, we have

$$\dim(P_0 z\dot{w} Q_0) = 2\dim(P_0) - \dim\big(\operatorname{Stab}_{P_0\times Q_0}(z\dot{w})\big).$$

The stabilizer $\operatorname{Stab}_{P_0\times Q_0}(z\dot{w})$ is the subgroup

$$\begin{aligned}\operatorname{Stab}_{P_0\times Q_0}(z\dot{w}) &= \{(a,b) \in P_0 \times Q_0,\ az\dot{w} = z\dot{w}b\}\\ &\simeq \{a \in P_0,\ (z\dot{w})^{-1}az\dot{w} \in Q_0\} = P_0 \cap {}^{z\dot{w}}Q_0.\end{aligned}$$

Hence, $Z_{P_0,w}$ has coarse closure if and only if $\dim(P_0/(P_0 \cap {}^{z\dot{w}}Q_0)) = \ell(w)$. Since the property is satisfied when $P_0 = {}^zB$, we have $\dim({}^zB/({}^zB \cap {}^{z\dot{w}}B)) = \ell(w)$, so we can rewrite the property as

$$\dim\big((P_0 \cap {}^{z\dot{w}}Q_0)/({}^zB \cap {}^{z\dot{w}}B)\big) = \dim(P_0/{}^zB). \tag{3.3.1}$$

Since (B, T, z) is a frame for $\mathcal{Z}_0$ and ${}^IW \subset {}^{I_0}W$, equation (1.2.2) shows that $P_0 \cap {}^zB = P_0 \cap {}^{z\dot{w}}B$, thus the inclusion $P_0 \cap {}^{z\dot{w}}Q_0 \subset P_0$ induces an embedding

$$(P_0 \cap {}^{z\dot{w}}Q_0)/({}^zB \cap {}^{z\dot{w}}B) \longrightarrow P_0/{}^zB.$$

Hence, (3.3.1) is satisfied if and only if the image of $P_0 \cap {}^{z\dot{w}}Q_0$ is open in $P_0/{}^zB$. Since $P_0/{}^zB \simeq L_0/({}^zB \cap L_0)$ it is also equivalent to $L_0 \cap {}^{z\dot{w}}Q_0$ having open image in $L_0/({}^zB \cap L_0)$.

Denote by B' the opposite Borel in G of B with respect to T. Then ${}^zB' \cap L_0$ is the opposite Borel of ${}^zB \cap L_0$ in L_0 with respect to ${}^zT$. Thus, the image of $L_0 \cap {}^{z\dot{w}}Q_0$ is open in $L_0/({}^zB \cap L_0)$ if and only if ${}^zB' \cap L_0 \subset {}^{z\dot{w}}Q_0$. It follows immediately from equation (1.2.2) that ${}^zB' \cap L_0 = {}^{z\dot{w}}B' \cap L_0$. Finally, we find that $Z_{P_0,w}$ has coarse closure if and only if

$$B' \cap {}^{(z\dot{w})^{-1}}L_0 \subset Q_0. \tag{3.3.2}$$

The groups $B' \cap {}^{(z\dot{w})^{-1}}L_0$ and $B \cap {}^{(z\dot{w})^{-1}}L_0$ are opposite Borel subgroups of ${}^{(z\dot{w})^{-1}}L_0$ containing T. Since $B \subset Q_0$, equation (3.3.2) is simply equivalent to ${}^{(z\dot{w})^{-1}}L_0 \subset Q_0$, which is equivalent to ${}^{(z\dot{w})^{-1}}L_0 = M_0$. This terminates the proof. ∎

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Proof of Theorem 3.1.3 The result follows immediately by combining Lemmas 3.2.1 and 3.3.1. ∎

Corollary 3.3.2 *Write $w = xw_J$ as in (1.2.1). The normalization of the Zariski closure $\overline{G}_w$ is the Stein factorization of the map $\widetilde{\pi}\colon \overline{G}_{P_w,w} \to \overline{G}_w$. It is isomorphic to $\widetilde{Z}_w :=$ $\operatorname{Spec}(\mathcal{O}(\overline{\mathbf{G}}_{P_w,x}))$, where*

$$\overline{\mathbf{G}}_{P_w,x} = \overline{G}_{P_w,w} = \left\{ (g, aP_w) \in G \times P/P_w,\ a^{-1} g\varphi(\overline{a}) \in \overline{P_w z\dot{w} Q_w} \right\},$$

and the first projection induces an isomorphism $G_{P_w,w} \simeq G_w$.

3.4 Shimura Varieties and Ekedahl–Oort Strata

Let X be the special fiber of a Hodge-type Shimura variety attached to a Shimura datum $(\mathbf{G}, \mathbf{X})$ with hyperspecial level at p. Write $G := G_{\mathbf{Z}_p} \times \mathbf{F}_p$, where $G_{\mathbf{Z}_p}$ is a reductive $\mathbf{Z}_p$-model of $\mathbf{G}_{\mathbf{Q}_p}$. By Zhang [Zha], there exists a smooth morphism of stacks $\zeta\colon X \to G\text{-}\mathsf{Zip}^{\mathcal{Z}}$, where $\mathcal{Z}$ is the zip datum attached to $(\mathbf{G}, \mathbf{X})$ as in [GKb, §6.2]. The Ekedahl–Oort stratification of X is defined as the fibers of ζ. For $w \in {}^I W$, set $X_w := \zeta^{-1}(Z_w)$. By the smoothness of ζ, this defines a stratification of X. Let ${}^z B \subset P_0 \subset P$ be a parabolic subgroup and define the partial flag space X_{P_0} as the fiber product

$$\begin{array}{ccc} X_{P_0} & \xrightarrow{\zeta_{P_0}} & G\text{-}\mathsf{ZipFlag}^{(\mathcal{Z},P_0)} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{P_0}} \\ X & \xrightarrow[\zeta]{} & G\text{-}\mathsf{Zip}^{\mathcal{Z}}. \end{array}$$

The map $\pi\colon X_{P_0} \to X$ is a P/P_0-bundle. For $w \in {}^{I_0}W$ and $x \in {}^{I_0}W^{J_0}$ define

$$X_{P_0,w} := \zeta_0^{-1}(Z_{P_0,w}) \quad \text{and} \quad \mathbf{X}_{P_0,x} := \zeta_0^{-1}(\mathbf{Z}_{P_0,x}).$$

We call $X_{P_0,w}$ the fine stratum attached to $w \in {}^{I_0}W$ and $\mathbf{X}_{P_0,x}$ the coarse stratum attached to $x \in {}^{I_0}W^{J_0}$. All coarse and fine strata are smooth and locally closed, they define stratifications of X_{P_0}, and the Zariski closure of a coarse stratum is normal. Recall that we defined $\widetilde{Z}_w := \operatorname{Spec}(\mathcal{O}(\overline{G}_{P_w,w}))$ (Corollary 3.3.2).

Corollary 3.4.1 *Let $w \in {}^I W$. The normalization of $\overline{X}_w$ is the Stein factorization of the map $\pi\colon \overline{X}_{P_w,w} \to \overline{X}_w$. It is isomorphic to $\overline{X}_w \times_{G\text{-}\mathsf{Zip}^{\mathcal{Z}}} \widetilde{Z}_w$.*

4 The Canonical Filtration

Most of the content of this section can be found in [PWZ11]. We merely unwind their proofs to make the link between the canonical filtration of a Dieudonné space and the group L_w defined previously. See also [Moo01, §4.4] and [Box15] for related results.

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4.1 Dieudonné Spaces and GL_n-zips

Let H be a truncated Barsotti–Tate groups of level 1 over k of height n. Set $d := \dim(\mathrm{Lie}(H))$ and write $D := \mathbf{D}(H)$ for its Dieudonné space. It is a k-vector space of dimension n endowed with a σ-linear endomorphism $\mathcal{F}: D \to D$, a σ^{-1}-linear endomorphism $\mathcal{V}: D \to D$ satisfying the conditions:

(a) $\mathrm{Ker}(\mathcal{F}) = \mathrm{Im}(\mathcal{V})$,
(b) $\mathrm{Ker}(\mathcal{V}) = \mathrm{Im}(\mathcal{F})$,
(c) $\mathrm{rk}(\mathcal{V}) = d$.

We say that $(D, \mathcal{F}, \mathcal{V})$ is a Dieudonné space of height n and dimension d. Let $M_n^{(r)}(k)$ be the set of matrices in $M_n(k)$ of rank r. After choosing a k-basis of D, we can write $\mathcal{F} = a \otimes \sigma$ and $\mathcal{V} = b \otimes \sigma^{-1}$, where (a, b) is in the set

$$\mathscr{X} := \left\{ (a,b) \in M_n^{(n-d)}(k) \times M_n^{(d)}(k),\ a\sigma(b) = \sigma(b)a = 0 \right\}.$$

Note that for $(a, b) \in \mathscr{X}$, we have

$$\mathrm{Ker}(a) = \mathrm{Im}\big(\sigma(b)\big) = \sigma\big(\mathrm{Im}(b)\big) \quad \text{and} \quad \mathrm{Im}(a) = \mathrm{Ker}\big(\sigma(b)\big) = \sigma\big(\mathrm{Ker}(b)\big).$$

It is easy to see that two such pairs (a, b) and (a', b') yield isomorphic Dieudonné spaces if and only if there exists $M \in GL_n(k)$ such that

$$a' = Ma\sigma(M)^{-1} \quad \text{and} \quad b' = Mb\sigma^{-1}(M)^{-1}.$$

This defines an action of $GL_n(k)$ on $\mathscr{X}$ and we obtain a bijection between isomorphism classes of Dieudonné spaces of height n and dimension d and the set of $GL_n(k)$-orbits in $\mathscr{X}$.

Let $(e_1, \dots, e_n)$ the canonical basis of k^n and define

$$V_1 := \mathrm{Span}(e_1, \dots, e_{n-d}) \quad \text{and} \quad V_2 := \mathrm{Span}(e_{n-d+1}, \dots, e_n)$$

Define $P := \mathrm{Stab}(V_2)$, $Q := \mathrm{Stab}(V_1)$, $L := P \cap Q$, $U := R_u(P)$, and $V := R_u(Q)$. Consider the set

$$\mathscr{Y} := \left\{ (a,b) \in \mathscr{X},\ \mathrm{Ker}(a) = V_2 \right\}.$$

The action of $GL_n(k)$ on $\mathscr{X}$ restricts to an action of $P(k)$ on $\mathscr{Y}$ and the inclusion $\mathscr{Y} \subset \mathscr{X}$ induces a bijection between $P(k)$-orbits in $\mathscr{Y}$ and $GL_n(k)$-orbits in $\mathscr{X}$.

Lemma 4.1.1 *There is a natural bijection $\Psi: \mathscr{Y} \to GL_n(k)/V$.*

Proof Let $(a, b) \in \mathscr{Y}$ and choose a subspace $H \subset k^n$ such that $\mathrm{Im}(a) \oplus H = k^n$. Define a matrix $f_H \in GL_n(k)$ by the following diagram

$$\text{(4.1.1)} \qquad \begin{array}{ccccc} k^n & = & V_1 & \oplus & V_2 \\ \big\downarrow{\scriptstyle f_H} & & \big\downarrow{\scriptstyle a} & & \big\downarrow{\scriptstyle \sigma(b)^{-1}} \\ k^n & = & \mathrm{Im}(a) & \oplus & H. \end{array}$$

In other words, $f_H v = av$ for $v \in V_1$, and if $v \in V_2$, then $f_H v$ is the only element $h \in H$ such that $\sigma(b)h = v$ (note that $\mathrm{Im}(a) = \mathrm{Ker}(\sigma(b))$, so this element is well defined). It is clear that f_H is invertible.

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If H' denotes another subspace such that $\mathrm{Im}(a) \oplus H' = k^n$, then we can write $f_{H'} = f_H \alpha$, for some $\alpha \in GL_n(k)$. It is clear that $\alpha(v) = v$ for all $v \in V_1$. Furthermore, for $v \in V_2$, one must have $\sigma(b)(f_{H'}v - f_H v) = 0$, thus $f_H(\alpha(v) - v) \in \mathrm{Ker}(\sigma(b)) = \mathrm{Im}(a)$, so $\alpha(v) - v \in V_1$. This shows that $\alpha \in V$. It follows that $(a, b) \mapsto f_H$ induces a well-defined map $\Psi: \mathscr{Y} \to GL_n(k)/V$. We leave it to the reader to check that this map is a bijection. ∎

Define a subgroup of $P \times Q$ by

$$E := \left\{ (M_1, M_2) \in P \times Q,\ \varphi(\overline{M}_1) = \overline{M}_2 \right\}.$$

Let this group acts on $GL_n(k)$ by the rule $(M_1, M_2) \cdot g := M_1 g M_2^{-1}$.

Proposition 4.1.2 *The map Ψ induces a bijection $P(k)\backslash\mathscr{Y} \to E\backslash GL_n(k)$. Hence there is a bijection between isomorphism classes of Dieudonné spaces of height n and dimension d and the set of E-orbits in $GL_n(k)$.*

Proof Let $M \in P(k)$, $(a, b) \in \mathscr{Y}$ and set $(a', b') := (Ma\sigma(M)^{-1}, Mb\sigma^{-1}(M)^{-1})$. Note that $\mathrm{Im}(a') = M(\mathrm{Im}(a))$. Choose a subspace H such that $\mathrm{Im}(a) \oplus H = k^n$ and set $H' := M(H)$. Let $\overline{M} \in L(k)$ denote the Levi component of $M \in P(k)$. Finally, write f_H and $f'_{H'}$ for the maps attached to (a, b, H) and (a', b', H'), respectively, by the previous construction. We claim that one has the relation

$$M f_H = f'_{H'} \sigma(\overline{M}).$$

First assume that $v \in V_1$. Then $f'_{H'}\sigma(\overline{M})v = Ma\sigma(M)^{-1}\overline{M}v$. Since $\sigma(M)^{-1}\overline{M} \in U$, we have $\sigma(M)^{-1}\overline{M}v - v \in V_2$, hence $f'_{H'}\sigma(\overline{M})v = Mav = Mf_H v$.

Now if $v \in V_2$, the element $f_H v$ is the only element $h \in H$ satisfying $\sigma(b)h = v$. Similarly, $f_{H'}\sigma(\overline{M})v$ is the only element $h' \in H' = M(H)$ such that $\sigma(b')h' = \sigma(\overline{M})v$. Hence, $\sigma(\overline{M})^{-1}\sigma(M)\sigma(b)M^{-1}h' = v$. But $\sigma(\overline{M})^{-1}\sigma(M) \in U$ and $\sigma(b)M^{-1}h' \in V_2$, so we deduce $\sigma(b)M^{-1}h' = v$, and finally $M^{-1}h' = h$ as claimed.

This shows that Ψ induces a well-defined map $P(k)\backslash\mathscr{Y} \to E\backslash GL_n(k)$. We leave it to the reader to check that it is bijective. ∎

4.2 The Canonical Filtration

Let $(D, \mathcal{F}, \mathcal{V})$ be a Dieudonné space. The operators $\mathcal{V}$ and $\mathcal{F}^{-1}$ act naturally on the set of subspaces of D. It can be shown that there exists a flag of D that is stable by $\mathcal{V}$ and $\mathcal{F}^{-1}$ and is coarsest among all such flags. This flag is called the *canonical filtration of* D. It is obtained by applying all finite combinations of $\mathcal{V}, \mathcal{F}^{-1}$ to the flag $0 \subset D$.

Choose a basis of D and write $\mathcal{F} = a \otimes \sigma$ and $\mathcal{V} = b \otimes \sigma^{-1}$ with $(a, b) \in \mathscr{X}$. By choosing an appropriate basis, we will assume that $(a, b) \in \mathscr{Y}$.

Remark 4.2.1 Actually, there exists a basis such that $(a, b) \in \mathscr{Y}$ and such that the coefficients of a, b are either 0 or 1 and each column and each row has at most one non-zero coefficient.

Let $H \subset k^n$ be a subspace such that $\mathrm{Im}(a) \oplus H = k^n$ and let $f_H \in GL_n(k)$ be the element defined in diagram (4.1.1). We have the following easy lemma.

Lemma 4.2.2 *For any subspace $W \subset k^n$, one has the relations*

$$\mathcal{V}(W) = V_2 \cap \left(\sigma^{-1}(f_H^{-1} W) + V_1 \right),$$
$$\mathcal{F}^{-1}(W) = V_2 + \left(\sigma^{-1}(f_H^{-1} W) \cap V_1 \right).$$

In particular, the right-hand terms are independent of the choice of H. This observation suggests the following definition.

Definition 4.2.3 For $f \in GL_n(k)$, there exists a unique coarsest flag $\mathrm{Fl}(f)$ of k^n satisfying the following properties:

(i) For any $W \in \mathrm{Fl}(f)$, the following inclusions hold

$$V_2 \cap \left(\sigma^{-1}(f^{-1} W) + V_1 \right) \subset W \subset V_2 + \left(\sigma^{-1}(f^{-1} W) \cap V_1 \right). \tag{4.2.1}$$

(ii) For any $W \in \mathrm{Fl}(f)$, all subspaces appearing in (4.2.1) are in $\mathrm{Fl}(f)$.

The flag $\mathrm{Fl}(f)$ is simply the canonical flag attached to the Dieudonné space corresponding to the left-coset fV under the bijection Ψ.

Lemma 4.2.4 *Let $f \in GL_n(k)$. The following assertions hold*

(i) *for all $v \in V$, one has $\mathrm{Fl}(fv) = \mathrm{Fl}(f)$;*
(ii) *for $M \in P$, one has $\mathrm{Fl}(M f \sigma(\overline{M})^{-1}) = M\,\mathrm{Fl}(f)$.*

We leave the verification of the lemma to the reader. In particular, the conjugation class of $\mathrm{Fl}(f)$ depends only on the E-orbit of f. Denote by $P(f) := \mathrm{Stab}(\mathrm{Fl}(f))$. Since $\mathrm{Fl}(f)$ contains $\mathrm{Im}(V) = V_2$, we have $P(f) \subset P$. Furthermore, for $v \in V$ and $M \in P$, one has

$$P(fv) = P(f) \quad \text{and} \quad P(M f \sigma(\overline{M})^{-1}) = {}^{M}P(f).$$

4.3 The Canonical Flag Versus P_w

Denote by T the diagonal torus of $G := GL_n$, and let B be the Borel subgroup of upper-triangular matrices. The Weyl group $W(G,T)$ is the symmetric group S_n, which we identify with a subgroup of $G(k)$ be letting it act on k^n by $\tau(e_i) = e_{\tau(i)}$ for all $\tau \in S_n$ and $i \in \{1, \dots, n\}$.

Using the notations of Section 1.2, define a permutation

$$z := w_0 w_{0,I} = \begin{pmatrix} 0 & I_{n-d} \\ I_d & 0 \end{pmatrix}.$$

Then (B, T, z) is a frame for the zip datum (G, P, Q, L, M, φ). For $w \in {}^I W$, set $f_w := zw$. By the parametrization (1.3.1), the set $\{zw, w \in {}^I W\}$ is a set of representatives of the E-orbits in G. For $w \in {}^I W$, it is easy to see that any $W \in \mathrm{Fl}(f_w)$ is spanned by $(e_i)_{i \in C_W}$ for some subset $C_W \subset \{1, \dots, n\}$. In particular we have $T \subset P(f_w)$. Note that for all $w \in {}^I W$, we have simplified formulas:

$$V_2 \cap \left(\sigma^{-1}(f_w^{-1} W) + V_1 \right) = V_2 \cap f_w^{-1} W,$$
$$V_2 + \left(\sigma^{-1}(f_w^{-1} W) \cap V_1 \right) = V_2 + f_w^{-1} W.$$

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There is a unique Levi subgroup $L(f_w) \subset P(f_w)$ containing T. Finally, for $w \in {}^I W$, denote by $L_w \subset L$ and $P_w \subset P$ the subgroups defined in (1.4.1) and (1.4.2).

Proposition 4.3.1 *We have $P(f_w) = P_w$ and $L(f_w) = L_w$.*

Proof We will show first that ${}^z B \subset P(f_w)$. Clearly, it suffices to show ${}^z B \cap L \subset P(f_w)$. Note that since $w \in {}^I W$, we have ${}^z B \cap L = B \cap L = {}^{zw} B \cap L$. From this it follows that if $W \subset k^n$ is a subspace such that ${}^z B \cap L \subset \operatorname{Stab}(W)$, then ${}^z B \cap L$ stabilizes also $\sigma^{-1}(f_w^{-1}(W))$. From this it follows easily by induction that ${}^z B \cap L$ stabilizes $\operatorname{Fl}(f_w)$; hence, ${}^z B \cap L \subset P(f_w)$, as claimed.

To finish the proof, it suffices to show the second assertion. By definition, we have ${}^{f_w}\varphi(L_w) = L_w$. Hence, if $W \subset k^n$ is a subspace such that $L_w \subset \operatorname{Stab}(W)$ then $L_w \subset \operatorname{Stab}(\sigma^{-1}(f_w^{-1}(W)))$. From this, it follows again by an easy induction that L_w stabilizes $\operatorname{Fl}(f_w)$, so $L_w \subset P(f_w)$. Since L_w contains the torus T, we deduce that $L_w \subset L(f_w)$.

Finally, we must show that ${}^{f_w}\varphi(L(f_w)) = L(f_w)$. Since $L(f_w)$ is clearly defined over $\mathbf{F}_p$, this is the same as ${}^{f_w}L(f_w) = L(f_w)$. Let $k^n = D_1 \oplus \cdots \oplus D_m$ denote the decomposition attached to $L(f_w)$, numbered so that the filtration $\operatorname{Fl}(f_w)$ is composed of the subspaces $W_j := \bigoplus_{j=1}^{i} D_j$ for $1 \le i \le m$. There exists an integer $1 \le r \le m$ such that $W_r = V_2$ (and then necessarily $V_1 = \bigoplus_{j=r+1}^{m} D_j$). We need to show that f_w permutes the $(D_i)_{1 \le i \le m}$. For this, it suffices to show that if $f_w(D_i) \cap D_j \neq 0$, then $D_j \subset f_w(D_i)$ for all $1 \le i, j \le m$.

First assume that $1 \le j \le r$ and let $1 \le i \le m$ be the smallest integer such that $D_j \cap f_w(D_i) \neq 0$. He have $0 \neq D_j \cap f_w(D_i) \subset V_2 \cap f_w(W_i)$, which implies $D_j \subset V_2 \cap f_w(W_i)$. By minimality of i, we deduce $D_j \subset f_w(D_i)$.

Now assume that $r < j \le m$ and let $1 \le i \le m$ be the smallest integer such that $D_j \cap f_w(D_i) \neq 0$. Then $0 \neq D_j \cap f_w(D_i) \subset V_2 + f_w(W_i)$, which implies

$$D_j \subset (V_2 + f_w(W_i)) \cap V_1 = f_w(W_i) \cap V_1 \subset f_w(W_i).$$

By minimality of i, we deduce $D_j \subset f_w(D_i)$, which terminates the proof. ∎

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