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REFLEXIVE INDEX OF A FAMILY OF SUBSPACES

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Abstract

A definition of the reflexive index of a family of (closed) subspaces of a complex, separable Hilbert space H is given, analogous to one given by D. Zhao for a family of subsets of a set. Following some observations, some examples are given, including: (a) a subspace lattice on H with precisely five nontrivial elements with infinite reflexive index; (b) a reflexive subspace lattice on H with infinite reflexive index; (c) for each positive integer n satisfying $\dim H \geq n+1$, a reflexive subspace lattice on H with reflexive index n. If H is infinite-dimensional and $\mathcal{B}$ is an atomic Boolean algebra subspace lattice on H with n equidimensional atoms and with the property that the vector sum $K+L$ is closed, for every $K, L \in \mathcal{B}$, then $\mathcal{B}$ has reflexive index at most n.

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1. Introduction and preliminaries

Throughout this paper H will denote a complex, separable Hilbert space. By 'subspace' we will mean 'closed linear manifold', and by 'operator', 'bounded linear transformation'. All scalars will be considered complex. The set of operators on H is denoted by $\mathcal{B}(H)$. If $\mathcal{V}$ is a set of vectors, we use $\langle \mathcal{V} \rangle$ to denote their linear span. If $e, f \in H$ we let $e \otimes f$ denote the operator of $\mathcal{B}(H)$ of rank at most one, acting according to $e \otimes f(x) = (x|e)f$, for all $x \in H$, where $(\cdot|\cdot)$ denotes the inner product on H. We let $C(H)$ denote the set of subspaces of H. For simplicity, the weak operator topology on $\mathcal{B}(H)$ will be referred to as the *weak topology*. If $\{M_\lambda\}_{\lambda\in\Lambda}$ is a family of subspaces, $\bigcap_{\lambda\in\Lambda} M_\lambda$ denotes their intersection and $\bigvee_{\lambda\in\Lambda} M_\lambda$ denotes their closed linear span. A family $\mathcal{L}$ of subspaces of H is called a *subspace lattice on* H if it contains (0) and H and it is closed under the formation of arbitrary intersections and arbitrary closed linear spans (of sets of elements of any cardinality). A subspace lattice $\mathcal{L}$ on a Hilbert space is called *commutative* if $P_M P_N = P_N P_M$ for all $M, N \in \mathcal{L}$, where P_K denotes the orthogonal projection with range $K \in C(H)$.

If $\mathcal{F}$ is a family of subspaces of H, then $\mathrm{Alg}\,\mathcal{F} = \{T \in \mathcal{B}(H) : T(M) \subseteq M,\ \forall M \in \mathcal{F}\}$, that is, $\mathrm{Alg}\,\mathcal{F}$ denotes the set of operators on H having every element of $\mathcal{F}$ as an invariant subspace. Also, if $\mathcal{A}$ is a family of operators on H, $\operatorname{Lat}\mathcal{A} = \{M \in C(H) : T(M) \subseteq M,\ \forall T \in \mathcal{A}\}$, that is, $\operatorname{Lat}\mathcal{A}$ is the set of common invariant subspaces of the elements of $\mathcal{A}$. Then $\operatorname{Lat}\mathcal{A}$ is a subspace lattice on H and $\operatorname{Alg}\mathcal{F}$ is a unital subalgebra of $\mathcal{B}(H)$, for every $\mathcal{A} \subseteq \mathcal{B}(H)$, $\mathcal{F} \subseteq C(H)$.

For any family of subspaces $\mathcal{F}$ of H we have $\mathcal{F} \subseteq \operatorname{LatAlg}\mathcal{F}$, and for any family $\mathcal{A}$ of operators on H we have $\mathcal{A} \subseteq \operatorname{AlgLat}\mathcal{A}$. A subspace lattice $\mathcal{L}$ on H is called *reflexive* if $\mathcal{L} = \operatorname{LatAlg}\mathcal{L}$. The notion, and notation, are due to Halmos [3, 4]. The reader interested in reflexive subspace lattices is referred to [5, 6, 8, 10–12, 14–18] for further reading. If $\mathcal{G} \subseteq C(H)$ and $\mathcal{B} \subseteq \mathcal{B}(H)$ then $\mathcal{F} \subseteq \mathcal{G}$ implies that $\operatorname{Alg}\mathcal{G} \subseteq \operatorname{Alg}\mathcal{F}$, and $\mathcal{A} \subseteq \mathcal{B}$ implies that $\operatorname{Lat}\mathcal{B} \subseteq \operatorname{Lat}\mathcal{A}$. It follows that $\operatorname{LatAlgLat}\mathcal{A} = \operatorname{Lat}\mathcal{A}$ and $\operatorname{AlgLatAlg}\mathcal{F} = \operatorname{Alg}\mathcal{F}$. So, $\operatorname{Lat}\mathcal{A}$ is a reflexive subspace lattice for any subset $\mathcal{A} \subseteq \mathcal{B}(H)$. In fact, a family $\mathcal{L} \subseteq C(H)$ is a reflexive subspace lattice on H if and only if $\mathcal{L} = \operatorname{Lat}\mathcal{A}$ for some family of operators $\mathcal{A} \subseteq \mathcal{B}(H)$.

The following definition is analogous to one given in [21] for a family of subsets of a set.

Definition 1.1. Let $\mathcal{F}$ be a family of subspaces of H. The reflexive index $\kappa_H(\mathcal{F})$ of $\mathcal{F}$ is

$$\kappa_H(\mathcal{F}) = \inf\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{B}(H) \text{ and } \operatorname{LatAlg}\mathcal{F} = \operatorname{Lat}\mathcal{A}\},$$

where $|\mathcal{F}|$ denotes the cardinality of $\mathcal{F}$.

Note that $\mathcal{F}$ and $\operatorname{LatAlg}\mathcal{F}$ always have the same reflexive index and, if $\mathcal{L}$ is a reflexive subspace lattice on H, then $\kappa_H(\mathcal{L}) = \inf\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{B}(H) \text{ and } \mathcal{L} = \operatorname{Lat}\mathcal{A}\}$.

In the remainder of this paper we frequently use ‘index’ to mean ‘reflexive index’.

An algebra $\mathcal{A} \subseteq \mathcal{B}(H)$ of operators is called *reflexive* if $\mathcal{A} = \operatorname{AlgLat}\mathcal{A}$, or equivalently, if $\mathcal{A} = \operatorname{Alg}\mathcal{F}$, for some family of subspaces $\mathcal{F} \subseteq C(H)$ (see [19, Section 9.2]).

If X is a Banach space and if $C(X)$ (respectively, $\mathcal{B}(X)$) denotes the set of closed linear manifolds of (respectively, the set of bounded linear transformations on) X, the operation ‘Alg’ (respectively, ‘Lat’) can be performed on subsets of $C(X)$ (respectively, $\mathcal{B}(X)$) and these give rise to a notion of reflexivity for subsets of $C(X)$ (respectively, $\mathcal{B}(X)$). From this follows a definition of ‘reflexive index’ for subsets of $C(X)$. We shall not consider this index here. Also, we will be primarily concerned here with the notion of reflexive index in the context of infinite-dimensional spaces.

2. Some observations

The author hopes that the more obvious of the following observations will not be without some interest to the reader.

2.1. O1. Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a family of operators. If $\overline{[\mathcal{A}]}$ denotes the closure in the weak (operator) topology of the algebra generated by $\mathcal{A}$, then $\overline{[\mathcal{A}]}$ is the weakly closed algebra generated by $\mathcal{A}$ and $\operatorname{Lat}\mathcal{A} = \operatorname{Lat}\overline{[\mathcal{A}]}$. For this, note that $\mathcal{A} \subseteq \overline{[\mathcal{A}]}$ implies that $\operatorname{Lat}\overline{[\mathcal{A}]} \subseteq \operatorname{Lat}\mathcal{A}$. On the other hand, if $M \in \operatorname{Lat}\mathcal{A}$, then $\mathcal{A} \subseteq \operatorname{Alg}\{M\}$.

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But $\mathrm{Alg}\,\{M\}$ is a weakly closed algebra, so $\overline{[\mathcal{A}]} \subseteq \mathrm{Alg}\,\{M\}$. Thus $M \in \mathrm{Lat}\,\overline{[\mathcal{A}]}$, and so $\mathrm{Lat}\,\mathcal{A} \subseteq \mathrm{Lat}\,\overline{[\mathcal{A}]}$.

The algebra $\mathrm{AlgLat}\,\mathcal{A}$ is also a weakly closed algebra containing $\mathcal{A}$, so $\overline{[\mathcal{A}]} \subseteq \mathrm{AlgLat}\,\mathcal{A}$. We need not have equality. Whereas $\mathrm{AlgLat}\,\mathcal{A}$ always contains the identity operator, $\overline{[\mathcal{A}]}$ need not.

2.2. O2. For any $T \in \mathcal{B}(H)$, $\mathrm{Lat}\,\{T\}$ is, of course, a reflexive subspace lattice of index one. An abstract lattice L is called *attainable* if there is an operator on a separable, infinite-dimensional Hilbert space with $\mathrm{Lat}\,\{T\}$ order isomorphic to L. This notion is due to Halmos. Every example of an attainable lattice leads to an example of a reflexive subspace lattice of index one (see [19, Section 4.1] for some examples).

If $\mathcal{A}$ is a family of operators on H, the index of $\mathrm{Lat}\,\mathcal{A}$ is at most $|\mathcal{A}|$, the cardinality of $\mathcal{A}$. In the latter situation, where we are given $\mathcal{A}$, we are interested in knowing if there is a (strictly) smaller set of operators with the same Lat as $\mathcal{A}$. On the other hand, we are also interested in deducing facts about the index of a reflexive subspace lattice for which only a description of the lattice structure and subspaces are provided.

A family $\mathcal{F}$ of subspaces of H is called *transitive* if $\mathrm{Alg}\,\mathcal{F} = \mathbb{C}I$, that is, if only scalar multiples of the identity operator leave all of the elements of $\mathcal{F}$ invariant. Of course, $C(H)$ is transitive. Halmos initiated the study of these in [3], where he gives an example of a transitive subspace lattice with five nontrivial elements; an example with four nontrivial elements, on separable infinite-dimensional Hilbert space, is given in [9] (see also [2, 19, Section 4.7]). Of course, every transitive family of subspaces has index one.

For an introduction to the dual notion of 'transitive algebra' the interested reader is referred to [19, Ch. 8].

2.3. O3. The invariant subspace problem (see [19, Section 0.2]) for a separable, infinite-dimensional Hilbert space H can be reformulated as: Does the reflexive subspace lattice $\{(0), H\}$ have reflexive index one? The index of the latter is at most two. Indeed, a well-known example of a pair of operators on H having no nontrivial common invariant subspaces is $\{A, B\}$, where the matrix of A is diagonal with respect to some orthonormal basis $\{e_k : k \geq 1\}$ of H, say $Ae_k = \alpha_k e_k$, where $\{\alpha_k\}_1^\infty$ is a decreasing sequence of positive real numbers converging to zero, and with $B = e \otimes f$, with e, f vectors of H having all Fourier coefficients nonzero with respect to the orthonormal basis $\{e_k : k \geq 1\}$. (See [19, Section 8.3].)

2.4. O4. The subspace lattice $\{(0), H\}$ is the simplest nest. A subspace lattice is called a *nest* if it is totally ordered by inclusion. Every nest is a reflexive subspace lattice. (This was first proved in [20].) A *nest algebra* is a subalgebra of $\mathcal{B}(H)$ of the form $\mathrm{Alg}\,\mathcal{N}$, for some nest $\mathcal{N}$. It is shown in [13] that every nest algebra on a separable Hilbert space is the weakly closed algebra generated by two operators. Thus, every nest on a separable Hilbert space has index at most two. (If $\{A_1, A_2\}$ generate $\mathrm{Alg}\,\mathcal{N}$ as a weakly closed algebra then $\mathrm{LatAlg}\,\mathcal{N} = \mathcal{N} = \mathrm{Lat}\,\{A_1, A_2\}$.) Many nests have index one. An operator whose lattice of invariant subspaces is a nest is called *unicellular*. There are several well-known examples of unicellular operators,

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for example, Donoghue operators and the Volterra operator (see [19, Section 4.4]). Of course, every abstract attainable totally ordered lattice gives rise to a nest with index one; for example, see [7].

2.5. O5. If $\dim H < \infty$, every family $\mathcal{F}$ of subspaces has finite index since $\operatorname{Alg}\mathcal{F}$ is finite-dimensional, so is finitely generated as a weakly closed algebra. It is still interesting to study the notion of reflexive index in the context of finite-dimensional spaces.

2.6. O6. The reflexive index is not invariant under lattice isomorphism. Every atomic Boolean algebra subspace lattice on H is reflexive [4]. Let $\mathcal{B}_2$ be the two-atom Boolean algebra subspace lattice on $\mathbb{C}^2$ with atoms $\langle(1,0)\rangle$, $\langle(0,1)\rangle$. Then $\mathcal{B}_2 = \operatorname{Lat}\{[\begin{smallmatrix}1&0\\0&2\end{smallmatrix}]\}$, so $\mathcal{B}_2$ has index one. No two-atom Boolean algebra subspace lattice on a space $\mathbb{C}^n$, $n \geq 3$, has index one since every operator on a nonzero finite-dimensional space has an eigenvector.

However, reflexive index is preserved under similarity. That is, if $S \in \mathcal{B}(H)$ is invertible and $\mathcal{F}$ is a family of subspaces of H, then $\mathcal{F}$ and $S\mathcal{F} = \{SM : M \in \mathcal{F}\}$ have the same reflexive index. This follows from the fact that $\operatorname{LatAlg}\mathcal{F} = \operatorname{Lat}\mathcal{A}$ if and only if $\operatorname{LatAlg}(S\mathcal{F}) = \operatorname{Lat} S\mathcal{A}S^{-1}$, for any family of operators $\mathcal{A} \subseteq \mathcal{B}(H)$.

Reflexive index is also invariant under orthogonal complements, that is, if $\mathcal{F} \subseteq C(H)$ and $\mathcal{F}^{\perp} = \{M^{\perp} : M \in \mathcal{F}\}$, then $\mathcal{F}$ and $\mathcal{F}^{\perp}$ have the same reflexive index. Indeed, if $\mathcal{A} \subseteq \mathcal{B}(H)$, then $\operatorname{LatAlg}\mathcal{F} = \operatorname{Lat}\mathcal{A}$ if and only if $\operatorname{LatAlg}(\mathcal{F}^{\perp}) = \operatorname{Lat}\mathcal{A}^*$, where $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$.

3. Some examples

3.1. E1. Let H be an infinite-dimensional separable Hilbert space. Let $\mathcal{B}$ be the atomic Boolean algebra subspace lattice on $H \oplus H$ with atoms $H \oplus (0)$ and $(0) \oplus H$. So $\mathcal{B} = \{(0) \oplus (0), H \oplus (0), (0) \oplus H, H \oplus H\}$. Then $\operatorname{Alg}\mathcal{B} = \{[\begin{smallmatrix}X&Y\\Z&T\end{smallmatrix}] : X, Y, Z, T \in \mathcal{B}(H)\}$. We show that $\mathcal{B}$ has index at most two.

Let $\{e_k : k \geq 1\}$ be an orthonormal basis for H and let A and B be operators of the type described in observation O3, with $Ae_k = (1/2^{k-1})e_k$, for all $k \geq 1$, and $B = e \otimes f$, where $e \perp f$ and with both e, f having no zero Fourier coefficients with respect to the orthonormal basis $\{e_k : k \geq 1\}$. (For example, take $e = \frac{1}{5}e_1 + \sum_{n=2}^{\infty}((-1)^{n-1}/2^{n-1})e_n$ and $f = \sum_{n=1}^{\infty}(1/2^{n-1})e_n$.) Then $B^2 = 0$ since $B^2 = e \otimes Bf = (f|e)B = 0$. It is easy to verify that A, B have no common nontrivial invariant subspaces, once it has been observed that $\operatorname{Lat}\{A\}$ is the atomic Boolean algebra subspace lattice with atoms $\{\langle e_k\rangle : k \geq 1\}$, so that $\operatorname{Lat}\{A\} = \{\bigvee_{k\in\mathcal{E}}\langle e_k\rangle : \mathcal{E} \subseteq \mathbb{Z}+\}$. (Let $\mathcal{D}$ be the weakly closed algebra generated by A. Since $A^k \to P_1$, in norm, where P_1 denotes the orthogonal projection onto $\langle e_1\rangle$, $P_1 \in \mathcal{D}$. Then, since $(2(A - P_1))^k \to P_2$, where P_2 denotes the orthogonal projection onto $\langle e_2\rangle$, $P_2 \in \mathcal{D}$. Next, consider $(A - P_1 - \frac{1}{2}P_2)^k$, and so on. It follows that $P_k \in \mathcal{D}$, for all $k \geq 1$, where P_k is the orthogonal projection onto $\langle e_k\rangle$.)

Let $X = [\begin{smallmatrix}A&0\\0&B\end{smallmatrix}]$ and $Y = [\begin{smallmatrix}B&0\\0&A\end{smallmatrix}]$. Clearly $\mathcal{B} \subseteq \operatorname{Lat}\{X, Y\}$. We claim that $\mathcal{B} = \operatorname{Lat}\{X, Y\}$.

We have $X^2 = [\begin{smallmatrix}A^2&0\\0&0\end{smallmatrix}]$ and $Y^2 = [\begin{smallmatrix}0&0\\0&A^2\end{smallmatrix}]$. Now the matrix of A^2 relative to the orthonormal basis $\{e_k : k \geq 1\}$ is diagonal with $A^2e_k = (1/4^{k-1})e_k$, for all $k \geq 1$. The weakly closed

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algebra generated by A^2 contains P_k, for all $k \geq 1$, where P_k denotes the orthogonal projection onto $\langle e_k \rangle$, so contains $Q_n = \sum_{k=1}^{n} P_k$. Since $Q_n \to I$, strongly, it follows that the weakly closed algebra generated by X contains $P = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$ (the orthogonal projection onto $H \oplus (0)$). Symmetrically, the weakly closed algebra generated by Y contains $Q = \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right]$ (the orthogonal projection onto $(0) \oplus H$). Let $M \in \mathrm{Lat}\{X, Y\}$. Then M is invariant under P and Q, so $(x, y) \in M$ implies $(x, 0) \in M$ and $(0, y) \in M$. It follows that $M = K \oplus L$ for some subspaces K, L of H. Since each of these subspaces is invariant under A and B, each is either (0) or H, so $M \in \mathcal{B}$.

The following theorem gives a more general result.

THEOREM 3.1. *Let $\mathcal{B}$ be an atomic Boolean algebra subspace lattice on a complex, separable, infinite-dimensional Hilbert space with the properties:*

(i) *for every $K, L \in \mathcal{B}$, the vector sum $K + L$ is closed;*
(ii) *$\mathcal{B}$ has n atoms, where $n \in \mathbb{Z}^+, n \geq 2$;*
(iii) *the atoms of $\mathcal{B}$ are equidimensional.*

Then the reflexive index of $\mathcal{B}$ is at most n.

PROOF. By [8, Theorem 3] there exists an invertible operator S acting on the given Hilbert space such that $\mathcal{B} = S\mathcal{L}$, where $\mathcal{L}$ is a commutative subspace lattice. By observation O6 above, we may as well suppose that $\mathcal{B}$ is commutative. We can then suppose that $\mathcal{B}$ is a subspace lattice on $H^{(n)} = H \oplus H \oplus \cdots \oplus H \oplus H$ with atoms $H_k = (0) \oplus (0) \oplus \cdots \oplus (0) \oplus H \oplus (0) \cdots \oplus (0) \oplus (0)$, where H occurs only in the kth position, $1 \leq k \leq n$, and where H is a complex, separable, infinite-dimensional Hilbert space. □

Let $\{e_k : k \geq 1\}$ be an orthonormal basis for H and let $A, B \in \mathcal{B}(H)$ be the operators as in the example immediately above. (So $Ae_k = (1/2^{k-1})e_k$, for all $k \geq 1$, and $B = e \otimes f$, where $e \perp f$ and with both e, f having no zero Fourier coefficients with respect to the orthonormal basis $\{e_k : k \geq 1\}$.) For each $1 \leq k \leq n$ let X_k be the operator on $H^{(n)}$ whose matrix is diagonal, $X_k = \mathrm{diag}\,(B, B, \ldots, B, A, B, \ldots, B, B)$, where the A occurs in the kth position. We show that $\mathcal{B} = \mathrm{Lat}\{X_k : 1 \leq k \leq n\}$. Clearly, $\mathcal{B} \subseteq \mathrm{Lat}\,\{X_k : 1 \leq k \leq n\}$. Now $X^2 = \mathrm{diag}\,(0, 0, \ldots, 0, A^2, 0, \ldots, 0, 0)$ and arguing as in E1 we get that $P_{H_k} = \mathrm{diag}\,(0, 0, \ldots, 0, I, 0, \ldots, 0, 0)$ belongs to the weakly closed algebra generated by X_k. Let $M \in \mathrm{Lat}\{X_k : 1 \leq k \leq n\}$. Then M is invariant under P_{H_k}, for every $1 \leq k \leq n$, so $M = K_1 \oplus K_2 \oplus \cdots \oplus K_{n-1} \oplus K_n$, for some subspaces K_k, $1 \leq k \leq n$, of H. Since each of the subspaces K_k is invariant under A and B, it is either (0) or H. Thus M is a span of atoms H_k of $\mathcal{B}$ and $M \in \mathcal{B}$.

3.2. E2. We next give examples of: (a) a subspace lattice on a separable Hilbert space with infinite reflexive index but with only five nontrivial elements; (b) a reflexive subspace lattice on a separable Hilbert space with infinite reflexive index.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be subspace lattices on the nonzero, complex, separable Hilbert spaces H_1 and H_2, respectively. Let $\mathcal{L}$ be the subspace lattice on $H_1 \oplus H_2$ defined by

$$\mathcal{L} = \{K \oplus (0) : K \in \mathcal{L}_1\} \cup \{H_1 \oplus L : L \in \mathcal{L}_2\}.$$

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Then

$$\operatorname{Alg}\mathcal{L} = \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} : A \in \operatorname{Alg}\mathcal{L}_1, C \in \operatorname{Alg}\mathcal{L}_2, B \in \mathcal{B}(H_2, H_1) \right\}$$

and

$$\operatorname{LatAlg}\mathcal{L} = \{K \oplus (0) : K \in \operatorname{LatAlg}\mathcal{L}_1\} \cup \{H_1 \oplus L : L \in \operatorname{LatAlg}\mathcal{L}_2\}.$$

Now let H_1 be infinite-dimensional and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be transitive subspace lattices. Then

$$\operatorname{LatAlg}\mathcal{L} = \{K \oplus (0) : K \in C(H_1)\} \cup \{H_1 \oplus L : L \in C(H_2)\}$$

and $\operatorname{Alg}\mathcal{L} = \{[\begin{smallmatrix} \alpha & B \\ 0 & \beta \end{smallmatrix}] : \alpha, \beta \in \mathbb{C}$ and $B \in \mathcal{B}(H_2, H_1)\}$. We show that $\mathcal{L}$ has infinite reflexive index.

Suppose that there exist operators X_k, $1 \le k \le n$, on $H_1 \oplus H_2$ with $\operatorname{LatAlg}\mathcal{L} = \operatorname{Lat}\{X_k : 1 \le k \le n\}$, where $n \in \mathbb{Z}^+$. Then $X_k = [\begin{smallmatrix} \alpha_k & B_k \\ 0 & \beta_k \end{smallmatrix}]$, for some scalars α_k, β_k and some operators B_k, $1 \le k \le n$. Let $y \in H_2$, $y \ne 0$. It is readily checked that the subspace $M = \langle B_1 y, B_2 y, \ldots, B_n y\rangle \oplus \langle y \rangle$ belongs to $\operatorname{Lat}\{X_k : 1 \le k \le n\}$. It does not belong to $\operatorname{Lat}\operatorname{Alg}\mathcal{L}$, however. This contradiction shows that $\mathcal{L}$ has infinite reflexive index.

(a) If we take $\mathcal{L}_1$ to be the transitive lattice with four nontrivial elements given in [9], and take $H_2 = \mathbb{C}$ in what is immediately above, we get a subspace lattice $\mathcal{L}$ with precisely five nontrivial elements, which has infinite reflexive index.

(b) With $\mathcal{L}$ the subspace lattice with seven elements described in (a), the reflexive subspace lattice $\operatorname{LatAlg}\mathcal{L}$ has infinite reflexive index.

3.3. E3. Finally, for every $n \in \mathbb{Z}^+$ we show that, on every Hilbert space H of dimension at least $n + 1$ (possibly infinity) there is a reflexive subspace lattice of reflexive index n.

Let $n \in \mathbb{Z}^+$ and suppose that $\dim H \ge n + 1$. We can suppose that $H = H_1 \oplus H_2$, where H_1 and H_2 are nonzero, complex, separable Hilbert spaces with $\dim H_1 \ge n$ (possibly infinity). Let $B_1, B_2, \ldots, B_n$ be operators in $\mathcal{B}(H_2, H_1)$ for which there exists a vector $e \in H_1$ such that $\{B_1 e, B_2 e, \ldots, B_n e\}$ is linearly independent. On $H_1 \oplus H_2$ let X_k, $1 \le k \le n$, be the operators defined by $X_k = [\begin{smallmatrix} 0 & B_k \\ 0 & 0 \end{smallmatrix}]$ and let $\mathcal{F}_n$ be the reflexive subspace lattice on $H_1 \oplus H_2$ given by $\mathcal{F}_n = \operatorname{Lat}\{X_k : 1 \le k \le n\}$.

We show that $\mathcal{F}_n$ has index n. Suppose that there were $n - 1$ operators $\{Y_k : 1 \le k \le n - 1\}$ such that $\mathcal{F}_n = \operatorname{Lat}\{Y_k : 1 \le k \le n - 1\}$. Then, since $\mathcal{L} = \{K \oplus (0) : K \in C(H_1)\} \cup \{H_1 \oplus L : L \in C(H_2)\} \subseteq \mathcal{F}_n$, each Y_k has the form $Y_k = [\begin{smallmatrix} \lambda_k & A_k \\ 0 & \mu_k \end{smallmatrix}]$, for some scalars λ_k, μ_k and some operators A_k, $1 \le k \le n - 1$. The subspace $\langle A_1 e, A_2 e, \ldots, A_{n-1} e\rangle \oplus \langle e \rangle$ belongs to $\operatorname{Lat}\{Y_k : 1 \le k \le n - 1\}$, so it belongs to $\operatorname{Lat}\{X_k : 1 \le k \le n\}$. Now $X_k(0, e) = (B_k e, 0)$, so $B_k e \in \langle A_1 e, A_2 e, \ldots, A_{n-1} e\rangle$, $1 \le k \le n$. This contradicts the fact that $\{B_1 e, B_2 e, \ldots, B_n e\}$ is linearly independent. Hence $\mathcal{F}_n$ has reflexive index n.

4. Some questions

Let H be a separable, infinite-dimensional Hilbert space. Every nest on H is reflexive, and so is every finite, distributive subspace lattice [5]. More generally, every completely distributive subspace lattice is reflexive [14]. A subspace lattice

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$\mathcal{L}$ is completely distributive if $M = \bigcap\{K_- : K \in \mathcal{L}$ and $K \nsubseteq M\}$, for every $M \in \mathcal{L}$, where $K_- = \vee\{L \in \mathcal{L} : K \nsubseteq L\}$. Every atomic Boolean subspace lattice is completely distributive. Also, all commutative subspace lattices are reflexive [1, 8].

Q1. Which completely distributive subspace lattices on H have finite reflexive index?

Q2. Does every finite distributive subspace lattice on H have finite reflexive index?

Q3. Does every commutative, atomic Boolean algebra subspace lattice on H with n equidimensional atoms have reflexive index n?

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