Canad. Math. Bull. Vol. **45** (3), 2002 pp. 364–377

Mellin Transforms of Whittaker Functions

Anton Deitmar

Abstract. In this note we show that for an arbitrary reductive Lie group and any admissible irreducible Banach representation the Mellin transforms of Whittaker functions extend to meromorphic functions. We locate the possible poles and show that they always lie along translates of walls of Weyl chambers.

Introduction

Whittaker functions occur naturally in the theory of automorphic forms as the Fourier coefficients of cusp forms. As a consequence, inner products of Poincaré series with cusp forms are Mellin transforms of Whittaker functions [14, 4]. In the case of groups of real rank one there are Poincaré series whose Mellin transforms give the Kloosterman zeta functions [2]. Thus the results of this paper (which in the rank one case are already in [2]) can be used to derive the meromorphicity of the Kloosterman zeta function for rank one groups. Mellin transforms of products of Whittaker functions are in certain cases equal to factors of Rankin-Selberg L-functions [11], [12]. In [15], E. Stade gives explicit expressions for the Mellin transforms of class one principal series Whittaker functions for the group $\mathrm{GL}_n(\mathbb{R})$ and thus verifies a conjecture of Goldfeld regarding the poles of this Mellin transform. In [8] S. Friedberg and D. Goldfeld show the meromorphicity of the Mellin transform of Whittaker functions attached to class one vectors of principal series representations in the case of a quasi-split group.

In the present paper we show the meromorphicity of the Mellin transform of the Whittaker function attached to an arbitrary differentiable vector of an arbitrary representation of an arbitrary reductive group. Since the Mellin transform is taken over the maximal split torus of the derived group it does not make a difference, for the purposes of this paper, to assume that the group is semisimple, and we will do so. It turns out, that the poles of the Mellin transform all lie along translates of walls of Weyl chambers.

1 Whittaker Functions

Let G be a semisimple connected Lie group with finite center. Fix a maximal compact subgroup K and let θ denote the Cartan involution with fixed point set K. We will write $\mathfrak{g}_0$ for the real Lie algebra of G and $\mathfrak{g}$ for its complexification. Next $U(\mathfrak{g})$ will denote the universal enveloping algebra of $\mathfrak{g}$. We will interpret $U(\mathfrak{g})$ as the algebra of all left invariant differential operators on G.

Received by the editors January 17, 2000.
AMS subject classification: Primary: 11F30; secondary: 22E30, 11F70, 22E45.

https://doi.org/10.4153/CMB-2002-039-5 Published online by Cambridge University Press

Let $P = MAN$ be a minimal parabolic subgroup, where we assume that A and hence also M are θ-stable. Let $\mathfrak{m}_0$, $\mathfrak{a}_0$, $\mathfrak{n}_0$ be the real Lie algebras of the groups M, A, N, and let $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$ be their complexifications. Let $\Phi^+ = \Phi^+(\mathfrak{a}, \mathfrak{g})$ be the set of positive roots of the pair $(\mathfrak{a}, \mathfrak{g})$ given by the choice of the parabolic P. For $\alpha \in \Phi^+(\mathfrak{a}, \mathfrak{g})$ let $\mathfrak{g}_\alpha$ denote its root space and let $m_\alpha = \dim \mathfrak{g}_\alpha$. Further let $\mathfrak{g}_{0,\alpha} = \mathfrak{g}_\alpha \cap \mathfrak{g}_0$. Then $\mathfrak{n}_0 = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{0,\alpha}$. Let $\rho_P = \frac{1}{2} \sum_{\alpha \in \Phi^+} m_\alpha \alpha$ be the modular shift of P.

The Killing form on $\mathfrak{g}$ is positive definite on $\mathfrak{a}_0$. It induces an identification of $\mathfrak{a}_0$ with its dual $\mathfrak{a}_0^*$ and also a bilinear form $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{a}_0^*$.

Let π be a continuous representation of G on some Banach space. Let π_∞ the Fréchet space of differentiable vectors for π. Fix a continuous linear functional $\psi = \psi_\eta$ on π_∞ such that

$$\psi\big(\pi(n)\phi\big) = \eta(n)\psi(\phi)$$

for every $\phi \in \pi_\infty$. Such a ψ is called a *Whittaker functional* to the character η. For $\phi \in \pi_\infty$ set

$$W_\phi(x) = \psi\big(\pi(x)\phi\big), \quad x \in G,$$

the corresponding *Whittaker function* on the group G. Since $G = NMAK$, it suffices, for growth questions, to consider W_ϕ restricted to MA.

Proposition 1.1 *Assume that π is admissible and of finite length. There is a finite set $\{D_j\} \subset U(\mathfrak{g})$ and a natural number k_0, depending on π, such that for all $\phi \in \pi_\infty$ we have*

$$|W_\phi(am)| \leq \sum_{w \in W} a^{k_0 w \rho_P} \sum_j \|D_j \phi\|.$$

Proof The continuity of ψ implies that there is a finite set $\{D'_j\} \subset U(\mathfrak{g})$ such that

$$|W_\phi(1)| \leq \sum_j \|D'_j \phi\|.$$

From [16], Lemma 2.2 we derive the existence of $c > 0$, $k' \in \mathbb{N}$ such that the operator norm can be estimated:

$$\|\pi(am)\| \leq c \sum_{w \in W} a^{k' w \rho_P}$$

for $a \in A$, $m \in M$. We thus get

$$\begin{aligned} W_\phi(am) &= W_{\pi(am)\phi}(1) \\ &\leq \sum_j \|D'_j \pi(am)\phi\| \\ &= \sum_j \|\pi(am)\, \mathrm{Ad}(am)^{-1} D'_j \phi\| \\ &\leq c \sum_{w \in W} a^{k' w \rho_P} \sum_j \|\, \mathrm{Ad}(am)^{-1} D'_j \phi\|. \end{aligned}$$

https://doi.org/10.4153/CMB-2002-039-5 Published online by Cambridge University Press

Let $U(\mathfrak{g})^\nu$ be the finite dimensional space of all $D \in U(\mathfrak{g})$ of degree $\leq \nu$. Let $\{D''_j\}$ be a basis of $U(\mathfrak{g})^\nu$. For ν large enough we get

$$\mathrm{Ad}(am)^{-1}D'_j = \sum_i a_{i,j}(am)D''_i.$$

By the properties of the adjoint action it follows that there is a constant $c_1 > 0$ with $a_{i,j}(am) \leq c_1 \sum_{w\in W} a^{2\nu\rho_P}$. The lemma follows with D_j being a multiple of D''_j. ∎

From the preceding lemma we will now conclude that the Whittaker function $W_\phi(am)$ actually is rapidly decreasing.

Proposition 1.2 *Let π be admissible and of finite length. Let $\alpha_0 \in \Phi^+(\mathfrak{a},\mathfrak{g})$ be a positive root such that $\log\eta$ is nontrivial on the root space $\mathfrak{g}_{0,\alpha_0} \subset \mathfrak{n}_0$. For every natural number N there are $D_1,\dots,D_m \in U(\mathfrak{g})$ such that for every $\phi \in \pi_\infty$ and every $a \in A$ we have*

$$|W_\phi(am)| \leq a^{-N\alpha_0}\Big(\sum_{w\in W} a^{k_0 w\rho_P}\Big)\Big(\sum_{j=1}^m \|D_j\phi\|\Big).$$

Proof Let $X_1,\dots,X_n$ be a basis of the root space $\mathfrak{g}_{0,\alpha_0}$. Since η is nontrivial on $\mathfrak{g}_{0,\alpha_0}$ the function

$$f(m) = \sum_j \big|\log\eta\big(\mathrm{Ad}(m)X_j\big)\big|$$

is nowhere vanishing on M. Since M is compact there is $c > 0$ with $f(m) \geq c$ for all $m \in M$. It follows

$$\begin{aligned}
W_{X_j\phi}(am) &= \frac{d}{dt}W_\phi\big(am\exp(tX_j)\big)\big|_{t=0}\\
&= \frac{d}{dt}W_\phi\Big(\exp\big(\mathrm{Ad}(am)tX_j\big)\,am\Big)\Big|_{t=0}\\
&= \frac{d}{dt}\eta\Big(\exp\big(t\,\mathrm{Ad}(am)X_j\big)\Big)\Big|_{t=0}W_\phi(am)\\
&= \log\eta\big(\mathrm{Ad}(am)X_j\big)\,W_\phi(am)\\
&= a^{\alpha_0}\log\eta\big(\mathrm{Ad}(m)X_j\big)\,W_\phi(am).
\end{aligned}$$

So for $a \in A$ it follows

$$\begin{aligned}
\sum_j |W_{X_j\phi}(am)| &= |W_\phi(am)|a^{\alpha_0}\sum_j\big|\log\eta\big(\mathrm{Ad}(m)X_j\big)\big|\\
&\geq c a^{\alpha_0}|W_\phi(am)|.
\end{aligned}$$

Iterating this process gives for arbitrary $N \in \mathbb{N}$ the existence of $\{D'_j\} \subset U(\mathfrak{n})$ such that

$$|W_\phi(am)| \leq a^{-N\alpha_0}\sum_j |W_{D'_j\phi}(am)|.$$

https://doi.org/10.4153/CMB-2002-039-5 Published online by Cambridge University Press

Applying the last lemma to $D'_j\phi$ gives the claim. ■

A character $N \to \mathbb{T}$ factors over $N^{\mathrm{ab}} = N/[N,N]$, where $[N,N]$ denotes the closed subgroup of N generated by all commutators $aba^{-1}b^{-1}$ for $a, b \in N$. Then N^{ab} is an abelian, simply connected Lie group with Lie algebra $\mathfrak{n}^{\mathrm{ab}} = \mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$, where in this case $[\cdot,\cdot]$ denotes the Lie bracket. It follows that N^{ab} is isomorphic to its Lie algebra and the characters of N thus identify with the linear functionals on $\mathfrak{n}^{\mathrm{ab}}$. Let

$$\mathfrak{n}_{\mathrm{simp}} = \bigoplus_{\substack{\alpha\in\Phi^+(\mathfrak{a},\mathfrak{g})\\ \alpha \text{ simple}}} \mathfrak{g}_\alpha$$

and $\mathfrak{n}_{0,\mathrm{simp}} = \mathfrak{n}_{\mathrm{simp}} \cap \mathfrak{g}_0$.

Lemma 1.3 *We have*

$$\mathfrak{n}_0 = \mathfrak{n}_{0,\mathrm{simp}} \oplus [\mathfrak{n}_0, \mathfrak{n}_0].$$

Proof This follows from Proposition 8.4d) of [10]. ■

It follows that each linear functional on $\mathfrak{n}_{0,\mathrm{simp}}$ extends to a character on N. This implies that there are characters η which are nontrivial on $\mathfrak{g}_{0,\alpha}$ for each simple root α. In this case we call η a *generic* character. Let $\Delta \subset \Phi^+$ be the set of simple roots.

Corollary 1.4 *Suppose that π is admissible of finite length and that the character η is generic. For every natural number N there are $D_1, \dots, D_m \in U(\mathfrak{g})$ such that for every $\phi \in \pi_\infty$ and every $a \in A$ we have*

$$|W_\phi(am)| \le \min_{\alpha\in\Delta} a^{-N\alpha}\Big(\sum_{w\in W} a^{k_0 w\rho_P}\Big)\Big(\sum_{j=1}^m \|D_j\phi\|\Big).$$

Proof This is a direct consequence of the proposition. ■

Let Γ be an arithmetic subgroup of G for which P is cuspidal. To be able to apply this corollary we have to make sure that there are generic characters which are trivial on $\Gamma \cap N$. By [13], Theorem 1.13 we infer that the image of Γ in N^{ab} is a lattice, which implies that N^{ab}/Γ is a torus. This implies the existence of an abundance of generic characters that are trivial on Γ.

2 The Mellin Transform, Rank One

This section rests on the method developed in [2] for the case $\mathrm{SO}(n,1)$.

Let τ be a finite dimensional unitary representation of the compact group M. Let ξ_τ be an arbitrary matrix coefficient of the representation τ. For $\phi \in \pi_\infty$ and $\lambda \in \mathfrak{a}^*$ let

$$I_\phi(\xi_\tau, \lambda) = \int_A\int_M W_\phi(am)\overline{\xi_\tau(m)}a^{\lambda-\rho_P}\,da\,dm.$$

https://doi.org/10.4153/CMB-2002-039-5 Published online by Cambridge University Press

For $\mu \in \mathfrak{a}_0^*$ we write $\mu > 0$ if $\langle \mu, \alpha \rangle > 0$ for all $\alpha \in \Phi^+$ and $\mu > \nu$ if $\mu - \nu > 0$. For the rest of this section we assume that π is admissible of finite length. From the last two lemmas we get

Proposition 2.1 *Suppose η is generic, then the integral $I_\phi(\xi_\tau, \lambda)$ converges absolutely for $\mathrm{Re}(\lambda) > k_0 \rho_P$, where k_0 is the number of Proposition 1.1. The linear functional $\phi \mapsto I_\phi(\xi_\tau, \lambda)$ is continuous on π_∞.*

Let (X_j) be a basis of $\mathfrak{n}$ such that each $X_j \in \mathfrak{g}_{0,\alpha_j}$ for some root α_j. Write $Z(\mathfrak{g})$ for the center of $U(\mathfrak{g})$.

Lemma 2.2 *Let $D \in Z(\mathfrak{g})$, then D can be written as*

$$D = D_{AM} + \sum_j X_j D_j$$

with $D_{AM} \in Z(\mathfrak{a} \oplus \mathfrak{m})$ and $D_j \in U(\mathfrak{g})$. Moreover we have $D_{AM} \in Z(\mathfrak{a} \oplus \mathfrak{m})^M$ the subalgebra of M-invariants.

Proof Write $\mathfrak{n}^- = \theta(n)$, then we have the decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}$$

and hence, by the Poincaré-Birkhoff-Witt Theorem:

$$U(\mathfrak{g}) = U(\mathfrak{a} \oplus \mathfrak{m}) \oplus \left(\mathfrak{n} U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{n}^- \right).$$

So $D \in Z(\mathfrak{g})$ can be written as $D_{AM} + f$ with $f \in \mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^-$. Since G is connected we get $Z(\mathfrak{g}) = U(\mathfrak{g})^G$, so for $m \in M$ and $D \in Z(\mathfrak{g})$ we have $\mathrm{Ad}(m)D = D$. The decomposition above is stable under M, so it follows that $\mathrm{Ad}(m)D_{AM} = D_{AM}$, which implies

$$Z(\mathfrak{g}) \subset (\mathfrak{a} \oplus \mathfrak{m})^M \oplus \left(\mathfrak{n} U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{n}^- \right).$$

Writing $X_j^- = \theta(X_j)$ we see by the Poincaré-Birkhoff-Witt Theorem that f is a sum of monomials of the form

$$X^a D_1 (X^-)^b,$$

where for $a, b \in \mathbb{Z}_+^d$ with $d = \dim N$:

$$X^a = X_1^{a_1} \cdots X_d^{a_d}, \quad (X^-)^b = (X_1^-)^{b_1} \cdots (X_d^-)^{b_d},$$

and $D_1 \in U(\mathfrak{a} \oplus \mathfrak{m})$. For $H \in \mathfrak{a}$ we compute that

$$[H, X^a D_1 (X^-)^b]$$

equals

$$\left(a_1 \alpha_1(H) + \cdots + a_d \alpha_d(H) - b_1 \theta(\alpha_1)(H) - \cdots - b_d \theta(\alpha_d)(H) \right) X^a D_1 (X^-)^b.$$

https://doi.org/10.4153/CMB-2002-039-5 Published online by Cambridge University Press

Since f lies in $\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^-$ it follows that only monomials occur with a and b not both zero. Next, since f commutes with each $H \in \mathfrak{a}$ it follows that for each monomial both a and b are nonzero, which implies the lemma. ■

Assume now that the representation π is *quasi-simple*, which means that the center $Z(\mathfrak{g})$ acts by scalars. Let $\wedge_\pi \colon Z(\mathfrak{g}) \to \mathbb{C}$ denote the *infinitesimal character* of π, *i.e.*

$$\pi(D)\phi = \wedge_\pi(D)\phi$$

holds for every $\phi \in \pi_\infty$, $D \in Z(\mathfrak{g})$. For $\sigma \in \hat{M}$ and $\nu \in \mathfrak{a}^*$ the representation

$$\pi_{\sigma,\nu} = \operatorname{Ind}_P^G\big(\sigma \otimes (\nu + \rho_P) \otimes 1\big)$$

is known to be quasi-simple. Let $\wedge_{\sigma,\nu}$ denote its infinitesimal character.

Proposition 2.3 *For $D \in Z(\mathfrak{g})$ write $D = D_{AM} + \sum_{j=1}^d X_j D_j$ as in the last lemma. Let r_j denote the matrix coefficient $r_j(m) = \overline{\eta\big(-\operatorname{Ad}(m)X_j\big)}$. Then, for $\langle \operatorname{Re}(\lambda), \rho_P\rangle > 0$ we have*

$$I_\phi(\xi_\tau, \lambda) = \frac{\sum_{j=1}^d I_{D_j\phi}(r_j\xi_\tau, \lambda + \alpha_j)}{\wedge_\pi(D) - \wedge_{\tau,\rho_P-\lambda}(D)}.$$

Proof For $D \in Z(\mathfrak{g})$ we have on the one hand

$$I_{D\phi}(\xi_\tau, \lambda) = \wedge_\pi(D) I_\phi(\xi_\tau, \lambda),$$

and on the other

$$I_{D\phi}(\xi_\tau, \lambda) = I_{D_{AM}\phi}(\xi_\tau, \lambda) + \sum_{j=1}^d I_{X_j D_j \phi}(\xi_\tau, \lambda).$$

We compute

$$\begin{aligned} I_{X_j\phi}(\xi_\tau, \lambda) &= \int_A \int_M W_{X_j\phi}(am)\overline{\xi_\tau(m)} a^{\lambda-\rho_P}\, dm\, da \\ &= \int_A \int_M a^{\alpha_j} \log \eta\big(\operatorname{Ad}(m)X_j\big) W_\phi(am)\overline{\xi_\tau(m)} a^{\lambda-\rho_P}\, dm\, da \\ &= I_\phi(r_j\xi_\tau, \lambda + \alpha_j) \end{aligned}$$

To prove the proposition it remains to show

$$I_{D_{AM}\phi}(\xi_\tau, \lambda) = \wedge_{\tau,\rho-\lambda}(D) I_\phi(\xi_\tau, \lambda).$$

Fix $\lambda \in \mathfrak{a}^*$ with $\operatorname{Re}(\lambda)$ large. Let $\check{\tau}$ denote the contragredient representation to τ and consider the representation γ of AM given by

$$\gamma = (\lambda - \rho_P) \otimes \check{\tau}.$$

https://doi.org/10.4153/CMB-2002-039-5 Published online by Cambridge University Press

Note that the function $am \mapsto \overline{\xi_\tau(m)} a^{\lambda-\rho_P}$ is a matrix coefficient of γ, we denote it by $\xi_\gamma(am)$.

On $U(\mathfrak{g})$ we have a unique $\mathbb{C}$-linear involuntary anti-automorphism given by $X' = -X$ for $X \in \mathfrak{g}$. For $D \in U(\mathfrak{g})$ we have

$$\int_G Df(x)g(x)\,dx = \int_G f(x)D'g(x)\,dx.$$

We compute

$$\begin{aligned} I_{D_{AM}\phi}(\xi_\tau, \lambda) &= \int_A \int_M W_{D_{AM}\phi}(am)\overline{\xi_\tau(m)} a^{\lambda-\rho_P}\,dm\,da \\ &= \int_A \int_M D_{AM} W_\phi(am)\xi_\gamma(am)\,dm\,da \\ &= \int_A \int_M W_\phi(am) D'_{AM}\xi_\gamma(am)\,dm\,da \\ &= \wedge_{\tau,\rho_P-\lambda}(D) I_\phi(\xi_\tau, \lambda) \end{aligned}$$

The proposition follows. ■

Let $\mathfrak{b} \subset \mathfrak{m}$ be a Cartan subalgebra and let $\mathcal{W} = W(\mathfrak{a} \oplus \mathfrak{b}, \mathfrak{g})$ be the big Weyl group. Via the Harish-Chandra homomorphism the infinitesimal character $\wedge_\pi$ can be viewed as (a Weyl group orbit of) an element of the dual space of $\mathfrak{a} \oplus \mathfrak{b}$.

Let $r = \dim A$. If $r = 1$ there are at most two positive roots in $\Phi^+(\mathfrak{a}, \mathfrak{g})$. Let α_0 be the short positive root in this case. Let R denote the adjoint representation of M on $\mathfrak{n}$. Let $\lambda_\tau \in \mathfrak{b}^*$ be the infinitesimal character of τ.

Theorem 2.4 *If $r = 1$ then the function $I_\phi(\xi_\tau, \lambda)$ has a meromorphic continuation to $\mathfrak{a}^*$. There is a possible pole at λ only if the following conditions are satisfied: There are integers $0 \le l \le k$ and an irreducible subrepresentation γ of $\tau \otimes R^{\otimes k}$ such that, λ_γ denoting its infinitesimal character we have that*

$$\lambda_\gamma + \rho_P - \big(\lambda + (k+l)\alpha_0\big)$$

lies in the Weyl group orbit of $\wedge_\pi$.

Proof The function $I_\phi(\xi_\tau, \lambda)$ is holomorphic in the region $\mathrm{Re}(\lambda) > k_0\rho_P$. Let α_1 be the short positive root. Now Proposition 2.3 shows that $I_\phi(\xi_\tau, \lambda)$ extends to a meromorphic function on $\{\mathrm{Re}(\lambda) > k_0\rho_P - \alpha_1\}$ with possible poles where

$$\wedge_\pi(D) = \wedge_{\tau,\rho_P-\lambda}(D)$$

for every $D \in Z(\mathfrak{g})$. This can only be when the two infinitesimal characters $\wedge_\pi$ and $\wedge_{\tau,\rho_P-\lambda}$ are in the same Weyl group orbit. We iterate this replacing ξ_τ by $r_j\xi_\tau$ which is a matrix coefficient of $\tau \otimes R$. Further iteration gives the claim in the case $r = 1$. ■

https://doi.org/10.4153/CMB-2002-039-5 Published online by Cambridge University Press

3 The Higher Rank Case

For every simple root $\alpha \in \Delta$ let G_α be the connected Lie subgroup of G with Lie algebra generated by $\mathfrak{g}_{0,-\alpha} \oplus \mathfrak{g}_{0,\alpha}$. Then G_α is semisimple of real rank one with split torus

$$A_\alpha = \{a \in A \mid a^\beta = 1 \text{ for all } \beta \in \Delta, \beta \neq \alpha\}.$$

Let $\mathfrak{a}_{\alpha,0}$ be the Lie algebra of A_α and $\mathfrak{a}_\alpha$ its complexification. Let $M_\alpha = M \cap G_\alpha$ and $\tau_\alpha = \tau|_{M_\alpha}$.

The set Δ is a basis of $\mathfrak{a}^*$, hence every $\lambda \in \mathfrak{a}^*$ can uniquely be written as $\lambda = \sum_{\alpha\in\Delta} \lambda_\alpha$, where $\lambda_\alpha \in \mathfrak{a}_\alpha^*$. Let m_α be the multiplicity of the root α and let $\rho_P^\Delta = \frac{1}{2}\sum_{\alpha\in\mathbb{N}\Delta} m_\alpha \alpha$.

Theorem 3.1 *Let π be an irreducible admissible Banach representation of G. Let $\phi \in \pi^\infty$, then the function $\lambda \mapsto I_\phi(\xi_\tau, \lambda)$ extends to a meromorphic function on $\mathfrak{a}^*$ with possible poles along the sets $(\lambda - 2\rho_P^\Delta)_\alpha = c_\alpha$, where $\alpha \in \Delta$ and $c_\alpha \in \mathfrak{a}_\alpha^*$ is such that there is an irreducible G_α-subquotient π_α of π and there are integers $0 \leq l \leq k$ and an irreducible subrepresentation γ of $\tau_\alpha \otimes (\mathrm{Ad}\,|\mathfrak{n}_\alpha)^{\otimes k}$ such that, $\wedge_\gamma$ denoting its infinitesimal character we have that*

$$\wedge_\gamma + \rho_P^\alpha - \big(c_\alpha + (k+l)\alpha\big)$$

lies in the Weyl orbit of $\wedge_{\pi_\alpha}$.

The proof of this theorem will occupy the rest of the section. We start by considering a special case. So let (σ, V_σ) be an irreducible unitary representation of M. Since M is compact it follows that σ is finite dimensional. Let $\nu \in \mathfrak{a}^*$ and let $\bar\pi_{\sigma,\nu}$ be the corresponding principal series representation induced from the parabolic $\bar P = MA\bar N$ opposite to P. The representation is defined to be the right regular representation on the Hilbert space $H_{\sigma,\nu}$ of all functions $f\colon G \to V_\sigma$ satisfying $f(ma\bar n x) = a^{\nu-\rho_P}\sigma(m)f(x)$ for $ma\bar n \in MA\bar N$ and $\int_K \|f(k)\|^2\,dk < \infty$, modulo nullfunctions. The space of smooth vectors $\bar\pi_{\sigma,\nu}^\infty$ coincides with the set of f which are smooth on G. We especially consider f of the form

$$f(ma\bar n n) = a^{\nu-\rho_P}\varphi(n)$$

for $n \in N$, where $\varphi \in C_c^\infty(N, V_\sigma)$. This function, defined on the open Bruhat cell $\bar P P$, extends by zero to a smooth function on G. Let $U_{\sigma,\nu}^\infty \subset H_{\sigma,\nu}^\infty$ denote the subset of all f of this form.

Let H_σ^∞ be the space of all smooth $f\colon K \to V_\sigma$ with $f(mk) = \sigma(m)f(k)$ for $m \in M, k \in K$. For $\nu \in \mathfrak{a}^*$ the function

$$f_\nu(ma\bar n k) = a^{\nu-\rho_P}\sigma(m)f(k)$$

defines an element of $H_{\sigma,\nu}^\infty$ and this attachment sets up an isomorphism of Frechét spaces $H_\sigma^\infty \to H_{\sigma,\nu}^\infty$ for any ν. Let $U_\sigma^\infty \subset H_\sigma^\infty$ be the inverse image of $U_{\sigma,\nu}^\infty$ then this space does not depend on ν, which justifies the notation.

From [17], Theorem 15.4.1 and Theorem 15.6.1 we take

Theorem 3.2 *Let η be generic and let $\nu \in \mathfrak{a}^*$ with $\mathrm{Re}(\nu) < 0$. Then for any $f \in H_\sigma^\infty$ the integral*

$$J_{\sigma,\nu}(f) = \int_N \eta(n)^{-1} f_\nu(n)\, dn$$

converges and extends to a holomorphic map on $\mathfrak{a}^*$.

Let ψ be any Whittaker functional on $\bar{\pi}_{\sigma,\nu}^\infty$, then $\psi(U_{\sigma,\nu}^\infty) = 0$ implies $\psi = 0$. Further, for any Whittaker functional ψ there is a functional μ on V_σ such that

$$\psi(f) = \mu\big(J_{\sigma,\nu}(f)\big).$$

Fix ν with $\mathrm{Re}(\nu) < 0$. Let $f \in U_\sigma^\infty$ and let $\varphi \in C_c^\infty(N, V_\sigma)$ be the function such that $f_\nu(ma\bar{n}n) = a^{\nu-\rho_P}\sigma(m)\varphi(n)$. Let ψ be a Whittaker functional and μ be the corresponding functional on V_σ. Then we have

$$\begin{aligned} W_{f_\nu}(am) &= \mu\Big(\int_N \eta(n)^{-1} f_\nu(nam)\, dn\Big) \\ &= \int_N \eta(n)^{-1} a^{\nu-\rho_P}\mu\big(\sigma(m)\varphi(n^{am})\big)\, dn. \end{aligned}$$

Thus for $\lambda \in \mathfrak{a}^*$ with $\mathrm{Re}(\lambda) > k_0\rho_P$ we get that $I_{f_\nu}(\xi_\tau, \lambda)$ equals

$$\tilde{I}_f(\xi_\tau, \lambda+\nu) = \int_A\int_N\int_M \eta(n)^{-1} a^{\lambda+\nu-2\rho_P}\mu\big(\sigma(m)\varphi(n^{am})\big)\,\overline{\xi_\tau(m)}\, dm\, dn\, da.$$

Recall that $\Delta \subset \Phi^+$ denotes the set of simple roots. Let $N_0 = \exp(\bigoplus_{\alpha\notin\mathbb{N}\Delta}\mathfrak{n}_\alpha)$ and $N_\alpha = \exp(\mathfrak{n}_\alpha \oplus \mathfrak{n}_{2\alpha})$ for $\alpha \in \Delta$. For $n \in N$ define

$$\tilde{\varphi}(n) = \tilde{\varphi}_{\mu,\sigma,\tau}(n) = \int_M \mu\big(\sigma(m)\varphi(n^m)\big)\,\overline{\xi_\tau(m)}\, dm.$$

We have a canonical identification $C_c^\infty(N, V_\sigma) \cong C_c^\infty(N) \otimes V_\sigma$. We equip $C_c^\infty(N)$ with the usual inductive limit topology, then the space $C_c^\infty(N)$ contains as a dense subspace the algebraic tensor product

$$T = \Big(\bigotimes_{\alpha\in\Delta} C_c^\infty(N_\alpha)\Big) \otimes C_c^\infty(N_0).$$

Let $T_{\sigma,\nu} \subset U_{\sigma,\nu}$ be the subspace of all f as above with $\varphi = f_\nu|_N$ in $T \otimes V_\sigma$. Now suppose φ lies in $T \otimes V_\sigma$, then $\tilde{\varphi}$ lies in T, since the spaces $\mathfrak{g}_{0,\alpha}$, $\alpha \in \Phi^+$ are stable under M. So suppose that $\tilde{\varphi}$ is a finite sum

$$\tilde{\varphi} = \sum_{i=1}^k \Big(\prod_{\alpha\in\Delta}\tilde{\varphi}_{i,\alpha}\Big)\tilde{\varphi}_{i,0},$$

where $\tilde{\varphi}_{i,\alpha} \in C_c^\infty(N_\alpha)$ and $\tilde{\varphi}_{i,0} \in C_c^\infty(N_0)$. Then

$$I_{f_\nu}(\xi_\tau, \lambda) = \int_A \int_N \eta(n)^{-1} a^{\lambda+\nu} \tilde{\varphi}(n^a)\, dn\, da$$
$$= \sum_{i=1}^{k} \int_A a^{\lambda+\nu-2\rho_P} \prod_{\alpha\in\Delta} \int_{N_\alpha} \eta(n_\alpha^{-1}) \tilde{\varphi}_{i,\alpha}(n_\alpha^a)\, dn_\alpha \int_{N_0} \tilde{\varphi}_{i,0}(n^a)\, dn\, da$$

Let $\rho_P^\Delta = \frac{1}{2}\sum_{\alpha\in\mathbb{N}\Delta} m_\alpha \alpha$, then

$$\int_{N_0} \tilde{\varphi}_{i,0}(n^a)\, dn = a^{2(\rho_P - \rho_P^\Delta)} \int_{N_0} \tilde{\varphi}_{i,0}(n)\, dn.$$

We may assume that

$$\int_{N_0} \tilde{\varphi}_{i,0}(n)\, dn = 1$$

for any i. Then we get

$$I_{f_\nu}(\xi_\tau, \lambda) = \sum_{i=1}^{k} \int_A a^{\lambda+\nu-2\rho_P^\Delta} \prod_{\alpha\in\Delta} \int_{N_\alpha} \eta(n_\alpha^{-1}) \tilde{\varphi}_{i,\alpha}(n_\alpha^a)\, dn_\alpha\, da.$$

We can write $A = \prod_{\alpha\in\Delta} A_\alpha$, where $A_\alpha = \{a \in A \mid a^\beta = 1 \text{ for all } \beta \in \Delta, \beta \neq \alpha\}$. Then we get

$$I_{f_\nu}(\xi_\tau, \lambda) = \sum_{i=1}^{k} \prod_{\alpha\in\Delta} \int_{A_\alpha} a^{\lambda+\nu-2\rho_P^\Delta} \int_{N_\alpha} \eta(n_\alpha^{-1}) \tilde{\varphi}_{i,\alpha}(n_\alpha^a)\, dn_\alpha\, da.$$

Fix an index i and a simple root α. Let G_α be the connected subgroup of G corresponding to the Lie subalgebra generated by

$$\mathfrak{n}_{-\alpha} \oplus \mathfrak{n}_\alpha.$$

Then G is a real rank one semisimple group with split torus A_α. Let $P_\alpha = P \cap G_\alpha$ and $M_\alpha = M \cap G_\alpha$, then $P_\alpha = M_\alpha A_\alpha N_\alpha$ is a minimal parabolic of G_α. Mapping a function h on M to $m_0 \mapsto \int_M h(m_0 m)\overline{\xi_\tau(m)}\, dm$ is an L^2-projection onto the part of $L^2(M)$ spanned by ξ_τ. Therefore it follows that for any $n \in N_\alpha$:

$$\int_{M_\alpha} \tilde{\varphi}_{i,\alpha}(n^m)\overline{\xi_\tau(m)}\, dm = \tilde{\varphi}_{i,\alpha}(n).$$

Writing ξ_τ also for $\xi_\tau|_{M_\alpha}$ we get

Lemma 3.3 *We have*

$$I_{f_\nu}(\xi_\tau, \lambda) = \sum_{i=1}^{k} \prod_{\alpha\in\Delta} I^{G_\alpha}_{f_{i,\alpha,\nu}}(\xi_\tau, \lambda - 2\rho_P^\Delta),$$

https://doi.org/10.4153/CMB-2002-039-5 Published online by Cambridge University Press

where $I^{G_\alpha}_{f_{i,\alpha,\nu}}(\xi_\tau, \lambda - 2\rho^\Delta_P)$ denotes the Mellin transform with respect to the group G_α and $f_{i,\alpha,\nu} \in \mathrm{Ind}^{G_\alpha}_{P_\alpha}(\sigma \otimes \nu \otimes 1)$ is given by

$$f_{i,\alpha,\nu}(m_\alpha a_\alpha \bar{n}_\alpha n_\alpha) = a_\alpha^{\nu+\rho^\alpha_P} \sigma(m_\alpha)\varphi_{i,\alpha}(n_\alpha),$$

and

$$\varphi_{i,\alpha}(n_\alpha) = \int_M \sigma(m)\varphi(n^m_\alpha)\overline{\xi_\tau(m)}\,dm.$$

Note that $I^{G_\alpha}_{f_{i,\alpha,\nu}}(\xi_\tau, \lambda - 2\rho^\Delta_P)$ only depends on the restriction of λ to A_α. By the results of Section 2 it follows that $I^{G_\alpha}_{f_{i,\alpha,\nu}}(\xi_\tau, \lambda - 2\rho^\Delta_P)$ extends to a meromorphic function and so then does $I_{f_\nu}(\xi_\tau, \lambda)$. The position of possible poles can be read off from Theorem 2.4

We next want to show that $I_{f_\nu}(\xi_\tau, \lambda)$ extends to a meromorphic function for any $f_\nu \in H^\infty_{\sigma,\nu}$. We need the

Lemma 3.4 *Let $f \in H^\infty_{\sigma,\nu}$, then for every $d \in \mathbb{N}$ there is a sequence $f_j \in T^\infty_{\sigma,\nu}$ such that $I_{Df_j}(\xi_\tau, \lambda)$ converges to $I_{Df}(\xi_\tau, \lambda)$ locally uniformly on $\mathrm{Re}(\lambda) > k_0\rho_P$ for every $D \in U(\mathfrak{g})$ of degree $\leq d$.*

Proof Fix $d \in \mathbb{N}$. We show the lemma in two steps. First assume that $f \in U^\infty_{\sigma,\nu}$, *i.e.* $f(ma\bar{n}n) = a^{\nu-\rho_P}\sigma(m)\varphi(n)$, where $\varphi \in C^\infty_c(N, V_\sigma)$. Then there is a sequence $(\varphi_j)_{j\in\mathbb{N}}$ in $T \otimes V_\sigma$ such that φ_j converges to φ in the inductive limit topology. Let f_j be defined by $f_j(ma\bar{n}n) = a^{\nu-\rho_P}\sigma(m)\varphi_j(n)$. Let $N \in \mathbb{N}$, then by Corollary 1.4 we know we know that there are $D_1, \dots, D_m \in U(\mathfrak{g})$ with

$$|W_{f_j}(am)| \leq \min_{\alpha\in\Delta} a^{-N\alpha}\Big(\sum_{w\in W} a^{k_0 w\rho_P}\Big)\Big(\sum_{k=1}^m \|D_k f_j\|\Big).$$

Since φ_j converges to φ we conclude that $\|D_k f_j\|$ converges to $\|D_k f\|$ for every k, which implies that there is a constant $C > 0$ such that

$$|W_{f_j}(am)| \leq C \min_{\alpha\in\Delta} a^{-N\alpha}\Big(\sum_{w\in W} a^{k_0 w\rho_P}\Big),$$

for all $j \in \mathbb{N}$. The claim follows by dominated convergence.

To prove the general case let now $f \in H^\infty_{\sigma,\nu}$ and $d \in \mathbb{N}$. By the first part of this proof it now suffices to show that there is a sequence $f_j \in U^\infty_{\sigma,\nu}$ such that $I_{Df_j}(\xi_\tau, \lambda)$ converges to $I_{Df}(\xi_\tau, \lambda)$ for every $D \in U(\mathfrak{g})$ of degree $\leq d$. For this let $(\delta_j)_j$ be a sequence of functions $N \to [0, 1]$ such that

- $\delta_j \in C^\infty_c(N)$ for every $j \in \mathbb{N}$, and
- $\delta_{j+1} \geq \delta_j$ for every $j \in \mathbb{N}$, and
- for every $n \in N$ there is a $j \in \mathbb{N}$ with $\delta_j(n) = 1$.

https://doi.org/10.4153/CMB-2002-039-5 Published online by Cambridge University Press

Now set $f_j(ma\bar{n}n) = a^{\nu-\rho_P}\sigma(m)f(n)\delta_j(n)$, then $f_j \in U^\infty_{\sigma,\nu}$ and f_j converges pointwise on $\bar{P}N$ to f.

For $f \in H^\infty_{\sigma,\nu}$ let

$$S(f) = \int_N \|f(n)\|\, dn.$$

On p. 382 of [17] it is shown that this integral converges. Then S defines a norm on $H^\infty_{\sigma,\nu}$ and

$$\|J_{\sigma,\nu}(f)\| \le S(f).$$

We compute

$$\int_N \|f(nam)\|\, dn = a^{\mathrm{Re}(\nu)+\rho_P} \int_N \|f(n)\|\, dn,$$

so that

$$W_f(am) \le a^{\mathrm{Re}(\nu)+\rho} S(f).$$

The proof of Proposition 1.2 now gives

Lemma 3.5 *There are $D_1, \dots, D_m \in U(\mathfrak{n})$ such that*

$$|W_{f_j}(am)| \le \min_{\alpha\in\Delta} a^{-N\alpha} a^{\mathrm{Re}(\nu)+\rho_P} \sum_{l=1}^m S(D_l f_j).$$

Finally the sequence δ_j can be chosen so that their $U(\mathfrak{n})$-derivatives remain bounded, so there is a constant $C > 0$ so that $S(D_l f_j) \le C$ for all l and all j. This implies Lemma 3.4. ■

We want to apply the preceding lemma to derive the meromorphic continuation of $I_{f_\nu}(\xi,\lambda)$ for general f_ν. In order to do this we need

Lemma 3.6 *Let $f_\nu \in T^\infty_{\sigma,\nu}$ and write as in Lemma 3.3:*

$$I_{f_\nu}(\xi_\tau,\lambda) = \sum_{i=1}^k \prod_{\alpha\in\Delta} I^{G_\alpha}_{f_{i,\alpha,\nu}}(\xi_\tau, \lambda - 2\rho_P^\Delta).$$

For $\alpha \in \Delta$ let $D_\alpha \in U\big(\mathrm{Lie}(G_\alpha)\big)$, then there is $D \in U(\mathfrak{g})$ such that

$$I_{Df_\nu}(\xi_\tau,\lambda) = \sum_{i=1}^k \prod_{\alpha\in\Delta} I^{G_\alpha}_{D_\alpha f_{i,\alpha,\nu}}(\xi_\tau, \lambda - 2\rho_P^\Delta).$$

Proof It suffices to consider the case $D_\alpha = Id$ if $\alpha \ne \alpha_0$ for some fixed α_0 and $D_{\alpha_0} = X \in \mathrm{Lie}(G_{\alpha_0})$. For this it suffices to assume that X lies in a generating set of $\mathrm{Lie}(G_{\alpha_0})$, so, one may take X inside $\mathfrak{n}_{-\alpha_0}$ or inside $\mathfrak{n}_{\alpha_0}$. In both cases we take $D = X$ and the claim follows since the factors associated to $\alpha \ne \alpha_0$ are annihilated by X. ■

The fact that $I_{f_\nu}(\xi_\tau, \lambda)$ admits a meromorphic continuation now follows from Lemma 3.4, Lemma 3.3 and Proposition 2.3. This all works for $\mathrm{Re}(\nu) > 0$. But since

$$I_{f_\nu}(\xi_\tau, \lambda) = \tilde{I}_f(\xi_\tau, \lambda + \nu),$$

we automatically get the meromorphicity for all ν.

We intend to generalize this result to an arbitrary Banach representation π. For this we need

Lemma 3.7 *Let π be an admissible irreducible Banach representation of G, then there are $\sigma \in \hat{M}$ and $\nu \in \mathfrak{a}^*$ such that the Frechét representation π^∞ is a quotient of $\pi^\infty_{\sigma,\nu}$.*

Proof Let π_K denote the admissible irreducible $(\mathfrak{g}, K)$-module of K-finite vectors in π. The dual $(\mathfrak{g}, K)$-module $\check{\pi}_K$ also is admissible and irreducible. By the subrepresentation theorem there exist $\sigma \in \hat{M}$ and $\nu \in \mathfrak{a}^*$ such that $\check{\pi}_K$ injects into $\bar{\pi}_{\check{\sigma},\nu,K}$. Dualizing we get a nontrivial and hence surjective $(\mathfrak{g}, K)$-map from $\bar{\pi}_{\sigma,\nu,K}$ to π_K, which proves the claim for the underlying $(\mathfrak{g}, K)$-modules. By Corollary 10.5 of [6] the claim follows. ∎

So let finally π be an arbitrary admissible irreducible Banach representation of G. Let $\phi \in \pi^\infty$ and let $F\colon \bar{\pi}^\infty_{\sigma,\nu} \to \pi^\infty$ be a surjective homomorphism. Pick some $f \in \bar{\pi}^\infty_{\sigma,\nu}$ such that $F(f) = \phi$. Then $\psi \circ F$ is a Whittaker functional on $\bar{\pi}^\infty_{\sigma,\nu}$. It then follows

$$I_\phi(\xi_\tau, \lambda) = I_f(\xi_\tau, \lambda),$$

so the meromorphic continuation of $I_\phi(\xi_\tau, \lambda)$ is established. Theorem 3.1 is proven.

References

[1] J. Cogdell, J. Li and I. Pyatetskii-Shapiro, *The meromorphic continuation of Kloosterman-Selberg Zeta functions*. Springer Lecture Notes **1422**(1990), 23–35.

[2] J. Cogdell, J. Li, I. Pyatetskii-Shapiro and P. Sarnak, *Poincaré series for* $\mathrm{SO}(n, 1)$. Acta Math. 167, 229-285 (1991).

[3] D. Bump, *Automorphic Forms on* $\mathrm{GL}(3, \mathbb{R})$. Lecture Notes in Math. **1083**, Springer-Verlag, 1984.

[4] D. Bump, S. Friedberg D. Goldfeld, *Poincaré series and Kloosterman sums for* $\mathrm{SL}(3, \mathbb{Z})$. Acta Arith. **50**(1988), 31–89.

[5] D. Bump and S. Friedberg, *On Mellin transforms of unramified Whittaker functions on* $\mathrm{GL}(3, \mathbb{C})$. J. Math. Anal. Appl. **139**(1989), 205–216.

[6] W. Casselman, *Canonical extensions of Harish-Chandra modules to representations of G*. Canad. J. Math. **41**(1989), 385–438.

[7] J. Dixmier and P. Malliavin, *Factorisations de fonctions et de vecteurs indefiniment differentiables*. Bull. Sci. Math. (2) **102**(1978), 305–330.

[8] S. Friedberg and D. Goldfeld, *Mellin transforms of Whittaker functions*. Bull. Soc. Math. France **121**(1993), 91–107.

[9] D. Goldfeld, *Kloosterman Zeta Functions for* $\mathrm{GL}(n, \mathbb{Z})$. Proceedings of the ICM, Berkeley, CA, 1986.

[10] J. Humphreys, *Introduction to Lie Algebras and Representation Theory*. Springer-Verlag, 1972.

[11] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic representations. I*. Amer. J. Math. **103**(1981), 499–558.

[12] ———, *On Euler products and the classification of automorphic forms. II*. Amer. J. Math. **103**(1981), 777–815.

[13] M. Raghunathan, *Lattices in Lie Groups*. Springer-Verlag, New York, 1972.

[14] A. Selberg, *On the estimation of Fourier coefficients of modular forms*. Proc. Sympos. Pure Math. **VIII**, Amer. Math. Soc., Providence, R.I. 1965, 1–15.
[15] E. Stade, *Mellin transforms of* $\mathrm{GL}(n, \mathbb{R})$ *Whittaker functions*. Amer. J. Math, to appear.
[16] N. Wallach, *Asymptotic expansions of generalized matrix coefficients of real reductive groups*. Lie Group Representations I. SLNM **1024**(1983), 287–369.
[17] ———, *Real Reductive Groups II*. Academic Press, 1993.

University of Exeter
Department of Mathematics
Exeter EX4 4QE
Devon
United Kingdom