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HOMOTOPY GROUPS AND H-MAPS

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The first nonvanishing homotopy group of a finite H-space X whose mod 2 homology ring is associative occurs in degrees 1, 3 or 7. Generators of these groups can be represented by maps $\alpha: S^n \to X$ for $n=1$, 3 or 7. In this note we prove that under some hypothesis on X there exists an H-structure on S^n, $n=1$, 3 or 7 such that α is an H-map.

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1. Introduction

Recently, J. P. Lin and the author [7] have proven that the first nonvanishing homotopy group of a finite H-space whose mod 2 homology ring is associative occurs in degrees 1, 3 or 7. (Recall that a finite H-space is an H-space whose integral homology is finitely generated as a graded abelian group). This result improves Adams' famous theorem [1] saying that a sphere S^n admits an H-structure if and only if $n=1$, 3 or 7.

The goal of this paper is to discuss the following question. If $(X;\mu)$ is a finite H-space, where μ is the multiplication on X, does there exist a set of generators of the first nonvanishing homotopy group $\{\alpha_i\} \subset \pi_n X$, $n=1$, 3 or 7 and a multiplication m on S^n such that

$$\alpha_i : (S^n; m) \to (X; \mu)$$

are H-maps? (We will always identify a map with its homotopy class). The question can be easily answered in the case where $n=1$. The proofs are given in the next sections.

Theorem 1.1. *If X is a finite H-space, then there exists a set of generators $\{\alpha_i\}_{i=1,\dots,n}$ of $\pi_1 X$ such that*

$$\alpha_i : S^1 \to X$$

are H-maps for all $i=1,\dots,n$.

We do need to specify the H-structure on S^1 because S^1 admits only one multiplication.

The case where $n=3$ was first investigated by J. Schiffmann [**13**]; more precisely he has proven the following theorem.

Theorem 1.2. *Let X be a finite H-space carrying an homotopy associative multiplication such that $\pi_3 X \cong \mathbf{Z}$. Let μ be a multiplication on X and $\alpha \in \pi_3 X$ a generator, then there exists a multiplication m on S^3 such that*

$$\alpha:(S^3;m)\to(X;\mu)$$

is an H-map.

Note that if X is a finite H-space, $\pi_3 X$ is always a free abelian group. Even if the associative multiplication does not appear in the conclusion of the theorem, it is strongly used to prove it.

We can improve Schiffmann's result for a wider class of H-spaces, namely the non-associative ones.

Theorem 1.3. *Let $(X;\mu)$ be a finite H-space with $\pi_3 X \cong \mathbf{Z}$ and $\alpha \in \pi_3 X$ a generator, then there exists a multiplication m on S^3 such that*

$$\alpha:(S^3;m)\to(X;\mu)$$

is an H-map.

As an obvious corollary we can offer:

Corollary 1.1. *Let $(X;\mu),(X_i;\mu_i), i=1,\dots,n$ be finite H-spaces with $\pi_3 X_i \cong \mathbf{Z}$ and $X \simeq \prod_{i=1}^n X_i$ as H-spaces. Then there exists a set of generators $\{\alpha_i\}_{i=1,\dots,n}$ of $\pi_3 X \cong \mathbf{Z}^n$ and multiplications $m_i, i=1,\dots,n$ on S^3 such that*

$$\alpha_i:(S^3;m_i)\to(X;\mu)$$

are H-maps.

Let us pause to comment:

(1) Many H-spaces do not admit homotopy associative multiplication. For example if G is a one-connected simple Lie group different from G_2 or $Spin(7)$, then $G\times S^7$ does not carry a homotopy associative multiplication [**4**].
(2) All finite H-spaces do not satisfy the splitting condition $X\simeq\prod X_i$ with $(X_i;\mu_i)$ a finite H-space, hence the corollary does not apply for all finite H-spaces.
(3) Unlike S^1 which has a unique H-structure, S^3 has many [**6**], therefore the multiplications m_i on S^3 depend on the generators α_i.
(4) The situation for compact Lie groups is much nicer because problems of type occurring in (2) and (3) do not appear, cf. below.

If G is a simply connected compact Lie group then G splits as $G \simeq \prod_{k=1}^{n} G_k$ with G_k simple Lie groups. Let H be any simple Lie group. It is well known that for any root ρ of H there exists a subgroup $SU(2) = S^3$ of H [**12**]. The inclusion $i: S^3 \to H$ is not necessarily a generator of $\pi_3 H \cong \mathbf{Z}$, but if the root ρ is dominant [**12**], then i is a generator of $\pi_3 H$. If we come back to the simply connected Lie group G with decomposition $G \cong \prod_{k=1}^{n} G_k$, it is obvious then there exists a set of generators $\{\alpha_k\}_{k=1,\ldots,n}$ of $\pi_3 G \cong \mathbf{Z}^n$ such that

$$\alpha_k : SU(2) = S^3 \to G$$

are group homomorphisms.

Before closing the introduction let us discuss the case of a 6-connected finite H-space. During many years the only known 6-connected finite H-spaces were products of 7-dimensional spheres. Recently Dwyer and Wilkerson have built up an "exotic" H-space at the prime 2 [**3**]. More precisely they constructed a mod 2 finite H-space $DI(4)$ whose mod 2 cohomology ring satisfies

$$H^* DI(4) \cong \mathbf{F}_2[x_7]/(x_7^4) \otimes E(x_{11}, x_{13}).$$

See [**3**] for the details. Using Zabrodsky's techniques of mixing homotopy types we can mix the 2-type of $DI(4)$ with $S^7 \times S^{11} \times S^{27}$ localised at the set of all odd primes to construct an H-space $X(4)$ whose mod 2 cohomology ring satisfies

$$H^* X(4) \cong H^* DI(4)$$

and which has no p-torsion in integral homology for odd primes p (note that $H^*(DI(4); \mathbf{Q}) \cong E(x_7, x_{11}, x_{27})$). In particular $\pi_7 X(4) \cong \mathbf{Z}$. We can state now our last proposition.

Proposition 1.1. *Let μ be a multiplication on $X(4)$ and α a generator of $\pi_7 X(4) \cong \mathbf{Z}$. Then there exists a multiplication m on S^7 such that*

$$\alpha : (S^7; m) \to (X(4); \mu)$$

is an H-map.

2. Proof of Theorem 1.1

The fundamental group of X is abelian and satisfies $\pi_1 X \cong \bigoplus_{k=1}^{n} C_k$ with C_k a cyclic group. We start with the situation where $\pi_1 X$ is an infinite cyclic group, i.e. $\pi_1 X \cong \mathbf{Z}$. If $\pi_1 X \cong \mathbf{Z}$, it is well known that $X \simeq S^1 \times Y$ with $\pi_1 Y = \pi_2 Y = 0$ cf. [**11**, page 63]. Therefore there exists a cellular decomposition of X such that its 2-skeleton, $X^{(2)}$, is just S^1 [**5**]. Let α be a geneator of $\pi_1 X$ that is identified with the inclusion $S^1 = X^{(2)} \to X$. The composition

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$$S^1 \times S^1 \xrightarrow{\alpha \times \alpha} X \times X \xrightarrow{\mu} X$$

factors through $X^{(2)} = S^1$. We have thus constructed a map $m: S^1 \times S^1 \to S^1$ such that the following diagram commutes

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{\alpha \times \alpha} & X \times X \\ \downarrow^{m} & & \downarrow^{\mu} \\ S^1 & \xrightarrow{\alpha} & X. \end{array}$$

Hence α is an H-map.

The second situation is when $\pi_1 X$ is a finite cyclic group i.e. $\pi_1 X \cong \mathbf{Z}/d$. In this case standard techniques show that $H_1(X;\mathbf{Z}) \cong \mathbf{Z}/d$, $H^1(X;\mathbf{Z}) = 0$, $H^2(X;\mathbf{Z}) \cong \mathbf{Z}/d$. Let x be a generator of $H^2(X;\mathbf{Z})$. For dimensional reasons x is a primitive class, and so the map $f: X \to K(\mathbf{Z};2)$ representing x is an H-map. The fiber F of f is then an H-space.

If α is a generator of $\pi_1 X$ then $\alpha: S^1 \to X$ lifts to $\tilde{\alpha}: S^1 \to F$. It is sufficient to prove that $\tilde{\alpha}$ is an H-map because the inclusion $i: F \to X$ is an H-map and by construction $\alpha = i \circ \tilde{\alpha}$. Now observe that F is a finite H-space with $\pi_1 F \cong \mathbf{Z}$ generated by $\tilde{\alpha}$. Using the first case discussed above we obtain that $\tilde{\alpha}$ is an H-map.

Let us consider now the general case $\pi_1 X \cong \bigoplus_{k=1}^{n} C_k$, C_k a cyclic group. Let $p_k: X_k \to X$ be the covering space such that $\pi_1 X_k \cong C_k$. The covering map p_k is an H-map. Then there exists a generator $\tilde{\alpha}_k$ of $\pi_1 X_k$ which is an H-map. The set $\{\alpha_k\}_{k=1,\dots,n}$ with $\alpha_k = p_k \circ \tilde{\alpha}_k$ is a set of generators of $\pi_1 X$ and all the α_k are H-maps.

3. Proof of Theorem 1.3

First remark that X is not assumed to be 1-connected. The universal cover $\tilde{X}$ of X satisfies $\pi_3 \tilde{X} \cong \pi_3 X$ and $\tilde{X}$ has the homotopy type of a finite H-space. We can therefore assume through all the proof that X is 2-connected and $\pi_3 X \cong \mathbf{Z}$.

A result of Lin **[9]** asserts that

$$QH^{even}(X;\mathbf{F}_p) \cong \sum_{i=1}^{\infty} \beta_1 \mathscr{P}^i H^{2n_i+1}(X;\mathbf{F}_p)$$

where p is an odd prime, QH^{even} is the module of indecomposables in even dimensions, β_1 is the first Bockstein and $\mathscr{P}^i$ is the i^{th} Steenrod power. In our situation this result implies that $H^k(X;\mathbf{Z})$ has no p-torsion for p an odd prime and $k \leqq 6$. Hence we are reduced to study the 2-torsion of the integral cohomology in low dimensions.

To simplify the notations, H^*X will stand for $H^*(X;\mathbf{F}_2)$. Recall two results due to Kane and Lin.

Theorem 3.1 [8]. *Let X be a simply connected* mod 2 *finite H-space, then*

$$QH^{even}X = 0.$$

Theorem 3.2 [10]. *Let X be a simply connected* mod 2 *finite H-space, then*

$$QH^{4k+1}X = \mathrm{Sq}^{2k}QH^{2k+1}X, k \geqq 1.$$

We apply these results in our particular case. The first result implies that $H^4X = QH^4X = 0$ because $H^2X = 0$, X being 2-connected. Moreover $H^6X \cong \xi H^3X$ with ξ the cup square map. The second result implies that $H^5X \cong \mathrm{Sq}^2H^3X$. Hence $\xi H^3X \cong \mathrm{Sq}^3H^3X \cong \mathrm{Sq}^1\mathrm{Sq}^2H^3X \cong \mathrm{Sq}^1H^5X$. So three different cases can occur:

1. $H^6X \neq 0$ and so $H^6X \cong H^5X \cong H^3X \cong \mathbf{F}_2$, if we set x_k a generator of H^kX then

$$\mathrm{Sq}^2x_3 = x_5, \mathrm{Sq}^1x_5 = x_6 = x_3^2.$$

2. $H^6X = 0, H^5X \neq 0$ and so

$$\mathrm{Sq}^2x_3 = x_5, \mathrm{Sq}^1x_5 = 0.$$

3. $H^5X = 0$ and so

$$\mathrm{Sq}^2x_3 = 0.$$

We will prove the theorem under the conditions stated in (1), the cases (2) and (3) being similar or simpler.

Let $K_0 = K(\mathbf{F}_2; 5)$ and $h_0: X \to K_0$ the classifying map of x_5 the generator of H^5X. Define E_0 to be the fiber of h_0. The Serre spectral sequence applied to the fibration

$$\begin{array}{ccc} \Omega K_0 & \xrightarrow{i_0} & E_0 \\ & & \downarrow p_0 \\ & & X \end{array}$$

implies that

$$H^jE_0 \cong \begin{cases} \mathbf{F}_2 & \textit{if } j = 3, 6 \\ 0 & \textit{if } j = 1, 2, 4, 5 \end{cases}$$

and $i_0^*: H^6E_0 \cong H^6\Omega K_0$. Since the class x_5 is primitive, the map h_0 is an H-map, hence E_0 is an H-space and i_0 and p_0 are H-maps too. We call a the generator of $H^6E_0 \cong \mathbf{F}_2$. It satisfies $i_0^*a = \mathrm{Sq}^2\iota_4$, where ι_4 is the fundamental class of $\Omega K_0 = K(\mathbf{F}_2; 4)$. We will prove in the next section the central lemma.

Lemma 3.1. *The class $a \in H^6E_0$ is primitive.*

Now let $K_1 = K(\mathbf{F}_2; 6)$ and $h_1: E_0 \to K_1$ be the classifying map for a. Let E_1 be the fiber of h_1. The Serre spectral sequence of the fibration

$$\begin{array}{ccc} \Omega K_1 & \xrightarrow{i_1} & E_1 \\ & & \downarrow p_1 \\ & & E_0 \end{array}$$

implies that

$$H^j E_1 \cong \begin{cases} \mathbf{F}_2 & \text{if } j=3 \\ 0 & \text{if } j=1,2,4,5,6. \end{cases}$$

In particular E_1 admits a cellular decomposition such that its 6-skeleton, $E_1^{(6)}$, is just S^3 [5]. The map h_1 is an H-map because the cohomology class a is primitive, therefore E_1 is an H-space; i_1 and p_1 are H-maps.

Let α be a generator of $\pi_3 X$, it is clear that α lifts in the following way:

$$\begin{array}{ccccc} & \tilde{\alpha} & E_1 & & \\ & \nearrow & \downarrow p_1 & & \\ S^3 & \xrightarrow{\bar{\alpha}} & E_0 & \xrightarrow{h_1} & K_1 \\ & \searrow & \downarrow p_0 & & \\ & \alpha & X & \xrightarrow{h_0} & K_0. \end{array}$$

The theorem will be proven as soon as we can exhibit a multiplication m on S^3 such that $\tilde{\alpha}$ is an H-map, because $\alpha = p_0 \circ p_1 \circ \tilde{\alpha}$ and p_0, p_1 are H-maps. The argument is the following:

As claimed above E_1 admits a cellular decomposition such that $E_1^{(6)} = S^3$. We identify $\tilde{\alpha}$ with the inclusion $S^3 = E_1^{(6)} \to E_1$. Let $\tilde{\mu}$ be a cellular multiplication on E_1 induced from the one on X. The composition

$$S^3 \times S^3 \xrightarrow{\tilde{\alpha} \times \tilde{\alpha}} E_1 \times E_1 \xrightarrow{\tilde{\mu}} E_1$$

factors through $E_1^{(6)} = S^3$. Therefore we have constructed a map $m: S^3 \times S^3 \to S^3$ such that the following diagram is commutative

$$\begin{array}{ccc} S^3 \times S^3 & \xrightarrow{\tilde{\alpha} \times \tilde{\alpha}} & E_1 \times E_1 \\ \downarrow m & & \downarrow \tilde{\mu} \\ S^3 & \xrightarrow{\tilde{\alpha}} & E_1. \end{array}$$

The restriction of m to $S^3 \vee S^3$ is homotopic to the folding map because the restriction of $\tilde{\mu}$ to $E_1 \vee E_1$ is homotopic to the folding map as $\tilde{\mu}$ is a multiplication on E_1. So m is a multiplication on S^3 and $\tilde{\alpha}$ is an H-map. The theorem is proven.

4. Proof of Lemma 3.1

The fibration defined in the previous section

$$\begin{array}{ccc} E_0 & & \\ \downarrow{\scriptstyle p_0} & & \\ X & \xrightarrow{h_0} & K_0 \end{array} \tag{*}$$

induces an exact sequence in cohomology with $\mathbf{F}_2$ coefficients up to dimension 7. Recall that if Y is an H-space then the projective plane of Y [2] denoted by P_2Y fits into a cofibration

$$\Sigma Y \xrightarrow{i_Y} P_2 Y \xrightarrow{k_Y} \Sigma Y \wedge \Sigma Y.$$

The naturality of the cofibration allows us to construct the following commutative diagram:

$$\begin{array}{ccccc} \Sigma E_0 & \xrightarrow{i_{E_0}} & P_2 E_0 & \xrightarrow{k_{E_0}} & \Sigma E_0 \wedge \Sigma E_0 \\ \downarrow{\scriptstyle \Sigma p_0} & & \downarrow{\scriptstyle P_2 p_0} & & \downarrow{\scriptstyle \Sigma p_0 \wedge \Sigma p_0} \\ \Sigma X & \xrightarrow{i_X} & P_2 X & \xrightarrow{k_X} & \Sigma X \wedge \Sigma X \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle j} & & \downarrow \\ A & \xrightarrow{i} & B & \xrightarrow{k} & C. \end{array} \tag{**}$$

The spaces A, B and C are the cofibres of the maps $\Sigma p_0, P_2 p_0, \Sigma p_0 \wedge \Sigma p_0$ respectively. The sequence

$$A \xrightarrow{i} B \xrightarrow{k} C$$

is again a cofibration.

As $(\Sigma p_0)^*: H^n\Sigma E_0 \cong H^n \Sigma X$ for $n \leqq 5$, we deduce that $H^nC = 0$ for $n \leqq 10$. In particular $H^nA \cong H^nB$ for $n \leqq 9$. The remark made after (**) allows us to identify

$$H^nA \cong H^nB \cong H^n\Sigma K_0$$

for $n \leqq 8$.

Let $x_3 \in H^3X$ be the generator and $y_3 \in H^3E_0$ be the class such that $p_0^*x_3 = y_3$. For dimensional reasons x_3 and y_3 are primitive, so there exist $u_4 \in H^4P_2X$ and $v_4 \in H^4P_2E_0$ satisfying $i_X^*u_4 = \sigma x_3$, $i_{E_0}^*v_4 = \sigma y_3$, σ standing as usual for the suspension isomorphism.

The top cofibration of (**) can be written as

$$P_2E_0 \xrightarrow{k_{E_0}} \Sigma E_0 \wedge \Sigma E_0 \xrightarrow{\Sigma^2\mu} \Sigma^2 E_0 \tag{***}$$

with μ the multiplication on E_0. Recall from [2] that $\Sigma^2\mu$ induces the reduced coproduct in cohomology. The only possible non-trivial reduced coproduct for $a \in H^6E_0$ is $y_3 \otimes y_3$ so $(\Sigma^2\mu)^*\sigma^2 a = \sigma y_3 \otimes \sigma y_3$ or $(\Sigma^2\mu)^*\sigma^2 a = 0$.

Browder and Thomas [2] have studied the cohomology ring structure of the projective plane. One of their results is that, in our situation, $(k_{E_0})^*\sigma y_3 \otimes \sigma y_3 = v_4^2$. In particular we deduce from the exactness in cohomology of (***) that a is primitive if and only if $v_4^2 \neq 0$ in $H^8 P_2 E_0$.

As X is a finite H-space we already know [2] that $u_4^2 \neq 0$ in $H^8 P_2 X$. The problem is that E_0 is not a finite H-space and so v_4^2 is not automatically non-trivial.

Using the exact sequence in cohomology induced from

$$P_2 E_0 \xrightarrow{p_2 p_0} P_2 X \xrightarrow{j} C$$

we therefore just need to prove that u_4^2 is not in the image of j^* (recall that $(P_2 p_0)^* u_4 = v_4$).

Under the identification $H^n A \cong H^n \Sigma K_0$, $n \leqq 8$ the homomorphism h^* coincides with $(\Sigma h_0)^*$ and j^* with $(P_2 h_0)^*$ up to dimension 8.

By definition $h_0^* \iota_5 = \mathrm{Sq}^2 x_3 = x_5$. So from commutativity of the diagram (**) we get $j^* \sigma \iota_5 = (P_2 h_0)^* \sigma \iota_5 = \mathrm{Sq}^2 u_4$, hence $j^* \sigma \mathrm{Sq}^2 \iota_5 = \mathrm{Sq}^2 \mathrm{Sq}^2 u_4 = 0$. As $\sigma \mathrm{Sq}^2 \iota_5$ is the only non-trivial element of $H^8 \Sigma K_0$, we conclude that $Im(P_2 h_0)^* = 0$ in dimension 8, which finishes the proof of the lemma.

5. Proof of Proposition 1.1

The proof of Proposition 1.1 is completely analogous to the one of Theorem 1.3. Let us just mention the 2 stage Postnikov tower needed:

$$\begin{array}{ccccc} & \tilde{\tilde{\alpha}} & E_1 & & \\ & \nearrow & \downarrow p_1 & & \\ S^7 & \xrightarrow{\tilde{\alpha}} & E_0 & \xrightarrow{h_1} & K(\mathbf{F}_2; 14) \\ & \searrow & \downarrow p_0 & & \\ & \alpha & X & \xrightarrow{h_0} & K(\mathbf{Z}; 11). \end{array}$$

The map h_0 is the classifying map for the class x_{11}, h_1 classifies the class $a \in H^{14} E_0$ whose restriction to the fiber of p_0, $\Omega K(\mathbf{Z}; 11) = K(\mathbf{Z}; 10)$ is $\mathrm{Sq}^4 \iota_{10}$. Again E_0 and E_1 are H-spaces, moreover the 14-skeleton of E_1 $E_1^{(14)}$ satisfies $E_1^{(14)} = S^7$. As before we construct the following commutative diagram

$$\begin{array}{ccc} S^7 = S^{(7)} & \xrightarrow{\tilde{\tilde{\alpha}} \times \tilde{\tilde{\alpha}}} & E_1 \times E_1 \\ \downarrow m & & \downarrow \tilde{\mu} \\ S^7 = E_1^{(14)} & \longrightarrow & E_1. \end{array}$$

In this way S^7 is endowed with a multiplication m such that $\tilde{\tilde{\alpha}}$ is an H-map, so α itself is an H-map.

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