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An Explicit Formula for the Generalized Cyclic Shuffle Map

Jiao Zhang and Qing-Wen Wang

Abstract. We provide an explicit formula for the generalized cyclic shuffle map for cylindrical modules. Using this formula we give a combinatorial proof of the generalized cyclic Eilenberg–Zilber theorem.

1 Introduction

The cyclic shuffle map on a tensor product of cyclic modules associated with algebras is defined by many authors, including Rinehart [8], Getzler-Jones [2], and Loday [6]. Later Kustermans–Rognes–Tuset [5] described the cyclic shuffle map for the case of a tensor product of cocyclic modules. It is the key map that used to define products and coproducts in cyclic (co)homology, to obtain Künneth type formula, and furthermore to prove the Eilenberg–Zilber theorem for cyclic (co)homology. This theorem has also been proved by several authors with various methods. For a comparison of these methods one can refer to Bauval's article [1].

The Eilenberg–Zilber theorem is further generalized by Getzler–Jones [3] with a topological proof to any cylindrical module. This extension is important because in many examples the cylindrical module may not be decomposable as a tensor product of two cyclic modules (*e.g.*, the cylindrical module defined in [9] and many concrete examples therein). This Eilenberg–Zilber theorem for cylindrical modules is called the generalized cyclic Eilenberg–Zilber theorem by Khalkhali–Rangipour [4]. They reprove the theorem using the homological perturbation lemma. In this paper we extend the cyclic shuffle map defined in [2, 5, 6, 8] to the case of cylindrical modules. An explicit formula for the generalized cyclic shuffle map for cylindrical modules is provided. Using this formula we give a combinatorial proof of the generalized cyclic Eilenberg–Zilber theorem.

In Section 2, we provide a reasonable and natural extension of the cyclic shuffle map to any cylindrical module that may not even be decomposable into a tensor product of cyclic modules. Using this generalized cyclic shuffle map we can construct a morphism of mixed complexes and prove the generalized cyclic Eilenberg–Zilber theorem directly. In Section 3, some combinatorial properties of shuffles are proved.

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In Section 4, using these properties, we prove the generalized cyclic Eilenberg–Zilber theorem.

Since the notions of cylindrical module, its total complex and diagonal module, and shuffle and shuffle map have been given numerous times (*cf.* [3, 4, 6, 7]), we collect these preliminaries in Appendix A at the end of this paper for the reader's convenience.

Notation

Let K be a commutative unital ring. We take K as our ground ring.

Let $(X_{\cdot,\cdot}, d_i, s_i, t, \delta_j, \sigma_j, \tau)$ be a cylindrical K-module. The total complex of X is denoted by $(\mathrm{Tot}(X), \mathbb{b}, \mathbb{B})$, and its normalized mixed complex is denoted by $(\overline{\mathrm{Tot}}(X), \mathbb{b}, \overline{\mathbb{B}})$. The diagonal cyclic module of X is $(\{\Delta_n(X)\}_{n\geq 0}, d_i\delta_i, s_j\sigma_j, t_n\tau_n)$. Denote its associated mixed complex by $(\Delta(X), \mathfrak{b}, \mathfrak{B})$ and the corresponding normalized mixed complex by $(\overline{\Delta}(X), \mathfrak{b}, \overline{\mathfrak{B}})$. One can refer to Appendix A.1–A.3 for definitions.

There are two equivalent definitions of shuffles. One uses the language of partitions (*cf.* [7]), and the other uses the language of permutations in a symmetric group (*cf.* [4]); both are widely used. In this paper we mainly use the first one (see *e.g.*, Definition A.3), since it clarifies our discussion of the combinatorial properties of shuffles. Denote the set of (p,q)-shuffles by $\mathfrak{S}_{p,q}$ and set $\mathfrak{S}_{0,q} = \mathfrak{S}_{p,0} = \{\mathrm{id}\}$. The (p,q)-shuffle map from $X_{p,q}$ to $X_{p+q,p+q}$ is denoted by $\zeta_{p,q}$, and the shuffle map is denoted by ζ.

In order to construct the generalized cyclic shuffle map, we define a subset $\mathcal{S}^{p,q}_{i,j}$ of $\mathfrak{S}_{p,q}$ by

$$\mathcal{S}^{p,q}_{i,j} = \{(\mu,\nu) \in \mathfrak{S}_{p,q} \mid \mu_i < \nu_j\}.$$

Moreover, fixing μ_i and ν_j in $\mathcal{S}^{p,q}_{i,j}$ we define a subset $\mathcal{T}^{p,q}_{i,j,k}$ of $\mathcal{S}^{p,q}_{i,j}$ by

$$\mathcal{T}^{p,q}_{i,j,k} = \{(\mu,\nu) \in \mathfrak{S}_{p,q} \mid \mu_i = k-1, \nu_j = k\}.$$

2 The Generalized Cyclic Shuffle Map

The cyclic shuffle is first described by Rinehart [8, p. 220, condition (10.14)]. The multiple version is stated by Getzler–Jones [2] using a lexicographical order. Independently, Loday [6, p. 128] also provided its explicit definition. However, there is some confusion in Loday's definition. Indeed, if a permutation

$$\{\omega(1), \dots, \omega(p+q)\}$$

of $\{1, \dots, p+q\}$ is a (p,q)-cyclic shuffle, then it should satisfy the condition $\omega(1) < \omega(p+1)$, not the condition that "1 appears before $p+1$" as in [6], which means $\omega^{-1}(1) < \omega^{-1}(p+1)$. Note that for an algebra A the permutation ω acting on an element $(1, a_1, \dots, a_{p+1})$ belonging to $A^{\otimes p+2}$ yields

$$\omega.(1, a_1, \dots, a_{p+1}) = (1, a_{\omega^{-1}(1)}, \dots, a_{\omega^{-1}(p+q)}).$$

If $\omega(1) < \omega(p+1)$, then a_1 appears before a_{p+1} after the action of ω. That is exactly what we need for a cyclic shuffle. For this reason the cyclic shuffles defined in [8] and in [2] coincide with each other. We will use cyclic operators and simplicial maps to express the action of any cyclic shuffle. The conclusion is that there exists a natural extension of cyclic shuffle map to the case of any cylindrical module which may not be decomposable to a tensor product.

Definition 2.1 Let $(X_{\cdot,\cdot}, d_i, s_i, t, \delta_j, \sigma_j, \tau)$ be a cylindrical module. For any $p, q \in \mathbb{N}$, define a map

$$\xi_{p,q}\colon X_{p,q} \to X_{p+q+2,p+q+2}$$

by

$$\xi_{p,q} = \sum_{\substack{0\le i\le p\\ 0\le j\le q}} \sum_{\substack{(\mu,\nu)\in\\ \mathcal{S}^{p+1,q+1}_{i+1,j+1}}} (-1)^{pi+qj+p}\operatorname{sgn}(\mu,\nu) s_{\nu_{q+1}}\cdots s_{\nu_1} s_{-1} t_p^i \sigma_{\mu_{p+1}}\cdots\sigma_{\mu_1}\sigma_{-1}\tau_q^j.$$

We call $\xi_{p,q}$ the *generalized* (p,q)*-cyclic shuffle map*. The *generalized cyclic shuffle map* $\xi\colon \operatorname{Tot}_n(X) \to \Delta_{n+2}(X)$ is defined by setting $\xi = \sum_{p+q=n} \xi_{p,q}$.

The generalized (p,q)-cyclic shuffle map is a natural extension of the (p,q)-cyclic shuffle map defined in [6], since the actions of cyclic operators can be regarded as cyclic permutations.

We will show that the generalized cyclic shuffle map plays the same role as the original one while constructing a morphism of mixed complexes from $\overline{\operatorname{Tot}}(X)$ to $\overline{\Delta}(X)$. Hence the generalized cyclic Eilenberg–Zilber theorem follows immediately. The key formula we should check is

$$\mathfrak{b}\xi - \xi\mathbb{b} + \overline{\mathfrak{B}}\zeta - \zeta\overline{\mathbb{B}} = 0.$$

Note that since X could be any cylindrical module, the operator $\overline{\mathbb{B}}$ contains T_ν in its expression (*cf.* Appendix A.2). The combinatorial method is used to prove the above formula directly.

3 Combinatorial Properties of Shuffles

In this section we provide some combinatorial properties of shuffles that will be needed in the proof of the generalized cyclic Eilenberg–Zilber theorem.

Lemma 3.1 *Let (μ,ν) be a (p,q)-shuffle.*

(i) *For all $1 \le r \le p$ and $1 \le l \le q$, we have $r-1 \le \mu_r \le q+r-1$ and $l-1 \le \nu_l \le p+l-1$.*

(ii) *For $1 \le r_1 < r_2 \le p$ and $1 \le l_1 < l_2 \le q$, we have $\mu_{r_1} + (r_2 - r_1) \le \mu_{r_2} \le \mu_{r_1} + (r_2 - r_1) + q$ and $\nu_{l_1} + (l_2 - l_1) \le \nu_{l_2} \le \nu_{l_1} + (l_2 - l_1) + p$.*

Lemma 3.1 follows directly from the definition of (p,q)-shuffles.

Recall $\mathcal{T}^{p,q}_{i,j,k}$ is the set $\{(\mu,\nu) \in \mathfrak{S}_{p,q} \mid \mu_i = k-1, \nu_j = k\}$, by the definition of shuffles, the set $\mathcal{T}^{p,q}_{i,j,k}$ is not empty if and only if $k = i+j-1$. First we explore some combinatorial properties of $\mathcal{T}^{p,q}_{i,j,k}$.

Proposition 3.2 *There is an isomorphism of sets* $\mathcal{T}^{p+1,q+1}_{p+1,q+1,p+q+1} \cong \mathfrak{S}_{p,q}$.

Proof Define a map

$$\Psi \colon \mathcal{T}^{p+1,q+1}_{p+1,q+1,p+q+1} \to \mathfrak{S}_{p,q}$$

by setting $\Psi(u,v) = (u',v')$ for any $(u,v) \in \mathcal{T}^{p+1,q+1}_{p+1,q+1,p+q+1}$ with $u'_r = u_r$ and $v'_l = v_l$ for $1 \le r \le p$ and $1 \le l \le q$. Since Ψ is a bijection, we get the isomorphism. ■

Proposition 3.3 *For* $1 \le k \le p+q+1$, *we have the following isomorphism of sets*

$$\bigcup_{\substack{0 \le i \le p \\ 0 \le j \le q}} \mathcal{T}^{p+1,q+1}_{i+1,j+1,k} \cong \mathcal{T}^{p+1,q+1}_{p+1,q+1,p+q+1}.$$

Proof Since $\mathcal{T}^{p+1,q+1}_{i+1,j+1,k} \neq \varnothing$ only when $k = (i+1)+(j+1)-1$, and $0 \le j \le q$, we get

$$\bigcup_{\substack{0 \le i \le p \\ 0 \le j \le q}} \mathcal{T}^{p+1,q+1}_{i+1,j+1,k} = \bigcup_{i=\max\{0,k-q-1\}}^{\min\{p,k-1\}} \mathcal{T}^{p+1,q+1}_{i+1,k-i,k}. \tag{3.1}$$

Note that if the fixed k equals $p+q+1$, then the right hand side of (3.1) is just $\mathcal{T}^{p+1,q+1}_{p+1,q+1,p+q+1}$.

First we will construct an injective map from $\mathcal{T}^{p+1,q+1}_{i+1,k-i,k}$ to $\mathcal{T}^{p+1,q+1}_{p+1,q+1,p+q+1}$. Define a permutation

$$\chi_{m,n} = \begin{pmatrix} 1 & 2 & \cdots & m & m+1 & m+2 & \cdots & m+n \\ n+1 & n+2 & \cdots & n+m & 1 & 2 & \cdots & n \end{pmatrix}.$$

For any $(\mu,\nu) \in \mathcal{T}^{p+1,q+1}_{i+1,k-i,k}$, let

$$\begin{cases} \mu'_r = \chi_{k+1,p+q+1-k}(\mu_{\chi_{p-i,i+1}(r)}+1)-1 & \text{for } 1 \le r \le p+1, \\ \nu'_l = \chi_{k+1,p+q+1-k}(\nu_{\chi_{q+i+1-k,k-i}(l)}+1)-1 & \text{for } 1 \le l \le q+1. \end{cases} \tag{3.2}$$

Then it is easy to check that (μ',ν') is a $(p+1,q+1)$-shuffle and $\mu'_{p+1} = p+q$, $\nu'_{q+1} = p+q+1$, *i.e.*, $(\mu',\nu') \in \mathcal{T}^{p+1,q+1}_{p+1,q+1,p+q+1}$. Hence the map

$$\Phi_{i,k} \colon \mathcal{T}^{p+1,q+1}_{i+1,k-i,k} \to \mathcal{T}^{p+1,q+1}_{p+1,q+1,p+q+1}$$

with $\Phi_{i,k}(\mu,\nu) = (\mu',\nu')$ defined by (3.2) is well defined. Since the permutation χ is invertible, the map $\Phi_{i,k}$ is an injection.

It is clear that if $i_1 \neq i_2$, then

$$\mathcal{T}^{p+1,q+1}_{i_1+1,k-i_1,k} \cap \mathcal{T}^{p+1,q+1}_{i_2+1,k-i_2,k} = \varnothing.$$

Moreover, we can derive that for a fixed k, if $0 \le i_1 < i_2 \le p$, then

$$\Phi_{i_1,k}(\mathcal{T}^{p+1,q+1}_{i_1+1,k-i_1,k}) \cap \Phi_{i_2,k}(\mathcal{T}^{p+1,q+1}_{i_2+1,k-i_2,k}) = \varnothing.$$

Indeed, if there exist $(\mu,\nu) \in \mathcal{T}^{p+1,q+1}_{i_1+1,k-i_1,k}$ and $(u,v) \in \mathcal{T}^{p+1,q+1}_{i_2+1,k-i_2,k}$, such that $\Phi_{i_1,k}(\mu,\nu) = \Phi_{i_2,k}(u,v)$, then for any $1 \le r \le p+1$, $\mu_{\chi_{p-i_1,i_1+1}(r)} = u_{\chi_{p-i_2,i_2+1}(r)}$. Especially for r equalling $p-i_2+1$ and $p-i_1+1$, we get

$$u_1 = u_{\chi_{p-i_2,i_2+1}(p-i_2+1)} = \mu_{\chi_{p-i_1,i_1+1}(p-i_2+1)} = \mu_{p+2+i_1-i_2}$$

and

$$u_{i_2-i_1+1} = u_{\chi_{p-i_2,i_2+1}(p-i_1+1)} = \mu_{\chi_{p-i_1,i_1+1}(p-i_1+1)} = \mu_1.$$

Since $i_2 - i_1 + 1 > 1$, $p+2+i_1-i_2 > 1$, and (μ,ν), (u,v) are $(p+1,q+1)$-shuffles, we deduce a contradiction from

$$u_1 < u_{i_2-i_1+1} = \mu_1 < \mu_{p+2+i_1-i_2} = u_1.$$

Therefore we can define an injection

$$\Phi\colon \bigcup_{i=\max\{0,k-q-1\}}^{\min\{p,k-1\}} \mathcal{T}^{p+1,q+1}_{i+1,k-i,k} \to \mathcal{T}^{p+1,q+1}_{p+1,q+1,p+q+1}$$

such that $\Phi|_{\mathcal{T}^{p+1,q+1}_{i+1,k-i,k}} = \Phi_{i,k}$.

Next, we will compare the cardinalities of the above two sets.

Denote the cardinality of the set $\mathcal{T}^{p+1,q+1}_{i+1,k-i,k}$ by $\#\mathcal{T}^{p+1,q+1}_{i+1,k-i,k}$. It is clear that

$$\#\mathcal{T}^{p+1,q+1}_{p+1,q+1,p+q+1} = \binom{p+q}{p},$$

and for $1 \le k \le p+q+1$, $0 \le i \le p$,

$$\#\mathcal{T}^{p+1,q+1}_{i+1,k-i,k} = \binom{k-1}{i}\binom{p+q+1-k}{p-i}.$$

So

$$\#\bigcup_{i=\max\{0,k-q-1\}}^{\min\{p,k-1\}} \mathcal{T}^{p+1,q+1}_{i+1,k-i,k} = \sum_{i=0}^{p}\binom{k-1}{i}\binom{p+q+1-k}{p-i}.$$

Indeed if $i > k-1$ or $i < k-q-1$, then

$$\binom{k-1}{i}\binom{p+q+1-k}{p-i} = 0.$$

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Using the Chu–Vandermonde identity, which states that

$$\binom{x+y}{z} = \sum_{w=0}^{z} \binom{x}{w}\binom{y}{z-w}$$

for all $x, y, z \in \mathbb{N}$, we get that

$$\sum_{i=0}^{p} \binom{k-1}{i}\binom{p+q+1-k}{p-i} = \binom{p+q}{p}.$$

Therefore,

$$\# \bigcup_{i=\max\{0,k-q-1\}}^{\min\{p,k-1\}} \mathcal{T}_{i+1,k-i,k}^{p+1,q+1} = \#\mathcal{T}_{p+1,q+1,p+q+1}^{p+1,q+1},$$

the map Φ is a bijection, and the isomorphism holds. ■

We can discuss in detail the change of the signature of a shuffle in $\mathcal{T}_{i+1,k-i,k}^{p+1,q+1}$ after the actions of Ψ and Φ.

Corollary 3.4 *Let Ψ and Φ be the two bijections defined in the proofs of the Propositions 3.2 and 3.3. For any $(\mu,\nu) \in \mathcal{T}_{i+1,k-i,k}^{p+1,q+1}$, where $\max\{0, k-q-1\} \le i \le \min\{p, k-1\}$, we have*

$$\operatorname{sgn}(\mu,\nu) = (-1)^{pi+q(k-i-1)+(k-1)(p+q)+p+k+1} \operatorname{sgn}\big(\Psi\Phi(\mu,\nu)\big). \tag{3.3}$$

Proof As $\operatorname{sgn}(\chi_{m,n}) = (-1)^{mn}$, we have

$$\begin{aligned}\operatorname{sgn}\big(\Phi(\mu,\nu)\big) &= \operatorname{sgn}(\chi_{p-i,i+1})\operatorname{sgn}(\chi_{q+i+1-k,k-i})\operatorname{sgn}(\chi_{k+1,p+q+1-k})\operatorname{sgn}(\mu,\nu)\\ &= (-1)^{pi+q(k-i-1)+(k-1)(p+q)+p+q+k+1}\operatorname{sgn}(\mu,\nu).\end{aligned}$$

It is clear that $\operatorname{sgn}(\Psi(u,v)) = (-1)^q \operatorname{sgn}(u,v)$ for any $(u,v) \in \mathcal{T}_{p+1,q+1,p+q+1}^{p+1,q+1}$. So the relation (3.3) holds. ■

Next we study some combinatorial properties of $\mathcal{S}_{i+1,j+1}^{p+1,q+1}$.

Lemma 3.5

(i) *For $1 \le i \le p$ and $0 \le j \le q$, we have the isomorphism of sets*

$$\big\{(\mu,\nu) \in \mathcal{S}_{i+1,j+1}^{p+1,q+1} \mid \mu_1 = 0\big\} \cong \mathcal{S}_{i,j+1}^{p,q+1}.$$

(ii) *For $0 \le j \le q$, we have*

$$\big\{(\mu,\nu) \in \mathcal{S}_{1,j+1}^{p+1,q+1} \mid \mu_1 = 0\big\} \cong \mathfrak{S}_{p,q+1}.$$

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(iii) *For* $0 \le i \le p$ *and* $1 \le j \le q$*, we have*

$$\{(\mu,\nu) \in \mathcal{S}^{p+1,q+1}_{i+1,j+1} \mid \nu_1 = 0\} \cong \mathcal{S}^{p+1,q}_{i+1,j}.$$

(iv) *For* $0 \le i \le p$*, we have*

$$\{(\mu,\nu) \in \mathcal{S}^{p+1,q+1}_{i+1,1} \mid \nu_1 = 0\} = \varnothing.$$

Proof For (i) and (ii), define $u_r = \mu_{r+1} - 1$ and $v_l = \nu_l - 1$ for $1 \le r \le p$ and $1 \le l \le q+1$. Let $\phi_1(\mu,\nu) = (u,v)$. We can easily check that ϕ_1 is bijective and $\operatorname{sgn}(\mu,\nu) = \operatorname{sgn}\phi_1(\mu,\nu)$.

For (iii), define $u'_r = \mu_r - 1$ and $v'_l = \nu_{l+1} - 1$ for $1 \le r \le p+1$ and $1 \le l \le q$. Let $\phi_2(\mu,\nu) = (u',v')$. We can easily check that ϕ_2 is bijective and $\operatorname{sgn}(\mu,\nu) = (-1)^{p+1}\operatorname{sgn}\phi_2(\mu,\nu)$. Case (iv) is clear. ∎

Similarly we have the following lemma.

Lemma 3.6

(i) *For* $0 \le i \le p$ *and* $0 \le j \le q-1$*, we have*

$$\{(\mu,\nu) \in \mathcal{S}^{p+1,q+1}_{i+1,j+1} \mid \nu_{q+1} = p+q+1\} \cong \mathcal{S}^{p+1,q}_{i+1,j+1}.$$

(ii) *For* $0 \le i \le p$*, we have*

$$\{(\mu,\nu) \in \mathcal{S}^{p+1,q+1}_{i+1,q+1} \mid \nu_{q+1} = p+q+1\} \cong \mathfrak{S}_{p+1,q}.$$

(iii) *For* $0 \le i \le p-1$ *and* $0 \le j \le q$*, we have*

$$\{(\mu,\nu) \in \mathcal{S}^{p+1,q+1}_{i+1,j+1} \mid \mu_{p+1} = p+q+1\} \cong \mathcal{S}^{p,q+1}_{i+1,j+1}.$$

(iv) *For* $0 \le j \le q$*, we have*

$$\{(\mu,\nu) \in \mathcal{S}^{p+1,q+1}_{p+1,j+1} \mid \mu_{p+1} = p+q+1\} = \varnothing.$$

Lemma 3.7 *For fixed i and j with* $0 \le i \le p$ *and* $0 \le j \le q$*,*

(i) *if* $2 \le l \le j+1$*, then we have*

$$\{(\mu,\nu) \in \mathcal{S}^{p+1,q+1}_{i+1,j+1} \mid \nu_l = \nu_{l-1} + 1\} \cong \mathcal{S}^{p+1,q}_{i+1,j};$$

(ii) *if* $j+2 \le l \le q+1$*, then we have*

$$\{(\mu,\nu) \in \mathcal{S}^{p+1,q+1}_{i+1,j+1} \mid \nu_l = \nu_{l-1} + 1\} \cong \mathcal{S}^{p+1,q}_{i+1,j+1}.$$

Proof For case (i), we can define a bijection

$$\varphi\colon \left\{ (\mu,\nu) \in \mathcal{S}^{p+1,q+1}_{i+1,j+1} \mid \nu_l = \nu_{l-1} + 1 \right\} \to \mathcal{S}^{p+1,q}_{i+1,j}.$$

Set $\varphi(\mu,\nu) = (u,v)$ with (u,v) defined by

$$u_r = \begin{cases} \mu_r & \text{for } \mu_r < \nu_{l-1}, \\ \mu_r - 1 & \text{for } \mu_r > \nu_l, \end{cases} \quad \text{and} \quad v_{r'} = \begin{cases} \nu_{r'} & \text{for } 1 \le r' \le l-1, \\ \nu_{r'+1} - 1 & \text{for } l \le r' \le q. \end{cases}$$

It is clear that $(u,v) \in \mathcal{S}^{p+1,q}_{i+1,j}$ and φ is a bijection. Also the signatures obey

$$\operatorname{sgn}(\mu,\nu) = (-1)^{p+1+l+\nu_{l-1}} \operatorname{sgn}(u,v) = (-1)^{p+1+l+\nu_{l-1}} \operatorname{sgn}(u,v) \tag{3.4}$$

with $\varphi(\mu,\nu) = (u,v)$. Case (ii) is proven similarly. ∎

We denote by $\mathcal{A}_l$ the set $\{(\mu,\nu) \in \mathcal{S}^{p+1,q+1}_{i+1,j+1} \mid \nu_l = \nu_{l-1}+1\}$ and let

$$\mathcal{B}_k = \left\{ (\mu,\nu) \in \mathcal{S}^{p+1,q+1}_{i+1,j+1} \mid k-1, k \in \{\nu_1,\dots,\nu_{q+1}\} \right\}.$$

We consider their disjoint unions. Recall the formal definition of a disjoint union of sets S_i, $i \in I$, is $\bigsqcup_{i\in I} S_i = \bigcup_{i \in I}\{(x,i) \mid x \in S_i\}$. Note that $\mathcal{A}_{l_1} \bigcap \mathcal{A}_{l_2}$ and $\mathcal{B}_{k_1} \bigcap \mathcal{B}_{k_2}$ may not be empty for $2 \le l_1 \ne l_2 \le q+1$ and $1 \le k_1 \ne k_2 \le p+q+1$; we can obtain the following lemma immediately by setting $\nu_l = k$.

Lemma 3.8

$$\bigsqcup_{2 \le l \le q+1} \mathcal{A}_l = \bigsqcup_{1 \le k \le p+q+1} \mathcal{B}_k.$$

4 Proof of the Generalized Cyclic Eilenberg–Zilber Theorem

We restate the generalized cyclic Eilenberg–Zilber theorem for the normalized complexes.

Theorem 4.1 ([3, 4]) *Let X be a cylindrical module, ζ the shuffle map, and ξ the generalized cyclic shuffle map. Then $(\zeta,\xi)\colon \overline{\operatorname{Tot}}(X) \to \overline{\Delta}(X)$ is a quasi-isomorphism of mixed complexes satisfying*

$$\mathfrak{b}\xi - \xi\mathbb{b} + \overline{\mathfrak{B}}\zeta - \zeta\overline{\mathbb{B}} = 0. \tag{4.1}$$

Hence (ζ,ξ) induces an isomorphism of cyclic homologies

$$\mathrm{HC}_*\big(\operatorname{Tot}(X)\big) \cong \mathrm{HC}_*\big(\Delta(X)\big).$$

Proof Explicitly, acting on $X_{p,q}$, the formula (4.1) yields the formula

$$\mathfrak{b}\xi_{p,q} = \xi_{p-1,q} b_h + (-1)^p \xi_{p,q-1} b_v + \zeta_{p+1,q} T_v \overline{B}_h + (-1)^p \zeta_{p,q+1} \overline{B}_v - \overline{\mathfrak{B}}\zeta_{p,q}. \tag{4.2}$$

One can refer to Appendix A for explicit definitions of the operators in the above formula. It is clear that $\zeta\mathbb{b} = \mathfrak{b}\zeta$. And, using the relation (A.5), we easily get $\xi\overline{\mathbb{B}} = \overline{\mathfrak{B}}\xi = 0$ in the normalized complexes. Therefore, if formula (4.2) holds, then the proof follows by using the classic Eilenberg–Zilber theorem for Hochschild homology and the Five Lemma.

In order to prove formula (4.2), we will often use the defining relations (A.1)–(A.5) of a paracyclic module. Also the following relations derived from relations (A.1)–(A.5) will be used frequently later:

$$s_{-1}s_k = s_{k+1}s_{-1} \quad \text{for } k \geq -1, \tag{4.3}$$

$$d_i t_n^j = \begin{cases} t_{n-1}^{j-1} d_{n+i+1-j} & \text{for } i < j, \\ t_{n-1}^{j} d_{i-j} & \text{for } j \leq i \leq n, \end{cases} \tag{4.4}$$

$$s_i t_n^j = \begin{cases} t_{n+1}^{j} s_{i-j} & \text{for } i \geq j > 0, \\ t_{n+1}^{j+1} s_{n+i+1-j} & \text{for } -1 \leq i < j \leq n, \end{cases} \tag{4.5}$$

$$d_i s_{-1} = \begin{cases} id_{M_n} & \text{for } i = 0, \\ s_{-1} d_{i-1} & \text{for } 0 < i < n+1, \\ t_n & \text{for } i = n+1. \end{cases} \tag{4.6}$$

Note that $\mathfrak{b}\,\xi_{p,q}$ is a sum of elements of the form

$$d_k s_{\nu_{q+1}} \cdots s_{\nu_1} s_{-1} t_p^i \delta_k \sigma_{\mu_{p+1}} \cdots \sigma_{\mu_1} \sigma_{-1} \tau_q^j,$$

where $(\mu, \nu) \in \mathcal{S}_{i+1,j+1}^{p+1,q+1}$. We can divide these elements into four parts:

(a) $k = 0$, or $k = p + q + 2$;
(b) $1 \leq k \leq p + q + 1$, and $k - 1, k \in \{\nu_1, \ldots, \nu_{q+1}\}$;
(c) $1 \leq k \leq p + q + 1$, and $k - 1, k \in \{\mu_1, \ldots, \mu_{p+1}\}$;
(d) $1 \leq k \leq p + q + 1$, and either $k - 1 \in \{\mu_1, \ldots, \mu_{p+1}\}$, $k \in \{\nu_1, \ldots, \nu_{q+1}\}$, or $k - 1 \in \{\nu_1, \ldots, \nu_{q+1}\}$, $k \in \{\mu_1, \ldots, \mu_{p+1}\}$.

We will show that the sum of elements in part (a) is $\zeta_{p+1,q} T_v \overline{B}_h + (-1)^p \zeta_{p,q+1} \overline{B}_v$; the sum of elements in part (b) is $(-1)^p \xi_{p,q-1} b_v$; the sum of elements in part (c) is $\xi_{p-1,q} b_h$, and the sum of elements in part (d) is $-\overline{\mathfrak{B}}\zeta_{p,q}$.

We further divide part (a) into six parts:

(a.1) $k = 0$, $i = 0$, and $\mu_1 = 0$;
(a.2) $k = 0$, $1 \leq i \leq p$, and $\mu_1 = 0$;
(a.3) $k = 0$, and $\nu_1 = 0$;
(a.4) $k = p + q + 2$, $j = q$, and $\nu_{q+1} = p + q + 1$;
(a.5) $k = p + q + 2$, $0 \leq j \leq q - 1$, and $\nu_{p+1} = p + q + 1$;
(a.6) $k = p + q + 2$, and $\mu_{p+1} = p + q + 1$.

For part (a.1), moving d_0 and δ_0 to the right until they meet s_{-1} and σ_0 respectively, by Lemma 3.5(ii) and relations (A.3), we obtain the sum of elements in part (a.1) is $(-1)^p \zeta_{p,q+1} \overline{B}_v$. Similarly, by Lemma 3.6(ii) and the relations (A.3), we can

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get that the sum of elements in part (a.4) is $\zeta_{p+1,q}T_v\overline{B}_h$. For parts (a.2) and (a.6), using Lemma 3.5(i), Lemma 3.6(iii), and the discussion about the signatures in their proofs, we get that the sum of elements in part (a.2) cancels with the sum of elements in part (a.6). Similarly, we can show that the sum of elements in part (a.3) cancels with the sum of elements in part (a.5).

For part (b), using relations (4.6) and (A.3) to move d_k and δ_k to the right, by Lemma 3.8 we obtain that the sum of elements in part (b) equals

$$(4.7)\quad \sum_{\substack{0\le i\le p\\ 0\le j\le q\\ 2\le l\le q+1}} \sum_{(\mu,\nu)\in\mathcal{A}_l} (-1)^{pi+qj+p+\nu_l}\operatorname{sgn}(\mu,\nu) \left(s_{\nu_{q+1}-1}\cdots s_{\nu_{l+1}-1}s_{\nu_{l-1}}\cdots s_{\nu_1}s_{-1}t_p^i\sigma_{\mu_{p+1}-1}\cdots\sigma_{\mu_{x+1}-1}\sigma_{\mu_x}\cdots\sigma_{\mu_1}\sigma_{-1}\delta_{l-2}\tau_q^j\right),$$

where $x=\nu_l-l+1$, since $\mu_r<\nu_{l-1}$ for $1\le r\le x$ and $\mu_{r'}>\nu_l$ for $x<r'\le p+1$.

Divide the sum (4.7) into two parts, one is summing over $2\le l\le j+1$, the other is summing over $j+2\le l\le q+1$. Then using Lemma 3.7, the bijection φ defined in its proof, and relations (3.4) and (4.5), we get that (4.7) equals $(-1)^p\xi_{p,q-1}b_v$.

Similarly, one can obtain the sum of elements in part (c) equals $\xi_{p-1,q}b_h$.

For part (d), if $(\mu,\nu)\in\mathcal{S}^{p+1,q+1}_{i+1,j+1}$, and $k-1\in\{\nu_1,\dots,\nu_{q+1}\}$, $k\in\{\mu_1,\dots,\mu_{p+1}\}$, then permuting $k-1$ and k, we get an other $(p+1,q+1)$-shuffle (μ',ν'). It is easy to see that

$$(\mu',\nu')\in\mathcal{S}^{p+1,q+1}_{i+1,j+1}\quad\text{and}\quad \operatorname{sgn}(\mu',\nu')=-\operatorname{sgn}(\mu,\nu).$$

Since $\mu_{i+1}<\nu_{j+1}$, we have $(\mu'_{i+1},\nu'_{j+1})\neq(k-1,k)$. On the other case, if $(u,v)\in\mathcal{S}^{p+1,q+1}_{i+1,j+1}$, and $k-1\in\{u_1,\dots,u_{p+1}\}$, $k\in\{v_1,\dots,v_{q+1}\}$, and moreover if $(u_{i+1},v_{j+1})\neq(k-1,k)$, then permuting $k-1$ and k, we get $(u',v')\in\mathcal{S}^{p+1,q+1}_{i+1,j+1}$, and $k-1\in\{\nu_1,\dots,\nu_{q+1}\}$, $k\in\{\mu_1,\dots,\mu_{p+1}\}$ with $\operatorname{sgn}(u',v')=-\operatorname{sgn}(u,v)$. Hence many elements in part (d) have disappeared; the rest is summing over $(\mu,\nu)\in\mathcal{S}^{p+1,q+1}_{i+1,j+1}$ with $\mu_{i+1}=k-1$ and $\nu_{j+1}=k$, *i.e.*, $(\mu,\nu)\in\mathcal{T}^{p+1,q+1}_{i+1,j+1,k}$. Note again that $\mathcal{T}^{p+1,q+1}_{i+1,j+1,k}\neq\varnothing$ only when $j=k-i-1$.

Consider the elements of the form $d_ks_{\nu_{q+1}}\cdots s_{\nu_1}s_{-1}t_p^i\delta_k\sigma_{\mu_{p+1}}\cdots\sigma_{\mu_1}\sigma_{-1}\tau_q^j$, where $(\mu,\nu)\in\mathcal{T}^{p+1,q+1}_{i+1,j+1,k}$. First moving d_k and δ_k to the right until they cancel with s_{k-1} and σ_k, then moving s_{-1} and σ_{-1} leftwards by the relation (4.3), we obtain

$$s_{-1}s_{\nu_{q+1}-2}\cdots s_{\nu_{j+2}-2}s_{\nu_j-1}\cdots s_{\nu_1-1}t_p^i\sigma_{-1}\sigma_{\mu_{p+1}-2}\cdots\sigma_{\mu_{i+2}-2}\sigma_{\mu_i-1}\cdots\sigma_{\mu_1-1}\tau_q^j.$$

Next we would like to use the relation (4.5) to move t_p and τ_q leftwards, so we should compare the indexes. For a $(p+1,q+1)$-shuffle (μ,ν), by Lemma 3.1(ii) we have $\nu_l+j+1-l\le\nu_{j+1}$ for $1\le l\le j$, and $\nu_{j+1}+r-j-1\le\nu_r$ for $j+2\le r\le q+1$. As $\nu_{j+1}=k=i+j+1$, we get that $\nu_l\le i+l$ and $\nu_r\ge r+i\ge j+2+i$. In fact we have $\nu_l<i+l$ for $1\le l\le j$. If there exists an $l\le j$ such that $\nu_l=i+l$, then ν_j should be

equal to $i+j$. This contradicts $\mu_{i+1} = i+j$. Then using relation (4.5) we get

$$(4.8) \quad s_{-1}s_{\nu_{q+1}-2}\cdots s_{\nu_{j+2}-2}s_{\nu_j-1}\cdots s_{\nu_1-1}t_p^i = \\ s_{-1}t_{p+q}^{i+j}(s_{\nu_{q+1}-2-i-j}\cdots s_{\nu_{j+2}-2-i-j})(s_{\nu_j+p-i}\cdots s_{\nu_2+p-i}s_{\nu_1+p-i}).$$

For any $(\mu,\nu) \in \mathfrak{S}_{p+1,q+1}$, by Lemma 3.1(ii) we have $\nu_l \le \nu_1 + l - 1 + p + 1$, for all $1 \le l \le q+1$. Then

$$\nu_{j+2+n} - 2 - i - j \le \nu_1 + p - i + n \le \cdots \le \nu_j + p - i + n, \text{ for } 0 \le n \le q - j - 1.$$

Using relation (A.2), we can move the left bracket in (4.8) to the right. Then (4.8) equals

$$s_{-1}t_{p+q}^{i+j}(s_{\nu_j+p+q-i-j}\cdots s_{\nu_2+p+q-i-j}s_{\nu_1+p+q-i-j})(s_{\nu_{q+1}-2-i-j}\cdots s_{\nu_{j+2}-2-i-j}).$$

Let

$$\nu_1' = \nu_{j+2} - 2 - i - j, \dots, \nu_{q-j}' = \nu_{q+1} - 2 - i - j,$$
$$\nu_{q-j+1}' = \nu_1 + p + q - i - j, \dots, \nu_q' = \nu_j + p + q - i - j,$$

i.e.,

$$\nu_l' = \chi_{i+j+2,p+q-i-j}(\nu_{\chi_{q-j,j+1}(l)} + 1) - 1, \forall\, 1 \le l \le q.$$

Then we obtain

$$s_{-1}s_{\nu_{q+1}-2}\cdots s_{\nu_{j+2}-2}s_{\nu_j-1}\cdots s_{\nu_1-1}t_p^i = s_{-1}t_{p+q}^{k-1}(s_{\nu_q'}\cdots s_{\nu_1'}).$$

Play the same game again. We get

$$\sigma_{-1}\sigma_{\mu_{p+1}-2}\cdots\sigma_{\mu_{i+2}-2}\sigma_{\mu_i-1}\cdots\sigma_{\mu_1-1}\tau_q^j = \sigma_{-1}\tau_{p+q}^{k-1}(\sigma_{\mu_p'}\cdots\sigma_{\mu_1'}),$$

where

$$\mu_r' = \chi_{i+j+2,p+q-i-j}(\mu_{\chi_{p-i,i+1}(r)} + 1) - 1, \forall\, 1 \le r \le p.$$

So using the maps Ψ and $\Phi_{i,k}$ defined in the proofs of Propositions 3.2 and 3.3, we get

$$(\mu',\nu') = \Psi\Phi_{i,k}(\mu,\nu) \in \mathfrak{S}_{p,q}.$$

By Propositions 3.2 and 3.3, when (μ,ν) runs over

$$\bigcup_{\substack{0\le i\le p\\ 0\le j\le q}} \mathcal{T}_{i+1,j+1,k}^{p+1,q+1}$$

for a fixed k, the corresponding (μ',ν') runs over all (p,q)-shuffles, with

$$(-1)^{pi+qj+p+k}\operatorname{sgn}(\mu,\nu) = (-1)^{(k-1)(p+q)+1}\operatorname{sgn}(\mu',\nu').$$

Hence we have

$$\sum_{k=1}^{p+q+1} \sum_{\substack{0\le i\le p\\ 0\le j\le q\\ (\mu,\nu)\in\mathfrak{T}^{p+1,q+1}_{i+1,j+1,k}}} (-1)^{pi+qj+p+k}\,\mathrm{sgn}(\mu,\nu) d_k s_{\nu_{q+1}}\cdots s_{\nu_1} s_{-1} t_p^i \delta_k \sigma_{\mu_{p+1}}\cdots\sigma_{\mu_1}\sigma_{-1}\tau_q^j$$

$$= \sum_{k=1}^{p+q+1} \sum_{(\mu',\nu')\in\mathfrak{S}_{p,q}} (-1)^{(k-1)(p+q)+1}\,\mathrm{sgn}(\mu',\nu') s_{-1} t_{p+q}^{k-1}(s_{\nu'_q}\cdots s_{\nu'_1})\sigma_{-1}\tau_{p+q}^{k-1}(\sigma_{\mu'_p}\cdots\sigma_{\mu'_1})$$

$$= -s_{-1}\sigma_{-1}\sum_{k=0}^{p+q}(-1)^{k(p+q)} t_{p+q}^k \tau_{p+q}^k \sum_{(\mu',\nu')\in\mathfrak{S}_{p,q}} \mathrm{sgn}(\mu',\nu')(s_{\nu'_q}\cdots s_{\nu'_1})(\sigma_{\mu'_p}\cdots\sigma_{\mu'_1})$$

$$= -\overline{\mathfrak{B}}\zeta_{p,q}.$$

This completes the proof. ■

A Appendix

The references for the notions introduced here are [3, 4, 6, 7].

A.1 Cylindrical Modules

We first recall the ingredient of a cylindrical module, which is a paracyclic module.

Definition A.1 A *paracyclic K-module M* is a family of K-modules $\{M_n\}_{n\ge 0}$ endowed, for each $n \ge 0$, with K-homomorphisms $d_i\colon M_{n+1} \to M_n$ for all $0 \le i \le n+1$, K-homomorphisms $s_j\colon M_n \to M_{n+1}$ for all $0 \le j \le n$, and K-automorphisms $t_n\colon M_n \to M_n$, satisfying the following relations

(A.1) $$d_i d_j = d_{j-1} d_i \quad \text{for } i < j,$$

(A.2) $$s_i s_j = s_{j+1} s_i \quad \text{for } i \le j,$$

(A.3) $$d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j,\\ id & \text{for } i = j,\ i = j+1,\\ s_j d_{i-1} & \text{for } i > j+1,\end{cases}$$

(A.4) $$d_i t_n = \begin{cases} t_{n-1} d_{i-1} & \text{for } 1 \le i \le n,\\ d_n & \text{for } i = 0,\end{cases}$$

(A.5) $$s_i t_n = \begin{cases} t_{n+1} s_{i-1} & \text{for } 1 \le i \le n,\\ t_{n+1}^2 s_n & \text{for } i = 0.\end{cases}$$

Here d_i, s_j, and t_n are called *face maps*, *degeneracy maps*, and *paracyclic operators* respectively.

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Let $s_{-1} = t_{n+1}s_n\colon M_n \to M_{n+1}$, for all $n \geq 0$, be its *extra degeneracy map*. Define the following operators

$$b = \sum_{i=0}^{n+1} (-1)^i d_i\colon M_{n+1} \to M_n, \tag{A.6}$$

$$T = t_n^{n+1}\colon M_n \to M_n, \tag{A.7}$$

$$N = \sum_{i=0}^{n} (-1)^{in} t_n^i\colon M_n \to M_n, \tag{A.8}$$

$$B = \big(1 + (-1)^n t_{n+1}\big) s_{-1}N\colon M_n \to M_{n+1}. \tag{A.9}$$

These operators satisfy the relations $b^2 = B^2 = 0$ and $Bb + Bb = 1 - T$.

Definition A.2 A *bi-paracyclic module* is a sequence of K-modules $(\{X_{m,n}\}_{m,n\geq 0}, d_i, s_i, t_m, \delta_j, \sigma_j, \tau_n)$, where

$$d_i\colon X_{m,n} \to X_{m-1,n},\ s_i\colon X_{m,n} \to X_{m+1,n},\ t_m\colon X_{m,n} \to X_{m,n}, \quad \forall\, 0 \leq i \leq m,$$
$$\delta_j\colon X_{m,n} \to X_{m,n-1},\ \sigma_j\colon X_{m,n} \to X_{m,n+1},\ \tau_n\colon X_{m,n} \to X_{m,n}, \quad \forall\, 0 \leq j \leq n,$$

such that, for each $m_0, n_0 \geq 0$, $(\{X_{m,n_0}\}_{m\geq 0}, d_i, s_i, t_m)$ and $(\{X_{m_0,n}\}_{n\geq 0}, \delta_j, \sigma_j, \tau_n)$ are two paracyclic modules and the operators d_i, s_i, t_m commute with the operators $\delta_j, \sigma_j, \tau_n$. Moreover, if in addition, $t_m^{m+1}\tau_n^{n+1} = id_{X_{m,n}}$ for all $m, n \geq 0$, then this bi-paracyclic module is called a *cylindrical module*.

Let b_h, T_h, N_h, B_h and b_v, T_v, N_v, B_v be the operators defined as in (A.6)–(A.9) with respect to the paracyclic modules $(\{X_{m,n}\}_{m\geq 0}, d_i, s_i, t_m)$ and $(\{X_{m,n}\}_{n\geq 0}, \delta_j, \sigma_j, \tau_n)$.

A.2 The Total Complex and the Diagonal Module

Let $(X_{\cdot,\cdot}, d_i, s_i, t, \delta_j, \sigma_j, \tau)$ be a cylindrical module. The total complex $(\mathrm{Tot}(X), \mathbb{b}, \mathbb{B})$ of X that is defined by $\mathrm{Tot}_n(X) = \bigoplus_{p+q=n} X_{p,q}$ with $\mathbb{b} = b_h + (-1)^p b_v$ and $\mathbb{B} = T_v B_h + (-1)^p B_v$ acting on $X_{p,q}$ is a mixed complex. The diagonal complex $(\{\Delta_n(X)\}_{n\geq 0}, d_i\delta_i, s_j\sigma_j, t_n\tau_n)$ of X is a cyclic module. Define the operators of this cyclic module $\Delta(X)$ as in (A.6)–(A.9), *i.e.*, acting on $X_{n,n}$, $\mathfrak{b} = \sum_{i=0}^n (-1)^i d_i\delta_i$, $\mathfrak{N} = \sum_{i=0}^n (-1)^{in} t_n^i \tau_n^i$, and $\mathfrak{B} = (1 + (-1)^n t_{n+1}\tau_{n+1}) s_{-1}\sigma_{-1}\mathfrak{N}$. Then we get a mixed complex $(\Delta(X), \mathfrak{b}, \mathfrak{B})$.

A.3 Normalized Complexes

Let $(X_{\cdot,\cdot}, d_i, s_i, t, \delta_j, \sigma_j, \tau)$ be a cylindrical module. Let $D(X)$ be the chain subcomplex of X generated by the images of the degeneracies s_i and σ_j, *i.e.*,

$$D_{p,q}(X) = \sum_{i=0}^{p-1} s_i(X_{p-1,q}) + \sum_{j=0}^{q-1} \sigma_j(X_{p,q-1}).$$

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Define $\overline{X}_{p,q} = X_{p,q}/D_{p,q}(X)$ and $\overline{\mathrm{Tot}}_n(X) = \bigoplus_{p+q=n} \overline{X}_{p,q}$. Thus $(\overline{\mathrm{Tot}}(X), \mathbb{b}, \overline{\mathbb{B}})$ is the normalized mixed complex of $(\mathrm{Tot}(X), \mathbb{b}, \mathbb{B})$, where $\overline{\mathbb{B}}$ is induced from $\mathbb{B}$ and defined by $\overline{\mathbb{B}} = T_v\overline{B}_h + (-1)^p\overline{B}_v$ with $\overline{B}_h = s_{-1}N_h$ and $\overline{B}_v = \sigma_{-1}N_v$.

Let $d_n(X) = \sum_{i=0}^{n-1} s_i\sigma_i(X_{n-1,n-1})$ and $\overline{\Delta}_n(X) = X_{n,n}/d_n(X)$. Then $(\overline{\Delta}(X), \mathfrak{b}, \overline{\mathfrak{B}})$ is the normalized mixed complex of $(\Delta(X), \mathfrak{b}, \mathfrak{B})$ with $\overline{\mathfrak{B}} = s_{-1}\sigma_{-1}\mathfrak{N}$.

The mixed complexes $(\mathrm{Tot}(X), \mathbb{b}, \mathbb{B})$ and $(\Delta(X), \mathfrak{b}, \mathfrak{B})$ are quasi-isomorphic to their normalized complexes respectively.

A.4 Shuffles and Shuffle Maps

Definition A.3 Let p and q be two positive integers. A (p,q)-*shuffle* (μ,ν) is a partition of the set of integers $\{0, 1, \dots, p+q-1\}$ into two disjoint subsets such that $\mu_1 < \cdots < \mu_p$ and $\nu_1 < \cdots < \nu_q$. So $\{\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_q\}$ determines a permutation of $\{0, \dots, p+q-1\}$. Let $\mathrm{sgn}(\mu,\nu)$ be the signature of the permutation.

Definition A.4 Let $(X_{\cdot,\cdot}, d_i, s_i, t, \delta_j, \sigma_j, \tau)$ be a cylindrical module. For any $p, q \in \mathbb{N}$, define a map $\zeta_{p,q} \colon X_{p,q} \to X_{p+q,p+q}$ by

$$\zeta_{p,q} = \sum_{(u,v)\in\mathfrak{S}_{p,q}} \mathrm{sgn}(u,v) s_{v_q}\cdots s_{v_1}\sigma_{u_p}\cdots\sigma_{u_1}.$$

Call $\zeta_{p,q}$ the (p,q)-*shuffle map*. The *shuffle map* $\zeta\colon \mathrm{Tot}_n(X) \to \Delta_n(X)$ is defined by $\zeta = \sum_{p+q=n} \zeta_{p,q}$.

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Department of Mathematics, Shanghai University, 99 Shangda Road, BaoShan District, 200444, Shanghai, P.R. China
e-mail: zhangjiao.math@gmail.com wqw858@yahoo.com.cn