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## Monoidal transforms and invariants of singularities in positive characteristic

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## ABSTRACT

The problem of resolution of singularities in positive characteristic can be reformulated as follows: fix a hypersurface  $X$ , embedded in a smooth scheme, with points of multiplicity at most  $n$ . Let an  $n$ -sequence of transformations of  $X$  be a finite composition of monoidal transformations with centers included in the  $n$ -fold points of  $X$ , and of its successive strict transforms. The open problem (in positive characteristic) is to prove that there is an  $n$ -sequence such that the final strict transform of  $X$  has no points of multiplicity  $n$  (no  $n$ -fold points). In characteristic zero, such an  $n$ -sequence is defined in two steps. The first consists of the transformation of  $X$  to a hypersurface with  $n$ -fold points in the so-called *monomial case*. The second step consists of the elimination of these  $n$ -fold points (in the monomial case), which is achieved by a simple combinatorial procedure for choices of centers. The invariants treated in this work allow us to present a notion of *strong monomial case* which parallels that of monomial case in characteristic zero: if a hypersurface is within the strong monomial case we prove that a resolution can be achieved in a combinatorial manner.

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### 1. Introduction

#### 1.1 Overview

The objective of this paper is to study invariants of singularities in positive characteristic. A particular motivation is to give invariants that would yield a sequence of monoidal transformations so as to eliminate the points of highest multiplicity of a hypersurface  $X$ . To be precise, let  $V$  be a smooth scheme of dimension  $d$  over a perfect field  $k$  of characteristic  $p > 0$ , and let  $X$  be a hypersurface in  $V$  with highest multiplicity  $n$ . The problem is to construct a sequence

$$\begin{array}{ccccccc}
 X & & X_1 & & & X_r & & X_N \\
 V \xleftarrow{\pi_{C_0}} & & V_1 \xleftarrow{\pi_{C_1}} & \dots & \xleftarrow{\pi_{C_{r-1}}} & V_r \xleftarrow{\pi_{C_r}} & \dots & \xleftarrow{\pi_{C_{N-1}}} V_N
 \end{array} \tag{1.1.1}$$

where each  $V_{i-1} \xleftarrow{\pi_i} V_i$  is a monoidal transformation with center  $C_{i-1}$  included in the  $n$ -fold points of  $X_{i-1}$ , so that  $X_N$  has no point of multiplicity  $n$ . Here each  $X_i \subset V_i$  denotes the strict transform of  $X_{i-1}$  by  $\pi_{C_{i-1}}$ . We require, in addition, that the exceptional locus of  $V \leftarrow V_N$  be a union of  $N$  hypersurfaces with normal crossings at  $V_N$ . A sequence with this property is said to be a *simplification of the  $n$ -fold points* of  $X$ .

In characteristic zero, simplifications of  $n$ -fold points of  $X$  are known to exist. This is usually done in two steps. The first step consists of a sequence of, say  $r$ , monoidal transformations, so that the set of points of highest multiplicity  $n$  of  $X_r$  is within the so-called *monomial case*. The second step consists of the elimination of the  $n$ -fold points of the hypersurface  $X_r$ , which is assumed to be in the monomial case. The latter step is rather simple, and it can be achieved by a combinatorial choice of centers.

Both steps rely on Hironaka’s main *inductive invariant*, say  $\text{ord}^{(d-1)}(x) \in \mathbb{Q}$ , defined for  $x$  in the highest multiplicity locus of the hypersurface. In fact, these invariants lead to the construction of a sequence in such a way that  $X_r$  is in the monomial case. The role of Hironaka’s main inductive function in both steps mentioned above, always in characteristic zero, will be recalled in § 1.3.

In this work we conduct a study of Hironaka’s inductive function over perfect fields of arbitrary characteristic. We will introduce the notion of *strong monomial case* for a hypersurface in positive characteristic. This notion will be characterized in terms of Hironaka’s inductive functions. It parallels that of monomial case in characteristic zero, i.e., if  $X_r$  is in the strong monomial case, then elimination of  $n$ -fold points is achieved in a combinatorial manner.

In the case of hypersurfaces in positive characteristic, a canonical sequence of transformations of  $X$  was defined in [BVS10]. This sequence transforms  $X$  to an embedded hypersurface, say  $X_r$ , which is closely related to the *monomial case*, but still weaker than the *strong monomial case* treated here. The simplification of the  $n$ -fold problem would be solved if one could fill the gap between the weak monomial case in [BVS10] and our strong monomial case. To be precise, the open problem of simplification (and of resolution of singularities) would be solved if one could construct a sequence of monoidal transformations that transforms a hypersurface in the monomial case into one in the strong monomial case.

This can be easily achieved in low dimension, and we prove resolution of singularities of 2-dimensional schemes by means of the invariants introduced here. A detailed proof of this fact can be found in [BV12].

#### 1.2 Reformulation in terms of algebras

Assume, for simplicity, that  $V$  is affine and  $X = V(f)$  is a hypersurface with highest multiplicity  $n$ . We will first attach to the previous data the algebra  $\mathcal{O}_V[fW^n] (\subset \mathcal{O}_V[W])$  with  $n$

as above, namely the  $\mathcal{O}_V$ -subalgebra of  $\mathcal{O}_V[W]$  generated by the element  $fW^n$ . The notion of transformation of hypersurfaces (with the center included in the subset of  $n$ -fold points) has a natural reformulation in the language of algebras. Moreover, the task of constructing a sequence (1.1.1) that eliminates the  $n$ -fold points of  $X$ , by means of monoidal transformations, can also be expressed in terms of algebras and transformations of algebras (see § 2.2).

This reformulation of the simplification problem in terms of algebras is well justified. In fact, the original algebra  $\mathcal{O}_V[fW^n]$  can be extended canonically to a so-called *differential algebra*, so that both are strongly linked; for the purpose of constructing a simplification of  $\mathcal{O}_V[fW^n]$ , there is no harm in replacing it by its differential extension. Over fields of characteristic zero, this procedure is well known. In fact, differential algebras (see § 2.5) are closely related to the theory of maximal contact in characteristic zero. In such context, hypersurfaces of maximal contact allow us to reformulate the problem of simplification with the simplification of a new algebra, defined over a smooth hypersurface  $\bar{V}$ , and hence in one less dimension. Here  $\bar{V}$  is called a *hypersurface of maximal contact*. This form of induction is formulated in the language of algebras, the correspondent algebra defined over  $\bar{V}$  is known as the *coefficient algebra*.

### 1.3 Monomial algebras and monomial case

In problems of resolution of singularities, it is natural to consider sequences of transformations of the form

$$V \xleftarrow{\pi_{C_0}} V_1 \xleftarrow{\pi_{C_1}} \dots \xleftarrow{\pi_{C_{r-1}}} V_r \tag{1.3.1}$$

with the additional condition that the exceptional locus of the sequence, say  $E_r = \{H_1, \dots, H_r\}$ , is given by hypersurfaces having only normal crossings at  $V_r$ . A *monomial algebra* in  $V_r$  is an algebra of the form  $\mathcal{O}_{V_r}[I(H_1)^{\alpha_1} \dots I(H_r)^{\alpha_r} W^s]$  for some  $s, \alpha_i \in \mathbb{Z}_{\geq 0}$ .

In the case of characteristic zero, the simplification of  $n$ -fold points can be achieved in two steps, both of them expressed in terms of algebras, once a hypersurface of maximal contact, say  $\bar{V}$ , is fixed.

*Step 1.* In which a sequence of monoidal transformations is constructed over the hypersurface of maximal contact, say

$$\bar{V} \xleftarrow{\pi_0} \bar{V}_1 \xleftarrow{\pi_1} \dots \xleftarrow{\pi_{r-1}} \bar{V}_r, \tag{1.3.2}$$

so that the coefficient algebra is transformed into a monomial algebra supported on the exceptional locus, say  $\mathcal{O}_{\bar{V}_r}[I(H_1)^{\alpha_1} \dots I(H_r)^{\alpha_r} W^s]$ . The point is that this sequence induces a sequence (1.3.1), and in this case, the  $n$ -fold points of  $X_r$  (the strict transform of  $X$ ) are said to be in the *monomial case*.

*Step 2.* In which a simplification of the  $n$ -fold points of  $X_r$  (monomial case) is constructed, say

$$V_r \xleftarrow{\pi_r} V_{r+1} \xleftarrow{\dots} \dots \xleftarrow{\pi_{N-1}} V_N. \tag{1.3.3}$$

This step is achieved in an easy combinatorial manner. This procedure of choice of centers is defined only in terms of the exponents  $\alpha_i$  of the monomial algebra obtained in Step 1.

All these arguments (always in characteristic zero) rely strongly on Hironaka’s inductive function  $\text{ord}^{(d-1)}$  (see (2.3.1)), defined in terms of the coefficient algebra. In fact, Hironaka’s function allows us to attach to an arbitrary sequence (1.3.1) a monomial algebra  $\mathcal{O}_{V_r}[I(H_1)^{\alpha_1} \dots I(H_r)^{\alpha_r} W^s]$ . To be precise, this is done by setting  $\alpha_i/s + 1 = \text{ord}_{i-1}^{(d-1)}(y_{i-1})$  ( $i = 1, \dots, r$ ); here the right-hand side is the evaluation of the inductive function at  $y_{i-1}$ , the generic point of the center  $C_{i-1}$ .

**1.4 Main objectives of this work**

In this work we consider schemes over perfect fields of positive characteristic. The two main objectives are as follows.

- (1) To define an analogue to Hironaka’s inductive function, called here  $\text{H-ord}_i^{(d-1)}$  (see Theorem 7.2 (Main Theorem 1)), with values in  $\mathbb{Q}$ . These functions enable us to attach a monomial algebra  $\mathcal{O}_{V_r}[I(H_1)^{h_1} \dots I(H_r)^{h_r} W^s]$  to a sequence of transformations (1.3.1), setting as before  $h_i/s + 1 = \text{H-ord}_{i-1}^{(d-1)}(y_{i-1})$  (see Theorem 7.6 (Main Theorem 2)).
- (2) To characterize, by numerical invariants, a case called here the *strong monomial case* (Definition 8.4), in which a combinatorial resolution of the monomial algebra defines, as in Step 2, a simplification of  $n$ -fold points (Theorem 8.14). This property will rely strongly on Main Theorem 2.

**1.5 Differences with characteristic zero**

In characteristic zero, Hironaka’s inductive function  $\text{ord}^{(d-1)}$  is upper-semi-continuous. This property follows from a form of coherence, and the proof of this property requires some form of patching of local data, and it is altogether quite involved. In positive characteristic the function  $\text{H-ord}^{(d-1)}$  is not upper-semi-continuous and therefore we do not go through this kind of discussion. In other words there is no coherence or patching to be proved in the positive characteristic case. Despite this fact, this function is essential in the study of singularities and we show that it leads to (1) and (2) in § 1.4.

In characteristic zero the value of the function  $\text{ord}^{(d-1)}$ , at a given point, is computed by fixing a hypersurface of maximal contact. As there is no maximal contact in positive characteristic, we replace reduction to hypersurfaces of maximal contact by *transversal projections*:  $V^{(d)} \rightarrow V^{(d-1)}$  defined in étale topology (Definition 2.10). In this setting, algebras over the smooth scheme  $V^{(d-1)}$  are defined; they are called *elimination algebras* (§ 2.11). In characteristic zero elimination algebras parallel the role of the coefficient algebras.

We use here transversal projections and elimination algebras to compute the value of the function  $\text{H-ord}^{(d-1)}$  at a given point, which is a rational number. To fix ideas let  $x$  be an  $n$ -fold point of  $X = V(f) \subset V^{(d)}$ . The Weierstrass preparation theorem ensures that one can choose a regular system of parameters  $\{z, x_1, \dots, x_{r-1}\}$  so that at the completion  $\widehat{\mathcal{O}}_{V^{(d)},x} = k'[[z, x_1, \dots, x_{r-1}]]$  ( $r = d$  if  $x$  is closed) we can take

$$f(z) = z^n + a_1 z^{n-1} + \dots + a_n \quad \text{with } a_i \in k'[[x_1, \dots, x_{r-1}]]. \tag{1.5.1}$$

A rational number  $\geq 1$  is defined as

$$\max_z \left\{ \min_{1 \leq i \leq n} \left\{ \frac{\nu_x(a_i)}{i} \right\} \right\} \in \mathbb{Q}, \quad (\nu_x(a_i) \text{ is the order at } k'[[x_1, \dots, x_{r-1}]]) \tag{1.5.2}$$

This is a rational number attached to the singularity  $x$  and to the particular projection. It is called the *maximum slope* as we choose the biggest value for the different choices of  $z$ , but always fix the inclusion of rings  $k'[[x_1, \dots, x_{r-1}]] \subset k'[[z, x_1, \dots, x_{r-1}]] = \widehat{\mathcal{O}}_{V^{(d)},x}$ . Fixing an inclusion of rings is formulated here by fixing a morphism of smooth schemes  $V^{(d)} \rightarrow V^{(d-1)}$  (the *projection*). In order to parallel the presentation in (1.5.1) (Weierstrass preparation theorem) we need to consider étale topology. To be precise, transversal projections to  $X$  are those for which the hypersurface can be expressed by an equation as in (1.5.1) (where  $n$  is the multiplicity of  $X$  at the point).

Our setting will be slightly more general. Once a transversal projection  $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$  is fixed, we will consider an expression

$$f(z) = z^n + a_1 z^{n-1} + \dots + a_n \in \mathcal{O}_{V^{(d-1)}}[z] \tag{1.5.3}$$

where the coefficients  $a_i$  are global functions on  $V^{(d-1)}$  and where  $z$  is a global function on  $V^{(d)}$  so that  $\{dz\}$  is a basis of  $\Omega_{\beta}^1$ , the sheaf of  $\beta$ -relative differentials. In this case, the smooth hypersurface  $\{z = 0\}$  is a section of  $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ . We will abuse the notation and say that the function  $z$  is a *transversal section* of  $\beta$ .

We will study conditions on  $z$  which ensure when the rational number in (1.5.2) is the maximum slope, and then we show that such value is independent of the chosen transversal projection, and hence intrinsic of the singularity (Main Theorem 1).

This defines an invariant attached to singular point  $x$ , denoted here by

$$\text{H-ord}^{(d-1)}(x).$$

If we fix two  $n$ -fold points  $x$  and  $y$ , so that  $x \in \bar{y}$ , then it will be shown that  $\text{H-ord}^{(d-1)}(x) \geq \text{H-ord}^{(d-1)}(y)$  (despite this property, the function is not upper-semi-continuous). This inequality will be used in the proof of the two main objectives (1) and (2) in § 1.4.

This invariant attached to the singularity has been studied by Cossart and Piltant in [CP08, Part II, ch. 1]. It has also been largely studied in positive characteristic for the particular case of equations of the form  $f_{p^e}(z) = z^{p^e} + a_{p^e} \in \mathcal{O}_{V^{(d-1)}}[z]$  (the purely inseparable case), e.g. [Cos87, Hau09, HW, Moh87, Moh96]. In our approach we also focus on how this invariant can be read from a particular projection, and on how it relates to other invariants attached to the projection as the so-called elimination algebra.

Equations of the form  $f_{p^e}(z) = z^{p^e} + a_{p^e} \in \mathcal{O}_{V^{(d-1)}}[z]$  involve a particular transversal projection, say  $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ . Note that pure inseparability fails to hold if the projection is changed (pure inseparability is not a property of the singularities of a hypersurface). In this work we draw attention to the fact that the invariant is independent of the projection, which shows that the rational number in (1.5.2), usually called the *slope of the singularity*, is independent of the projection and hence intrinsic of the singularity.

### 1.6 Organization and further comments

#### Part I: $p$ -presentations, adaptations and the tight monomial algebra

The objective of this first part is the definition of the inductive function and the study of its main properties mentioned in § 1.4. This leads to the two main theorems stated in § 7; we suggest a first look at this section for an overall view of the preliminary results that are needed.

This first part is developed so as to introduce gradually the inductive function  $\text{H-ord}^{(d-1)}$  in positive characteristic, and to pave the way to the study of the strong monomial case in Part II. This part has been organized so as to present only those technical aspects which are crucial in the first two parts, whereas other technical arguments are gathered in Part III.

Section 2 encompasses several notions used throughout the paper, such as Rees algebras and Rees algebras endowed with a suitable compatibility with differential operators. This will lead us to the notion of *simple* differential algebras, which will be essential for the definition of our invariants.

In our approach the study of  $n$ -fold points of the hypersurface  $X = V(f)$  is reformulated here in terms of the Rees algebra  $\mathcal{O}_{V^{(d)}}[fW^n]$ . This is our first example of *simple* algebra. Attached to this Rees algebra is a well-defined *differential algebra*.

Simple algebras which are differential will lead us naturally to the study of monic polynomials (1.5.3), where  $n = p^e$  is now a power of the characteristic.

We also discuss here the notion of *elimination algebras*. These are defined in terms of differential algebras and transversal projections. Elimination algebras will play a central role in the definition of invariants. A first step in this direction will be given by our notion of *p-presentation* in Definition 2.14.

We shall make use of a fundamental property of *stability of transversality* with monoidal transformations, a property that parallels the stability of the maximal contact in characteristic zero: to fix ideas, set  $X = \{f = 0\} \subset V^{(d)}$  and a transversal projection  $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$  as in (1.5.3). Now consider an arbitrary sequence of monoidal transformations

$$\begin{array}{ccccccc} X & & X_1 & & & & X_r \\ V^{(d)} & \xleftarrow{\pi_{C_0}} & V_1^{(d)} & \xleftarrow{\pi_{C_1}} & \dots & \xleftarrow{\pi_{C_{r-1}}} & V_r^{(d)}, \end{array} \tag{1.6.1}$$

where each  $X_{i+1}$  denotes the strict transform of  $X_i$ , and each  $\pi_{C_i}$  is a monoidal transformation with center  $C_{i-1}$  included in the  $n$ -fold points of  $X_i$ . The stability property of the transversality is that (1.6.1) induces a sequence

$$V^{(d-1)} \longleftarrow V_1^{(d-1)} \longleftarrow \dots \longleftarrow V_r^{(d-1)} \tag{1.6.2}$$

together with projections  $V_i^{(d)} \xrightarrow{\beta_i} V_i^{(d-1)}$  which are transversal to  $X_i$  along the  $n$ -fold points (the  $\beta_i$  are defined in an open neighborhood of the  $n$ -fold points of  $X_i$  in  $V_i^{(d)}$ ).

This will lead us to some form of *transformations* of the monic polynomial in (1.5.3):

$$f^{(i)}(z_i) = z_i^n + a_1^{(i)} z_i^{n-1} + \dots + a_n^{(i)} \in \mathcal{O}_{V_i^{(d-1)}}[z_i]. \tag{1.6.3}$$

Here, the polynomials in (1.6.3) are not the strict transform of the first expression in (1.5.3). Changes of the transversal parameter  $z_i$  will be required in the definition of each expression.

In § 3 sequences in the form of (1.6.1) are expressed as transformations of Rees algebras. In this context each transversal projection  $\beta_i$  will define an elimination algebra on  $V_i^{(d-1)}$ . In this section, we also discuss a form of compatibility of elimination with monoidal transformations. This, in turn, will lead to Theorem 3.8 in which monomial algebras appear in a natural manner [BVS10]. The proof of the theorem relies on a form of induction that will be clarified.

One of the objectives of this first part is to assign a monomial algebra, say  $\mathcal{O}_{V_r^{(d)}}[I(H_1)^{h_1} \dots I(H_r)^{h_r} W^s]$  (see § 1.4(1)), to a sequence of transformations of  $X$  in the form of (1.6.1). This monomial algebra, to be assigned to (1.6.1) in a canonical manner, will relate to the coefficients of  $f^{(r)}(z_r) = z_r^n + a_1^{(r)} z_r^{n-1} + \dots + a_n^{(r)} \in \mathcal{O}_{V_r^{(d-1)}}[z_r]$ . In fact, we show that such an expression can be chosen so each coefficient  $a_i^{(r)}$  is divisible, in some weighted manner, by this monomial algebra (see Definition 3.10).

A first step in the definition of our inductive function  $\text{H-ord}^{(d-1)}$  is addressed in § 4, where a rational number is assigned to a  $p$ -presentation (slope at a point). A notion of *well-adapted p-presentation* at a point is introduced in § 5. It will be ultimately shown, in a further section, that the slope of  $p$ -presentations, which are well-adapted at a point  $x$ , is optimal, and that such a slope is the value of Hironaka’s inductive function at  $x$ , namely  $\text{H-ord}^{(d-1)}(x)$  (the value of the inductive function at  $x$ ). All this highlights the importance of the notion of well-adapted presentations in the following.

Both §§ 4 and 5 are focused in giving, in an explicit manner, the value of the Hironaka’s inductive function at a singular point.

In § 6, monoidal transforms of  $p$ -presentations are defined. This leads to the statement of the two main results of this first part: Main Theorems 1 and 2, stated in § 7. Main Theorem 1 (Theorem 7.2) asserts that the previously defined inductive function is independent of the chosen transversal projection  $\beta$ . Main Theorem 2 (Theorem 7.6) characterizes the monomial algebra, called here  $\mathcal{M}_r W^s$ , defined by the inductive functions. Proofs will be addressed in Part III.

**Part II: Strong monomial case**

In Part I we define the inductive functions,  $\text{H-ord}_i^{(d-1)}$ , and a monomial algebra, say  $\mathcal{M}_r W^s$ , will be assigned to a sequence of transformations (1.6.1). It can be shown that for any  $n$ -fold point

$$\text{H-ord}^{(d-1)}(x) \geq \text{ord}(\mathcal{M}_r W^s)(x),$$

where the order function in the right-hand side is the usual order defined for an arbitrary algebra (see (2.3.1)). The function  $\text{ord}(\mathcal{M}_r W^s)$  is a nicely behaved upper-semi-continuous function as opposed to the function on the left-hand side. The previous inequality between the previous functions will lead us to the numerical characterization of the strong monomial case, expressed by the condition

$$\text{H-ord}^{(d-1)}(x) = \text{ord}(\mathcal{M}_r W^s)(x),$$

for any  $n$ -fold point  $x$ .

It is proved in Theorem 8.14 that if such equality holds, then a combinatorial resolution of  $\mathcal{M}_r W^s$  can be lifted to a simplification of the  $n$ -fold points. This settles § 1.4(2).

**1.7 Final comments**

The invariants studied in this paper make use of transversal projections  $V^{(d)} \rightarrow V^{(d-1)}$  and of elimination algebras defined in  $V^{(d-1)}$ . There are other approaches in the definition of invariants along  $n$ -fold points of a hypersurface. The bibliography indicates some, but certainly not all the effort done in this way. An account of the problem, due to Hauser, appears in [Hau10]. There is an alternative approach of Włodarczyk; his presentation in [W108] includes an important study of pathologies in positive characteristic. There are also recent contributions by Kawanoue–Matsuki [Kaw07, KM10, KM12], Hironaka [Hir08], Cutkosky [Cut09, Cut11], Piltant [Pil03], and a fundamental contribution of Cossart–Jannsen–Saito in [CJS09] which proves embedded resolution for 2-dimensional arithmetical schemes. Some important and remarkable results on the resolution of singularities appear in [Abh56, Abh59, EV00, Hir64a, Hir64b, Lip78, Vil89, Vil08b, W105]. We have profited from discussions with S. Encinas, V. Cossart, H. Hauser, H. Kawanoue, J. Lipman, K. Matsuki, O. Piltant and from ideas of A. Bravo which will be discussed elsewhere.

PART I. INDUCTIVE FUNCTIONS AND THE TIGHT MONOMIAL ALGEBRA

**2. Differential algebras, elimination and local presentations**

**2.1 Motivation**

The initial motivation is the study of the highest multiplicity locus of an embedded hypersurface  $X$ . Here we begin in § 2.2 by showing how to reformulate this study in terms of algebras. This reformulation will enable us to consider algebras with more structure and, in fact, algebras with a form of compatibility with differential operators are studied in §§ 2.3–2.5, where the notions of *absolute* and *relative differential algebras* are discussed (following the classical ideas introduced by Giraud, [Gir75, Gir83a, Gir83b]).



It is in the context of differential algebras in which the fundamental notions of transversal projections and elimination algebras will be introduced (see § 2.7 and § 2.11, respectively).

The main objective of this section is to show that given a differential algebra, together with a transversal projection, the algebra can be entirely reconstructed in terms of two ingredients:

- (i) the elimination algebra; and
- (ii) a monic polynomial.

This is the main result in this section, which is collected in Proposition 2.12. This form of presentation of the algebra will be essential throughout this work. In the case of characteristic zero the monic polynomial can be chosen of degree one. In the case of positive characteristic one can choose the monic polynomial so as to have as degree a power of the characteristic. This will lead to the definition of  $p$ -presentations in Definition 2.14.

The particular role of positive characteristic is played by the coefficients of this monic polynomial as will be shown in this development. The definition of the main invariant will rely entirely on these two ingredients.

### 2.2 Rees algebras and the resolution problem

Here we introduce the notion of Rees algebras, which will play a prominent role in our development. Let  $V^{(d)}$  be a smooth scheme over a perfect field  $k$  of dimension  $d$ . The problem of resolution of singularities of a singular scheme embedded in  $V^{(d)}$  can be stated in terms of Rees algebras over  $V^{(d)}$ . These are algebras of the form  $\mathcal{G} = \bigoplus_{n \in \mathbb{N}} I_n W^n$ , where  $I_0 = \mathcal{O}_{V^{(d)}}$  and each  $I_n$  is a coherent sheaf of ideals. Here  $W$  stands for a dummy variable introduced simply to keep track of the degree. It will be assumed that, locally at any point of  $V$ ,  $\mathcal{G}$  is a finitely generated  $\mathcal{O}_{V^{(d)}}$ -algebra.

A non-zero sheaf of ideals  $J \subset \mathcal{O}_{V^{(d)}}$  defines an upper-semi-continuous function  $\nu(J) : V^{(d)} \rightarrow \mathbb{Z}$ , where  $\nu_x(J)$  denotes the order of the stalk  $J_x$  at the local regular ring  $(\mathcal{O}_{V^{(d)},x}, m_x)$ . Recall that the order of  $J_x$  in  $\mathcal{O}_{V^{(d)},x}$  is the highest integer  $n$  so that  $J_x \subset m_x^n$ . The *singular locus* of  $\mathcal{G}$  is the closed set

$$\text{Sing}(\mathcal{G}) = \{x \in V^{(d)} \mid \nu_x(I_n) \geq n \text{ for each } n \in \mathbb{N}\}. \tag{2.2.1}$$

In the setting of § 1.2 in which  $X = V(f)$ , we will first attach to  $X$  the algebra  $\mathcal{G} = \mathcal{O}_{V^{(d)}}[fW^n]$ . The set  $\text{Sing}(\mathcal{G})$  consists of the points of multiplicity  $n$  of the hypersurface  $X = V(f)$ .

Fix a monoidal transformation  $V^{(d)} \xleftarrow{\pi_C} V_1^{(d)}$  along the closed smooth center  $C \subset \text{Sing}(\mathcal{G})$ . If  $H \subset V_1^{(d)}$  denotes the exceptional hypersurface, then for each integer  $n \geq 0$ , there is a factorization:

$$I_n \mathcal{O}_{V_1^{(d)}} = I(H)^n I_n^{(1)}.$$

This defines a new Rees algebra,  $\mathcal{G}_1 = \bigoplus_{n \in \mathbb{N}} I_n^{(1)} W^n$ , called the *transform of  $\mathcal{G}$* . The transformation is denoted by

$$\begin{array}{ccc} \mathcal{G} & & \mathcal{G}_1 \\ V^{(d)} & \xleftarrow{\pi_C} & V_1^{(d)}. \end{array}$$

A sequence of transformations will be denoted by

$$\begin{array}{ccccccc} \mathcal{G} & & \mathcal{G}_1 & & & & \mathcal{G}_r \\ V^{(d)} & \xleftarrow{\pi_{C_1}} & V_1^{(d)} & \xleftarrow{\pi_{C_2}} & \dots & \xleftarrow{\pi_{C_r}} & V_r^{(d)}, \end{array} \tag{2.2.2}$$

and herein we always assume that the exceptional locus of the composite morphism  $V^{(d)} \leftarrow V_r^{(d)}$  is a union of hypersurfaces with only normal crossings.

The sequence (2.2.2) is said to be a *resolution of  $\mathcal{G}$*  if  $\text{Sing}(\mathcal{G}_r) = \emptyset$ . For  $\mathcal{G} = \mathcal{O}_{V^{(d)}}[fW^n]$ , a resolution (2.2.2) defines a simplification of  $n$ -fold points as in (1.1.1).

A Rees algebra  $\mathcal{G}$  is said to be *simple at  $x \in \text{Sing}(\mathcal{G})$*  if there is an index  $n \in \mathbb{N}$  so that  $\nu_x(I_n) = n$ . It is said to be *simple* if this condition holds for any  $x \in \text{Sing}(\mathcal{G})$ . Such is the case for  $\mathcal{G} = \mathcal{O}_{V^{(d)}}[f_n W^n]$ , when  $f_n$  defines a hypersurface, say  $X$ , of maximum multiplicity  $n$ .

**2.3 Sections of projections**

Here  $\beta : V^{(d)} \rightarrow V^{(d-1)}$  will denote a smooth morphism of relative dimension one, between smooth schemes of dimensions  $d$  and  $d - 1$ . Locally at a point  $x \in V^{(d)}$ ,  $V^{(d)}$  is étale over  $V^{(d-1)} \times \mathbb{A}^1$  (where  $\mathbb{A}^1$  denotes the affine line), and such a map is compatible with the projection on  $V^{(d-1)}$  [AK70, p. 128]. Consequently, the local ring  $\mathcal{O}_{V^{(d)},x}$  is étale over a localization of a polynomial ring in one variable, say  $\mathcal{O}_{V^{(d-1)},\beta(x)}[Z]$ . After restriction to a neighborhood of  $x$ ,  $Z$  gives rise to a global function at  $V^{(d)}$ , say  $z$ . So there is an inclusion  $\mathcal{O}_{V^{(d-1)}}[z] \subset \mathcal{O}_{V^{(d)}}$ , where  $z \in \Gamma(\mathcal{O}_{V^{(d)}}, V^{(d)})$ , and the closed set  $\{z = 0\}$  is a section of  $\beta : V^{(d)} \rightarrow V^{(d-1)}$ .

Given a ring  $S[Z]$ , a morphism of  $S$ -algebras, say  $\text{Tay} : S[Z] \rightarrow S[Z, T]$ , is defined by setting  $\text{Tay}(Z) = Z + T$  (Taylor expansion). Here

$$\text{Tay}(f(Z)) = f(Z + T) = \sum \Delta^{(r)}(f(Z))T^r,$$

for some operator  $\Delta^{(r)} : S[Z] \rightarrow S[Z]$  defined from this morphism. It is well known that  $\{\Delta^{(0)}, \Delta^{(1)}, \dots, \Delta^{(r)}\}$  is a basis of the free module of  $S$ -differential operators of order  $\leq r$ . The same applies here for  $\mathcal{O}_{V^{(d-1)}}[z]$  if we assume that  $\{dz\}$  is a basis of  $\Omega^1_\beta (= \Omega^1(\mathcal{O}_{V^{(d)}} | \mathcal{O}_{V^{(d-1)}}))$ . Namely,  $\{\Delta^{(0)}, \Delta^{(1)}, \dots, \Delta^{(r)}\}$  spans the sheaf of differential operators of order  $r$  relative to the smooth morphism  $\beta : V^{(d)} \rightarrow V^{(d-1)}$ .

Throughout this paper we will slightly abuse the notation; here  $\beta : V^{(d)} \rightarrow V^{(d-1)}$  is called a *local projection*, and the function  $z$  is said to be a *section of  $\beta$* , or a  *$\beta$ -section*.

Let  $\mathcal{G} = \bigoplus_{n \geq 0} I_n W^n$  be a Rees algebra on a  $d$ -dimensional smooth scheme  $V^{(d)}$ . We always assume that  $I_0 = \mathcal{O}_{V^{(d)}}$  and that  $\mathcal{G}$  is a locally finite generated  $\mathcal{O}_{V^{(d)}}$ -algebra, namely that

$$\mathcal{G} = \mathcal{O}_{V^{(d)}}[f_{n_1} W^{n_1}, \dots, f_{n_s} W^{n_s}] (\subset \mathcal{O}_{V^{(d)}}[W]),$$

locally at any point of  $V^{(d)}$ .

Given two such algebras  $\mathcal{G}_1$  and  $\mathcal{G}_2$ ,  $\mathcal{G}_1 \odot \mathcal{G}_2$  will denote the smallest algebra containing  $\mathcal{G}_1$  and  $\mathcal{G}_2$ . In terms of local generators, if  $\{f_1 W^{n_1}, \dots, f_r W^{n_r}\}$  generates  $\mathcal{G}_1$  and  $\{g_1 W^{m_1}, \dots, g_s W^{m_s}\}$  generates  $\mathcal{G}_2$ , then  $\mathcal{G}_1 \odot \mathcal{G}_2$  is generated by  $\{f_1 W^{n_1}, \dots, f_r W^{n_r}, g_1 W^{m_1}, \dots, g_s W^{m_s}\}$ .

A function  $\text{ord}(\mathcal{G})(-) : V^{(d)} \rightarrow \mathbb{Q}$  is defined

$$\text{ord}(\mathcal{G})(x) = \min_{n \geq 0} \left\{ \frac{\nu_x(I_n)}{n} \right\} \tag{2.3.1}$$

where  $\nu_x$  denotes the order at the local regular ring  $\mathcal{O}_{V^{(d)},x}$ . It takes only finitely many values. Note that the singular locus is  $\text{Sing}(\mathcal{G}) = \{x \in V^{(d)} | \text{ord}(\mathcal{G})(x) \geq 1\}$ .

*Remark 2.4.* It is a general fact that objects treated by resolution techniques are gathered in equivalence classes. Such is the case, for instance, with log-resolutions of ideals on smooth schemes. If two ideals have the same integral closure they undergo the same log-resolution, so ideals are considered up to integral closure. A similar situation applies here, where the objects are algebras; two algebras with the same integral closure will not be distinguishable. For instance, if  $\mathcal{G}$

and  $\mathcal{G}'$  are two algebras on  $V^{(d)}$  with the same integral closure, then they define the same functions  $\text{ord}(\mathcal{G}) = \text{ord}(\mathcal{G}')$  (in particular,  $\text{Sing}(\mathcal{G}) = \text{Sing}(\mathcal{G}')$ , [Vil08a, Proposition 4.4]). The reader should keep aware of this fact, as it also affects the notation. The expression  $fW^t \in \mathcal{G} = \bigoplus_{n \geq 0} I_n W^n$  means that  $f^r \in I_{t,r}$  for some positive integer  $r$ .

A Rees algebra can be defined by fixing an ideal  $I$  and a positive integer  $s$ , say  $\mathcal{O}_V[IW^s]$  ( $\subset \mathcal{O}_V[W]$ ), which we denote simply as  $IW^s$ . Moreover, up to integral closure, any Rees algebra is of this kind [EV07, Remark 1.3]. In this case,  $fW^t \in \mathcal{O}_V[IW^s]$  means that  $f^s \in I^t$ .

**2.5 Differential Rees algebras**

An algebra  $\mathcal{G} = \bigoplus_{n \geq 0} I_n W^n$  over  $V^{(d)}$  is said to be a *differential algebra* if  $D_r(I_n) \subset I_{n-r}$  for any  $r < n$  and for any differential operator  $D_r$  of order  $r$ , whenever we restrict to an affine open subset of  $V^{(d)}$ .

$\mathcal{G}$  is said to be an *absolute differential algebra*, if this property holds for all  $k$ -linear differential operators. Fix a smooth morphism  $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ . Then, if the previous property holds for differential operators which are  $\mathcal{O}_{V^{(d-1)}}$ -linear, or say,  $\beta$ -relative operators, then  $\mathcal{G}$  is said to be a  $\beta$ -relative differential algebra, or simply  $\beta$ -differential.

If  $\mathcal{G}$  is an absolute differential algebra, then it is also a  $\beta$ -relative differential algebra for any smooth morphism  $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$  defined over  $k$ . The  $\beta$ -relative structure has an advantage: the transform of an absolute differential algebra is not absolute differential, but the notion of  $\beta$ -differential algebra will turn out to be well suited with transformations.

If  $\mathcal{G}$  is not a differential algebra, then it has a natural extension to a differential algebra [Vil08a, Theorem 3.4]. The same holds if  $\mathcal{G}$  is not a  $\beta$ -differential algebra. These natural extensions are compatible with integral closure: if  $\mathcal{G}_1$  and  $\mathcal{G}_2$  have the same integral closure, the same holds for their extensions to differential algebras or to  $\beta$ -differential algebras [Vil08a, Theorem 6.14].

*Remark 2.6.* If  $\mathcal{G}$  is a  $\beta$ -differential Rees algebra, then, locally, there is a finite set of elements of  $\mathcal{G}$ , say  $\{f_1 W^{n_1}, \dots, f_s W^{n_s}\}$ , so that

$$\mathcal{G} = \mathcal{O}_{V^{(d)}}[f_i W^{n_i}, \Delta^{(\alpha_i)}(f_i) W^{n_i - \alpha_i}]_{1 \leq \alpha_i \leq n_i - 1, 1 \leq i \leq s},$$

with  $\Delta^{(\alpha_i)}$  as in § 2.3. Conversely, these local presentations characterize  $\beta$ -differential algebras [Vil08a, Theorem 2.9].

**2.7 Hironaka’s  $\tau$ -invariant**

The graded algebra of the maximal ideal  $m_x$  of a point  $x \in V^{(d)}$ , say  $\text{Gr}_x(\mathcal{O}_{V^{(d)},x})$ , is isomorphic to a polynomial ring. When  $x$  is a closed point, it is a polynomial ring in  $d$ -variables, which is the coordinate ring associated to the tangent space of  $V^{(d)}$  at  $x$ , namely  $\text{Spec}(\text{Gr}_x(\mathcal{O}_{V^{(d)},x})) = \mathbb{T}_{V^{(d)},x}$ . The *initial ideal* or *tangent ideal* of  $\mathcal{G}$  at  $x \in \text{Sing } \mathcal{G}$ , say  $\text{In}_x(\mathcal{G})$ , is the ideal of  $\text{Gr}_x(\mathcal{O}_{V^{(d)},x})$  generated by the elements  $\text{In}_x(I_n)$  for all  $n \geq 1$ , where  $\text{In}_x(I_n)$  is the class of  $I_n$  at  $m_x^n/m_x^{n+1}$ . Observe that  $\text{In}_x(\mathcal{G})$  is zero unless  $\text{ord}(\mathcal{G})(x) = 1$ . The zero set of the tangent ideal  $\text{In}_x(\mathcal{G})$  in  $\text{Spec}(\text{Gr}_x(\mathcal{O}_{V^{(d)},x}))$  is the *tangent cone* of  $\mathcal{G}$  at  $x$ , denoted by  $\mathcal{C}_{\mathcal{G},x}$ .

Given a vector space  $\mathbb{V}$ , a vector  $v \in \mathbb{V}$  defines a translation, say  $\text{tr}_v(w) = w + v$  for  $w \in \mathbb{V}$ . There is a largest linear subspace, denoted by  $\mathcal{L}_{\mathcal{G},x}$ , so that  $\mathcal{C}_{\mathcal{G},x}$  is invariant under translations of  $\mathcal{L}_{\mathcal{G},x}$ , that is,  $\text{tr}_v(\mathcal{C}_{\mathcal{G},x}) = \mathcal{C}_{\mathcal{G},x}$  for any  $v \in \mathcal{L}_{\mathcal{G},x}$ . This subspace  $\mathcal{L}_{\mathcal{G},x}$  is called the *linear space of vertices*.

DEFINITION 2.8 (Hironaka’s  $\tau$ -invariant). We will denote by  $\tau_{\mathcal{G},x}$  the minimum number of variables required to express generators of the tangent ideal  $\text{In}_x(\mathcal{G})$ . This algebraic definition

can be reformulated geometrically as follows:  $\tau_{\mathcal{G},x}$  is the codimension of the linear subspace  $\mathcal{L}_{\mathcal{G},x}$  in  $\mathbb{T}_{V^{(d)},x}$ .

Throughout this paper the invariant  $\tau_{\mathcal{G},x}$  is only to be used when  $x \in \text{Sing}(\mathcal{G})$  is a closed point.

**2.9 Transversal projections**

Now fix a closed point  $x \in V^{(d)}$ . Let  $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$  be smooth and set  $\beta(x) = y \in V^{(d-1)}$ . A regular system of parameters  $\{y_1, \dots, y_s\}$  in  $\mathcal{O}_{V^{(d-1)},y}$ , extends to  $\{y_1, \dots, y_s, z\}$ , a regular system of parameters in  $\mathcal{O}_{V^{(d)},x}$ . Here  $x$  is a point of  $\beta^{-1}(y)$ , and the tangent space of this subscheme at  $x$ , say  $\mathbb{T}_{\beta^{-1}(y),x}$ , is identified with the subscheme in  $\mathbb{T}_{V^{(d)},x}$  defined by the linear forms  $\langle \text{In}_x(y_1), \dots, \text{In}_x(y_s) \rangle \subset \text{Gr}_x(\mathcal{O}_{V^{(d)},x})$  (i.e., a one-dimensional subspace in  $\mathbb{T}_{V^{(d)},x}$ ).

DEFINITION 2.10. A local projection  $\beta : V^{(d)} \rightarrow V^{(d-1)}$  is said to be *transversal* to  $\mathcal{G}$  at a simple point  $x \in \text{Sing}(\mathcal{G})$  (see § 2.2), if  $\mathcal{C}_{\mathcal{G},x} \cap \mathbb{T}_{\beta^{-1}(y),x} = \mathcal{O}$ , the origin of  $\mathbb{T}_{V^{(d)},x}$ . The local projection is said to be *transversal* to  $\mathcal{G}$  if it is so at any point of  $\text{Sing } \mathcal{G}$ . Transversality is an open condition so we are led to consider this condition only at closed points (see [BVS10, Remark 8.5]).

**2.11 Elimination algebras**

Set a local projection  $\beta : V^{(d)} \rightarrow V^{(d-1)}$ . Let  $x \in \text{Sing}(\mathcal{G})$  be a closed point in  $V^{(d)}$ , so  $y = \beta(x)$  is closed in  $V^{(d-1)}$ . A regular system of parameters  $\{y_1, \dots, y_{d-1}\} \subset \mathcal{O}_{V^{(d-1)},y}$  extends to a regular system of parameters  $\{y_1, \dots, y_{d-1}, z\}$  in  $\mathcal{O}_{V^{(d)},x}$ . In this case,  $z$  defines a section of  $\beta : V^{(d)} \rightarrow V^{(d-1)}$  after suitable restrictions. We view this projection locally. Let us stress here that  $\text{Sing}(\mathcal{G})$  is not included in the section of  $\beta$  defined by  $z = 0$ . When the characteristic is zero one can choose  $z$  with such condition, but this does not hold in positive characteristic.

Take  $\mathcal{G}$  to be a simple algebra, and let  $\beta : V^{(d)} \rightarrow V^{(d-1)}$  be transversal to  $\mathcal{G}$ . Fix a closed point  $x \in \text{Sing}(\mathcal{G})$ . The Weierstrass preparation theorem ensures that, taking restrictions in étale topology,  $\mathcal{G}$  has the same integral closure as an algebra  $\mathcal{O}_{V^{(d)}}[f_1(z)W^{n_1}, \dots, f_s(z)W^{n_s}]$ , where each

$$f_i(z) = z^{n_i} + a_1^{(i)}z^{n_i-1} + \dots + a_{n_i}^{(i)} \in \mathcal{O}_{V^{(d-1)}}[z] \tag{2.11.1}$$

is a monic polynomial of degree  $n_i \in \mathbb{Z}_{\geq 0}$  (see [Vil07, 4.7]).

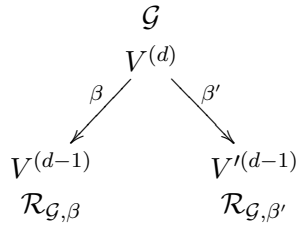
The following properties are known to hold within this setting.

**(P0)** The restriction of  $\beta$  to  $\text{Sing}(\mathcal{G})$ , say  $\beta : \text{Sing}(\mathcal{G}) \rightarrow \beta(\text{Sing}(\mathcal{G}))$ , is a set theoretical bijection and two corresponding points have the same residue fields. Namely,  $k(x) \cong k(\beta(x))$ . To this end note first that once we fix a monic polynomial as in (2.11.1), it defines a hypersurface, and the restriction of  $\beta$  defines a finite map on  $V^{(d-1)}$ . In addition,  $\text{Sing}(\mathcal{G})$  is included in the closed set of  $n_i$ -fold points of this hypersurface. The statements in **(P0)** follow from the fact that they hold for the  $n_i$ -fold points of this hypersurface, and their image in  $V^{(d-1)}$  (see [BVS10, 7.1], or [Vil07, 1.15 and Theorem 4.11]).

If  $\mathcal{G}$  is a  $\beta$ -relative differential algebra, then a Rees algebra on  $V^{(d-1)}$ , say  $\mathcal{R}_{\mathcal{G},\beta} \subset \mathcal{O}_{V^{(d-1)}}[W]$ , is defined. It is called the *elimination algebra of  $\mathcal{G}$* , and has the following properties.

**(P1)** The projection  $\beta(\text{Sing}(\mathcal{G})) \subset \text{Sing}(\mathcal{R}_{\mathcal{G},\beta})$ ; moreover if  $C$  is a closed and smooth scheme included in  $\text{Sing}(\mathcal{G})$ , then  $\beta(C) \subset V^{(d-1)}$  is smooth, isomorphic to  $C$ , and  $\beta(C) \subset \text{Sing}(\mathcal{R}_{\mathcal{G},\beta})$  [BVS10, Theorem 9.1].

(P2) [Vil07, Theorem 5.5]. Fix two projections:



where both  $\beta$  and  $\beta'$  are transversal to  $\mathcal{G}$ . This defines an algebra  $\mathcal{R}_{\mathcal{G},\beta}$  over  $V^{(d-1)}$  and an algebra  $\mathcal{R}_{\mathcal{G},\beta'}$  over  $V^{(d-1)}$ . At any point  $x \in \text{Sing}(\mathcal{G})$ ,

$$\text{ord}(\mathcal{R}_{\mathcal{G},\beta})(\beta(x)) = \text{ord}(\mathcal{R}_{\mathcal{G},\beta'})(\beta'(x)).$$

(P3) [Vil07, Theorem 1.16]. If  $\text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y) > 0$  at a point  $y \in V^{(d-1)}$ , the restriction of (2.11.1) to  $\beta^{-1}(y)$ , say

$$\overline{f_i}(Z) = Z^{n_i} + \overline{a_1^{(i)}} Z^{n_i-1} + \dots + \overline{a_{n_i}^{(i)}} \in k(y)[Z],$$

is a power of a purely inseparable polynomial. Namely,  $\overline{f_i}(Z) = (Z^{p^{r_i}} + b_i)^{m_i}$  at  $k(y)[Z]$ . Moreover, there is at most one point  $x \in V^{(d)}$  so that  $\beta(x) = y$  and  $\text{ord}(\mathcal{G})(x) > 0$ .

A particular feature of characteristic zero is that  $z$  can be chosen to be of maximal contact, in particular this ensures the inclusion  $\text{Sing}(\mathcal{G}) \subset \{z = 0\}$  mentioned before. However this is not always the case in positive characteristic, and the relative differential structure will partially fill in this gap.

PROPOSITION 2.12 (Local presentation). *Set  $x \in \text{Sing}(\mathcal{G})$  a closed point and  $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$  transversal to  $\mathcal{G}$  at  $x$ . Assume that  $\mathcal{G}$  is a  $\beta$ -relative differential algebra, that there is an element  $f_n W^n \in \mathcal{G}$ ,  $f_n$  of order  $n$  at  $\mathcal{O}_{V^{(d)},x}$ , and that  $f_n = f_n(z)$  is a monic polynomial of degree  $n$  in  $\mathcal{O}_{V^{(d-1),\beta(x)}}[z]$ , where  $z$  is a  $\beta$ -section and an element at  $\mathcal{O}_{V^{(d)},x}$ . Then, at a neighborhood of  $x$ ,  $\mathcal{G}$  has the same integral closure as*

$$\mathcal{O}_{V^{(d)}}[f_n(z)W^n, \Delta^{(\alpha)}(f_n(z))W^{n-\alpha}]_{1 \leq \alpha \leq n-1} \odot \mathcal{R}_{\mathcal{G},\beta}, \tag{2.12.1}$$

where  $\mathcal{R}_{\mathcal{G},\beta}$  is identified with  $\beta^*(\mathcal{R}_{\mathcal{G},\beta})$ , and the  $\Delta^{(\alpha)}$  are as in § 2.3. Moreover,  $\mathcal{R}_{\mathcal{G},\beta}$  is non-zero whenever  $\text{Sing}(\mathcal{G})$  is not of codimension one locally at  $x$ .

*Proof.* The last assertion follows from [Vil07, Theorem 4.11(i)]. Take  $f_n(z)W^n \in \{f_1 W^{n_1}, \dots, f_s W^{n_s}\}$  as in (2.11.1). For ease of notation we consider the case  $s = 2$ , i.e.,  $\mathcal{G} = \mathcal{O}_{V^{(d)}}[f_n(z)W^n, g_m(z)W^m]$ .

We follow here the arguments and notation as in [Vil07, ch. 1], particularly Proposition 1.29. Rees algebras are endowed with a natural graded structure. Elimination algebras are also Rees algebras. They are defined as a specialization of the so-called *universal elimination algebras*, which are graded subalgebras in a polynomial ring.

Take variables  $Z, Y_1, \dots, Y_n$  and  $V_1, \dots, V_m$  over a field  $k$ , and set

$$F_n(Z) = (Z - Y_1) \cdot (Z - Y_2) \dots (Z - Y_n).$$

This is the so-called *universal polynomial of degree  $n$* , and  $f_n = f_n(z)$  can be obtain as a specialization of  $F_n(Z)$ . Similarly, let

$$G_m(Z) = (Z - V_1) \cdot (Z - V_2) \dots (Z - V_m)$$

be the universal polynomial of degree  $m$  which will specialize to  $g_m(z)$ .

The natural action of the permutation groups  $\mathbb{S}_n$  on  $k[Y_1, \dots, Y_n]$ , and of  $\mathbb{S}_m$  on  $k[V_1, \dots, V_m]$ , induces an action of the product  $\mathbb{S}_n \times \mathbb{S}_m$  on  $k[Z, Y_1, \dots, Y_n, V_1, \dots, V_m]$  by fixing  $Z$ . This group also acts on the subring

$$S = k[Z - Y_1, Z - Y_2, \dots, Z - Y_n, Z - V_1, Z - V_2, \dots, Z - V_m].$$

The subring of invariants of  $S$ , say  $S^{\mathbb{S}_n \times \mathbb{S}_m}$ , is

$$k[\Delta^{(\alpha)}(F_n(Z)), \Delta^{(\alpha')}(G_m(Z))]_{0 \leq \alpha \leq n-1, 0 \leq \alpha' \leq m-1},$$

where  $\Delta^{(\alpha)}(F_n(Z))$  is a homogeneous polynomial of degree  $n - \alpha$ , obtained as in § 2.3. Similarly  $\Delta^{(\alpha')}(G_m(Z))$  is homogeneous of degree  $m - \alpha'$ .

As these actions are linear,  $S^{\mathbb{S}_n \times \mathbb{S}_m}$  inherits the grading of the polynomial ring  $k[Z, Y_i, V_j]$ . We add a dummy variable  $W$  that will simply express the degree of each homogeneous element. Hence, the subring of invariants  $S^{\mathbb{S}_n \times \mathbb{S}_m}$  is now

$$k[\Delta^{(\alpha)}(F_n(Z))W^{n-\alpha}, \Delta^{(\alpha')}(G_m(Z))W^{m-\alpha'}]_{0 \leq \alpha \leq n-1, 0 \leq \alpha' \leq m-1}.$$

Consider the subring

$$S' = k[(Z - Y_2) - (Z - Y_1), \dots, (Z - Y_n) - (Z - Y_1), (Z - V_1) - (Z - Y_1), \dots, (Z - V_m) - (Z - Y_1)],$$

of  $S$ . Note that  $\mathbb{S}_n \times \mathbb{S}_m$  acts on  $S'$ . The universal elimination algebra is, in this case of  $s = 2$ , defined as the invariant ring  $S'^{\mathbb{S}_n \times \mathbb{S}_m}$ .

The key observation to prove the assertion is that  $S$  is spanned by two subrings:  $k[Z - Y_1, \dots, Z - Y_n]$  and  $S'$ , and  $\mathbb{S}_n \times \mathbb{S}_m$  acts on both.

Recall that the subring of invariants in the first is

$$T = k[(F_n(Z))W^n, \Delta^{(\alpha)}(F_n(Z))W^{n-\alpha}]_{1 \leq \alpha \leq n-1},$$

and that of the second is the universal elimination algebra, say  $R(\subset S')$ .

Thus both invariant algebras,  $T$  and  $R$ , are included in  $S^{\mathbb{S}_n \times \mathbb{S}_m}$ . Let  $T \odot R$  denote the smallest algebra containing both rings. We now claim that  $T \odot R \subset S^{\mathbb{S}_n \times \mathbb{S}_m}$  is a finite extension of graded subalgebras of  $S$ . In order to prove this last assertion note that  $S$  is a finite extension of both subalgebras.

The statement follows now from the previous observation. In fact,  $\mathcal{G}$  and (2.12.1) are obtained by specialization of the previous subrings. This specialization preserves the grading. On the other hand, integral extension of rings is preserved by specialization (change of base rings).  $\square$

*Remark 2.13.* Fix a Rees algebra  $\mathcal{G} = \bigoplus_{n \geq 0} I_n W^n$ . If the setting of Proposition 2.12 holds at a closed point  $x \in \text{Sing}(\mathcal{G})$ , then it holds globally after taking suitable restrictions of  $V^{(d-1)}$  to a neighborhood of  $\beta(x)$ , and of  $V^{(d)}$  to a neighborhood of  $x$ . Moreover,  $z$  defines a  $\beta$ -section.

If the characteristic is zero,  $I_1$  has order one at  $\mathcal{O}_{V^{(d)},x}$ , and  $z \in I_1$  can be chosen as an element of order one at this local ring. This is not always the case in positive characteristic. However, as  $\mathcal{G}$  is a simple  $\beta$ -relative differential algebra, one can check that there is a power of the characteristic, say  $p^e$ , so that  $I_{p^e}$  has order  $p^e$  at  $\mathcal{O}_{V^{(d)},x}$ . This follows from Proposition 2.12 that required the existence of an element  $f_n W^n \in \mathcal{G}$  so that  $f_n = f_n(z)$  is a monic polynomial of degree  $n$  in  $\mathcal{O}_{V^{(d-1)},\beta(x)}[z]$ . Note that one can always lower the degree  $n$  by replacing  $f_n(z)$  by  $\Delta^{(\alpha)}(f_n(z))W^{n-\alpha}$ , for suitable  $\alpha$ , except when  $n$  is a power of the characteristic. Therefore the integer  $n$  in the last proposition can be chosen as a power of the characteristic. This integer  $p^e$  is defined in terms of  $\mathcal{G}$  and the closed point  $x \in \text{Sing}(\mathcal{G})$ . This leads to the following definition.

DEFINITION 2.14 (*p*-Presentations). Fix, after suitable restriction in étale topology, a projection  $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$  transversal to a simple  $\beta$ -relative differential Rees algebra  $\mathcal{G}$ . Assume that  $\text{Sing}(\mathcal{G})$  has no components of codimension one.

Assume also that:

- (i) there is a  $\beta$ -section  $z$ , a global function on  $V^{(d)}$ , and  $\{dz\}$  is a basis of  $\Omega_{\beta}^1$ ;
- (ii) there is an element  $f_{p^e}(z)W^{p^e} \in \mathcal{G}$ , where  $f_{p^e}(z)$  is a monic polynomial of order  $p^e$ , say

$$f_{p^e}(z) = z^{p^e} + a_1z^{p^e-1} + \dots + a_{p^e} \in \mathcal{O}_{V^{(d-1)}}[z],$$

where each  $a_i$  is a global function on  $V^{(d-1)}$ ;

- (iii) locally at any closed point in  $\text{Sing}(\mathcal{G})$ , the conditions in (2.12.1) hold for  $\mathcal{G}$  and

$$\mathcal{O}_{V^{(d)}}[f_{p^e}(z)W^{p^e}, \Delta^{(\alpha)}(f_{p^e}(z))W^{p^e-\alpha}]_{1 \leq \alpha \leq p^e-1} \odot \beta^*(\mathcal{R}_{\mathcal{G},\beta}). \tag{2.14.1}$$

That is, that  $\mathcal{G}$  and (2.14.1) have the same integral closure.

In this case, we say that  $\beta: V^{(d)} \rightarrow V^{(d-1)}$ , the  $\beta$ -section  $z$ , and  $f_{p^e}(z) = z^{p^e} + a_1z^{p^e-1} + \dots + a_{p^e}$  define a *p*-presentation of  $\mathcal{G}$ . These data will be denoted by:

$$p\mathcal{P}(\beta: V^{(d)} \rightarrow V^{(d-1)}, z, f_{p^e}(z) = z^{p^e} + a_1z^{p^e-1} + \dots + a_{p^e}), \tag{2.14.2}$$

or simply by  $p\mathcal{P}(\beta, z, f_{p^e}(z))$ .

### 2.15 Higher codimension

Note that a Rees algebra  $\mathcal{G}$  is simple when  $\tau_{\mathcal{G},x} \geq 1$  for any closed point  $x \in \text{Sing}(\mathcal{G})$  (see § 2.2 and Definition 2.8). Simple algebras are further classified in accordance to the  $\tau$ -invariant as follows. Given a positive integer  $e$ ,  $e \geq 1$ , a simple algebra is said to be of *codimensional type*  $\geq e$  if  $\tau_{\mathcal{G},x} \geq e$  for any closed point  $x \in \text{Sing}(\mathcal{G})$  [BVS10, Definition 6.4].

It is natural to face the problem of resolution of a Rees algebra defined over a smooth scheme of dimension  $d$  by increasing induction on  $d$ . In our strategy we will fix  $d$  and we will try to prove resolution of Rees algebras of codimensional type  $e$ , by decreasing induction on  $e$ . In fact  $e \leq d$ , and in the case in which  $e = d$  the singular locus is a union of isolated points, and resolution is achieved simply by blowing up these isolated singularities. In this paper we are going to discuss only the case  $e = 1$ . In fact, as we will clarify below, this case turns out to be the most relevant one.

Resolution of singularities can be formulated in terms of resolution of *simple* Rees algebras in any dimension  $d$ . Recall that  $\mathcal{G}$  is simple when  $\tau_{\mathcal{G},x} \geq 1$  for any closed point  $x \in \text{Sing}(\mathcal{G})$ . In the case in which  $\tau_{\mathcal{G},x} \geq 2$ , and using the notion of elimination algebras, one can attach to  $\mathcal{G}$  a simple Rees algebra with  $\tau$ -invariant  $\geq 1$ , now in dimension  $d - 1$  (see [Ben10, Theorem 6.4]).

More generally, in the case  $\tau_{\mathcal{G},x} \geq e$  we can attach to  $\mathcal{G}$  a simple algebra with  $\tau$ -invariant  $\geq e - 1$ , now in dimension  $d - 1$ . This form of reduction is very strong in characteristic zero, and it leads to resolution of singularities by induction on  $d$ .

This reduction also holds in positive characteristic, although the link with the  $(d - 1)$ -dimensional algebra is not as strong as it is in characteristic zero. However, in positive characteristic, it leads to a form of simplification of Rees algebras by increasing induction on  $d - e$ . This simplification is called *reduction to the monomial case* of an algebra  $\mathcal{G}$  with  $\tau$ -invariant  $\geq e$ . We aim to present numerical conditions under which this reduction extends to a resolution of the Rees algebra. This is what we call the *strong monomial case*.

As we have said, once  $d$  is fixed, the relevant case is the one with  $\tau$ -invariant  $\geq 1$ , namely the case  $e = 1$ . This is the case to be treated here in Theorem 8.14. The general case, or say

the case in which  $e \geq 2$ , is discussed in [BV11, Theorem 7.6]. The extension to the general case relies on a generalization of the Weierstrass preparation theorem, which was introduced in [BV11, Theorem 6.8]. This generalization of the Weierstrass theorem will lead us to a notion of  $p$ -presentations for higher values of  $e$ . This will enable us to extend the results and invariants discussed in this paper.

In [BV12], and using the techniques introduced in this paper, we prove the resolution of Rees algebras with  $\tau$ -invariant  $\geq 1$ , defined in an ambient space of dimension  $d = 3$  of positive characteristic  $p$ . This will prove resolution of singularities of a surface embedded in dimension  $d = 3$ . Moreover, this last result combined with the one in [BV11, Theorem 6.8] leads to the resolution of singularities of any reduced 2-dimensional scheme over a perfect field, with independence of the embedding. This shows that the invariants introduced in this paper lead to resolution of singularities of schemes of small dimension ( $\leq 2$ ) over a perfect field.

This induction, involving the value  $e$ , also appears in works of Kawanoue and Matsuki [Kaw07, KM10]. This approach is also used in the resolution of 2-dimensional schemes presented in [CJS09, KM12].

### 3. Monomial algebras and the behavior of elimination under monoidal transformations

#### 3.1 Overview

The definition of elimination algebras makes use of the notion of the relative differential structure. We now discuss some results that grow from a form of compatibility of the relative differential structure with monoidal transformations.

Recall that a *sequence of transformations* of  $\mathcal{G}$  is defined in (2.2.2) as

$$\mathcal{G} \qquad \mathcal{G}_1 \qquad \mathcal{G}_r \\ V^{(d)} \xleftarrow{\pi_0} V_1^{(d)} \xleftarrow{\pi_1} \dots \xleftarrow{\pi_{r-1}} V_r^{(d)} \tag{3.1.1}$$

where we always assume that the exceptional locus of  $V^{(d)} \xleftarrow{\pi} V_r^{(d)}$  is a union of hypersurfaces with normal crossings. This last condition will hold because such a sequence is constructed by choosing, at each step, the center of the monoidal transformation having normal crossings with the exceptional hypersurfaces introduced in the previous steps.

In the first part of this section we study the compatibility of transversality and elimination algebras with monoidal transformations. Sequences such as (3.1.1) will also give rise to the definition of the so-called monomial algebras (Definition 3.5), and to a notion of monomial contact introduced in Definition 3.10. This notion appears in the formulation of Main Theorem 2.

#### 3.2 Transversality and monoidal transformations

Transversal projections are defined only for simple algebras (Definition 2.10). When  $\mathcal{G}$  is a simple algebra, we claim that all the  $\mathcal{G}_i$  defined in (3.1.1) are also simple. It suffices to check this property locally. Fix a closed point  $x \in C \subset \text{Sing}(\mathcal{G})$ , where  $C$  is a smooth center. There is an integer  $n$  and an element  $f_n \in I_n$  so that  $\nu_x(f_n) = n$ . Note that  $\nu_C(f_n) = n$  and  $f_n$  is equimultiple at  $C$  locally at  $x$ , so the strict transform of  $f_n$  has multiplicity at most  $n$  on points on the exceptional locus, and hence  $\mathcal{G}_1$  is simple.

Take  $\mathcal{G}$  to be a simple algebra on  $V^{(d)}$ , together with a transversal projection  $\beta: V^{(d)} \rightarrow V^{(d-1)}$ . Assume that  $\mathcal{G}$  is a  $\beta$ -relative differential algebra. A notion of compatibility of these properties with monoidal transformations can be formulated as follows [BVS10].



In [BVS10, Theorem 9.1], it is shown that after suitable restrictions to an étale cover of  $V^{(d)}$ , the sequence (3.1.1) induces a diagram

$$\begin{array}{ccccc}
 \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_r \\
 V^{(d)} \xleftarrow{\pi_0} & & V_1^{(d)} \xleftarrow{\pi_1} & \dots & \xleftarrow{\pi_{r-1}} V_r^{(d)} \\
 \downarrow \beta & & \downarrow \beta_1 & & \downarrow \beta_r \\
 V^{(d-1)} \xleftarrow{\pi'_0} & & V_1^{(d-1)} \xleftarrow{\pi'_1} & \dots & \xleftarrow{\pi'_{r-1}} V_r^{(d-1)} \\
 \mathcal{R}_{\mathcal{G},\beta} & & (\mathcal{R}_{\mathcal{G},\beta})_1 & & (\mathcal{R}_{\mathcal{G},\beta})_r
 \end{array} \tag{3.2.1}$$

where:

- (i) each vertical morphism  $\beta_i : V_i^{(d)} \rightarrow V_i^{(d-1)}$  is transversal to  $\mathcal{G}_i$ , and each  $\mathcal{G}_i$  is a  $\beta_i$ -differential algebra, with the  $\beta_i$  defined only in an neighborhood of  $\text{Sing}(\mathcal{G}_i)$ ;
- (ii) the lower sequence induces transformations of the elimination algebra  $\mathcal{R}_{\mathcal{G},\beta}$ , and furthermore, each  $(\mathcal{R}_{\mathcal{G},\beta})_i$  is the elimination algebra of  $\mathcal{G}_i$  relative to  $\beta_i : V_i^{(d)} \rightarrow V_i^{(d-1)}$ , that is,  $(\mathcal{R}_{\mathcal{G},\beta})_i = \mathcal{R}_{\mathcal{G}_i,\beta_i}$ .

DEFINITION 3.3. A local projection  $V_r^{(d)} \xrightarrow{\beta_r} V_r^{(d-1)}$  is said to be *r-transversal* to  $\mathcal{G}_r$  if there is a transversal morphism  $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ , as in Definition 2.10, and a simple  $\beta$ -differential algebra  $\mathcal{G}$  over  $\mathcal{O}_{V^{(d)}}$ , so that  $\mathcal{G}_r$  and  $\beta_r$  arise from a diagram such as that in (3.2.1).

Remark 3.4. In characteristic zero, given a simple differential algebra  $\mathcal{G}$ , there are hypersurfaces of maximal contact at  $V^{(d)}$ . We fix one such hypersurface, and given a sequence of transformations of  $\mathcal{G}$  as in (3.1.1), we consider the strict transforms of that fixed hypersurface. Here hypersurfaces of maximal contact will be replaced by transversal projections. We shall fix a transversal projection at  $V^{(d)}$  and for any sequence (3.1.1) we will make use of the lifting of this fixed projection in (3.2.1).

Local  $p$ -presentations of  $\mathcal{G}_r$  will be defined in terms of  $\beta_r$ , where  $\beta_r$  arises from the fixed smooth transversal morphism  $\beta$ . In § 6, a notion of transformation of  $p$ -presentations will be defined. This together with the theorems in § 5 will show that given a simple algebra  $\mathcal{G}$ , if  $V^{(d)}$  can be covered by  $p$ -presentations of the form  $p\mathcal{P}(\beta, z, f_{p^e}(z) = z^{p^e} + a_1 z^{p^e-1} + \dots + a_{p^e})$ , with the same exponent  $p^e$ , then the same holds for  $\mathcal{G}_r$  at  $V_r^{(d)}$ . Namely, that there is a covering of  $V_r^{(d)}$  by presentations of the form  $p\tilde{\mathcal{P}}(\beta_r, \tilde{z}, \tilde{f}_{p^e}(\tilde{z}) = \tilde{z}^{p^e} + \tilde{a}_1 \tilde{z}^{p^e-1} + \dots + \tilde{a}_{p^e})$ , where  $\beta_r$  is  $r$ -transversal, with the same exponent  $p^e$  on any such  $p$ -presentation.

DEFINITION 3.5. Let  $E = \{H_1, \dots, H_r\}$  be a set of smooth hypersurfaces with normal crossings. A *monomial ideal* supported on  $E$  is an invertible sheaf of ideals of the form  $\mathcal{M} = I(H_1)^{\alpha_1} \dots I(H_2)^{\alpha_2} \dots I(H_r)^{\alpha_r}$ , for some integers  $\alpha_i \geq 0$ .

A *monomial algebra* will be a Rees algebra of the form  $\mathcal{O}_V[\mathcal{M}W^s]$  for some monomial ideal  $\mathcal{M}$  and some positive integer  $s$ . This algebra will be denoted by  $\mathcal{M}W^s$ .

Remark 3.6. (1) Fix a monomial algebra  $\mathcal{M}W^s = \mathcal{O}_V[\mathcal{M}W^s]$ . Locally at a point  $x \in V$ ,  $\mathcal{M}_x$  is the ideal spanned by a monomial on a regular system of parameters of  $\mathcal{O}_{V^{(d)},x}$ . Recall that Rees algebras are to be considered up to integral closures. Given  $f_n \in \mathcal{O}_{V^{(d)},x}$ ,  $f_n W^n \in \mathcal{M}W^s$  if and only if  $f_n^s$  is divisible by  $\mathcal{M}_x^n$  at  $\mathcal{O}_{V^{(d)},x}$  for any  $x$  at  $V^{(d)}$ .

(2) Assume that  $\mathcal{M}W^s$  is the Rees algebra generated by the monomial ideal  $\mathcal{M} = I(H_1)^{h_1} \dots I(H_r)^{h_r}$  at degree  $s$ . Rees algebras are considered up to integral closure, so we shall

now describe the integral closure of  $\mathcal{M}W^s$ , say  $\widetilde{\mathcal{M}W^s} = \bigoplus J_t W^t$ . Given a positive integer  $t$ , the ideal corresponding to the degree  $t$ , say  $J_t$ , is generated by a monomial, say

$$\mathcal{M}^{[t]} = J_t = I(H_1)^{\lfloor h_1 t/s \rfloor} \dots I(H_r)^{\lfloor h_r t/s \rfloor}. \tag{3.6.1}$$

Moreover,  $\text{ord}(\mathcal{M}W^s)(x) \leq \text{ord}(\mathcal{M}^{[t]}W^t)(x)$ , and equality holds if and only if  $\mathcal{M}W^s$  and  $\mathcal{M}^{[t]}W^t$  have the same integral closure.

### 3.7 The monomial case

Let  $\pi : V' \rightarrow V$  be a smooth morphism. Note that the pull-backs of the hypersurfaces of  $E$  have normal crossings at  $V'$  and a monomial ideal supported on  $E$  has a natural lifting to  $V'$ .

In our setting, we fix a transversal projection  $\beta : V^{(d)} \rightarrow V^{(d-1)}$  as in Definition 2.10, a sequence (3.1.1) induces a diagram (3.2.1) with smooth morphisms  $\beta_i$  defined in a neighborhood of  $\text{Sing}(\mathcal{G}_i)$ . Note that at each such neighborhood, the exceptional hypersurfaces in  $V_i^{(d)}$  are pull-backs of the exceptional hypersurfaces at  $V_i^{(d-1)}$ . In particular, a monomial algebra supported on the exceptional locus of the composite map  $V^{(d-1)} \leftarrow V_r^{(d-1)}$ , say

$$\mathcal{M}_r W^s = I(H_1)^{h_1} \dots I(H_r)^{h_r} W^s, \tag{3.7.1}$$

can be naturally lifted to a monomial algebra supported on the exceptional locus of  $V^{(d)} \leftarrow V_r^{(d)}$ .

In the following theorem we state the *reduction to the monomial case* as presented in [BVS10, Part 5], but only for the case  $e = 1$  (see § 2.15).

**THEOREM 3.8** [BVS10, 10.4 and Part 5]. *Let  $\mathcal{G}$  be a differential algebra such that  $\tau_{\mathcal{G},x} \geq 1$  at any closed point  $x \in \text{Sing}(\mathcal{G})$  (in particular,  $\mathcal{G}$  is simple). Assume that  $\text{Sing}(\mathcal{G})$  has no component of codimension one, and assume also, by inductive hypothesis on the codimensional type, that it is known how to resolve simple algebras in which the  $\tau$ -invariant is  $\geq 2$  at any closed point (algebras of codimensional type  $\geq 2$ ). Then there is a sequence of transformations as in (3.1.1), so that for any local transversal projection  $\beta : V^{(d)} \rightarrow V^{(d-1)}$  (defined by restriction to an étale covering of  $V^{(d)}$ ) the induced sequence (3.2.1) is such that the sequence in the lower row is a resolution or  $(\mathcal{R}_{\mathcal{G},\beta})_r$  is a monomial algebra supported on the exceptional locus. Furthermore, in the latter case the monomial algebra  $\beta_r^*((\mathcal{R}_{\mathcal{G},\beta})_r)$  is independent of  $\beta$ .*

In what follows, we consider, étale locally, a sequence (3.2.1) which fulfills the conditions as in the formulation of Theorem 3.8 [BVS10, Main Theorem]. So here,  $(\mathcal{R}_{\mathcal{G},\beta})_r \subset \mathcal{O}_{V_r^{(d-1)}}[W]$  is monomial and supported on the exceptional locus, and so is its pull-back to  $V_r^{(d)}$ . The same holds if we enlarge the sequence of transformations as this condition is stable.

We identify  $(\mathcal{R}_{\mathcal{G},\beta})_r$  with its pull-back, say

$$(\mathcal{R}_{\mathcal{G},\beta})_r = I(H_1)^{\alpha_1} \dots I(H_r)^{\alpha_r} W^s = \mathcal{N}_r W^s.$$

The previous theorem, together with the notion of *transformation* of  $p$ -presentations in (2.14.1), to be discussed later, will lead us to the existence, locally at any closed point of  $\text{Sing}(\mathcal{G}_r)$ , of a  $\beta_r$ -section  $z'$ , and a monic polynomial, say  $f_{p^e}^{(r)}$ , so that  $\mathcal{G}_r$  has the same integral closure as

$$\mathcal{O}_{V_r^{(d)}}[f_{p^e}^{(r)}(z)W^{p^e}, \Delta^\alpha(f_{p^e}^{(r)}(z))W^{p^e-\alpha}]_{1 \leq \alpha \leq p^e-1} \odot \mathcal{N}_r W^s.$$

### 3.9 Monomial contact

The outcome of Theorem 3.8, in the case of fields of characteristic zero, is known as the reduction to the monomial case (here in the specific case of  $e = 1$ ). In that context it is simple to extend (3.1.1) to a resolution. Unfortunately, the reduction to the monomial case in positive characteristic does not lead easily to resolution of singularities. This task has been accomplished in [BV12] for the case  $d = 3$  and  $e = 1$ .

In the following definition we fix a sequence of transformations and discuss the role of the exceptional divisors that has been introduced.

DEFINITION 3.10. (1) Fix a sequence of transformations as in (3.1.1). We say that a monomial algebra  $\mathcal{M}_r W^s$  (3.7.1) has *monomial contact* with  $\mathcal{G}_r$  at a point  $x \in \text{Sing}(\mathcal{G}_r)$  if there is a  $\beta_r$ -section  $z$  which vanishes at  $x$  (of order one at  $\mathcal{O}_{V_r^{(d)},x}$ ), so that

$$\mathcal{G}_r \subset \langle z \rangle W \odot \mathcal{M}_r W^s.$$

(2) A local  $p$ -presentation of  $\mathcal{G}_r$ , say  $p\mathcal{P}(\beta_r, z, f_{p^e}^{(r)}(z))$  (with  $f_{p^e}^{(r)} = z^{p^e} + a_1 z^{p^e-1} + \dots + a_{p^e}$ ) is said to be *compatible with the monomial algebra*  $\mathcal{O}_{V_r^{(d-1)}}[\mathcal{M}_r W^s]$  locally at  $x \in \text{Sing}(\mathcal{G}_r)$  if the previous condition holds for the  $\beta_r$ -section  $z$ . It follows from Proposition 2.14 that this is equivalent to the conditions:

- (i)  $z$  vanishes at  $x$  (i.e.,  $x \in \{z = 0\}$ );
- (ii)  $(\mathcal{R}_{\mathcal{G},\beta})_r \subset \mathcal{O}_{V_r^{(d-1)}}[\mathcal{M}_r W^s]$ ;
- (iii)  $a_i W^i \in \mathcal{O}_{V_r^{(d-1)}}[\mathcal{M}_r W^s]$ , for  $1 \leq i \leq p^e$  (Remark 2.4).

### 3.11 Tight monomial algebra

We will show that given a simple algebra  $\mathcal{G}$  and a sequence of transformations as in (3.1.1), there is a monomial algebra  $\mathcal{M}_r W^s$  supported on the exceptional locus which has monomial contact with  $\mathcal{G}_r$ . That is, locally at any point  $x \in \text{Sing}(\mathcal{G}_r)$  there is a  $\beta_r$ -section  $z$  which vanishes at  $x$ , so that  $\mathcal{G}_r \subset \langle z \rangle W \odot \mathcal{M}_r W^s$ . Main Theorem 2 will show that this monomial algebra will be defined in terms of the sequence (3.1.1), with independence of the choice of  $\beta$  (of (3.2.1)).

### 3.12 Higher codimension

As indicated before, Theorem 3.8 is the restriction of a more general result, stated in [BVS10, Part 5]. We focus on  $e = 1$ , which is the case required in the definition of the inductive function  $\text{H-ord}^{(d-1)}$  introduced in Part I of § 1.6.

Let us indicate that the notion of  $p$ -presentation in Definition 2.14 will play a decisive role in the definition of the function  $\text{H-ord}^{(d-1)}$ . The main result in [BV11] is Theorem 6.8 which provides an extension of the notion of  $p$ -presentation for algebras of codimensional type  $e$ . This will lead to the definition of a function  $\text{H-ord}^{(d-e)}$  which follows easily from the case  $e = 1$  treated here (see [BV11, 2.23]). Moreover, the properties of the strong monomial case, studied for  $e = 1$  in Theorem 8.14, are extended to the general case in [BV11, Theorem 2.21].

## 4. Invariants defined in terms of $p$ -presentations

### 4.1 $p$ -presentations and the slope

Fix a transversal projection  $\beta : V^{(d)} \rightarrow V^{(d-1)}$  (Definition 2.10) and a simple  $\beta$ -differential algebra  $\mathcal{G}$ . In Definition 2.14 we introduced the notion of  $p$ -presentation, say  $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p^e}(z))$ . The aim of this section is to define two functions:

- (i) a function  $\text{Sl}(p\mathcal{P})(-) : V^{(d-1)} \rightarrow \mathbb{Q}$  (Definition 4.2); and
- (ii) a function  $\beta\text{-ord}^{(d-1)}(\mathcal{G})(-) : V^{(d-1)} \rightarrow \mathbb{Q}$  (Definition 4.9).

There are many  $p$ -presentations  $p\mathcal{P}$  which make use of the fixed projection  $\beta$ . Each  $p$ -presentation will define a function  $\text{Sl}(p\mathcal{P})$ . The value of the new function  $\beta\text{-ord}^{(d-1)}(\mathcal{G})$  at a given point  $y \in V^{(d-1)}$  will be given by the biggest value of the form  $\text{Sl}(p\mathcal{P})(y)$  among all  $p$ -presentations making use of the fixed transversal projection  $\beta$ .

Over fields of characteristic zero, the function  $\beta\text{-ord}^{(d-1)}(\mathcal{G})$  coincides with the upper-semi-continuous function  $\text{ord}(\mathcal{R}_{\mathcal{G},\beta})$  (see 2.3.1). The situation in positive characteristic is quite different, for example  $\beta\text{-ord}^{(d-1)}(\mathcal{G})$  is not upper-semi-continuous. Theorem 4.4 features a peculiar behavior of the function  $\text{Sl}(p\mathcal{P})$ , which also leads to a simplification which will be crucial in our further development.

The function in (2) is a first step in the definition of our inductive function  $\text{H-ord}^{(d-1)}$  in § 7. In this section we fix a transversal projection  $\beta$  and study different rational numbers, attached to a point, defined by choosing different transversal sections  $z = 0$ . We focus here, essentially, on how the function in (1) varies for different choices of  $z$ .

DEFINITION 4.2. Fix  $\mathcal{G}, \beta : V^{(d)} \rightarrow V^{(d-1)}$ , a  $\beta$ -section  $z$ , and  $f_{p^e}(z)$  as in 2.14. Namely, fix a  $p$ -presentation  $p\mathcal{P}(\beta, z, f_{p^e}(z))$  with  $f_{p^e}(z) = z^{p^e} + a_1 z^{p^e-1} + \dots + a_{p^e}$  as in (2.14.2), so that  $\mathcal{G}$  has the same integral closure as

$$\mathcal{O}_{V^{(d)}}[f_{p^e}(z)W^{p^e}, \Delta^{(\alpha)}(f_{p^e}(z))W^{p^e-\alpha}]_{1 \leq \alpha \leq p^e-1} \odot \mathcal{R}_{\mathcal{G},\beta}.$$

Define a function  $\text{Sl}(p\mathcal{P})(-) : V^{(d-1)} \rightarrow \mathbb{Q}$ , by setting

$$\text{Sl}(p\mathcal{P})(y) := \min_{1 \leq j \leq p^e} \left\{ \frac{\nu_y(a_j)}{j}, \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y) \right\}.$$

This value is called the *slope of  $\mathcal{G}$  relative to  $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p^e}(z))$  at  $y \in V^{(d-1)}$ .*

Remark 4.3. The function  $\text{ord}(\mathcal{R}_{\mathcal{G},\beta})(-) : V^{(d-1)} \rightarrow \mathbb{Q}$  takes values with denominators in  $(1/n)\mathbb{Z}$ , for some integer  $n > 0$ . Thus the same holds for the slope function: it takes values in  $(1/n(p^e!))\mathbb{Z}$ . Moreover, both functions take only finitely many values.

THEOREM 4.4. Fix  $\mathcal{G}$  and  $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p^e}(z))$  as in Definition 2.14. If  $\text{Sl}(p\mathcal{P})(y) = \nu_y(a_j)/j$  for some index  $j \in \{1, \dots, p^e - 1\}$ , then  $\text{Sl}(p\mathcal{P})(y) = \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y)$ . In particular,

$$\text{Sl}(p\mathcal{P})(y) = \min \left\{ \frac{\nu_y(a_{p^e})}{p^e}, \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y) \right\}.$$

Proof. Let  $n \in \{1, \dots, p^e - 1\}$  be the smallest index for which  $\text{Sl}(p\mathcal{P})(y) = \nu_y(a_n)/n$ . That is,

$$\frac{\nu_y(a_n)}{n} < \frac{\nu_y(a_i)}{i} \quad \text{for } i \leq n-1 \quad \text{and} \quad \frac{\nu_y(a_n)}{n} \leq \frac{\nu_y(a_\ell)}{\ell} \quad \text{for } \ell \geq n+1. \tag{4.4.1}$$

Recall the definition of the  $\beta$ -differential operators  $\Delta^{(r)}$  in § 2.3. As  $\mathcal{G}$  is assumed to be a  $\beta$ -differential algebra, it follows that  $\Delta^{(p^e-n)}(f_{p^e}(z))W^n \in \mathcal{G}$ .

Note that

$$\Delta^{(p^e-n)}(f_{p^e}(z))W^n = (c_1 a_1 z^{n-1} + \dots + c_{n-1} a_{n-1} z + a_n)W^n \in \mathcal{G}$$

for some elements  $c_i \in k$ , and  $i = 1, \dots, n-1$ .

Let  $\overline{\Delta^{p^e-n}(f_{p^e}(z))W^n}$  denote the class of  $\Delta^{(p^e-n)}(f_{p^e}(z))W^n$  in  $\mathcal{O}_{V^{(d)}}/\langle f_{p^e}(z) \rangle[W]$ . The scheme  $\mathcal{O}_{V^{(d)}}/\langle f_{p^e}(z) \rangle[W]$  is a finite and free extension of  $\mathcal{O}_{V^{(d-1)}}[W]$ . The norm of the element

$$\overline{\Delta^{p^e-n}(f_{p^e}(z))W^n} = (c_1 a_1 \bar{z}^{n-1} + \dots + c_{n-1} a_{n-1} \bar{z} + a_n)W^n$$

over  $\mathcal{O}_{V^{(d-1)}}[W]$  is an element of the elimination algebra of  $f_{p^e}(z)$ , and hence of  $\mathcal{R}_{\mathcal{G},\beta}$  (see [Vil07]). Denote this element by  $G(a_1, \dots, a_{p^e})W^t \in \mathcal{R}_{\mathcal{G},\beta}$ . In addition, in this case  $t = np^e$ , and  $G(V_1, \dots, V_{p^e}) \in k[V_1, \dots, V_{p^e}]$  is a weighted homogeneous of degree  $t = p^e$  provided each  $V_i$  is given weight  $i$ .

Note that:

- (i)  $G(a_1, \dots, a_{p^e}) = a_n^{p^e} + \tilde{G}(a_1, \dots, a_{p^e})$ ;
- (ii)  $\tilde{G}(a_1, \dots, a_{p^e}) \in \langle a_1, \dots, a_{n-1} \rangle$ .

To check the last assertion set formally  $a_1 = 0, \dots, a_{n-1} = 0$ , in which case  $\overline{\Delta^{p^e-n}(f_{p^e}(z))}W^n = a_n W^n$ , which has norm  $a_n^{p^e} W^{np^e}$ .

Here  $\tilde{G}$  is a weighted homogeneous polynomial of degree  $np^e$ , and each monomial in  $\tilde{G}$  is of the form  $a_1^{\alpha_1} \dots a_{p^e}^{\alpha_{p^e}}$  with  $\sum_{j=1}^{p^e} j\alpha_j = np^e$ , and  $\alpha_j \neq 0$  for some  $j < n$  (as  $\tilde{G}(a_1, \dots, a_{p^e}) \in \langle a_1, \dots, a_{n-1} \rangle$ ).

We claim that  $\nu_y(a_1^{\alpha_1} \dots a_{p^e}^{\alpha_{p^e}}) > \nu_y(a_n^{p^e}) = p^e \nu_y(a_n)$  for any monomial in  $\tilde{G}$ . In fact,

$$\nu_y(a_1^{\alpha_1} \dots a_{p^e}^{\alpha_{p^e}}) = \sum_{j=1}^{p^e} \alpha_j \nu_y(a_j) > \sum_{j=1}^{p^e} \alpha_j j \frac{\nu_y(a_n)}{n} = np^e \frac{\nu_y(a_n)}{n} = \nu_y(a_n^{p^e}),$$

where the inequality follows from the hypotheses in (4.4.1). In particular,  $\nu_y(\tilde{G}) > \nu_y(a_n^{p^e})$ .

This proves that the order of  $GW^{np^e} \in \mathcal{R}_{\mathcal{G},\beta}$  is  $\nu_y(G)/np^e = \nu_y(a_n^{p^e})/np^e = \nu_y(a_n)/n$ . Hence  $\text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y) \leq \nu_y(G)/np^e = \nu_y(a_n)/n = \text{Sl}(p\mathcal{P})(y)$ . Finally, this inequality together with  $\text{Sl}(p\mathcal{P})(y) \leq \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y)$  implies that  $\text{Sl}(p\mathcal{P})(y) = \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y)$ .  $\square$

*Remark 4.5.* Let  $p\mathcal{P}$  be a  $p$ -presentation defined in a neighborhood of a closed point  $x \in V^{(d-1)}$  and assume  $x \in \bar{y}$  for some  $y \in V^{(d-1)}$ . Then,

$$\text{Sl}(p\mathcal{P})(y) \leq \text{Sl}(p\mathcal{P})(x).$$

Recall that  $\text{Sl}(p\mathcal{P})(y) = \min\{\nu_y(a_{p^e})/p^e, \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y)\}$ . Since  $p\mathcal{P}$  is defined in a neighborhood of  $x$ , it follows that  $\nu_y(a_{p^e}) \leq \nu_x(a_{p^e})$ . The upper-semi-continuity of  $\text{ord}(\mathcal{R}_{\mathcal{G},\beta})$  implies that  $\text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y) \leq \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(x)$ . Thus  $\text{Sl}(p\mathcal{P})(y) \leq \text{Sl}(p\mathcal{P})(x)$ .

**PROPOSITION 4.6.** Fix  $\mathcal{G}$  and  $\beta: V^{(d)} \rightarrow V^{(d-1)}$  together with a  $p$ -presentation  $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p^e}(z))$  as in 2.14 and a point  $y \in V^{(d-1)}$ .

(i) Suppose that  $\text{Sl}(p\mathcal{P})(y) > 0$ . Then there is a unique point  $x$  in  $V(\langle f_{p^e} \rangle)$  mapping to  $y$ . Moreover,

$$x \in \text{Sing}(\mathcal{G}) \text{ if and only if } \text{Sl}(p\mathcal{P})(y) \geq 1.$$

(ii) If  $y \in \beta(\text{Sing}(\mathcal{G}))$ , then  $\beta^{-1}(y) \cap \text{Sing}(\mathcal{G})$  is a unique point, say  $q$ , and:

- (iia) if  $\text{Sl}(p\mathcal{P})(y) > 0$ , then  $q$  is the unique point in  $V(\langle f_{p^e} \rangle)$  that maps to  $y$ ;
- (iib) if  $\text{Sl}(p\mathcal{P})(y) = 0$ , then the class of  $a_{p^e}$  is a  $p^e$ th power in  $k(y)$ , say  $\bar{a}_{p^e} = \alpha^{p^e}$ , and the class of  $a_i$  is zero for  $i = 1, \dots, p^e - 1$ . Namely,

$$\overline{f_{p^e}(z)} = Z^{p^e} + \alpha^{p^e} \in k(y)[Z].$$

*Proof.* (i) As  $\text{Sl}(p\mathcal{P})(y) > 0$ , it follows that the restriction of the equation to the fiber is, say,  $\overline{f_{p^e}(z)} = Z^{p^e} \in k(y)[Z]$ . So there is a unique point  $x$  on the fiber, and  $z$  vanishes at  $x$ .

Fix a regular system of parameters  $\{y_1, \dots, y_s\}$  in  $\mathcal{O}_{V^{(d-1)},y}$ . In this case,  $\{y_1, \dots, y_s, z\}$  is a regular system of parameters in  $\mathcal{O}_{V^{(d)},x}$ , so  $f_{p^e}(z) = z^{p^e} + a_1 z^{p^e-1} + \dots + a_{p^e} \in m_x^{p^e}$  if and only if  $a_i \in m_y^i$ . That the equivalence now follows is straightforward.

(ii) Note that  $\text{Sing}(\mathcal{G}) \subset V(\langle f_{p^e} \rangle)$ . Moreover,  $\text{Sing}(\mathcal{G}) \subset \mathcal{F}_{p^e}$ , the closed set of points of multiplicity  $p^e$  of the hypersurface  $V(\langle f_{p^e} \rangle)$ . A theorem of Zariski states that  $\beta$  induces a set theoretical bijection:  $\beta : \mathcal{F}_{p^e} \rightarrow \beta(\mathcal{F}_{p^e})$ , and matching points have the same residue field (see **(P0)** in § 2.11, or [BVS10, 8.4]). In particular  $\beta^{-1}(y) \cap \text{Sing}(\mathcal{G})$  is a unique point.

(iia) Here  $q \in \text{Sing}(\mathcal{G})$ , so  $q \in V(\langle f_{p^e} \rangle)$ . The assertion follows from the first part of (i).

(iib) In this case,  $y = \beta(q) \in \text{Sing}(\mathcal{R}_{\mathcal{G},\beta})$  (see **(P3)** in § 2.11), so  $\text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y) \geq 1$ . On the other hand, as  $q \in \text{Sing}(\mathcal{G})$ ,  $q \in \mathcal{F}_{p^e}$ , and it follows that  $k(q) = k(y)$ . This, together with Theorem 4.4, implies that

$$\overline{f_{p^e}(z)} = Z^{p^e} + \bar{a}_{p^e} \in k(y)[Z].$$

Finally, the result of Zariski says that this purely inseparable polynomial is a  $p^e$ th power of a monic polynomial of degree 1, say  $Z^{p^e} + \bar{a}_{p^e} = (Z + \alpha)^{p^e}$  in  $k(y)[Z]$ , as it defines a unique  $k(y)$ -rational point on the fiber. □

**COROLLARY 4.7.** Fix two  $p$ -presentations for  $\mathcal{G}$  on  $V^{(d)}$ . Say,  $p\mathcal{P}$ , defined in terms of  $\beta : V^{(d)} \rightarrow V^{(d-1)}$ , a  $\beta$ -section  $z$ , and a monic polynomial  $f_{p^e}(z)$ ; and another  $p$ -presentation  $p\mathcal{P}'$  defined by  $\beta' : V^{(d)} \rightarrow V^{(d-1)}$ , a  $\beta'$ -section  $z'$ , and a polynomial  $f'_{p^e}(z')$ .

Fix points  $y \in V^{(d-1)}$ ,  $y' \in V^{(d-1)}$ , and assume that:

- (1)  $\text{Sl}(p\mathcal{P})(y) > 0$  and  $\text{Sl}(p\mathcal{P}')(y') > 0$ ;
- (2) there is a point  $q \in V^{(d)}$  which is the unique point mapping to both. Namely,  $\beta(q) = y$  and  $\beta'(q) = y'$ .

Then,  $\text{Sl}(p\mathcal{P})(y) \geq 1$  if and only if  $\text{Sl}(p\mathcal{P}')(y) \geq 1$ . In fact, this condition holds when both  $y$  and  $y'$  are images of a point  $q \in \text{Sing}(\mathcal{G})$ .

### 4.8 $\beta$ -order

In what follows we fix the simple algebra  $\mathcal{G}$  on a smooth scheme  $V^{(d)}$ , together with a transversal projection  $\beta : V^{(d)} \rightarrow V^{(d-1)}$ , and define different  $p$ -presentations of the form  $p\mathcal{P}(\beta, z, f_{p^e}(z))$ , (for different choices of sections  $z$ ).

Let us denote by  $\mathcal{F}(\mathcal{G}, \beta)$  the set of all such  $p$ -presentations. Namely,

$$\mathcal{F}(\mathcal{G}, \beta) = \{p\mathcal{P}(\beta, z, f_{p^e}(z)) \text{ for which (2.14.1) holds}\}.$$

There is a natural notion of restriction on local presentations. Let  $\mathcal{U}^{(d-1)}$  be an open subset in  $V^{(d-1)}$ , and set  $\mathcal{U}^{(d)}$  as the inverse image of  $\mathcal{U}^{(d-1)}$ . There is a natural restriction of  $\mathcal{G}$ , say  $\mathcal{G}|_{\mathcal{U}^{(d)}}$ , of  $\beta$ , say  $\beta|_{\mathcal{U}^{(d)}} : \mathcal{U}^{(d)} \rightarrow \mathcal{U}^{(d-1)}$ , and of the  $p$ -presentation  $p\mathcal{P}$ , so that (2.14.1) holds at the restriction. For each open  $\mathcal{U}^{(d-1)} \subset V^{(d-1)}$ , we take all  $p$ -presentations  $\mathcal{F}(\mathcal{G}|_{\mathcal{U}^{(d)}}, \beta|_{\mathcal{U}^{(d)}})$ .

Finally, fix a point  $y \in V^{(d-1)}$ , and set

$$\mathcal{F}(\mathcal{G}, \beta, y) = \bigcup \mathcal{F}(\mathcal{G}|_{\mathcal{U}^{(d)}}, \beta|_{\mathcal{U}^{(d)}}),$$

where the union is over all restrictions  $\mathcal{U}^{(d-1)} \subset V^{(d-1)}$  containing  $y$ .

**DEFINITION 4.9.** Fix  $\beta : V^{(d)} \rightarrow V^{(d-1)}$  and  $\mathcal{G}$  as in § 2.3. Define the  $\beta$ -order at  $y \in V^{(d-1)}$  as

$$\beta\text{-ord}^{(d-1)}(\mathcal{G})(y) = \max_{p\mathcal{P} \in \mathcal{F}(\mathcal{G}, \beta, y)} \{\text{Sl}(p\mathcal{P})(y)\}. \tag{4.9.1}$$

5. Well-adapted  $p$ -presentations

5.1 Sketch of the cleaning process

Let  $p\mathcal{P}$  be a  $p$ -presentation involving  $\beta$ . Here we sketch a criteria which will allow us to decide when, for a given point  $y \in V^{(d-1)}$ , the  $p$ -presentation  $p\mathcal{P}$  is such that  $\beta\text{-ord}^{(d-1)}(\mathcal{G})(y) = \text{Sl}(p\mathcal{P})(y)$ . These  $p$ -presentations, called well-adapted at  $y$ , will ultimately give us the value of the inductive function at such a point (see Corollary 7.3).

The following cases can occur:

- (A)  $\text{Sl}(p\mathcal{P})(y) = \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y)$ ;
- (B)  $\text{Sl}(p\mathcal{P})(y) = \nu_y(a_{p^e})/p^e < \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y)$  (see Theorem 4.4); and:
  - (B1)  $\nu_y(a_{p^e})/p^e \notin \mathbb{Z}_{>0}$ ;
  - (B2)  $\nu_y(a_{p^e})/p^e \in \mathbb{Z}_{\geq 0}$  and  $\text{In}_y(a_{p^e})$  is not a  $p^e$ th power at  $\text{Gr}_y(\mathcal{O}_{V^{(d-1)},y})$ ;
  - (B3)  $\nu_y(a_{p^e})/p^e \in \mathbb{Z}_{\geq 0}$  and  $\text{In}_y(a_{p^e})$  is a  $p^e$ th power at  $\text{Gr}_y(\mathcal{O}_{V^{(d-1)},y})$ .

We shall prove that a new  $p$ -presentation  $p\mathcal{P}'$  can be defined with the condition  $\text{Sl}(p\mathcal{P}')(y) > \text{Sl}(p\mathcal{P})(y)$ , only in case (B3). This leads to the *cleaning process* developed in Proposition 5.3.

This cleaning process relies on suitable changes of the transversal section  $z$ . The finiteness of this process will be addressed in Remark 5.6. In Proposition 5.7 we show that these changes of  $z$ , in this cleaning process, can be done so as to be compatible with the notion of monomial contact; a property that will be used in the proof of Main Theorem 2.

Proposition 5.8 will be useful in the study of  $p$ -presentations and their compatibility with monomial transformations.

5.2 The cleaning process and well adaptation

Let  $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p^e}(z))$  be a  $p$ -presentation and fix  $y \in V^{(d-1)}$ . We study changes of the  $p$ -presentation  $p\mathcal{P}$  obtained by changing the  $\beta$ -section  $z$  by another of the form  $uz + \alpha$ . Here  $u$  and  $\alpha$  are in  $\mathcal{O}_{V^{(d-1)},y}$  and  $u$  is a unit, so the change is a composition of  $z_1 = uz$  and  $z_2 = z + \alpha$ . The function  $u$  is a unit (invertible) at any point in an open neighborhood of  $y$ , say  $\mathcal{U}^{(d-1)}$ . This is to be interpreted as a new  $p$ -presentation, defined at the restriction of both  $\mathcal{G}$  and  $V^{(d)}$  over  $\mathcal{U}^{(d)} = \beta^{-1}(\mathcal{U}^{(d-1)})$  as in § 4.8.

For a change of the form  $z_1 = uz$ , set  $p\mathcal{P}_1$  with

$$f'_{p^e}(z_1) = u^{p^e} f_{p^e}(z) = z_1^{p^e} + ua_1 z_1^{p^e-1} + \dots + u^{p^e} a_{p^e} \in \mathcal{O}_{V^{(d-1)},y}[z_1].$$

Clearly,  $\text{Sl}(p\mathcal{P})(y) = \text{Sl}(p\mathcal{P}_1)(y)$  and also cases (A), (B1), (B2), and (B3) in § 5.1 are preserved. Henceforth we study only changes of the form  $z' = z + \alpha$ .

At  $\mathcal{O}_{V^{(d-1)},y}[z] = \mathcal{O}_{V^{(d-1)},y}[z']$ ,

$$f_{p^e}(z) = f'_{p^e}(z') = z'^{p^e} + a'_1 z'^{p^e-1} + \dots + a'_{p^e} \in \mathcal{O}_{V^{(d-1)},y}[z'] \quad \text{and} \tag{5.2.1}$$

$$a'_{p^e} = \alpha^{p^e} + a_1 \alpha^{p^e-1} + \dots + a_{p^e}. \tag{5.2.2}$$

Define, as before, a new presentation, say  $p\mathcal{P}'$ , with these data at a suitable restriction to a neighborhood of  $y$ .

PROPOSITION 5.3 (Cleaning process). *Fix the setting and notation as above. Assume that  $\text{Sl}(p\mathcal{P})(y) = \nu_y(a_{p^e})/p^e < \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y)$ . There will be a change of the form  $z' = z + \alpha$ , defining a new presentation  $p\mathcal{P}'$  as in § 5.2, so that*

$$\text{Sl}(p\mathcal{P})(y) < \text{Sl}(p\mathcal{P}')(y) \quad \text{if and only if case (B3) holds in § 5.1 for } p\mathcal{P}.$$

*Proof.* Theorem 4.4 ensures that if  $\text{Sl}(p\mathcal{P})(y) = \nu_y(a_{p^e})/p^e < \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y)$ , then

$$\frac{\nu_y(a_{p^e})}{p^e} < \frac{\nu_y(a_i)}{i} \quad \text{for } i = 1, \dots, p^e - 1. \tag{5.3.1}$$

Set  $z' = z + \alpha$  as above. If  $\nu_y(\alpha) < \nu_y(a_{p^e})/p^e$  then the previous inequalities applied to (5.2.2) show that  $\nu_y(a'_{p^e}) = \nu_y(\alpha^{p^e})$ , so  $\text{Sl}(p\mathcal{P}')(y) < \text{Sl}(p\mathcal{P})(y)$ .

Assume that  $\nu_y(\alpha) \geq \nu_y(a_{p^e})/p^e$ . For each summand in (5.2.2) of the form  $a_i\alpha^{p^e-i}$ ,  $i = 1, \dots, p^e - 1$ ,

$$\begin{aligned} \nu_y(a_i\alpha^{p^e-i}) &= (p^e - i)\nu_y(\alpha) + \nu_y(a_i) \\ &> (p^e - i)\nu_y(\alpha) + i\frac{\nu_y(a_{p^e})}{p^e} \geq (p^e - i)\frac{\nu_y(a_{p^e})}{p^e} + i\frac{\nu_y(a_{p^e})}{p^e} = \nu_y(a_{p^e}). \end{aligned} \tag{5.3.2}$$

Therefore (5.2.2) can be expressed as

$$a'_{p^e} = \alpha^{p^e} + A + a_{p^e}, \tag{5.3.3}$$

where  $\nu_y(\alpha^{p^e}) \geq \nu_y(a_{p^e})$  and  $\nu_y(A) > \nu_y(a_{p^e})$ .

On the other hand,

$$a'_n = \Delta^{(p^e-n)}(f_{p^e})(\alpha) = c_1\alpha^{n-1}a_1 + \dots + c_{n-1}\alpha a_{n-1} + a_n,$$

where  $c_j \in k$  for  $j = 1, \dots, n - 1$ .

For each summand of the form  $a_j\alpha^{n-j}$ ,  $j = 1, \dots, n$ ,

$$\begin{aligned} \nu_y(a_j\alpha^{n-j}) &= (n - j)\nu_y(\alpha) + \nu_y(a_j) \\ &> (n - j)\nu_y(\alpha) + j\frac{\nu_y(a_{p^e})}{p^e} \geq (n - j)\frac{\nu_y(a_{p^e})}{p^e} + j\frac{\nu_y(a_{p^e})}{p^e} = \frac{n\nu_y(a_{p^e})}{p^e}. \end{aligned} \tag{5.3.4}$$

In particular,  $\nu_y(a'_n)/n > \nu_y(a_{p^e})/p^e$ .

One can easily check now that if (B1) holds, then  $f'_{p^e}(z') = z'^{p^e} + a'_1z'^{p^e-1} + \dots + a'_{p^e}$  in (5.2.1) is also in case (B1), and  $\text{Sl}(p\mathcal{P})(y) = \text{Sl}(p\mathcal{P}')(y)$ .

The same arguments apply if (B2) holds, namely  $f'_{p^e}(z')$  is also in case (B2), and  $\text{Sl}(p\mathcal{P})(y) = \text{Sl}(p\mathcal{P}')(y)$ .

Conversely, in case (B3) it suffices to choose  $\alpha$  so that  $\nu_y(\alpha^{p^e} + a_{p^e}) > \nu_y(a_{p^e})$  to get  $\text{Sl}(p\mathcal{P})(y) < \text{Sl}(p\mathcal{P}')(y)$ . □

**DEFINITION 5.4.** Let  $p\mathcal{P}$  be a  $p$ -presentation. We say that  $p\mathcal{P}$  is *well-adapted to  $\mathcal{G}$  at  $y \in V^{(d-1)}$*  if either case (A), (B1), or (B2) in §5.1 hold.

*Remark 5.5.* Fix  $q \in \text{Sing}(\mathcal{G})$  and a  $p$ -presentation  $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p^e}(z))$  which is well-adapted to  $\beta(q)$ . Then  $z$  vanishes at  $q$  and  $\text{Sl}(p\mathcal{P})(\beta(q)) \geq 1$  (Proposition 4.6); moreover, if  $q \in \text{Sing}(\mathcal{G})$  is a closed point, then  $\tau_{\mathcal{G},q} = 1$  if and only if  $\text{Sl}(p\mathcal{P})(\beta(q)) > 1$ .

*Remark 5.6* (Finiteness of the cleaning process). When case (B3) occurs,  $\nu_y(a_{p^e}) = \ell p^e$  for some integer  $\ell \geq 1$ , and

$$\text{In}_y(a_{p^e}) = F^{p^e}$$

for some homogenous polynomial  $F$  of degree  $\ell$  at  $\text{Gr}_y(\mathcal{O}_{V^{(d-1)}})$ . In this case, set  $z' = z + \alpha$  for some  $\alpha \in \mathcal{O}_{V^{(d-1)},y}$  such that  $\text{In}_y(\alpha) = F$ . Thus  $\text{Sl}(p\mathcal{P})(y) < \text{Sl}(p\mathcal{P}')(y)$ .



If this new presentation  $p\mathcal{P}' = p\mathcal{P}'(\beta, z', f'_{p^e}(z'))$  is within case (A), (B1) or (B2) then stop. If, on the contrary,  $f'_{p^e}(z')$  is in case (B3), then:

- (i)  $\nu_y(a'_{p^e}) = \ell'p^e$  (with  $\ell' > \ell$ ); and
- (ii)  $\text{In}_y(a'_{p^e}) = (F')^{p^e}$  for some homogeneous element  $F'$  of degree  $\ell'$  at  $\text{Gr}_y(\mathcal{O}_{V^{(d-1)},y})$ .

So again we can set  $z'' = z' + \alpha'$  for some  $\alpha' \in \mathcal{O}_{V^{(d-1)},y}$  with  $\text{In}_y(\alpha') = F'$ ; and  $\text{Sl}(p\mathcal{P}')(y) < \text{Sl}(p\mathcal{P}'')(y)$ . This shows that with this procedure of modification of the transversal section, locally over  $y$ , the slope will increase every time we come to case (B3). Finally, Remark 4.3 guarantees that case (B3) can arise only finitely many times throughout this procedure. So ultimately the procedure leads to a well-adapted  $p$ -presentation.

**PROPOSITION 5.7.** *Assume that  $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p^e}(z))$  is a  $p$ -presentation of  $\mathcal{G}$ , locally at  $x \in \text{Sing}(\mathcal{G})$ , which is compatible with a monomial algebra  $\mathcal{O}_{V^{(d)}}[\mathcal{M}W^s]$  as in Definition 3.10. Then the cleaning process to obtain a well-adapted  $p$ -presentation at the point  $\beta(x)$  can be done so as to preserve the compatibility with  $\mathcal{O}_{V^{(d)}}[\mathcal{M}W^s]$ .*

**PROPOSITION 5.8** (Simultaneous adaptation). *Let  $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p^e}(z))$  be a  $p$ -presentation of  $\mathcal{G}$ , defined locally at  $x \in \text{Sing}(\mathcal{G})$ , which is compatible with a monomial algebra  $\mathcal{M}W^s$ . Let  $y$  be a point in  $\text{Sing}(\mathcal{G})$  so that  $x \in C = \bar{y}$ , and assume that  $\mathcal{O}_{C,x}$  is regular. Then:*

- (A) *the  $p$ -presentation  $p\mathcal{P}$  can be modified so as to be well-adapted to  $\mathcal{G}$  at  $\beta(y)$ , and still defined in a neighborhood of  $x$ .*

Moreover:

- (B) *there is a  $p$ -presentation which is well-adapted to  $\mathcal{G}$  both at  $y$  and  $x$ , and also compatible with  $\mathcal{M}W^s$ .*

*Proofs of Propositions 5.7 and 5.8.* Once we fix a  $p$ -presentation, say  $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p^e})$  and  $\beta(y) \in V^{(d-1)}$ , cleaning applies as indicated in Remark 5.6, when  $\text{In}_y(a_{p^e})$  is a  $p^e$ th power. In such a case cleaning consists of finding  $\alpha \in \mathcal{O}_{V^{(d)},\beta(y)}$  so that  $(\text{In}_{\beta(y)}(\alpha))^{p^e} = \text{In}_{\beta(y)}(a_{p^e})$ .

We proceed with the proof of Proposition 5.8. Fix a  $p$ -presentation  $p\mathcal{P}$  locally defined at  $\mathcal{O}_{V^{(d-1)},\beta(x)}$ . Set  $\mathfrak{p} \subset \mathcal{O}_{V^{(d-1)},\beta(x)}$  to be the regular prime ideal corresponding to  $\beta(y)$  (so that localization at  $\mathfrak{p}$  is  $\mathcal{O}_{V^{(d-1)},\beta(y)}$ ). Cleaning is necessary at  $\mathcal{O}_{V^{(d-1)},\beta(y)}$  if and only if  $\text{In}_{\beta(y)}(a_{p^e})$  is a  $p^e$ th power in  $\text{gr}_{\beta(y)}(\mathcal{O}_{V^{(d-1)},\beta(y)})$ . Since  $x$  is a smooth point at  $\bar{y}$ ,  $\text{gr}_{\mathfrak{p}}(\mathcal{O}_{V^{(d-1)}})$  is a regular ring and  $\text{In}_{\mathfrak{p}}(a_{p^e}) \in \text{gr}_{\mathfrak{p}}(\mathcal{O}_{V^{(d-1)},x})$ . The ring  $\text{gr}_{\beta(y)}(\mathcal{O}_{V^{(d-1)}})$  can be obtained from  $\text{gr}_{\mathfrak{p}}(\mathcal{O}_{V^{(d-1)},\beta(x)})$  by localization, namely by passing from  $[\text{gr}_{\mathfrak{p}}(\mathcal{O}_{V^{(d-1)},\beta(x)})]_0$  to the total quotient field. Notice that  $\text{In}_{\mathfrak{p}}(a_{p^e})$  maps to  $\text{In}_{\beta(y)}(a_{p^e})$ , and that  $\text{In}_{\mathfrak{p}}(a_{p^e})$  is a  $p^e$ th power at  $\text{gr}_{\mathfrak{p}}(\mathcal{O}_{V^{(d-1)},\beta(x)})$  if and only if  $\text{In}_{\beta(y)}(a_{p^e})$  is a  $p^e$ th power at  $\text{gr}_{\beta(y)}(\mathcal{O}_{V^{(d-1)}})$ . This ensures that the element  $\alpha$ , used in the cleaning process at  $\beta(y)$ , can be chosen to be an element in  $\mathcal{O}_{V^{(d-1)},\beta(x)}$ , and hence the cleaning process at  $\beta(y)$  can be done so as to obtain a new  $p$ -presentation with coefficients in  $\mathcal{O}_{V^{(d-1)},\beta(x)}$ .

This settles (A) as we can assume that  $p\mathcal{P}$  is well-adapted at  $\beta(y)$ . We want now to study the ‘adaptability’ of  $p\mathcal{P}$  at  $\beta(x)$ . Note here that the only case to be considered occurs when  $\text{Sl}(p\mathcal{P})(\beta(x)) = \nu_{\beta(x)}(a_{p^e})/p^e < \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(\beta(x))$  and  $\text{In}_{\beta(x)}(a_{p^e})$  is a  $p^e$ th power. Here, we prove, under this last assumption, that the cleaning process at  $\beta(x)$  can be done without affecting the fact that the presentation is already well-adapted at  $\beta(y)$ .

Consider a regular system of parameters  $\{y_1, \dots, y_\ell, y_{\ell+1}, \dots, y_{d-1}\}$  at  $\mathcal{O}_{V^{(d-1)},\beta(x)}$  so that  $\mathfrak{p} = \langle y_1, \dots, y_\ell \rangle$ . There are two cases to consider, case  $\text{Sl}(p\mathcal{P})(\beta(y)) = \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(\beta(y))$  and case  $\text{Sl}(p\mathcal{P})(\beta(y)) = \nu_{\beta(y)}(a_{p^e})/p^e < \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(\beta(y))$ . Assume that the latter case holds and set  $\nu_{\beta(y)}(a_{p^e})/p^e = n/p^e$ . Theorem 4.4 says that  $\nu_{\beta(y)}(a_{p^e})/p^e < \nu_{\beta(y)}(a_j)/j$  for  $j = 1, \dots, p^e - 1$ .

At the completion,  $a_{p^e}$  is a sum of monomials of the form  $y_1^{\alpha_1} \dots y_\ell^{\alpha_\ell} y_{\ell+1}^{\alpha_{\ell+1}} \dots y_{d-1}^{\alpha_{d-1}}$  with  $\alpha_1 + \dots + \alpha_\ell \geq n$ . We can identify  $\text{In}_{\beta(x)}(a_{p^e})$  with a sum of some of these terms. If there is an element  $\alpha \in \mathcal{O}_{V^{(d-1)},x}$  so that  $(\text{In}_x(\alpha))^{p^e} = -\text{In}_x(a_{p^e})$ , then it can be chosen so that  $\alpha \in \langle y_1, \dots, y_\ell \rangle^{\lceil n/p^e \rceil}$  and, in particular,  $\nu_y(\alpha) \geq n/p^e$ .

A change of variables of the form  $z \mapsto z + \alpha$  produces a new independent coefficient of the form

$$a'_{p^e} = \alpha^{p^e} + a_1 \alpha^{p^e-1} + \dots + a_{p^e},$$

where  $\nu_y(a_j \alpha^{p^e-j}) > j \cdot (n/p^e) + (p^e - j)(n/p^e) = n$ , for  $j = 1, \dots, p^e - 1$ ; so  $\nu_y(a'_{p^e}) \geq \min\{\nu_y(\alpha^{p^e}), \nu_y(a_{p^e})\} \geq n$  and the new presentation is still well-adapted at  $y$ .

If  $\text{Sl}(p\mathcal{P})(y) = \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y) = n/s$ , then  $\nu_y(a_i)/i \geq n/s$  for  $1 \leq i \leq p^e$ , and the same arguments will lead to the existence of  $\alpha \in \mathcal{O}_{V^{(d-1)}}$  so that  $\nu_y(\alpha) \geq n/s$ , which again ensures that the change of variables  $z \mapsto z + \alpha$  (needed for the cleaning process at  $x$ ) does not affect the slope at  $y$ .

To prove Proposition 5.7 just notice that such change can be achieved with  $\alpha W \in \mathcal{O}_{V^{(d-1)}}[\mathcal{M}W^s]$ . Applying similar arguments as those used before, one checks that the new coefficients  $a'_n W^n \in \mathcal{O}_{V^{(d-1)}}[\mathcal{M}W^s]$  ( $n = 1, \dots, p^e$ ).  $\square$

### 6. Transformations of $p$ -presentations

#### 6.1 Overview

In the previous sections some invariants were defined in terms of  $p$ -presentations. In this section we discuss a form of compatibility of these invariants when applying a monoidal transformation along a smooth center  $C$ .

The starting point will be the notion of transformation of  $p$ -presentations in § 6.2. A monoidal transformation defined by blowing up a smooth center  $C$  introduces an exceptional hypersurface, say  $H$ . The aim of the section is to relate the value of the slope  $\text{Sl}$  at the generic point of  $H$  with the value of  $\text{Sl}$  at the generic point of  $C$  (see Proposition 6.6). This result will be an essential ingredient for the proofs of the main theorems in this work.

#### 6.2 $p$ -presentations and monoidal transformations

Take a  $p$ -presentation  $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p^e})$  of a simple  $\beta$ -differential algebra  $\mathcal{G}$  on  $V^{(d)}$ , namely a projection  $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ , a  $\beta$ -section  $z$  and a monic polynomial  $f_{p^e}(z) = z^{p^e} + a_1 z^{p^e-1} + \dots + a_{p^e}$ . Assume that  $C \subset \text{Sing}(\mathcal{G})$  is a closed and smooth center, and that  $z \in I(C)$ . Locally at a closed point  $x \in C$ , there is a regular system of parameters  $\{z, x_1, \dots, x_{d-1}\}$  and, after restriction to a suitable neighborhood of  $x$ ,  $I(C) = \langle z, x_1, \dots, x_\ell \rangle$ . Consider the commutative diagram

$$\begin{array}{ccc}
 \mathcal{G} & & \mathcal{G}_1 \\
 V^{(d)} \xleftarrow{\pi_C} & & V_1^{(d)} \\
 \downarrow \beta & & \downarrow \beta_1 \\
 V^{(d-1)} \xleftarrow{\pi_{\beta(C)}} & & V_1^{(d-1)} \\
 \mathcal{R}_{\mathcal{G},\beta} & & (\mathcal{R}_{\mathcal{G},\beta})_1 = \mathcal{R}_{\mathcal{G}_1,\beta_1}
 \end{array} \tag{6.2.1}$$

and recall that  $\text{Sing}(\mathcal{G}_1) \subset V_1^{(d)}$  can be covered by affine charts  $U_{x_i}$ ,

$$U_{x_i} = \text{Spec} \left( \mathcal{O}_{V^{(d)}} \left[ \frac{z}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, x_i, \frac{x_{i+1}}{x_i}, \dots, \frac{x_\ell}{x_i}, x_{\ell+1}, \dots, x_{d-1} \right] \right),$$

for  $i = 1, \dots, \ell$ ; and also  $V_1^{(d-1)}$  is covered by charts  $U'_{x_i}$ ,

$$U'_{x_i} = \text{Spec} \left( \mathcal{O}_{V^{(d-1)}} \left[ \frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, x_i, \frac{x_{i+1}}{x_i}, \dots, \frac{x_\ell}{x_i}, x_{\ell+1}, \dots, x_{d-1} \right] \right).$$

Note that the strict transform of  $z$ , say  $z_1 = z/x_i$ , is a transversal parameter for  $U_{x_i} \rightarrow U'_{x_i}$ .

The hypersurface defined by  $f_{p^e}$  at  $V^{(d)}$  has multiplicity  $p^e$  along points of  $C$ . Let

$$f_{p^e}^{(1)}(z_1) = z_1^{p^e} + a_1^{(1)} z_1^{p^e-1} + \dots + a_{p^e}^{(1)}$$

denote the strict transform of  $f_{p^e}(z)$ . These data define, locally, a  $p$ -presentation of  $\mathcal{G}_1$ , say  $p\mathcal{P}_1 = p\mathcal{P}_1(\beta_1, z_1, f_{p^e}^{(1)})$ , which we call the *transform* of  $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p^e})$ .

*Remark 6.3.* (i) In the previous discussion we have assumed that  $z \in I(C)$ . If  $C$  is irreducible, this condition will hold for any  $p$ -presentation  $p\mathcal{P}(\beta, z, f_{p^e})$  well-adapted at  $\xi_{\beta(C)}$ , (the generic point of  $\beta(C)$  in  $V^{(d-1)}$ ). In fact, after a suitable restriction to a neighborhood of the closed point  $x \in C$ , the simultaneous cleaning procedure at  $\beta(x)$  and  $\xi_{\beta(C)}$ , and the fact that  $C \subset \text{Sing}(\mathcal{G})$  will allow us to modify  $z$  so that  $z \in I(C)$ . This latter fact follows from property (P1) in § 2.11 and from Proposition 5.3.

(ii) Note that the exponent  $p^e$  (the degree of the monic polynomial), is also preserved by transformations of  $p$ -presentations.

*Remark 6.4.* A point  $y \in V_1^{(d-1)}$  has an image in  $V^{(d-1)}$ , say  $\pi_{\beta(C)}(y)$ . If  $y$  is not in the exceptional locus of  $\pi_{\beta(C)}$ , there is an open neighborhood, say  $U$ , of  $\pi_{\beta(C)}(y)$  over which both  $\pi_C$  and  $\pi_{\beta(C)}$  are the identity map. Thus the restriction of both  $p$ -presentations  $p\mathcal{P}$  and  $p\mathcal{P}_1$  to  $U$  coincide.

In particular,

$$\text{Sl}(p\mathcal{P}_1)(y) = \text{Sl}(p\mathcal{P})(\pi_{\beta(C)}(y))$$

whenever  $y \in V_1^{(d-1)}$  is not on the exceptional locus. Moreover, if  $p\mathcal{P}$  is well-adapted to  $\mathcal{G}$  at  $\pi_{\beta(C)}(y)$ , then the same holds for  $p\mathcal{P}_1$  at  $y$ .

*Remark 6.5.* Fix  $x \in \text{Sing}(\mathcal{G})$  a closed point so that  $\tau_{\mathcal{G},x} = 1$  and assume that  $C$  is a permissible center containing  $x$ . Let  $p\mathcal{P}(\beta, z, f_{p^e})$  be a  $p$ -presentation. Denote by  $y$  the generic point of  $\beta(C)$ . Assume that  $p\mathcal{P}$  is well-adapted simultaneously at  $\beta(x)$  and  $y$  and, in particular, that  $z \in I(C)$  (Remark 6.3(1)).

The intersection of the strict transform of  $f_{p^e}$  with the exceptional locus  $\pi_C^{-1}(C)$  is the projective hypersurface defined by  $\text{In}_C(f_{p^e}) \in \text{gr}_{I(C)}(\mathcal{O}_{V^{(d)}})$ . As  $C$  is an equimultiple center for  $f_{p^e}$ , the intersection of the strict transform with points of  $\pi_C^{-1}(x)$  is determined by  $\text{In}_x(f_{p^e})$ .

Finally as  $\tau_{\mathcal{G},x} = 1$  and  $p\mathcal{P}$  is well-adapted at  $x$ , it follows that  $\text{In}_x(f_{p^e}) = Z^{p^e}$ , and hence  $x' \in \{z_1 = 0\}$  for any  $x' \in \text{Sing}(\mathcal{G}_1)$  mapping to  $x$ . Here  $z_1$  denotes the strict transform of  $z$ .

In the following proposition we study the transform of a  $p$ -presentation, say  $p\mathcal{P}$ , well-adapted at a closed point  $x \in \text{Sing}(\mathcal{G})$ , when applying a monoidal transformation. It shows that if  $p\mathcal{P}$  is compatible with a monomial algebra, then the transform of the presentation is compatible with a new monomial algebra at any closed  $x' \in \text{Sing}(\mathcal{G}_1)$  mapping to  $x$ . Recall that the definition in Definition 3.10 of compatibility of a  $p$ -presentation, say  $p\mathcal{P}(\beta, z, f_{p^e})$ , with a monomial algebra at a given point  $x$  requires that  $x \in \{z = 0\}$ .

**PROPOSITION 6.6.** *Let  $C$  be a permissible center passing through a closed point  $x \in \text{Sing}(\mathcal{G})$  and assume that  $\tau_{\mathcal{G},x} = 1$ . Fix a  $p$ -presentation  $p\mathcal{P}(\beta, z, f_{p^e})$ . Let  $y$  denote the generic point of  $\beta(C)$  and assume that  $p\mathcal{P}$  is well-adapted to  $\mathcal{G}$  both at  $\beta(x)$  and at  $y$ . Define a monoidal transformation with center  $C$ . Then we have the following.*

(i) The transform  $p\mathcal{P}_1$  is well-adapted to  $\mathcal{G}_1$  at  $\xi_H$ , (the generic point of the exceptional hypersurface  $H \subset V^{(d-1)}$ ). Moreover,

$$\text{Sl}(p\mathcal{P}_1)(\xi_H) = \text{Sl}(p\mathcal{P})(y) - 1. \tag{6.6.1}$$

(ii) If, in addition,  $p\mathcal{P}$  is compatible with a monomial algebra, say  $\mathcal{O}_{V^{(d-1)}}[I(H_1)^{h_1} \dots I(H_r)^{h_r} W^s]$ , then  $p\mathcal{P}_1$  is compatible with the monomial algebra

$$\mathcal{O}_{V^{(d-1)}}[I(H_1)^{h_1} \dots I(H_r)^{h_r} I(H)^\gamma W^s],$$

where  $\gamma/s = \text{Sl}(p\mathcal{P}_1)(\xi_H) = \text{Sl}(p\mathcal{P})(y) - 1$ , at any closed point  $x' \in \text{Sing}(\mathcal{G}_1)$  mapping to  $x$ .

*Proof.* Fix the setting as in § 6.2. Note that (2) follows from (1) as  $z_1$ , the strict transform of  $z$ , vanishes at  $x'$ . Now set

$$\begin{array}{ccc} p\mathcal{P} & & p\mathcal{P}_1 \\ V^{(d)} & \xleftarrow{\pi_C} & V_1^{(d)} \\ \downarrow \beta & & \downarrow \beta_1 \\ V^{(d-1)} & \xleftarrow{\pi_{\beta(C)}} & V_1^{(d-1)} \\ y = \xi_{\beta(C)} & & \xi_H \end{array} \tag{6.6.2}$$

where  $H$  is the exceptional hypersurface, and

$$f_{p^e}^{(1)}(z_1) = z_1^{p^e} + a_1^{(1)}z^{p^e-1} + \dots + a_{p^e}^{(1)}$$

is the strict transform of  $f_{p^e}(z)$ . At points of  $U_{x_i}$ , we have that  $z_1 = z/x_i$ , and that the coefficients  $a_n^{(1)}$  factor as

$$a_n^{(1)} = x_i^{\nu_y(a_n)-n} a'_n = x_i^{r_n} a'_n, \tag{6.6.3}$$

where  $a'_n$  denotes the strict transform of  $a_n$  and  $r_n = \nu_y(a_n) - n$ , for  $n = 1, \dots, p^e$ .

Different cases can arise under these assumptions, we classify them as in § 5.1.

(A) Suppose that  $\text{Sl}(p\mathcal{P})(y) = \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y)$  and, in particular, that  $\nu_y(a_j)/j \geq \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y)$ . At the points of  $\text{Sing}(\mathcal{G}_1) \cap U_{x_i}$ ,

$$\frac{\nu_{\xi_H}(a_{p^e}^{(1)})}{p^e} = \frac{\nu_y(a_{p^e})}{p^e} - 1 \geq \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y) - 1 = \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_1)(\xi_H).$$

Thus  $p\mathcal{P}_1$  is well-adapted to  $\mathcal{G}_1$  at  $\xi_H$  (case (A) in § 5.1 and Definition 5.4). So equality in (6.6.1) holds.

(B) Suppose that

$$\text{Sl}(p\mathcal{P})(y) = \frac{\nu_y(a_{p^e})}{p^e} < \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y).$$

(B.1) Assume now that  $\nu_y(a_{p^e})/p^e \notin \mathbb{Z}_{>0}$ . In this case,  $\nu_y(a_{p^e}) > p^e$  and, in addition,  $\nu_y(a_j) > j$  for  $j = 1, \dots, p^e - 1$ . In particular,  $\nu_{\beta(x)}(a_j) > j$  for  $j = 1, \dots, p^e$  and hence  $\text{In}_x(f_{p^e}) = \mathbb{Z}^{p^e}$ .

Therefore Remark 6.5 applies, so  $\text{Sing}(\mathcal{G}_1) \cap H \subset \{z_1 = 0\}$ . Under these assumptions,

$$\frac{r_{p^e}}{p^e} = \frac{\nu_y(a_{p^e})}{p^e} - 1 < \frac{\nu_y(a_j)}{j} - 1 = \frac{r_j}{j}$$

for  $j = 1, \dots, p^e - 1$ , and  $r_{p^e}/p^e \notin \mathbb{Z}_{\geq 0}$ . So, locally at any closed point  $x' \in \text{Sing}(\mathcal{G}_1)$  mapping to  $x$ ,  $p\mathcal{P}_1 = p\mathcal{P}_1(\beta_1, z_1, f_{p^e}^{(1)})$  is of the form (B1) in § 5.1 and therefore well-adapted to  $\mathcal{G}_1$  at  $\xi_H$ . So (6.6.1) holds.

(B.2) Suppose that  $\nu_y(a_{p^e})/p^e = r \in \mathbb{Z}_{>0}$  and  $\tau_{\mathcal{G},x} = 1$ . Any singular point, at the exceptional locus, mapping to  $x$ , is contained in the strict transform of  $z$  as indicated in Remark 6.5. Consider  $\text{In}_{\beta(C)}(a_{p^e}) \in \text{Gr}_{I(\beta(C))}(\mathcal{O}_{V^{(d-1)}}) = \mathcal{O}_{\beta(C)}[X_1, \dots, X_\ell]$ , and set

$$\text{In}_{\beta(C)}(a_{p^e}) = \sum_{|\alpha|=rp^e} b_\alpha M^\alpha,$$

which, by assumption, is not a  $p^e$ th power. Here  $b_\alpha \in \mathcal{O}_{\beta(C)}$  and  $M^\alpha$  is a monomial in  $X_1, \dots, X_\ell$  of order  $rp^e$ .

(B.2.a) First assume that  $\text{In}_{\beta(C)}(a_{p^e}) \notin \mathcal{O}_{\beta(C)}[X_1^{p^e}, \dots, X_\ell^{p^e}]$ . In this case, as the degree of  $\text{In}_{\beta(C)}(a_{p^e})$  is a multiple of  $p^e$ , there is a multi-index  $\alpha = (\alpha_1, \dots, \alpha_\ell)$  with at least two integers  $\alpha_i$  which are not multiples of  $p^e$ , and  $b_\alpha \neq 0$ .

We claim now that  $a'_{p^e}$  restricted to the exceptional hypersurface  $x_i = 0$ , say  $\overline{a'_{p^e}}$ , is not a  $p^e$ th power. This can be checked using the existence of the previous multi-index  $\alpha = (\alpha_1, \dots, \alpha_\ell)$ .

This ensures that  $p\mathcal{P}_1$  is in the case (B2) in § 5.1, so  $p\mathcal{P}_1$  is well-adapted to  $\mathcal{G}_1$  at  $\xi_H$ , and (6.6.1) holds.

(B.2.b) Suppose now that any  $M^\alpha$  is a  $p^e$ th power whenever  $|\alpha| = rp^e$ . Recall that  $\text{In}_{\beta(C)}(a_{p^e})$  is not a  $p^e$ th power, so some  $b_\alpha$  is not a  $p^e$ th power. Setting as before  $\overline{a'_{p^e}}$  as the restriction of  $a'_{p^e}$  to the exceptional hypersurface, one checks that  $\overline{a'_{p^e}}$  is not a  $p^e$ th power since  $b_\alpha$  is not a  $p^e$ th power. So again  $p\mathcal{P}_1$  is well-adapted at  $\xi_H$ , and (6.6.1) holds.  $\square$

### 7. On the two main theorems

#### 7.1 Main Theorem 1

Fix a smooth scheme  $V^{(d)}$  and a simple algebra  $\mathcal{G}$  which we assume to be an absolute differential algebra. This ensures that  $\mathcal{G}$  is a  $\beta$ -differential algebra for any transversal morphism  $\beta : V^{(d)} \rightarrow V^{(d-1)}$  (§ 2.5). It is under this last condition that a function  $\beta\text{-ord}^{(d-1)}(\mathcal{G}) : V^{(d-1)} \rightarrow \mathbb{Q}$  was defined in Definition 4.9. The same holds for any other  $\beta' : V^{(d)} \rightarrow V'^{(d-1)}$  transversal to  $\mathcal{G}$ .

A sequence (3.1.1) of permissible transformations of  $\mathcal{G}$  induces the following two diagrams.

$$\begin{array}{ccccc} \mathcal{G} & \mathcal{G}_1 & \mathcal{G}_r & \mathcal{G} & \mathcal{G}_1 & \mathcal{G}_r \\ V^{(d)} \xleftarrow{\pi_{C_1}} V_1^{(d)} \xleftarrow{\dots} \xleftarrow{\pi_{C_r}} V_r^{(d)} & & & V^{(d)} \xleftarrow{\pi_{C_1}} V_1^{(d)} \xleftarrow{\dots} \xleftarrow{\pi_{C_r}} V_r^{(d)} & & \\ \downarrow \beta & \downarrow \beta_1 & \downarrow \beta_r & \downarrow \beta' & \downarrow \beta'_1 & \downarrow \beta'_r \\ V^{(d-1)} \xleftarrow{\pi_{\beta(C_1)}} V_1^{(d-1)} \xleftarrow{\dots} \xleftarrow{\pi_{\beta(C_r)}} V_r^{(d-1)} & & & V'^{(d-1)} \xleftarrow{\pi_{\beta'(C_1)}} V_1'^{(d-1)} \xleftarrow{\dots} \xleftarrow{\pi_{\beta'(C_r)}} V_r'^{(d-1)} & & \\ \mathcal{R}_{\mathcal{G},\beta} & (\mathcal{R}_{\mathcal{G},\beta})_1 & (\mathcal{R}_{\mathcal{G},\beta})_r & \mathcal{R}_{\mathcal{G},\beta'} & (\mathcal{R}_{\mathcal{G},\beta'})_1 & (\mathcal{R}_{\mathcal{G},\beta'})_r \end{array} \tag{7.1.1}$$

THEOREM 7.2 (Main Theorem 1). Assume that the previous setting holds. Then, at any  $q \in \text{Sing}(\mathcal{G}_r)$ ,

$$\beta_r\text{-ord}(\mathcal{G}_r)(\beta(q)) = \beta'_r\text{-ord}(\mathcal{G}_r)(\beta'(q)).$$

Moreover, if  $p\mathcal{P} = p\mathcal{P}(\beta_r, z, f_{p^e})$  is well-adapted to  $\mathcal{G}_r$  at  $\beta(q)$ , then

$$\beta_r\text{-ord}(\mathcal{G}_r)(\beta_r(q)) = \text{Sl}(p\mathcal{P})(\beta_r(q)).$$

COROLLARY 7.3. The previous result enables us to define a function along  $\text{Sing}(\mathcal{G}_r)$ :

$$\text{H-ord}^{(d-1)}(\mathcal{G}_r)(-) : \text{Sing}(\mathcal{G}_r) \rightarrow \mathbb{Q}.$$

Moreover, if  $p\mathcal{P} = p\mathcal{P}(\beta_r, z, f_{p^e} = z^{p^e} + a_1 z^{p^e-1} + \dots + a_{p^e})$  is well-adapted to  $\mathcal{G}_r$  at  $\beta_r(x)$  ( $x \in \text{Sing}(\mathcal{G}_r)$ ), then

$$\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) = \min \left\{ \frac{\nu_{\beta_r(x)}(a_{p^e})}{p^e}, \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(\beta_r(x)) \right\}.$$

**7.4**

Recall that the exceptional locus of the composite map  $V^{(d)} \xleftarrow{\pi} V_r^{(d)}$  in (3.1.1), say  $\{H_1, \dots, H_r\}$ , is a set of hypersurfaces at  $V_r^{(d)}$  and it is assumed that the union has only normal crossings.

DEFINITION 7.5. We now attach to a sequence of transformations as in (3.1.1) a monomial algebra supported on the exceptional locus,

$$\mathcal{M}_r W^s = \mathcal{O}_{V_r^{(d)}}[I(H_1)^{h_1} \dots I(H_r)^{h_r} W^s], \tag{7.5.1}$$

with exponents  $h_i \in \mathbb{Z}_{\geq 0}$  defined so that

$$q_{H_i} := \frac{h_i}{s} = \text{H-ord}^{(d-1)}(\mathcal{G}_{i-1})(\xi_{C_i}) - 1$$

where  $\xi_{C_i}$  denotes the generic point of each center  $C_i$  ( $i = 1, \dots, r$ ).

Here  $s$  is a positive integer so that  $\{q_{H_1}, \dots, q_{H_r}\} \subset (1/s)\mathbb{Z}$ . As Rees algebras are considered up to integral closure,  $\mathcal{M}_r W^s$  is independent of the choice of  $s$ , and will be called the *tight monomial algebra* of  $\mathcal{G}_r$  or the *tight monomial algebra* defined by a sequence (3.1.1).

THEOREM 7.6 (Main Theorem 2). *Fix a sequence of permissible transformations as in (3.1.1), and let  $\mathcal{M}_r W^s$  denote the tight monomial algebra defined in Definition 7.5. Then, at any closed point  $x \in \text{Sing}(\mathcal{G}_r)$ ,  $\mathcal{M}_r W^s$  has monomial contact with  $\mathcal{G}_r$ , i.e., there is a  $\beta_r$ -transversal section  $z$  vanishing at  $x$  (i.e., of order one at  $\mathcal{O}_{V_r^{(d)},x}$ ) for which*

$$\mathcal{G}_r \subset \langle z \rangle W \odot \mathcal{M}_r W^s.$$

The two main theorems, to be proved in Part III, will lead us to the notion of the strong monomial case, to be discussed now in Part II. The main result concerning the strong monomial case will be given by Theorem 8.14 which ensures resolution of singularities in positive characteristic if one can achieve some numerical conditions. These conditions are achievable for two dimensional schemes [BV12].

PART II. STRONG MONOMIAL CASE

**8. The strong monomial case and resolution of singularities**

**8.1 Overview and general strategy**

In this second part we address the proof of § 1.4(2). Given a simple differential algebra  $\mathcal{G}$  in  $V^{(d)}$  and a sequence of transformations, say

$$\begin{array}{ccccccc} \mathcal{G} & & \mathcal{G}_1 & & & & \mathcal{G}_r \\ V^{(d)} & \xleftarrow{\pi_1} & V_1^{(d)} & \xleftarrow{\quad} & \dots & \xleftarrow{\pi_r} & V_r^{(d)}, \end{array} \tag{8.1.1}$$

we have defined:

- a function  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(-) : \text{Sing}(\mathcal{G}_r) \rightarrow \mathbb{Q}$  (see Corollary 7.3);

- a monomial algebra  $\mathcal{M}_r W^s$  in  $V_r^{(d)}$ , called the tight monomial algebra, supported on the exceptional locus of the sequence (see § 7.4).

We begin this part by showing that the inequality

$$\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) \geq \text{ord}(\mathcal{M}_r W^s)(x) \tag{8.1.2}$$

holds at any closed point  $x \in \text{Sing}(\mathcal{G}_r)$ . The main objective is to study the case in which equality holds at any closed point of  $\text{Sing}(\mathcal{G}_r)$ . This will be called the *strong monomial case* in Definition 8.4. We will assume in our definition that  $\mathcal{G}_r$  is in the monomial case, meaning that the elimination algebra is monomial, or that the conditions of Theorem 3.8 hold for this sequence of transformations. The objective of this section is to prove that:

- (i) the strong monomial case is stable under transformations (Proposition 8.13);
- (ii) it parallels the so-called monomial case in characteristic zero, namely that if  $\mathcal{G}_r$  is in the strong monomial case, then a combinatorial resolution of  $\mathcal{M}_r W^s$  leads to a resolution of  $\mathcal{G}$  (Theorem 8.14).

*Remark 8.2.* Let  $p\mathcal{P}(\beta_r, z, f_{p^e})$  be a  $p$ -presentation compatible with  $\mathcal{M}_r W^s$  (Definition 3.10) and well-adapted to  $\mathcal{G}_r$  at  $\mathbf{x} = \beta_r(x)$  for  $x \in \text{Sing}(\mathcal{G}_r)$  (Definition 5.4). We denote  $\mathbf{x} = \beta_r(x)$  along this section. In this case,  $z$  must vanish at  $x$ , defining an element of order one at the local ring  $\mathcal{O}_{V^{(d)},x}$  (see Remark 5.5), and  $f_{p^e}(z) = z^{p^e} + a_1 z^{p^e-1} + \dots + a_{p^e}$ , where  $a_j W^j \in \mathcal{M}_r W^s$  for  $j = 1, \dots, p^e$ . In addition,  $(\mathcal{R}_{\mathcal{G},\beta})_r \subset \mathcal{M}_r W^s$  (Definition 3.10(2)).

It follows from the previous lines that  $\text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\mathbf{x}) \geq \text{ord}(\mathcal{M}_r W^s)(\mathbf{x})$  and that  $\nu_{\mathbf{x}}(a_j)/j \geq \text{ord}(\mathcal{M}_r W^s)(\mathbf{x})$  for  $j = 1, \dots, p^e$ . In particular,

$$\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) = \min \left\{ \frac{\nu_{\mathbf{x}}(a_{p^e})}{p^e}, \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\mathbf{x}) \right\} \geq \text{ord}(\mathcal{M}_r W^s)(\mathbf{x}).$$

This proves (8.1.2) for any  $p\mathcal{P}(\beta_r, z, f_{p^e})$  as above.

### 8.3 Definition of the strong monomial case

We shall say that, after a sequence of transformations as in (8.1.1), the transform of a Rees algebra is within the *monomial case* when its elimination algebra is monomial, as stated in Theorem 3.8. We shall assume here, in this section, that  $\mathcal{G}_r$  is in the monomial case. Namely, that a sequence of transformations of  $\mathcal{G}$  has been defined so that  $(\mathcal{R}_{\mathcal{G},\beta})_r = \mathcal{N}_r W^s$  is a monomial algebra supported on the exceptional hypersurfaces. Without loss of generality fix  $s \in \mathbb{Z}$  as in Definition 7.5 so that

$$\mathcal{N}_r W^s = I(H_1)^{\alpha_1} \dots I(H_r)^{\alpha_r} W^s \quad \text{and} \quad \mathcal{M}_r W^s = I(H_1)^{h_1} \dots I(H_r)^{h_r} W^s \quad (\text{see Definition 7.5}), \tag{8.3.1}$$

where  $\{H_1, \dots, H_r\}$  are the components of the exceptional locus of the composite map  $V^{(d)} \xleftarrow{\pi} V_r^{(d)}$  in (8.1.1). Note that the monomial  $\mathcal{M}_r$  divides  $\mathcal{N}_r$  (i.e.,  $\alpha_i \geq h_i$  for any  $i = 1, \dots, r$ ).

**DEFINITION 8.4.** The Rees algebra  $\mathcal{G}_r$  is said to be within the *strong monomial case at a closed point*  $x \in \text{Sing}(\mathcal{G}_r)$  if

$$\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) = \text{ord}(\mathcal{M}_r W^s)(x).$$

We say that  $\mathcal{G}_r$  is within the *strong monomial case* if this condition holds at any closed point  $x \in \text{Sing}(\mathcal{G}_r)$ .

The following provides a characterization of this case.

**THEOREM 8.5** (Characterization of the strong monomial case). *Fix a closed point  $x \in \text{Sing}(\mathcal{G}_r)$ . Let  $p\mathcal{P}(\beta_r, z, f_{p^e})$  be well-adapted to  $\mathcal{G}_r$  at  $\mathbf{x} = \beta_r(x)$  and compatible with the tight monomial algebra  $\mathcal{M}_r W^s$ . The algebra  $\mathcal{G}_r$  is in the strong monomial case at  $x$  if and only if one of the following conditions holds in an open neighborhood; either:*

- (i)  $(\mathcal{R}_{\mathcal{G},\beta})_r = \mathcal{M}_r W^s$ ; or
- (ii) the  $\mathcal{O}_{V^{(d-1)}}$ -algebra spanned by  $a_{p^e} W^{p^e}$ , namely  $\mathcal{O}_{V^{(d-1)}}[a_{p^e} W^{p^e}]$ , has the same integral closure as  $\mathcal{O}_{V^{(d-1)}}[\mathcal{M}_r W^s]$ .

The first condition holds if and only if  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) = \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\mathbf{x})$ .

*Proof.* (i) Fix  $x \in \text{Sing}(\mathcal{G}_r)$  and denote by  $E_x = \{H_{i_1}, \dots, H_{i_\ell}\}$  the set of exceptional hypersurfaces containing  $x$ . Let  $\Lambda_x = \{i_1, \dots, i_\ell\}$  be the set indexing  $E_x$ .

If  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) = \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\mathbf{x})$  and  $\mathcal{G}_r$  is within the strong monomial case at  $x \in \text{Sing}(\mathcal{G}_r)$ , then

$$\sum_{i \in \Lambda_x} \alpha_i = \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\mathbf{x}) = \text{ord}(\mathcal{M}_r W^s)(\mathbf{x}) = \sum_{i \in \Lambda_x} h_i.$$

Since  $\alpha_i \geq h_i$  for any  $1 \leq i \leq s$ , then  $h_i = \alpha_i$  for any  $i \in \Lambda_x$ . So  $\mathcal{M}_r W^s = (\mathcal{R}_{\mathcal{G},\beta})_r$  at  $\mathbf{x}$ .

Conversely, if  $\mathcal{M}_r W^s = (\mathcal{R}_{\mathcal{G},\beta})_r$  locally at  $\mathbf{x}$ , then the inequality  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) \geq \text{ord}(\mathcal{M}_r W^s)(x)$  in (8.1.2) must be an equality since  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) = \min\{\nu_{\mathbf{x}}(a_{p^e})/p^e, \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\mathbf{x})\}$ , and  $\nu_{\mathbf{x}}(a_j)/j \geq \text{ord}(\mathcal{M}_r W^s)(\mathbf{x}) = \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(\mathbf{x})$  (see Remark 8.2).

(ii) Suppose now that  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) = \nu_{\mathbf{x}}(a_{p^e})/p^e < \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\mathbf{x})$ . Recall that  $a_{p^e} W^{p^e} \in \mathcal{M}_r W^s$ , so  $a_{p^e} W^{p^e} \in \mathcal{M}_r^{[p^e]} W^{p^e}$ , where  $\mathcal{M}_r^{[p^e]}$  is the monomial ideal defined in Remark 3.6.

Set  $a_{p^e} = \mathcal{M}_r^{[p^e]} a'$ , for some  $a' \in \mathcal{O}_{V^{(d-1)}, \mathbf{x}}$ . We claim that  $a'$  is a unit, and that the algebras  $\mathcal{M}_r W^s$  and  $\mathcal{M}_r^{[p^e]} W^{p^e}$  have the same integral closure.

Assume first that  $\mathcal{G}_r$  is in the strong monomial case at  $x \in \text{Sing}(\mathcal{G}_r)$ , so  $\text{ord}(\mathcal{M}_r W^s)(x) = \nu_{\mathbf{x}}(a_{p^e})/p^e$ . Then  $\text{ord}(\mathcal{M}_r W^s)(\mathbf{x}) = \nu_{\mathbf{x}}(a_{p^e})/p^e \geq \text{ord}(\mathcal{M}_r^{[p^e]} W^{p^e})(\mathbf{x})$ . Remark 3.6 implies that equality must hold and both monomial algebras,  $\mathcal{M}_r^{[p^e]} W^{p^e}$  and  $\mathcal{M}_r W^s$ , have the same integral closure. In particular, we have that  $a'$  is a unit, and the algebra spanned by  $a_{p^e} W^{p^e}$  has the same integral closure as that of  $\mathcal{M}_r W^s$ .

Conversely, if the algebras generated by  $a_{p^e} W^{p^e}$  and  $\mathcal{M}_r W^s$  have the same integral closure, we claim that  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) = \nu_{\mathbf{x}}(a_{p^e})/p^e$ . To show this, use (8.1.2) to check that  $\nu_{\mathbf{x}}(a_{p^e})/p^e \geq \text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) \geq \text{ord}(\mathcal{M}_r W^s)(\mathbf{x}) = \nu_{\mathbf{x}}(a_{p^e})/p^e$ . So, finally,  $\mathcal{G}_r$  is in the strong monomial case at  $x \in \text{Sing}(\mathcal{G}_r)$ .  $\square$

*Remark 8.6.* Let  $\mathcal{G}_r$  be in the strong monomial case at the closed point  $x \in \text{Sing}(\mathcal{G}_r)$ . Then we claim that the following conditions hold for a  $p$ -presentation  $p\mathcal{P}$  in the conditions of Theorem 8.5.

(1) In case (i), the transversal parameter  $z$  defines a hypersurface of maximal contact. In particular, there exists an open neighborhood of  $\mathbf{x}$  where  $(\mathcal{R}_{\mathcal{G},\beta})_r = \mathcal{N}_r W^s = \mathcal{M}_r W^s$ .

(2) In case (ii), the monomial algebra can be described as  $\mathcal{M}_r W^{p^e}$  (i.e.,  $s = p^e$ ) where  $\mathcal{M}_r$  is not a  $p^e$ th power.

For (1), note that each  $a_j W^j \in (\mathcal{R}_{\mathcal{G},\beta})_r$  and that  $zW$  fulfills the integral condition

$$\lambda^{p^e} + (a_1 W^1) \lambda^{p^e-1} + \dots + (a_{p^e} - f_{p^e}(z)) W^{p^e} = 0.$$

This says that  $zW$  is integral over  $\mathcal{G}_r$  locally at the closed point  $x \in \text{Sing}(\mathcal{G}_r)$ , and hence, that  $z = 0$  defines a hypersurface of maximal contact.



(2) Follows from the proof of case (ii) in the previous theorem. Note that  $p\mathcal{P}$  is well-adapted to  $\mathcal{G}_r$  at  $\mathbf{x}$  and compatible with  $\mathcal{M}_r W^s$  (see §5). In terms of local coordinates, case (ii) says that there is a regular system of parameters, say  $\{x_1, \dots, x_{d-1}\}$ , at  $\mathcal{O}_{V^{(d-1)}, \mathbf{x}}$ , so that  $a_{p^e} W^{p^e} = u \cdot x_1^{h_1} \cdots x_l^{h_l} W^{p^e}$  where  $u$  is a unit at the local ring, and, in addition one can take  $\mathcal{M}_r W^s$  of the form  $x_1^{h_1} \cdots x_l^{h_l} W^{p^e}$ .

As  $p\mathcal{P}$  is well-adapted at the point,  $\text{In}_{\mathbf{x}}(a_{p^e})$  is not a  $p^e$ th power and therefore some exponent  $h_i$  is not a multiple of  $p^e$ .

LEMMA 8.7. *Let  $\mathcal{G}_r$  be in the strong monomial case, and set  $(\mathcal{R}_{\mathcal{G}, \beta})_r = \mathcal{N}_r W^s$  as in §8.3. Fix  $y \in \text{Sing}(\mathcal{G}_r)$ .*

(A) *If  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(y) = \text{ord}((\mathcal{R}_{\mathcal{G}, \beta})_r)(\beta_r(y))$ , then:*

(A1) *there is a dense open set  $U \subset \bar{y}$  so that  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x') = \text{ord}((\mathcal{R}_{\mathcal{G}, \beta})_r)(\beta_r(x'))$  and  $\mathcal{N}_r W^s = \mathcal{M}_r W^s$  at any  $x' \in U$ ;*

(A2)  $\text{ord}((\mathcal{R}_{\mathcal{G}, \beta})_r)(\beta_r(y)) = \text{ord}(\mathcal{M}_r W^s)(\beta_r(y))$ .

(B) *If  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(y) < \text{ord}((\mathcal{R}_{\mathcal{G}, \beta})_r)(\beta_r(y))$ , then at each smooth closed point  $x \in \bar{y}$  ( $\subset \text{Sing}(\mathcal{G}_r)$ ), and given a simultaneously adapted  $p$ -presentation  $p\mathcal{P}(\beta_r, z, f_{p^e})$ ,*

$$\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) = \frac{\nu_{\mathbf{x}}(a_{p^e})}{p^e} < \text{ord}((\mathcal{R}_{\mathcal{G}, \beta})_r)(\mathbf{x}).$$

*Proof.* (A) Note that (A2) follows from (A1) (since  $\mathcal{N}_r W^s = \mathcal{M}_r W^s$  at any  $x' \in U$ ).

Let  $E_y = \{H_{j_1}, \dots, H_{j_\ell}\}$  denote the set of exceptional hypersurfaces containing  $y$  with set of indexes  $\Lambda_y = \{j_1, \dots, j_\ell\}$ . Set  $\Lambda_y^- = \{1, \dots, r\} \setminus \Lambda_y$ . Consider a closed point  $x'$  so that  $x' \in \bar{y} \setminus \bigcup_{i \in \Lambda_y^-} H_i$ . Assume that  $x'$  is a smooth point of  $\bar{y}$ . Fix a  $p$ -presentation  $p\mathcal{P}(\beta_r, z, f_{p^e})$  well-adapted to  $\mathcal{G}_r$  both at  $\beta_r(x')$  and  $\beta_r(y)$ . Note that, for such  $x'$ , and since  $(\mathcal{R}_{\mathcal{G}, \beta})_r$  is a monomial algebra supported on the exceptional components

$$\text{ord}((\mathcal{R}_{\mathcal{G}, \beta})_r)(\beta_r(x')) = \text{ord}((\mathcal{R}_{\mathcal{G}, \beta})_r)(\beta_r(y)) \leq \frac{\nu_{\beta_r(y)}(a_{p^e})}{p^e} \leq \frac{\nu_{\beta_r(x')}(a_{p^e})}{p^e}.$$

Here the first equality follows from the choice of  $x'$ , the first inequality follows from the hypothesis, and the second is straightforward. So in this case,  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x') = \text{ord}((\mathcal{R}_{\mathcal{G}, \beta})_r)(\beta_r(x'))$ , and since  $\mathcal{G}_r$  is in the strong monomial case,  $\text{ord}(\mathcal{M}_r W^s)(\beta_r(x')) = \text{H-ord}^{(d-1)}(\mathcal{G}_r)(x') = \text{ord}((\mathcal{R}_{\mathcal{G}, \beta})_r)(\beta_r(x'))$ . Finally, argue as in the proof of Theorem 8.5 to conclude that  $\alpha_i = h_i$  for all  $i \in \Lambda_y$ . Hence  $(\mathcal{R}_{\mathcal{G}, \beta})_r = \mathcal{N}_r W^s = \mathcal{M}_r W^s$ . In particular,

$$\text{ord}((\mathcal{R}_{\mathcal{G}, \beta})_r)(\beta_r(y)) = \text{ord}(\mathcal{M}_r W^s)(\beta_r(y)).$$

(B) Fix a smooth closed point  $x \in \bar{y}$  and a  $p$ -presentation  $p\mathcal{P}(\beta, z, f_{p^e})$  well-adapted at  $\mathbf{x} = \beta_r(x)$  and  $\beta_r(y)$ , which we assume, in addition, to be compatible with  $\mathcal{M}_r W^s$ . Assume, on the contrary, that  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) = \text{ord}((\mathcal{R}_{\mathcal{G}, \beta})_r)(\mathbf{x}) (= \text{ord}(\mathcal{M}_r W^s)(\mathbf{x}))$ . Remark 8.6(i) ensures that  $(\mathcal{R}_{\mathcal{G}, \beta})_r = \mathcal{M}_r W^s$  in a neighborhood of  $\mathbf{x}$ . In particular,  $\text{ord}((\mathcal{R}_{\mathcal{G}, \beta})_r)(\beta_r(y)) = \text{ord}(\mathcal{M}_r W^s)(\beta_r(y))$ , so

$$\text{H-ord}^{(d-1)}(\mathcal{G}_r)(y) = \frac{\nu_{\beta_r(y)}(a_{p^e})}{p^e} < \text{ord}((\mathcal{R}_{\mathcal{G}, \beta})_r)(\beta_r(y)) = \text{ord}(\mathcal{M}_r W^s)(\beta_r(y)),$$

which is in contradiction with the fact that  $a_{p^e} W^{p^e} \in \mathcal{M}_r W^s$  locally at  $y$  (see Remark 8.2).  $\square$

COROLLARY 8.8. *Let  $\mathcal{G}_r$  be within the strong monomial case at a closed point  $x \in \text{Sing}(\mathcal{G}_r)$ . Let  $y \in \text{Sing}(\mathcal{G}_r)$  be such that  $x \in \bar{y}$  is a smooth point. Then,*

$$\text{H-ord}^{(d-1)}(\mathcal{G}_r)(y) = \text{ord}(\mathcal{M}_r W^s)(y).$$

*Proof.* (A) When  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(y) = \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\beta_r(y))$  the assertion is (A2) in Lemma 8.7.

(B) Assume that  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(y) = \nu_{\beta_r(y)}(a_{p^e})/p^e < \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\beta_r(y))$ . At the closed point  $x \in \bar{y}$ , Lemma 8.7(B) ensures that  $\text{ord}(\mathcal{M}_r W^s)(\mathbf{x}) = \text{H-ord}^{(d-1)}(\mathcal{G}_r)(\mathbf{x}) = \nu_{\mathbf{x}}(a_{p^e})/p^e < \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\mathbf{x})$ . Theorem 8.5 asserts that, locally at  $\mathbf{x}$ , and given a  $p$ -presentation in the conditions of the theorem, the algebras generated by  $a_{p^e} W^{p^e}$  and  $\mathcal{M}_r W^s$  have the same integral closure, so  $\nu_{\beta_r(y)}(a_{p^e})/p^e = \text{ord}(\mathcal{M}_r W^s)(\beta_r(y))$ .  $\square$

**8.9 The strong monomial case and proof of Theorem 8.14**

We now extend Definition 8.4 to the full spectrum.

Assume that  $\mathcal{G}_r$  is in the strong monomial case. Corollary 8.8 says that both functions  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(-)$  and  $\text{ord}(\mathcal{M}_r W^s)(-)$  take the same value at any point of  $\text{Sing}(\mathcal{G}_r)$ . In fact, given  $y \in \text{Sing}(\mathcal{G}_r)$ , one can always choose a smooth closed point  $x \in \bar{y}$ .

In particular, whenever  $\mathcal{G}_r$  is in the strong monomial case, the function  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)$  is upper-semi-continuous.

We now prove that the strong monomial case is stable under transformations.

*Remark 8.10.* When  $\mathcal{G}_r$  is within the strong monomial case, and  $C \subset \text{Sing}(\mathcal{G}_r)$  is a permissible center, Corollary 8.8 shows that  $\beta_r(C)$  is also a permissible center for the algebra generated by  $\mathcal{M}_r W^s$  (i.e.,  $\beta_r(C) \subset \text{Sing}(\mathcal{M}_r W^s)$ ). In particular, the transform of the tight monomial algebra can be defined.

We claim now that the transform of the tight monomial is the tight monomial of the transform.

LEMMA 8.11. *Assume that  $\mathcal{G}_r$  is in the strong monomial case. Let  $C \subset \text{Sing}(\mathcal{G}_r)$  be an irreducible permissible center. Let  $V_r^{(d)} \xleftarrow{\pi_C} V_{r+1}^{(d)}$  be the monoidal transformation with center  $C$ . Denote by  $\mathcal{M}'_{r+1} W^s$  the transform of  $\mathcal{M}_r W^s$  and by  $\mathcal{M}_{r+1} W^s$  the tight monomial algebra of  $\mathcal{G}_{r+1}$ . Then,  $\mathcal{M}'_{r+1} W^s = \mathcal{M}_{r+1} W^s$ .*

*Proof.* By definition the tight monomial algebra of the transform, say  $\mathcal{G}_{r+1}$ , is of the form

$$\mathcal{M}_{r+1} W^s = I(H_1)^{h_1} \dots I(H_r)^{h_r} I(H_{r+1})^{h_{r+1}} W^s,$$

where the  $H_j$  are the strict transforms of the previous exceptional hypersurfaces ( $j = 1, \dots, r$ ) and  $H_{r+1}$  is the new exceptional hypersurface introduced by  $\pi_C$ . Recall that  $h_{r+1} = q_{H_{r+1}} \cdot s$  where  $q_{H_{r+1}} = \text{H-ord}^{(d-1)}(\mathcal{G}_r)(\xi_C) - 1$ , and  $\xi_C$  denotes the generic point of  $C$ . On the other hand,

$$\mathcal{M}'_{r+1} W^s = I(H_1)^{h_1} \dots I(H_r)^{h_r} I(H_{r+1})^\gamma W^s,$$

where  $\gamma/s = \text{ord}(\mathcal{M}_r W^s)(y) - 1$  and  $y$  denotes the generic point of  $\beta_r(C)$ . Corollary 8.8 asserts that in the strong monomial case  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(\xi_C) = \text{ord}(\mathcal{M}_r W^s)(y)$ . Thus,  $\gamma = q_{H_{r+1}}$ , and hence  $\mathcal{M}'_{r+1} W^s = \mathcal{M}_{r+1} W^s$ .  $\square$

Fix a closed point  $x \in \text{Sing}(\mathcal{G}_r)$ . Theorem 8.5 characterizes when the algebra  $\mathcal{G}_r$  is in the strong monomial case at  $x$  under the assumption that there is a  $p$ -presentation, say  $p\mathcal{P}(\beta_r, z, f_{p^e})$ , which is well-adapted to  $\mathcal{G}_r$  at  $\mathbf{x} = \beta_r(x)$  and compatible with the tight monomial algebra  $\mathcal{M}_r W^s$ .

The following lemma provides conditions on a  $p$ -presentation  $p\mathcal{P}(\beta_r, z, f_{p^e})$ , which *a priori* is not necessarily well-adapted at  $\mathbf{x}$ , and yet ensures that  $\mathcal{G}_r$  is in the strong monomial case at  $x$ . This result will be useful in the forthcoming Proposition 8.13.

Recall that  $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p^e})$  is defined by a transversal projection  $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$ , a  $\beta$ -section  $z$  and a monic polynomial  $f_{p^e}(z) = z^{p^e} + a_1 z^{p^e-1} + \dots + a_{p^e}$ .

LEMMA 8.12. Assume that  $\mathcal{G}_r$  is in the monomial case (i.e., that the elimination algebra is monomial), and that a  $p$ -presentation  $p\mathcal{P}(\beta_r, z, f_{p^e})$  is defined in a neighborhood of closed point  $x \in \text{Sing}(\mathcal{G}_r)$ . Assume that  $p\mathcal{P}$  fulfills the following properties.

(i) The point  $x \in \{z = 0\}$  ( $x$  is in the  $\beta_r$ -section), and moreover that  $p\mathcal{P}$  is compatible with the tight monomial algebra  $\mathcal{M}_r W^s$ .

(ii) The  $\mathcal{O}_{V_r^{(d-1)}}$ -algebra spanned by  $a_{p^e} W^{p^e}$ , namely  $\mathcal{O}_{V_r^{(d-1)}}[a_{p^e} W^{p^e}]$ , has the same integral closure as  $\mathcal{O}_{V_r^{(d-1)}}[\mathcal{M}_r W^s]$ .

Then:

- (a)  $\mathbf{x} = \beta_r(x) \in V(\mathcal{M}_r)$ ;
- (b) furthermore, if locally at  $\mathbf{x}$  we set  $s = p^e$  and  $\mathcal{M}_r W^{p^e} = x_1^{h_1} \cdots x_{r'}^{h_{r'}} W^{p^e}$  where  $\{x_1, \dots, x_{r'}\}$  extend to a regular system of parameters, and if some  $h_i$  is not a multiple of  $p^e$ , then  $p\mathcal{P}$  is well-adapted at  $\mathbf{x}$ , and  $\mathcal{G}_r$  is in the strong monomial case at  $x$ .

*Proof.* Here (a) follows easily from (i) and (ii).

Note that (ii) ensures that we can take  $\mathcal{M}_r W^{p^e} = a_{p^e} W^{p^e}$ , and  $a_{p^e} = u x_1^{h_1} \cdots x_{r'}^{h_{r'}}$  at  $\mathcal{O}_{V_r^{(d-1)}, \mathbf{x}}$ , where  $u$  is a unit at such a ring. If we assume that some  $h_i$  is not a multiple of  $p^e$  then  $\text{In}_{\mathbf{x}}(a_{p^e})$  is not a power of  $p^e$  at the graded ring of  $\mathcal{O}_{V_r^{(d-1)}, \mathbf{x}}$ . Hence  $p\mathcal{P}$  is well-adapted at  $\mathbf{x}$ , and one can easily check that  $\mathcal{G}_r$  is in the strong monomial at  $x$ . □

PROPOSITION 8.13 ( $\tau = 1$ -stability of the strong monomial case). Suppose that  $\mathcal{G}_r$  is within the strong monomial case. Let  $C$  be a permissible center. Assume that  $\tau_{\mathcal{G}_r, x} = 1$  at a closed point  $x \in C$ . Consider the monoidal transformation of center  $C$ , say  $V_r^{(d)} \xleftarrow{\pi_C} V_{r+1}^{(d)}$ . Then, over a neighborhood of  $x \in \text{Sing}(\mathcal{G}_r)$ , the transform of  $\mathcal{G}_r$ , say  $\mathcal{G}_{r+1}$ , is within the strong monomial case.

*Proof.* Fix a  $p$ -presentation  $p\mathcal{P}(\beta_r, z, f_{p^e})$  well-adapted to  $\mathcal{G}_r$  both at  $\mathbf{x} = \beta_r(x)$  and at  $\xi_{\beta(C)}$ , and compatible with  $\mathcal{M}_r W^s$ . Here  $\xi_{\beta_r(C)}$  denotes the generic point of  $\beta_r(C)$ . Set  $f_{p^e}(z) = z^{p^e} + a_1 z^{p^e-1} + \cdots + a_{p^e}$ .

Proposition 6.6 asserts that  $p\mathcal{P}_1$  is well-adapted to  $\mathcal{G}_{r+1}$  at  $\xi_{H_{r+1}}$ , the generic point of  $H_{r+1}$  (i.e., the new exceptional hypersurface), and that  $p\mathcal{P}_1$  is compatible with the tight monomial algebra of  $\mathcal{G}_{r+1}$ , namely with  $\mathcal{M}_{r+1} W^s$ .

We claim that  $p\mathcal{P}_1$  is well-adapted to  $\mathcal{G}_{r+1}$  at  $\mathbf{x}' = \beta_{r+1}(x')$  for any closed point  $x' \in \text{Sing}(\mathcal{G}_{r+1})$  mapping to  $x$ . In particular, that  $\text{H-ord}^{(d-1)}(\mathcal{G}_{r+1})(x') = \min\{\nu_{\mathbf{x}'}(a_{p^e}^{(1)})/p^e, \text{ord}((\mathcal{R}_{\mathcal{G}, \beta})_{r+1})(\mathbf{x}')\}$ , where  $a_{p^e}^{(1)}$  denotes the independent term of  $f_{p^e}^{(1)}(z_1)$ , the strict transform of  $f_{p^e}(z)$ .

As  $\mathcal{G}_r$  is in the strong monomial case at  $x$ , either  $\mathcal{M}_r W^s = (\mathcal{R}_{\mathcal{G}, \beta})_r = \mathcal{N} W^s$ , or the algebras spanned by  $a_{p^e} W^{p^e}$  and  $\mathcal{M}_r W^s$  have the same integral closure. Lemma 8.11 asserts that the transform  $\mathcal{M}_r W^s$  (the tight monomial algebra of  $\mathcal{G}_r$ ) is  $\mathcal{M}_{r+1} W^s$ , the tight monomial algebra of  $\mathcal{G}_{r+1}$ . Hence, at  $V_{r+1}^{(d)}$ , conditions (i) or (ii) in Theorem 8.5 are preserved (this also holds for  $p\mathcal{P}_1$ ). Thus, if the claim is true, then the cited theorem ensures that  $\mathcal{G}_{r+1}$  is also in the strong monomial case at  $x'$ . So, in what follows, we focus on the claim.

We may assume that, locally at  $x \in \text{Sing}(\mathcal{G}_r)$ , the tight monomial algebra  $\mathcal{M}_r W^s$  is of the form

$$I(H_1)^{h_1} \cdots I(H_{r'})^{h_{r'}} W^s \quad \text{with } 0 < h_i < s. \tag{8.13.1}$$

In order to achieve this (namely, that all  $h_i < s$ ), it suffices to consider a finite sequence of permissible transformations with centers of codimension 2 at  $V_r^{(d)}$ .

Of course one must check that the strong monomial case is preserved by blowing up at such centers of codimension 2. To check this fact, fix, as in Theorem 8.5, locally at a closed point  $x \in \text{Sing}(\mathcal{G}_r)$ , a presentation  $p\mathcal{P}(\beta_r, z, f_{p^e})$  well-adapted to  $\mathcal{G}_r$  at  $\mathbf{x} = \beta_r(x)$  and compatible with the tight monomial algebra  $\mathcal{M}_r W^s$ . The statement is clear in case (i) of Theorem 8.5, namely if  $(\mathcal{R}_{\mathcal{G},\beta})_r = \mathcal{M}_r W^s$ , as this is the case of maximal contact. It remains to consider case (ii) of this theorem, namely when one can take  $\mathcal{M}_r W^s$  with  $s = p^e$ , and in addition  $a_{p^e} = u \cdot \mathcal{M}_r$ . Note here that locally at  $\mathbf{x}$ ,  $\mathcal{M}_r W^{p^e}$  has an expression as in (8.13.1), with the additional condition that some  $h_i \neq 0$  (see Remark 8.6). In this case Lemma 8.12 ensures that the transform of the presentation (by blowing up at such centers of codimension 2) is well-adapted, and that the transform of the algebra is in the strong monomial case at  $\mathbf{x}$  after such a sequence of permissible transformations.

We now address the proof of the previous claim: fix a closed point  $x' \in \text{Sing}(\mathcal{G}_{r+1}) \cap H_{r+1}$  mapping to  $x$  (i.e.,  $\pi_C(x') = x$ ). Recall that, under the hypothesis  $\tau_{\mathcal{G}_r,x} = 1$ , Remark 6.5 asserts that  $x' \in \{z_1 = 0\}$ , where  $z_1$  denotes the strict transform of  $z$ . We now prove our claim, namely that  $p\mathcal{P}_1$  is well-adapted to  $\mathcal{G}_{r+1}$  at  $\mathbf{x}'$ .

We divide the proof of the claim into the following cases.

(1) Assume that  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) = \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\mathbf{x})$  ( $= \text{ord}(\mathcal{M}_r W^s)(\mathbf{x})$ ). Theorem 8.5 ensures that  $(\mathcal{R}_{\mathcal{G},\beta})_r = \mathcal{N}_r W^s = \mathcal{M}_r W^s$  in a neighborhood of  $\mathbf{x}$  and, in particular, that  $\text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\xi_C) = \text{ord}(\mathcal{M}_r W^s)(\xi_C)$ . Note that  $\mathcal{M}_{r+1} W^s = (\mathcal{R}_{\mathcal{G},\beta})_{r+1}$ , and again Theorem 8.5 says that  $\mathcal{G}_{r+1}$  is in the strong monomial case (in particular,  $p\mathcal{P}_1$  is well-adapted to  $\mathcal{G}_{r+1}$  at  $\mathbf{x}'$ ).

(2) Now suppose that  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(x) = \nu_{\mathbf{x}}(a_{p^e})/p^e < \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\mathbf{x})$ . In this case, Theorem 8.5 says that the algebras spanned by  $a_{p^e} W^{p^e}$  and  $\mathcal{M}_r W^s$  have the same integral closure. In particular, we can take  $s = p^e$  and assume that  $a_{p^e} W^{p^e} = u \mathcal{M}_r W^{p^e}$ , where  $u$  is a unit (see Remark 8.6).

The equality  $a_{p^e} W^{p^e} = u \mathcal{M}_r W^{p^e}$  implies that

$$\frac{\nu_{\xi_{\beta_r(C)}}(a_{p^e})}{p^e} = \text{ord}(\mathcal{M}_r W^s)(\xi_{\beta_r(C)}) \leq \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\xi_{\beta_r(C)}), \tag{8.13.2}$$

and, as  $p\mathcal{P}$  is well-adapted at  $\xi_{\beta_r(C)}$ ,  $\text{H-ord}^{(d-1)}(\mathcal{G}_r)(\xi_C) = \nu_{\xi_{\beta_r(C)}}(a_{p^e})/p^e$ , and hence  $h_{r+1} = \nu_{\xi_{\beta_r(C)}}(a_{p^e}) - p^e$  is the exponent of  $I(H_{r+1})$  in  $\mathcal{M}_{r+1} W^s$  ( $s = p^e$ ).

(2.A) If  $\nu_{\xi_C}(a_{p^e})/p^e \notin \mathbb{Z}_{>0}$ , then  $h_{r+1} = \nu_{\xi_{H_{r+1}}}(a_{p^e}^{(1)}) = \nu_{\xi_{\beta_r(C)}}(a_{p^e}) - p^e (\not\equiv 0 \pmod{p^e})$  and  $h_{r+1}/s \leq \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_{r+1})(\xi_{H_{r+1}})$ .

Notice here that  $\text{In}_{\mathbf{x}'}(a_{p^e}^{(1)})$  cannot be a  $p^e$ th power since  $h_{r+1} \not\equiv 0 \pmod{p^e}$ . Hence, Lemma 8.12 applies here to ensure that  $p\mathcal{P}_1$  is well-adapted to  $\mathcal{G}_{r+1}$  at  $\mathbf{x}'$  (see Definition 5.4).

Let us introduce some useful notation for the proof of the claim in the remaining cases. Fix a regular system of parameters in the regular local ring  $\mathcal{O}_{V_r^{(d)},x}$ , say  $\{z, x_1, \dots, x_{d-1}\}$ , such that:

- (i)  $\{x_1, \dots, x_{d-1}\}$  are parameters at  $\mathcal{O}_{V_r^{(d-1)},x}$ , and the tight monomial algebra is locally generated by a monomial in  $x_1, \dots, x_{r'}$  ( $r' \leq d-1$ ), say  $x_1^{h_1} \dots x_{r'}^{h_{r'}}$ ; and
- (ii) the permissible center is  $I(C) = \langle z, x_1, \dots, x_\ell, y_1, \dots, y_m \rangle$ , where  $\ell \leq r'$  and  $y_j = x_{r'+j}$  for  $j = 1, \dots, m$ .

As  $a_{p^e} W^{p^e} = u \cdot \mathcal{M}_r W^s$  where  $u$  is invertible,

$$a_{p^e} = u x_1^{h_1} \dots x_\ell^{h_\ell} x_{\ell+1}^{h_{\ell+1}} \dots x_{r'}^{h_{r'}} \quad \text{with } 0 < h_i < p^e.$$

In all the remaining cases we will show that the conditions of Lemma 8.12(b), hold for the transform of the  $p$ -presentation  $p\mathcal{P}(\beta, z, f_{p^e})$ .

(2.B) Assume that  $\nu_{\xi_{\beta_r(C)}}(a_{p^e})/p^e \in \mathbb{Z}_{>0}$  and that  $\ell < r'$ . In this case, one can check that  $\text{In}_{\mathbf{x}'}(a_{p^e}^{(1)})$ , which is also monomial, is not a  $p^e$ th power. In fact, at each chart

$$a_{p^e}^{(1)} = ux_1^{h_1+\dots+h_\ell-p^e} \left(\frac{x_2}{x_1}\right)^{h_2} \dots \left(\frac{x_\ell}{x_1}\right)^{h_\ell} x_{\ell+1}^{h_{\ell+1}} \dots x_{r'}^{h_{r'}} \quad \text{in the } U_{x_1}\text{-chart, or}$$

$$a_{p^e}^{(1)} = uy_1^{h_1+\dots+h_\ell-p^e} \left(\frac{x_1}{y_1}\right)^{h_1} \dots \left(\frac{x_\ell}{y_1}\right)^{h_\ell} x_{\ell+1}^{h_{\ell+1}} \dots x_{r'}^{h_{r'}} \quad \text{in the } U_{y_1}\text{-chart}$$

and  $0 < h_{\ell+1} < p^e$  (i.e.,  $h_{\ell+1} \not\equiv 0 \pmod{p^e}$ ). This ensures that  $\text{In}_{\mathbf{x}'}(a_{p^e}^{(1)})$  is not a  $p^e$ th power, and hence  $p\mathcal{P}_1$  is well-adapted at  $\mathbf{x}'$ .

(2.C) Assume that  $\nu_{\xi_{\beta_r(C)}}(a_{p^e})/p^e \in \mathbb{Z}_{>0}$  and  $\ell = r'$ .

Note here that  $\ell = r' \geq 2$ , since  $\mathcal{M}_r W^{p^e}$  is not a  $p^e$ th power and  $h_1 + \dots + h_{r'} \equiv 0 \pmod{p^e}$ .

We now prove that  $p\mathcal{P}_1$  is well-adapted at  $\mathbf{x}'$  by considering two cases.

(2.C.1) Firstly suppose that  $\nu_{\xi_{\beta_r(C)}}(a_{p^e})/p^e < \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\xi_{\beta_r(C)})$ . After a finite number of monoidal transformations over  $V_{r+1}^{(d)}$  at centers of codimension 2, we can assume that  $h_{r+1} = 0$ . Thus, the independent term, say  $a_{p^e}^{(1)}$ , is

$$a_{p^e}^{(1)} = u \left(\frac{x_2}{x_1}\right)^{h_2} \dots \left(\frac{x_{r'}}{x_1}\right)^{h_{r'}} \quad \text{in the } U_{x_1}\text{-chart, or}$$

$$a_{p^e}^{(1)} = u \left(\frac{x_1}{y_1}\right)^{h_1} \dots \left(\frac{x_{r'}}{y_1}\right)^{h_{r'}} \quad \text{in the } U_{y_1}\text{-chart.}$$

Both cases are analogous, so it suffices to consider the problem at the  $U_{x_1}$ -chart. The difference with the discussion in (2.B) appears when considering a closed exceptional point where  $a_{p^e}^{(1)}$  is a unit. We now address this case. Let

$$f_{p^e}^{(1)} = z_1^{p^e} + a_1^{(1)} z_1^{p^e-1} + \dots + a_{p^e}^{(1)}$$

be the strict transform of  $f_{p^e}$ . The assumption  $\nu_{\xi_{\beta_r(C)}}(a_{p^e})/p^e < \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\xi_{\beta_r(C)})$  ensures that  $\text{ord}((\mathcal{R}_{\mathcal{G},\beta})_{r+1})(\xi_{H_{r+1}}) > 0$ , and hence that  $x_1$  divides  $a_j^{(1)}$  for  $j = 1, \dots, p^e - 1$  (see Theorem 4.4).

We claim that if  $x' \in \text{Sing}(\mathcal{G}_{r+1})$ , then  $x' \in \{x_1 = 0\} \cap \{x_j/x_1 = 0\}$  for some  $j \in \{2, \dots, r'\}$ . Let  $\bar{f}_{p^e}^{(1)} = z_1^{p^e} + \bar{a}_{p^e}^{(1)}$  be the restriction of  $f_{p^e}^{(1)}$  to  $x_1 = 0$ , where  $\bar{a}_{p^e}^{(1)} = \bar{u}(x_2/x_1)^{h_1} \dots (x_{r'}/x_1)^{h_{r'}}$ . We identify  $\bar{a}_{p^e}^{(1)}$  with an element of  $\mathcal{O}_{\beta_r(C)}[(x_2/x_1), \dots, (x_{r'}/x_1), (y_1/x_1), \dots, (y_m/x_1)]$ . This is a polynomial ring in  $r' - 1 + m$  variables. Consider the Taylor expansion of  $\bar{a}_{p^e}^{(1)}$  at this ring, say

$$\text{Tay}(\bar{a}_{p^e}^{(1)}) = \sum_{\alpha \in \mathbb{N}^{r'-1+m}} \Delta^\alpha(\bar{a}_{p^e}^{(1)}) T^\alpha.$$

The operators  $\Delta^\alpha$  in this expansion are differential operators in  $\mathcal{O}_{\beta_r(C)}[(x_2/x_1), \dots, (x_{r'}/x_1), (y_1/x_1), \dots, (y_m/x_1)]$ , relative to the ring  $\mathcal{O}_{\beta_r(C)}$ , defined in terms of the  $r' - 1 + m$  variables.

Note here that  $\bar{u} \in \mathcal{O}_{\beta_r(C)}$ , so in particular,  $\Delta^\alpha(\bar{a}_{p^e}^{(1)}) = \bar{u} \Delta^\alpha((x_2/x_1)^{h_2} \dots (x_{r'}/x_1)^{h_{r'}})$ .

Since it is assumed that  $h_j < p^e$ , it follows that

$$\Delta^{\alpha_j}(\bar{a}_{p^e}^{(1)}) = \bar{u} \left(\frac{x_2}{x_1}\right)^{h_2} \dots \left(\frac{x_{j-1}}{x_1}\right)^{h_{j-1}} \left(\frac{x_{j+1}}{x_1}\right)^{h_{j+1}} \dots \left(\frac{x_{r'}}{x_1}\right)^{h_{r'}}$$

for  $\alpha_j = (0, \dots, h_j, \dots, 0) \in \mathbb{N}^{r'-1+m}$ . Moreover, if  $\Delta^{\alpha_j}(\bar{a}_{p^e}^{(1)})(\mathbf{x}') = 0$  for all  $j = 2, \dots, r'$ , then  $\mathbf{x}' \in \{x_j/x_1 = 0\}$  for some  $j$ . So, if  $x' \in \text{Sing}(\mathcal{G}_1) \cap H_{r+1} \cap U_{x_1}$ , then  $\mathbf{x}' \in \bigcup_{2 \leq j \leq r'} \{x_j/x_1 = 0\}$ .

In this case we can argue as in (2.B) to show that  $\text{In}_{\mathbf{x}'}(a_{p^e}^{(1)})$  is not a  $p^e$ th power, and hence that  $p\mathcal{P}_1$  is well-adapted at  $\mathbf{x}'$ .

(2.C.2) According to (8.13.2), the only case left is  $\nu_{\xi_{\beta_r(C)}}(a_{p^e})/p^e = \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\xi_{\beta_r(C)})$  within the case  $\ell = r'$ .

The equality  $\text{ord}(\mathcal{M}_r W^s)(\xi_{\beta_r(C)}) = \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_r)(\xi_{\beta_r(C)})$ , implies that  $h_i = \alpha_i$  for  $i = 1, \dots, r'$  (see (8.3.1)).

By the assumption in case (2),  $\text{ord}(\mathcal{M}_r W^s)(\mathbf{x}) < \text{ord}(\mathcal{N}_r W^s)(\mathbf{x})$ . Thus, there must be an exceptional hypersurface, say  $H$ , so that  $\mathbf{x} \in H$ ,  $H$  is not a component of the support of  $\mathcal{M}_r$  (of  $V(\mathcal{M}_r)$ ), and  $H$  is a component of  $V(\mathcal{N}_r W^s)$ . That is,  $H \neq H_j$  for  $j = 1, \dots, r'$  and  $\text{ord}(\mathcal{N}_r W^s)(\xi_H) > 0$ .

Consider the monoidal transformation along  $C$ . We may assume, after a finite number of monoidal transformations at centers of codimension 2, that the new exceptional hypersurface, say  $H_{r+1}$ , is not a component of  $V(a_{p^e}^{(1)})$ . Here  $a_{p^e}^{(1)}$  is essentially monomial and admits expressions as in (2.C.1), both in  $U_{x_i}$ -charts or in  $U_{y_j}$ -charts. In addition,  $H_{r+1}$  is not a component of  $V(\mathcal{N}_{r+1} W^s)$ , where  $\mathcal{N}_{r+1} W^s$  is the transform of  $\mathcal{N}_r W^s$ . On the contrary, the strict transform of  $H$  is a component of  $V(\mathcal{N}_{r+1} W^s)$  and is not a component of  $V(a_{p^e}^{(1)})$ .

Note here that  $x \in (C \cap H)$  and that  $C$  is not included in  $H$ . Therefore, the full fiber over  $x$  of the monoidal transformation, say  $\pi_C^{-1}(x)$ , is included in the strict transform of  $H$ . We argue as in (2.C.1), now considering the restriction of  $\mathcal{G}_{r+1}$  to the strict transform of  $H$ , instead of restrictions to  $H_{r+1}$ . Note now that in fact the same argumentation as used in (2.C.1) applies here to show that  $p\mathcal{P}_1$  is well-adapted at  $\mathbf{x}'$ . □

We may assume that resolution of simple Rees algebras can be achieved by decreasing induction on  $\tau$ . The following theorem shows how to increase the invariant  $\tau$ , under the assumption that  $\mathcal{G}_r$  is in the strong monomial case.

**THEOREM 8.14.** *Let  $\mathcal{G}_r$  be within the strong monomial case. Then, any combinatorial resolution of  $\mathcal{M}_r W^s$  can be lifted to a sequence of transformations of  $\mathcal{G}_r$ , say*

$$\mathcal{G}_r \quad \mathcal{G}_{r+1} \quad \mathcal{G}_N$$

$$V_r^{(d)} \xleftarrow{\pi_{r+1}} V_{r+1}^{(d)} \xleftarrow{\pi_{r+2}} \dots \xleftarrow{\pi_N} V_N^{(d)} \tag{8.14.1}$$

and if  $x \in \text{Sing}(\mathcal{G}_r)$  is a closed point so that  $\tau_{\mathcal{G}_r,x} = 1$ , then  $\pi^{-1}(x) \cap \text{Sing}(\mathcal{G}_N) = \emptyset$ , or consists of points with  $\tau_{\mathcal{G}_N,x'} \geq 2$ . Hence,  $\tau_{\mathcal{G}_N,x'} \geq 2$  for any  $x' \in \text{Sing}(\mathcal{G}_N)$  if not empty.

*Proof.* Recall that  $\mathcal{M}_r W^s (\subset \mathcal{O}_{V_r^{(d)}}[W])$  is the pull-back of a monomial algebra, say  $\mathcal{M}_r W^s$  again ( $\subset \mathcal{O}_{V_r^{(d-1)}}[W]$ ). What we mean here is that a combinatorial resolution of  $\mathcal{M}_r W^s$  in dimension  $d - 1$  can be lifted to a permissible sequence in dimension  $d$ .

Fix a closed point  $x \in \text{Sing}(\mathcal{G}_r)$  so that  $\tau_{\mathcal{G}_r,x} = 1$ . Proposition 8.13 ensures that after a permissible sequence of transformations as in (8.14.1), the transform  $\mathcal{G}_N$  is in the strong monomial case. In particular,  $\text{H-ord}^{(d-1)}(\mathcal{G}_N)(x') = \text{ord}(\mathcal{M}_N W^s)(x')$  for any closed point  $x' \in \text{Sing}(\mathcal{G}_N)$  mapping to  $x$ , for which  $\tau_{\mathcal{G}_N,x'} = 1$ . Moreover, by assumption  $\text{ord}(\mathcal{M}_N W^s)(x') < 1$ . That is,  $\pi^{-1}(x) \cap \text{Sing}(\mathcal{G}_N) = \emptyset$ , or consists of points with  $\tau_{\mathcal{G}_N,x'} \geq 2$ . □

PART III. PROOFS OF THEOREMS

Appendix A. Proof of Main Theorem 1

A.1 Hironaka’s weak equivalence

There are two natural operations on Rees algebras, and both are crucial in understanding Hironaka’s fundamental notion (or meaning) of *invariance*. Fix a smooth scheme  $V^{(d)}$  and a set, say  $E = \{H_1, \dots, H_r\}$ , of smooth hypersurfaces so that  $\cup H_i$  has only normal crossings. Let  $\mathcal{G} = \bigoplus I_n W^n$  be a Rees algebra in  $V^{(d)}$ . Now, let

$$V^{(d)} \xleftarrow{\pi} U \tag{A.1.1}$$

be defined either by:

- (A) an open set  $U$  of  $V^{(d)}$  in Zariski or étale topology;
- (B) the projection of  $U = V^{(d)} \times \mathbb{A}_k^n$  on the first coordinate. Here,  $\mathbb{A}_k^n$  denotes the  $n$ -dimensional affine scheme (with  $n \in \mathbb{Z}_{\geq 1}$ ).

In both cases, there is a naturally defined pull-back of the Rees algebra  $\mathcal{G}$  and of the set  $E$ . This defines a Rees algebra  $\mathcal{G}_U$  and a set  $E_U$ . Here  $E_U$  consists of the pull-backs of the hypersurfaces in  $E$ . The Rees algebra  $\mathcal{G}_U$  is defined as:

- (A) the restriction to  $U$  in case (A), i.e.,  $\mathcal{G}_U = \bigoplus (I_n)_U W^n$ ;
- (B) the total transforms of each ideal  $I_n$ , say  $I_n^*$ , in case (B), i.e.,  $\mathcal{G}_U = \bigoplus I_n^*$ .

The pull-back defined by  $V^{(d)} \xleftarrow{\pi} U$  is denoted by the following diagram.

$$\begin{array}{ccc} \mathcal{G} & & \mathcal{G}_U \\ (V^{(d)}, E) & \xleftarrow{\pi} & (U, E_U) \end{array}$$

Observe here that  $\text{Sing}(\mathcal{G}_U) = \pi^{-1}(\text{Sing}(\mathcal{G}))$ .

DEFINITION A.2. A *local sequence* of a Rees algebra  $\mathcal{G}$  and a set  $E$  is a sequence

$$\begin{array}{ccccc} \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_r \\ (V^{(d)}, E) & \xleftarrow{\pi_1} & (\tilde{V}_1^{(d)}, E_1) & \xleftarrow{\pi_2} \dots \xleftarrow{\pi_r} & (\tilde{V}_r^{(d)}, E_r) \end{array} \tag{A.2.1}$$

where each  $\tilde{V}_i^{(d)} \xleftarrow{\pi_{i+1}} \tilde{V}_{i+1}^{(d)}$  is a pull-back, in which case  $E_{i+1}$  is the pull-back of hypersurfaces in  $E_i$ , or a monoidal transformation at a center  $C_i \subset \text{Sing}(\mathcal{G}_i)$  with normal crossing with the exceptional hypersurfaces in  $E_i$  for  $i = 0, \dots, r - 1$ .

DEFINITION A.3. Fix two Rees algebras  $\mathcal{G}$  and  $\mathcal{G}'$  and a set of hypersurfaces with normal crossings  $E$  in the smooth scheme  $V^{(d)}$ . We say that  $\mathcal{G}$  and  $\mathcal{G}'$  are *weakly equivalent* if:

- (i)  $\text{Sing}(\mathcal{G}) = \text{Sing}(\mathcal{G}')$ ;
- (ii) any local sequence of  $\mathcal{G}$ , say (A.2.1), defines a local sequence of  $\mathcal{G}'$  (and vice versa), and  $\text{Sing}(\mathcal{G}_i) = \text{Sing}(\mathcal{G}'_i)$  for  $i = 0, \dots, r$ .

*Remark A.4.* Note that if  $\mathcal{G}$  and  $\mathcal{G}'$  are weakly equivalent, then their transforms  $\mathcal{G}_i$  and  $\mathcal{G}'_i$  are also weakly equivalent, so the weak equivalence is preserved after any local sequence. Two algebras with the same integral closure are weakly equivalent.

**A.5 On Main Theorem 1**

PROPOSITION A.6. Fix a Rees algebra  $\mathcal{G}$  and a presentation  $p\mathcal{P} = \mathcal{P}(\beta, z, f_n(z))$ . Let  $H$  be a smooth irreducible hypersurface in  $V^{(d-1)}$ . Denote by  $y$  the generic point of  $H$  and assume that  $p\mathcal{P}$  is well-adapted at  $y$ . Then,  $H$  is a component of  $\beta(\text{Sing}(\mathcal{G}))$  if and only if  $\text{Sl}_y(p\mathcal{P}) \geq 1$ .

*Proof.* It follows from Proposition 4.6, and from the assumption that  $p\mathcal{P}$  is well-adapted at  $y$ .  $\square$

THEOREM A.7 (Main Theorem 1). Fix a Rees algebra  $\mathcal{G}$ . Consider a point  $x \in \text{Sing}(\mathcal{G})$  and a  $p$ -presentation, say  $p\mathcal{P}$ , well-adapted at  $\beta(x)$ . The value  $\text{Sl}(p\mathcal{P})(\beta(x))$  is completely determined by the weak equivalence class of  $\mathcal{G}$ .

*Proof.* Here  $x$  stands for an arbitrary singular point, not necessarily closed. Fix  $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p^e}(z))$ , as above, well-adapted to  $\mathcal{G}$  at  $\mathbf{x} = \beta(x)$ . Fix  $f_{p^e}(z) = z^{p^e} + a_1 z^{p^e-1} + \dots + a_{p^e}$  and set  $r_j = \nu_{\mathbf{x}}(a_j)$  for  $j = 1, \dots, p^e$  and  $\text{ord}(\mathcal{R}_{\mathcal{G},\beta})(\mathbf{x}) = \alpha/s$ . Set  $q = \text{Sl}(p\mathcal{P})(\mathbf{x})$ , and recall from Theorem 4.4 that  $\text{Sl}(p\mathcal{P})(\mathbf{x}) = \min\{\nu_{\mathbf{x}}(a_{p^e})/p^e(\mathbf{x}), \text{ord}^{(d-1)}(\mathcal{R}_{\mathcal{G},\beta})(\mathbf{x})\}$ . Note that  $q \geq 1$  and that  $z$  is an element of order 1 in  $\mathcal{O}_{V^{(d)},x}$  (see Remark 5.5).

Consider  $V^{(d)} \times \mathbb{A}^1$ , the product of  $V^{(d)}$  with the affine line. Locally, in a neighborhood of  $(x, 0) \in V^{(d)} \times \mathbb{A}^1$ , we identify  $f_{p^e}(z)$  with its pull-back. Consider, in addition, the natural projection  $\tilde{\beta}_0 = \beta \times \text{id} : V^{(d)} \times \mathbb{A}^1 \rightarrow V^{(d-1)} \times \mathbb{A}^1$ , mapping  $(x, 0)$  to  $(\mathbf{x}, 0)$ . Finally, identify  $\mathcal{R}_{\mathcal{G},\beta}$  with its pull-back in  $V^{(d-1)} \times \mathbb{A}^1$ . This defines a  $p$ -presentation of the pull-back of  $\mathcal{G}$  at  $V^{(d)} \times \mathbb{A}^1$ , say  $p\mathcal{P}$  again. Note that

$$\text{Sl}(p\mathcal{P})((\mathbf{x}, 0)) = \text{Sl}(p\mathcal{P})(\mathbf{x})$$

and that  $\text{In}_{(\mathbf{x},0)}(a_{p^e})$  can be naturally identified with  $\text{In}_{\mathbf{x}}(a_{p^e})$ .

Fix coordinates  $\{z, x_1, \dots, x_a, t\}$  locally at  $(x, 0)$ , where  $\{z, x_1, \dots, x_a\}$  is a regular system of parameters at  $\mathcal{O}_{V^{(d)},x}$ , and  $\{x_1, \dots, x_a\}$  is a regular system of parameters at  $\mathcal{O}_{V^{(d-1)},\mathbf{x}}$ . Consider the monoidal transformation with center  $q_0 = (x, 0)$  and let  $q_1$  be the intersection of the new exceptional hypersurface, say  $H_1$ , and the strict transform of  $x \times \mathbb{A}^1$ . This monoidal transformation at  $q_0$  induces a monoidal transformation, say  $V^{(d-1)} \times \mathbb{A}^1 \leftarrow V_1^{(d)}$ , at  $(\beta(x), 0) = \tilde{\beta}_0(q_0)$ . Moreover, one can define a smooth morphism  $\tilde{\beta}_1 : V_1^{(d+1)} \rightarrow V_1^{(d)}$ . The exceptional hypersurface  $H_1 \subset V_1^{(d+1)}$  is the pull-back of the exceptional hypersurface in  $V_1^{(d)}$ . To simplify notation, we denote both by  $H_1$ . As  $x$  might not be closed,  $q_0$  and  $q_1$  are not necessarily closed. But as restrictions to open sets are permitted in Hironaka’s notion of weak equivalence, we argue as if these points would be closed, in the sense that  $q_0$  is the generic point of a smooth scheme when restricting to a suitable open set.

The point  $q_1$  is the origin of the  $U_t$ -chart, ( $U_t = \text{Spec}(\mathcal{O}_{V^{(d)},x}[z/t, \dots, x_a/t, t])$ ). The transform of  $p\mathcal{P}$ , say  $p\mathcal{P}_1 = p\mathcal{P}_1(\tilde{\beta}_1, z_1, f_{p^e}^{(1)})$ , is defined by

$$f_{p^e}^{(1)}(z_1) = z_1^{p^e} + t^{r_1-1} a'_1 z_1^{p^e-1} + \dots + t^{r_{p^e}-p^e} a'_{p^e} \quad \text{and} \quad (\mathcal{R}_{\mathcal{G},\beta})_1$$

where the  $a'_j$  are not divisible by  $t$ , and  $\text{ord}((\mathcal{R}_{\mathcal{G},\beta})_1)(\xi_{H_1}) = (\alpha - s)/s$ , where  $\xi_{H_1}$  denotes the generic point of  $H_1 \subset V_1^{(d)}$ .

This process can be iterated  $N$ -times, defining a sequence of monoidal transformations at  $q_0, q_1, \dots, q_{N-1}$ , where each  $q_j$  is the intersection of the new exceptional component, say  $H_j (\subset V_j^{(d+1)})$ , with the strict transform of  $x \times \mathbb{A}^1$ . Let  $q_N$  be the intersection of the latest exceptional hypersurface, say  $H_N$ , and the strict transform of  $x \times \mathbb{A}^1$ . The transform of  $p\mathcal{P}$  at



the final  $U_t$ -chart, say  $p\mathcal{P}_N = p\mathcal{P}_N(\tilde{\beta}_N, z_N, f_{p^e}^{(N)})$ , is given by

$$f_{p^e}^{(N)}(z_N) = z_N^{p^e} + t^{N(r_1-1)} a'_1 z_N^{p^e-1} + \dots + t^{N(r_{p^e}-p^e)} a'_{p^e} \quad \text{and} \quad (\mathcal{R}_{\mathcal{G},\beta})_N, \tag{A.7.1}$$

with  $\text{ord}((\mathcal{R}_{\mathcal{G},\beta})_N)(\xi_{H_N}) = N(\alpha - s)/s$ , where  $\xi_{H_N}$  denotes the generic point of  $H_N \subset V_N^{(d)}$ .

Fix  $N \gg 0$ , it may occur that  $\text{Sing}(\mathcal{G}_N) \cap H_N$  has codimension 2 in the  $d + 1$ -dimensional ambient space (i.e., that  $H_N \subset V_N^{(d)}$  is a component of  $\tilde{\beta}_N(\text{Sing}(\mathcal{G}_N))$ ). This does occur, clearly, if  $(r_j - j) > 0$  ( $j = 1, \dots, p^e$ ) and  $(\alpha - s) > 0$ , and we claim that this is the only case in which it occurs. In other words, we claim that  $\text{Sing}(\mathcal{G}_N) \cap H_N$  has codimension 2 in the  $d + 1$ -dimensional ambient space if and only if  $(r_j - j) > 0$  ( $j = 1, \dots, p^e$ ) and  $(\alpha - s) > 0$ .

In fact, these strict inequalities do not hold if either  $r_{p^e} = p^e$ , or if  $\alpha = s$ . If  $\alpha = s$ , then  $H_N \subset V_N^{(d)}$  is not a component of  $\tilde{\beta}_N(\text{Sing}(\mathcal{G}_N))$ , as it is not a component of  $\text{Sing}((\mathcal{R}_{\mathcal{G},\beta})_N)$  (see (A.7.1)). So  $\text{Sing}(\mathcal{G}_N) \cap H_N$  is not of codimension 2 in this case. Finally, assume that  $(\alpha - s) > 0$  and that  $r_{p^e} = p^e$ . In this case we consider the restriction of  $\mathcal{G}_N$  to  $H_N$ . More precisely, consider the restriction of the data in (A.7.1) to  $H_N$ . Note here that the restriction of  $(\mathcal{R}_{\mathcal{G},\beta})_N$  is zero, and the restriction of  $f_{p^e}^{(N)}(z_N)$  is of the form

$$f_{p^e}^{(N)}(z_N) = z_N^{p^e} + \bar{a}'_{p^e},$$

where  $\bar{a}'_{p^e}$  denotes the restriction of  $a'_{p^e}$  to  $H_N$ . If  $\text{Sing}(\mathcal{G}_N) \cap H_N$  would be a component of codimension 2, then this restricted equation would have order at least  $p^e$  at any point where it vanishes. In this case, the restricted equation should be a  $p^e$ th power of an equation, say

$$f_{p^e}^{(N)}(z_N) = z_N^{p^e} + \bar{a}'_{p^e} = (z_N + \bar{b}'_{p^e})^{p^e}. \tag{A.7.2}$$

We now show that this last condition cannot hold by analyzing the presentation  $p\mathcal{P}$  at  $\mathbf{x}$ . In fact in this case  $\text{Sl}(p\mathcal{P})(\mathbf{x}) = 1$ , and, by assumption

$$1 = \frac{\nu_{\mathbf{x}}(a_{p^e})}{p^e} < \text{ord}^{(d-1)}(\mathcal{R}_{\mathcal{G},\beta})(\mathbf{x}).$$

Finally, note that  $\bar{a}'_{p^e}$  can be identified with  $\text{In}_{\mathbf{x}}(a_{p^e})$ , and this is a homogeneous element of degree  $p^e$ , which is not a  $p^e$ th power as  $p\mathcal{P}$  is well-adapted at  $\mathbf{x}$ . So, a factorization as in (A.7.2) cannot hold, and hence, in this case also,  $\text{Sing}(\mathcal{G}_N) \cap H_N$  is not of codimension 2.

The previous discussion shows that  $\text{Sing}(\mathcal{G}_N) \cap H_N$  has codimension 2 if and only if it is defined by  $\langle z_N, t \rangle$ , and in particular it is a smooth center. If this is the case, we look for further transformations constructed by blowing up at such centers, as we explain below.

Firstly consider the monoidal transformation of  $V_N^{(d+1)}$  with center  $\langle z_N, t \rangle$ . Set  $z_{N+1} = z_N/t$ . At the  $U_t$ -chart, the transform of  $p\mathcal{P}_N$ , say  $p\mathcal{P}_{N+1}$ , is defined by

$$f_{p^e}^{(N+1)}(z_{N+1}) = z_{N+1}^{p^e} + t^{N(r_1-1)-1} a'_1 z_{N+1}^{p^e-1} + \dots + t^{N(r_{p^e}-p^e)-p^e} a'_{p^e} \quad \text{and} \quad (\mathcal{R}_{\mathcal{G},\beta})_{N+1},$$

with  $\text{ord}((\mathcal{R}_{\mathcal{G},\beta})_{N+1})(\xi_{H_{N+1}}) = (N(\alpha - s) - s)/s$ .

Now  $\text{Sing}(\mathcal{G}_{N+1}) \cap H_{N+1}$  has codimension 2 in  $V_{N+1}^{(d+1)}$  if and only if it is described by  $\langle z_{N+1}, t \rangle$ , which is a smooth center.

Now consider, if possible, a sequence of  $\ell$  monoidal transformations at centers of codimension 2 of the form  $\langle z_{N+i}, t \rangle$ . It gives rise to a sequence

$$\mathcal{G}_N \quad \mathcal{G}_{N+1} \quad \mathcal{G}_{N+\ell} \\ V_N^{(d+1)} \xleftarrow{\pi_N} V_{N+1}^{(d+1)} \xleftarrow{\pi_{N+1}} \dots \xleftarrow{\pi_{N+\ell-1}} V_{N+\ell}^{(d+1)}. \tag{A.7.3}$$

Geometrically, this sequence is constructed as follows: let  $H_{N+i-1}$  denote the exceptional hypersurface introduced by  $\pi_{N+i-1}$ . The center of the  $(N+i)$ th monoidal transformation, say  $\pi_{N+i}$ , is constructed by blowing up at  $H_{N+i} \cap \text{Sing}(\mathcal{G}_{N+i})$ , which is assumed to be of codimension 2. The sequence (A.7.3) induces a sequence

$$\begin{array}{ccccccc} V_N^{(d)} & \longleftarrow & V_{N+1}^{(d)} & \longleftarrow & \dots & \longleftarrow & V_{N+\ell}^{(d)} \\ (\mathcal{R}_{\mathcal{G},\beta})_N & & (\mathcal{R}_{\mathcal{G},\beta})_{N+1} & & & & (\mathcal{R}_{\mathcal{G},\beta})_{N+\ell} \end{array}$$

where each transformation is the blow-up at the exceptional hypersurface  $H_{N+i} \subset V_{N+i}^{(d)}$ . Hence each transformation is the identity map.

After the  $N + \ell$  monoidal transformations, the exponent of  $t$  in the  $j$ th coefficient of  $f_{p^e}^{N+\ell}$  is  $N(r_j - j) - \ell j$ , and  $\text{ord}((\mathcal{R}_{\mathcal{G},\beta})_{N+\ell})(\xi_{H_{N+\ell}}) = N(\alpha - s)/s - \ell$ . Therefore,  $\langle z_{N+\ell}, t \rangle$  is a permissible center if  $N(r_j - j) - \ell j \geq j$  ( $j = 1, \dots, p^e$ ) and  $N(\alpha - s) - \ell s \geq s$ . In particular, this requires that

$$\ell \leq \min_{1 \leq j \leq n} \left\{ N \left( \frac{r_j}{j} - 1 \right) - 1, N \left( \frac{\alpha}{s} - 1 \right) - 1 \right\} = N(\text{Sl}(p\mathcal{P})(\mathbf{x}) - 1) - 1.$$

Set

$$\ell_N = \lfloor N(q - 1) - 1 \rfloor,$$

where the lower-bracket stands for the biggest integer  $\leq N(q - 1) - 1$ .

We claim that this is the highest length of a sequence as in (A.7.3), namely, that  $H_{N+\ell_N}$  is not a component of  $\tilde{\beta}_{N+\ell_N}(\text{Sing}(\mathcal{G}_{N+\ell_N}))$  in  $V_{N+\ell_N}^{(d)}$ .

One can check that  $\text{Sl}(p\mathcal{P}_{N+\ell_N})(\xi_{H_{N+\ell_N}}) < 1$ , where  $\xi_{H_{N+\ell_N}}$  is the generic point of  $H_{N+\ell_N}$ . We now show that  $p\mathcal{P}_{N+\ell_N}$  is well-adapted at  $\xi_{H_{N+\ell_N}}$ . This will be proved now in three steps, and this, together with Proposition A.6, will ensure that the previous claim holds.

Firstly, suppose that  $q = \text{Sl}(p\mathcal{P})(\mathbf{x}) = \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(\mathbf{x}) = \alpha/s$ . In this case,  $N(\alpha - s) - \ell_N \cdot s \leq N(r_{p^e} - p^e) - \ell_N \cdot p^e$ . So,  $\text{Sl}(p\mathcal{P}_{N+\ell_N})(\xi_{H_{N+\ell_N}}) = \text{ord}((\mathcal{R}_{\mathcal{G},\beta})_{N+\ell_N})(\xi_{H_{N+\ell_N}})$  and hence  $p\mathcal{P}_{N+\ell_N}$  is well-adapted to  $\xi_{H_{N+\ell_N}}$ .

Now assume that  $q = \text{Sl}(p\mathcal{P})(\mathbf{x}) = \nu_{\mathbf{x}}(a_{p^e})/p^e = r_{p^e}/p^e < \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(\mathbf{x})$  and that  $Nr_{p^e}/p^e \notin \mathbb{Z}$ . Then,  $N(r_{p^e} - p^e) - \ell_N \cdot p^e \leq N(\alpha - s) - \ell_N \cdot s$ , so  $\text{Sl}(p\mathcal{P}_{N+\ell_N})(\xi_{H_{N+\ell_N}}) = \nu_{\xi_{H_{N+\ell_N}}}(a_{p^e})/p^e$  and  $0 < \text{Sl}(p\mathcal{P}_{N+\ell_N})(\xi_{H_{N+\ell_N}}) < 1$ , so  $p\mathcal{P}_{N+\ell_N}$  is well-adapted to  $\xi_{H_{N+\ell_N}}$ .

Finally, assume that  $q = \text{Sl}(p\mathcal{P})(\mathbf{x}) = \nu_{\mathbf{x}}(a_{p^e})/p^e = r_{p^e}/p^e < \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(\mathbf{x})$  and that  $Nr_{p^e}/p^e \in \mathbb{Z}$ . Note that  $\text{Sl}(p\mathcal{P}_{N+\ell})(\xi_{H_{N+\ell}}) = \nu_{\xi_{H_{N+\ell}}}(a'_{p^e})/p^e = 0 < \text{ord}^{(d-1)}((\mathcal{R}_{\mathcal{G},\beta})_{N+\ell})(\xi_{H_{N+\ell}})$ , and that  $\text{In}_{\xi_{H_{N+\ell_N}}}(a'_{p^e})$  can be naturally identified with  $\text{In}_{\mathbf{x}}(a_{p^e})$ , which is not a  $p^e$ th power (as  $p\mathcal{P}$  is well-adapted at  $\mathbf{x}$ ). So in this last case also,  $p\mathcal{P}_{N+\ell_N}$  is well-adapted to  $\xi_{H_{N+\ell_N}}$ . This proves the previous claim.

The previous discussion already proves the theorem, as it shows that the rational number  $q = \text{Sl}(p\mathcal{P})(\mathbf{x})$  is completely characterized by the weak equivalence class of  $\mathcal{G}$ . To this end, note that

$$\lim_{N \rightarrow \infty} \frac{\ell_N}{N} = q - 1. \quad \square$$

Further consequences of the previous discussion are the following.

**COROLLARY A.8.** *Let  $\mathcal{G}$  be a Rees algebra. Fix a  $p$ -presentation  $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p^e}(z))$  which is well-adapted to  $\mathcal{G}$  at  $x \in \text{Sing}(\mathcal{G})$ . Then,*

$$\text{H-ord}^{(d-1)}(\mathcal{G})(x) = \text{Sl}(p\mathcal{P})(\beta(x)).$$

COROLLARY A.9. Let  $\mathcal{G}$  be a Rees algebra. Fix two transversal projections  $V^{(d)} \xrightarrow{\beta} V^{(d-1)}$  and  $V^{(d)} \xrightarrow{\beta'} V^{(d-1)}$ . For any  $x \in \text{Sing}(\mathcal{G})$

$$\beta - \text{ord}(\mathcal{G})(\beta(x)) = \beta' - \text{ord}(\mathcal{G})(\beta'(x)) \quad (= \text{H-ord}^{(d-1)}(\mathcal{G})(x)).$$

**Appendix B. The tight monomial algebra and the proof of Main Theorem 2**

**B.1 Proof of Main Theorem 2**

Here, we address the proof of Main Theorem 2 (Theorem 7.6).

THEOREM B.2 (Main Theorem 2). Fix a sequence of permissible transformations as in (3.1.1). Let  $\mathcal{M}_r W^s$  denote the tight monomial algebra defined in Definition 7.5. Then, at any closed point  $x \in \text{Sing}(\mathcal{G}_r)$ ,  $\mathcal{M}_r W^s$  has monomial contact with  $\mathcal{G}_r$ , i.e., there is a  $\beta_r$ -transversal section  $z$  vanishing at  $x$  for which

$$\mathcal{G}_r \subset \langle z \rangle W \odot \mathcal{M}_r W^s.$$

*Proof.* We argue by induction on the length of the sequence of transformations. Assume by induction in  $r$  that, locally at any closed point  $x \in \text{Sing}(\mathcal{G}_r)$  the algebra  $\mathcal{M}_r W^s$  has monomial contact with  $\mathcal{G}_r$ , i.e., for some  $\beta_r$ -transversal section  $z'$  vanishing at  $x$ ,

$$\mathcal{G}_r \subset \langle z' \rangle W \odot \mathcal{M}_r W^s.$$

The condition is vacuous for  $r = 0$ .

Let  $C$  be a permissible center, and consider the monoidal transformation at  $C$ , say  $V_r^{(d)} \xleftarrow{\pi_C} V_{r+1}^{(d)}$ . The task is to prove that  $\mathcal{G}_{r+1}$  has monomial contact with the new tight monomial algebra, say

$$\mathcal{M}_{r+1} W^s = \mathcal{O}_{V^{(d-1)}} [I(H_1)^{h_1} \dots I(H_r)^{h_r} I(H_{r+1})^{h_{r+1}} W^s]. \tag{B.2.1}$$

Fix a  $p$ -presentation  $p\mathcal{P}'_r = p\mathcal{P}'_r(\beta_r, z', f'_{p^e})$  involving  $z'$  at  $V_r^{(d)}$ . Note that the hypothesis ensures that  $p\mathcal{P}'_r$  is compatible with  $\mathcal{M}_r W^s$ . Proposition 5.8(B) applies here to show that  $p\mathcal{P}'_r$  can be modified into a new  $p$ -presentation, say  $p\mathcal{P}_r$ , (doing a change of variables of the form  $z = z' + \alpha$  with  $\alpha \in \mathcal{O}_{V_r^{(d-1)}}$ ), so that  $p\mathcal{P}_r$  is compatible with  $\mathcal{M}_r W^s$  and also well-adapted to  $\mathcal{G}_r$  both at  $x$  and  $\xi_{\beta(C)}$ , the generic point of  $\beta(C)$ .

We claim that, locally at any closed point  $x' \in \text{Sing}(\mathcal{G}_{r+1})$  mapping to  $x$ , there is a  $p$ -presentation with the properties:

- it is compatible with the strict transform of the monomial algebra  $\mathcal{M}_r W^s$ ;
- it is well-adapted to  $\mathcal{G}_{r+1}$  at  $\xi_{H_{r+1}^{(d)}}$ .

That is, locally at any closed point  $x' \in \text{Sing}(\mathcal{G}_{r+1})$ , there is a  $p$ -presentation which is well-adapted simultaneously to every  $\xi_{H_i^{(d-1)}}$  ( $i = 1, \dots, r + 1$ ). This, in particular, ensures our task.

If  $x' \notin H_{r+1}^{(d)}$ , then Remark 6.4 shows that there is an identification between the  $p$ -presentations  $p\mathcal{P}_r$  of  $\mathcal{G}_r$  and  $p\mathcal{P}_{r+1}$  of  $\mathcal{G}_{r+1}$  (in an open subset). Thus the claim follows in a straightforward manner in this case.

Suppose that  $x' \in \text{Sing}(\mathcal{G}_{r+1}) \cap H_{r+1}^{(d)}$ .

Firstly, we address the claim under the assumption that  $\text{In}_x(f_{p^e}) = Z^{p^e}$ . In this case,  $\pi^{-1}(x) \cap \text{Sing}(\mathcal{G}_{r+1}) \subset \{z_1 = 0\}$ , where  $z_1$  denotes the strict transform of  $z$  (see Remark 6.5). Moreover,  $p\mathcal{P}_{r+1}$  is well-adapted to  $\mathcal{G}_{r+1}$  at  $\xi_{H_{r+1}^{(d)}}$  (see Proposition 6.6). Let

$$f_{p^e}^{(1)}(z_1) = z_1^{p^e} + a_1^{(1)} z_1^{p^e-1} + \dots + a_{p^e}^{(1)}$$

be the strict transform of  $f_{p^e}(z) = z^{p^e} + a_1 z^{p^e-1} + \dots + a_{p^e}$ . Since  $a_i W^i \in \mathcal{M}_r W^s$ , it follows that  $a_i^{(1)} W^i \in \mathcal{M}' W^s$  for  $i = 1, \dots, p^e$ . Here  $\mathcal{M}' W^s$  denotes the strict transform of  $\mathcal{M}_r W^s$ . On the other hand,  $a_i^{(1)} W^i \in I(H_{r+1}^{(d)})^{h_{r+1}} W^s$ , since  $p\mathcal{P}_{r+1}$  is well-adapted at  $\xi_{H_{r+1}^{(d)}}$  (recall that  $q_{H_{r+1}^{(d)}} = h_{r+1}/s$ ). Thus  $a_i^{(1)} W^i \in \mathcal{M}_{r+1} W^s$  (the new tight monomial algebra).

The same arguments apply here to show that  $\mathcal{M}_{r+1} W^s \subset (\mathcal{R}_{\mathcal{G},\beta})_{r+1}$ . Then,  $p\mathcal{P}_{r+1}$  is compatible with  $\mathcal{M}' W^s$  and well-adapted to  $\mathcal{G}_{r+1}$  at  $\xi_{H_{r+1}^{(d-1)}}$ . Therefore,  $p\mathcal{P}_{r+1}$  is compatible with  $\mathcal{M}_{r+1} W^s$ . Hence,

$$\mathcal{G}_{r+1} \subset \langle z_1 \rangle W \odot \mathcal{M}_{r+1} W^s$$

in case  $\text{In}_x(f_{p^e}) = Z^{p^e}$ .

Now, assuming that  $\text{In}_x(f_{p^e}) \neq Z^{p^e}$ , two different cases can occur.

- Suppose firstly that  $\text{In}_x(f_{p^e}) = Z^{p^e} + A_{p^e}$  where  $A_{p^e}$  is not a  $p^e$ th power and free of the variable  $Z$ . In this case,  $\text{Sl}(p\mathcal{P}_r)(\beta_r(x)) = 1$  and then, also,  $\text{Sl}(p\mathcal{P}_r)(\xi_{\beta_r(C)}) = 1$  (see Remark 4.5). This would ensure that  $h_{r+1} = 0$  in (B.2.1).

Let  $x' \in \text{Sing}(\mathcal{G}_{r+1}) \cap H_{r+1}^{(d)}$  be a closed point such that  $\pi_C(x') = x$ . Assume that  $\beta_{r+1}(x') \in V(\mathcal{M}'_r)(\subset V_{r+1}^{(d-1)})$ , where  $\mathcal{M}'_r W^s$  denotes the strict transform of  $\mathcal{M}_r W^s$  in  $V_{r+1}^{(d-1)}$ . One can check that  $x' \in \{z_1 = 0\}$  (the strict transform of  $z$ ) as all coefficients  $a_i^{(1)}$  vanish at  $\beta_{r+1}(x')$  for  $i = 1, \dots, p^e$ . The same argument as used before shows that  $p\mathcal{P}_{r+1}$  is compatible with  $\mathcal{M}_{r+1} W^s$ .

Now assume that  $\beta_{r+1}(x') \notin V(\mathcal{M}'_r)$ . Locally at  $\beta_{r+1}(x')$  the monomial algebra  $\mathcal{M}_{r+1} W^s$  has integral closures  $\mathcal{O}_{V_{r+1}^{(d-1)}}[W]$  (or say,  $\mathcal{M}_{r+1} = 1$ ), and there is nothing to prove in this case.

- Finally, suppose that  $\text{In}_x(f_{p^e}(z)) = Z^{p^e} + A_j Z^{p^e-j} + \dots$  with  $A_j \neq 0$  and  $j < p^e$ . In this case,  $\text{ord}(\mathcal{R}_{\mathcal{G},\beta})(\xi_{\beta(C)}) = 1$  and hence  $h_{r+1} = 0$  in (B.2.1) (see Theorem 4.4).

Similar arguments as those used before apply here to show the compatibility of the strict transform with the monomial algebra: whenever the point  $x' \in V(\mathcal{M}'_r)$ ,  $x' \in \{z_1 = 0\}$ . If not, the monomial algebra is locally of the form  $\mathcal{O}_{V_{r+1}^{(d-1)}}[W]$ . This concludes the proof.  $\square$

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