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Measures with Fourier Transforms in L^2 of a Half-space

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Abstract. We prove that if the Fourier transform of a compactly supported measure is in L^2 of a half-space, then the measure is absolutely continuous to Lebesgue measure. We then show how this result can be used to translate information about the dimensionality of a measure and the decay of its Fourier transform into geometric information about its support.

1 Introduction

Let $M(\mathbb{R}^n)$ be the space of all complex Borel measures on $\mathbb{R}^n$. The Fourier transform of a measure $\mu \in M(\mathbb{R}^n)$ is defined for $\xi \in \mathbb{R}^n$ by

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} d\mu(x).$$

Our starting point is the following well-known result.

Theorem 1.1 *Suppose $\mu \in M(\mathbb{R}^n)$. If $\widehat{\mu} \in L^2(\mathbb{R}^n)$, then μ is absolutely continuous to Lebesgue measure (in symbols, $d\mu \ll dx$.)*

The first goal of this paper is to prove the following stronger version of Theorem 1.1.

Theorem 1.2 (i) *Suppose $\mu \in M(\mathbb{R}^n)$ and u is a unit vector in $\mathbb{R}^n$. If μ is compactly supported and $\widehat{\mu} \in L^2(\{\xi \cdot u \leq 0\})$, then $d\mu \ll dx$.*
(ii) *Suppose $\mu \in M(\mathbb{R})$. If $\widehat{\mu} \in L^2((-\infty, 0])$, then $d\mu \ll dx$.*

The simple proof of the fact that Theorem 1.2 implies Theorem 1.1 will be given in Section 2.

The second goal of this paper is to use this result to show that if $\mu \in M(\mathbb{R}^n)$ is a positive compactly supported measure such that μ satisfies an appropriate dimensionality condition and $\widehat{\mu}$ satisfies an appropriate decay condition, and if $K \subset \mathbb{R}^n$ is a symmetric convex body such that $\operatorname{bd} K$ is smooth and has a nowhere vanishing Gaussian curvature, then the set

$$\{R \in \mathbb{R} : (\operatorname{bd} RK) \cap (\operatorname{supp} \mu) \neq \varnothing\}$$

has a positive one-dimensional Lebesgue measure. This application of Theorem 1.2 is closely related to Mattila's work [6] on Falconer's distance set conjecture.

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The proof of Theorem 1.2 relies on two papers of Frank Forelli; it uses the main result of [2] and was inspired by the argument of [3]. Both papers generalize the F. and M. Riesz theorem, but in different directions.

The F. and M. Riesz theorem *Suppose $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\lambda \in M(\mathbb{T})$, and $\|\lambda\| > 0$. If $\widehat{\lambda}(n) = 0$ for all $n < 0$, then $d\lambda$ and dx have the same null sets:*

$$\int_E d|\lambda| = 0 \Leftrightarrow \int_E dx = 0.$$

The result of the above theorem remains true when the circle $\mathbb{T}$ is replaced by the real line $\mathbb{R}$. Even more generally,[1] we can replace $\mathbb{T}$ by $\mathbb{R}^n$.

Theorem 1.3 (Forelli [2]) *Suppose $\lambda \in M(\mathbb{R}^n)$ and u is a unit vector in $\mathbb{R}^n$. If $\widehat{\lambda}(\xi) = 0$ for all ξ in the half-space $\{\xi \cdot u \le 0\}$, then λ is quasi-invariant in the direction of u:*

$$\int_E d|\lambda| = 0 \Leftrightarrow \int_{E+tu} d|\lambda| = 0 \forall t \in \mathbb{R}.$$

Notice that if $\lambda \in M(\mathbb{R})$ is quasi-invariant and $\|\lambda\| > 0$, then $d|\lambda|$ and dx are mutually absolutely continuous, *i.e.*, $d|\lambda|$ and dx have the same null sets. This follows immediately from the observation that

$$\begin{aligned}\int_{-\infty}^{\infty} |\lambda|(E+x)dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_{E+x}(t)d|\lambda|(t)dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_{-E+t}(x)dxd|\lambda|(t) \\ &= \int_{-\infty}^{\infty} |-E+t|d|\lambda|(t) \\ &= |E|\|\lambda\|\end{aligned}$$

for all Borel sets $E \subset \mathbb{R}$, where $|E|$ denotes the Lebesgue measure of E.

2 The Support Condition on μ

The support condition on μ in Theorem 1.2(i) can be relaxed to assuming that μ is compactly supported only in the direction of u. This will be easily seen from the proof of Theorem 1.2. But the support condition on μ cannot be completely removed in dimensions $n \ge 2$. Let $f \in H^1(\mathbb{R})$ be a nonzero function and define the measure $\mu \in M(\mathbb{R}^2)$ by

$$\int_{\mathbb{R}^2} \phi d\mu = \int_{\mathbb{R}} \phi(x,0) f(x)dx \qquad (\phi \in C_0(\mathbb{R}^2)).$$

[1] In [2], Theorem 1.3 is stated and proved in the more general setting of locally compact Hausdorff spaces on which $\mathbb{R}$ acts as a topological transformation group.

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Then $\widehat{\mu}$ vanishes in the left half-plane while μ is singular to the two-dimensional Lebesgue measure.

In spite of the support condition on the measure, Theorem 1.2 implies Theorem 1.1. This can be seen as follows.

Let $\mu \in M(\mathbb{R}^n)$ be such that $\widehat{\mu} \in L^2(\mathbb{R}^n)$. We need to show that if $E \subset \mathbb{R}^n$ has Lebesgue measure zero, then $\mu(E) = 0$. Let B_k be a sequence of concentric balls in $\mathbb{R}^n$ such that the radius of B_k is k. Let $\{\phi_k\}$ be a sequence in $C_0^\infty(\mathbb{R}^n)$ such that $\phi_k = 1$ on B_k, and let $d\mu_k = \phi_k d\mu$. Then by Young's inequality,

$$\widehat{\mu_k} = \widehat{\phi_k} * \widehat{\mu} \in L^2(\mathbb{R}^n)$$

for all k. So by Theorem 1.2, $d\mu_k \ll dx$ for all k. Since $\mu_k(E) \to \mu(E)$ as $k \to \infty$ for all Borel sets $E \subset \mathbb{R}^n$, it follows that $d\mu \ll dx$.

3 Proof of Theorem 1.2

By Plancherel's theorem, there is a function $f \in L^2(\mathbb{R}^n)$ such that

$$\widehat{f}(\xi) = \begin{cases} \widehat{\mu}(\xi) & \text{if } \xi \cdot u \leq 0, \\ 0 & \text{if } \xi \cdot u > 0. \end{cases}$$

Let ϕ be a C_0^∞ function on $\mathbb{R}^n$ such that $\widehat{\phi} \geq 1$ on $\operatorname{supp} \mu$. Clearly, $\widehat{\phi} d\mu \in M(\mathbb{R}^n)$. Also, by the Cauchy–Schwarz inequality, $\widehat{\phi} f dx \in M(\mathbb{R}^n)$. So, if we define the measure λ by

$$d\lambda = \widehat{\phi} d\mu - \widehat{\phi} f dx, \tag{3.1}$$

then $\lambda \in M(\mathbb{R}^n)$ and $\widehat{\lambda} = 0$ on a half-space of the form $\{\xi \cdot u < a\}$ for some $a < 0$. Applying Theorem 1.3, we conclude that λ is quasi-invariant in the direction of u.

Suppose $E \subset \mathbb{R}^n$ is a Borel set of Lebesgue measure zero. Let $E' = E \cap (\operatorname{supp} \mu)$ and choose a real number t such that $(E' + tu) \cap (\operatorname{supp} \mu) = \varnothing$. Then $|\mu|(E' + tu) = 0$ and $|E' + tu| = |E'| = 0$, and hence $|\lambda|(E' + tu) = 0$. Since λ is quasi-invariant in the direction of u, it follows that $|\lambda|(E') = 0$. Since $\widehat{\phi} \geq 1$ on $\operatorname{supp} \mu$, it follows by (3.1) that

$$\int_{E'} d|\mu| \leq \int_{E'} \widehat{\phi} d|\mu| \leq \int_{E'} d|\lambda| + \int_{E'} |\widehat{\phi}(x) f(x)| dx = 0.$$

Thus $|\mu|(E) = |\mu|(E') = 0$. This proves part (i).

To prove part (ii), we let f be as above (with $u = 1$) and choose a C_0^∞ function ϕ on $\mathbb{R}$ such that $\widehat{\phi}(\xi) \geq 1$ for $|\xi| \leq 1$. For $R > 0$, we then define the function $\phi_R \in C_0^\infty(\mathbb{R})$ by $\phi_R(x) = R\phi(Rx)$ and the measure $\lambda_R \in M(\mathbb{R})$ by

$$d\lambda_R = \widehat{\phi_R} d\mu - \widehat{\phi_R} f dx. \tag{3.2}$$

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Then $\widehat{\lambda_R} = 0$ on an interval of the form $(-\infty, a]$ for some $a < 0$, and $\widehat{\phi_R}(\xi) \geq 1$ for all $|\xi| \leq R$. So applying Theorem 1.3, and recalling that quasi-invariant measures on $\mathbb{R}$ are absolutely continuous to Lebesgue measure, we conclude that $d\lambda_R \ll dx$.

Suppose $E \subset \mathbb{R}$ is a Borel set with $|E| = 0$. Let $E_R = E \cap (-R, R)$. Then $|E_R| = 0$, and so $|\lambda_R|(E_R) = 0$. Since $\widehat{\phi_R} \geq 1$ on $(-R, R)$, it follows by (3.2) that

$$\int_{E_R} d|\mu| \leq \int_{E_R} \widehat{\phi_R} d|\mu| \leq \int_{E_R} d|\lambda_R| + \int_{E_R} |\widehat{\phi_R}(x) f(x)| dx = 0.$$

Thus $|\mu|(E_R) = 0$. Letting $R \to \infty$, we get $|\mu|(E) = 0$, as desired.

4 The Application

A convex body in $\mathbb{R}^n$ is a compact convex subset of $\mathbb{R}^n$ with nonempty interior. A convex body K is symmetric if $K = -K$. The polar body K^* of a symmetric convex body K is defined as

$$K^* = \{x \in \mathbb{R}^n : |x \cdot y| \leq 1 \text{ for all } y \in K\},$$

and the support function of K is defined as

$$h_K(x) = \sup\{x \cdot y : y \in K\} \qquad (x \in \mathbb{R}^n).$$

It is easy to see that

$$h_K(x) = \frac{1}{\sup\{R \geq 0 : Rx \in K^*\}} \qquad (x \in \mathbb{R}^n).$$

It is also easy to see that

$$x \in \operatorname{bd} RK^* \Leftrightarrow h_K(x) = R. \tag{4.1}$$

Let $n \geq 2$ and fix a symmetric convex body K in $\mathbb{R}^n$ such that $\operatorname{bd} K$ is smooth and has a nowhere vanishing Gaussian curvature. Let σ be surface measure on $\operatorname{bd} K$ and define the measure σ_R on $\operatorname{bd} RK$ by

$$\int_{\operatorname{bd} RK} f d\sigma_R = \int_{\operatorname{bd} K} f(R\theta) R^{n-1} d\sigma(\theta) \qquad (f \in C(\operatorname{bd} RK)).$$

If $\theta \in \mathbb{S}^{n-1}$ and $x \in \operatorname{bd} K$ are such that θ is the outer unit normal vector to $\operatorname{bd} K$ at x, we define $\kappa(\theta)$ to be the Gaussian curvature of $\operatorname{bd} K$ at x. Then a result of Herz [4] says that

$$\widehat{\sigma_R}(\xi) = \frac{2R^{n-1}}{(R|\xi|)^{(n-1)/2}} \kappa(\xi/|\xi|)^{-1/2} \cos\Big(2\pi\Big(h_K(R\xi) - \frac{n-1}{8}\Big)\Big) + R^{n-1} O\Big(\frac{1}{(R|\xi|)^{(n+1)/2}}\Big). \tag{4.2}$$

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For positive compactly supported $\mu \in M(\mathbb{R}^n)$, define

$$\Delta_K(\mu) = \{h_K(x) : x \in \operatorname{supp}\mu\}.$$

Then by (4.1),

$$\Delta_K(\mu) = \{R \in \mathbb{R} : (\operatorname{bd} RK^*) \cap (\operatorname{supp}\mu) \neq \varnothing\}.$$

The aim of this section is to show how Theorem 1.2 can be used to translate information about the dimensionality of μ and the decay of $\widehat{\mu}$ into information about the Lebesgue measure of $\Delta_K(\mu)$.

Throughout this section, the ball in $\mathbb{R}^n$ of center x and radius r will be denoted by $B(x, r)$, the Lebesgue measure of a Borel set $E \subset \mathbb{R}^n$ will be denoted by $|E|$, and the notation $A \approx B$ will mean that $C^{-1}A \leq B \leq CA$ for some appropriate positive constant C. For example, $|B(x, r)| \approx r^n$.

Let $\mu \in M(\mathbb{R}^n)$ be a positive compactly supported measure with $\|\mu\| > 0$. Suppose

$$\int_1^\infty \int_{\operatorname{bd} RK} |\widehat{\mu}|^2 d\sigma_R dR < \infty.$$

Then, since $\operatorname{bd} K$ has a nowhere vanishing Gaussian curvature,

$$\|\widehat{\mu}\|^2_{L^2(\mathbb{R}^n)} \approx \int_0^\infty \int_{\operatorname{bd} RK} |\widehat{\mu}|^2 d\sigma_R dR < \infty,$$

and it follows by Theorem 1.1 that $d\mu \ll dx$. Since $\|\mu\| > 0$, it follows that $|\operatorname{supp}\mu| > 0$, and hence that $|\Delta_K(\mu)| > 0$.

Now by the Cauchy–Schwarz inequality,

$$\int_1^\infty \Big| \int_{\operatorname{bd} RK} \widehat{\mu} d\sigma_R \Big|^2 \frac{dR}{R^{n-1}} \leq \|\sigma\| \int_1^\infty \int_{\operatorname{bd} RK} |\widehat{\mu}|^2 d\sigma_R dR,$$

so it is natural to ask if $|\Delta_K(\mu)| > 0$ will continue to be true under the weaker assumption

$$\int_1^\infty \Big| \int_{\operatorname{bd} RK} \widehat{\mu} d\sigma_R \Big|^2 \frac{dR}{R^{n-1}} < \infty.$$

It turns out that to answer this question in the affirmative one needs to also assume that

$$\int_{\mathbb{R}^n} \frac{d\mu(x)}{|x|^\alpha} < \infty$$

for some $\alpha > n/2$, and use Theorem 1.2 instead of Theorem 1.1. Notice that if μ satisfies the dimensionality condition

$$\mu(B(0, r)) \leq Cr^\beta \qquad (r > 0)$$

for some positive constants β and C, then

$$\int_{\mathbb{R}^n} \frac{d\mu(x)}{|x|^\alpha} < \infty$$

for all $0 < \alpha < \beta$.

Theorem 4.1 *Suppose $\mu \in M(\mathbb{R}^n)$ is a positive compactly supported measure such that*

$$0 < \int_{\mathbb{R}^n} \frac{d\mu(x)}{|x|^\alpha} < \infty \tag{4.3}$$

for some $\alpha > n/2$, and

$$\int_1^\infty \Big| \int_{\mathrm{bd}\, RK} \widehat{\mu} d\sigma_R \Big|^2 \frac{dR}{R^{n-1}} < \infty. \tag{4.4}$$

Then $|\Delta_K(\mu)| > 0$.

Proof By (4.2),

$$\widehat{\sigma_R}(\xi) = \frac{2R^{n-1}}{(R|\xi|)^{(n-1)/2}} \kappa(\xi/|\xi|)^{-1/2} \cos\Big(2\pi\Big(h_K(R\xi) - \frac{n-1}{8}\Big)\Big) + R^{n-1}E(\xi)$$

with

$$|E(\xi)| \le \begin{cases} \frac{C}{(R|\xi|)^{(n+1)/2}} & \text{if } R|\xi| \ge 1, \\ \frac{C}{(R|\xi|)^{(n-1)/2}} & \text{if } R|\xi| \le 1. \end{cases}$$

Thus,

$$\begin{aligned}
&\int_{\mathrm{bd}\, RK} \widehat{\mu} d\sigma_R \\
&\quad = \int_{\mathbb{R}^n} \widehat{\sigma_R}(\xi) d\mu(\xi) \\
&\quad = \int_{\mathbb{R}^n} \frac{2R^{n-1}}{(R|\xi|)^{(n-1)/2}} \kappa(\xi/|\xi|)^{-1/2} \cos\Big(2\pi\Big(h_K(R\xi) - \frac{n-1}{8}\Big)\Big) d\mu(\xi) \\
&\qquad + \int_{\mathbb{R}^n} R^{n-1}E(\xi) d\mu(\xi) \\
&\quad = R^{(n-1)/2} \int_{\mathbb{R}^n} \frac{2}{|\xi|^{(n-1)/2}} \kappa(\xi/|\xi|)^{-1/2} \cos\Big(2\pi\Big(h_K(R\xi) - \frac{n-1}{8}\Big)\Big) d\mu(\xi) \\
&\qquad + R^{n-1}E_1(R)
\end{aligned}$$

with

$$\begin{aligned}
|E_1(R)| &\le \int_{|\xi| \ge 1/R} \frac{C}{(R|\xi|)^{(n+1)/2}} d\mu(\xi) + \int_{|\xi| \le 1/R} \frac{C}{(R|\xi|)^{(n-1)/2}} d\mu(\xi) \\
&\le \int_{\mathbb{R}^n} \frac{2C}{(R|\xi|)^\alpha} d\mu(\xi) \\
&= 2CR^{-\alpha} \int_{\mathbb{R}^n} \frac{d\mu(\xi)}{|\xi|^\alpha}
\end{aligned}$$

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for any $\alpha \in [(n-1)/2, (n+1)/2]$. Thus,

$$\frac{1}{R^{(n-1)/2}} \int_{\mathrm{bd}\,RK} \widehat{\mu} d\sigma_R = \int_{\mathbb{R}^n} \frac{2}{|\xi|^{(n-1)/2}} \kappa(\xi/|\xi|)^{-1/2} \cos\Big(2\pi\Big(h_K(R\xi) - \frac{n-1}{8}\Big)\Big)\, d\mu(\xi) + E_2(R)$$

with

$$|E_2(R)| \le 2CR^{(n-1)/2-\alpha} \int_{\mathbb{R}^n} \frac{d\mu(\xi)}{|\xi|^{\alpha}}$$

for any $\alpha \in [(n-1)/2, (n+1)/2]$.

Now define a positive measure $\nu_0 \in M(\mathbb{R})$ by[2]

$$\int_{\mathbb{R}} f d\nu_0 = \int_{\mathbb{R}^n} \frac{f(h_K(x))}{|x|^{(n-1)/2}} \kappa(x/|x|)^{-1/2} d\mu(x) \qquad (f \in C_0(\mathbb{R}))$$

(notice that $\int_{\mathbb{R}} d\nu_0 \approx \int_{\mathbb{R}^n} |x|^{-(n-1)/2} d\mu(x) < \infty$) and a complex measure $\nu \in M(\mathbb{R})$ by

$$d\nu(t) = e^{i\pi(n-1)/4} d\nu_0(t) + e^{-i\pi(n-1)/4} d\nu_0(-t).$$

Then

$$\begin{aligned}
\widehat{\nu}(s) &= \int_{\mathbb{R}} e^{-2\pi i s t} e^{i\pi(n-1)/4} d\nu_0(t) + \int_{\mathbb{R}} e^{-2\pi i s t} e^{-i\pi(n-1)/4} d\nu_0(-t) \\
&= \int_{\mathbb{R}} \Big(e^{-2\pi i (st-(n-1)/8)} + e^{2\pi i (st-(n-1)/8)} \Big)\, d\nu_0(t) \\
&= \int_{\mathbb{R}} 2\cos\Big(2\pi\Big(st - \frac{n-1}{8}\Big)\Big)\, d\nu_0(t) \\
&= \int_{\mathbb{R}^n} 2\cos\Big(2\pi\Big(sh_K(x) - \frac{n-1}{8}\Big)\Big) \frac{1}{|x|^{(n-1)/2}} \kappa(x/|x|)^{-1/2} d\mu(x).
\end{aligned}$$

Thus,

$$\frac{1}{R^{(n-1)/2}} \int_{\mathrm{bd}\,RK} \widehat{\mu} d\sigma_R = \widehat{\nu}(R) + E_2(R).$$

Since $\int_1^\infty (R^{(n-1)/2-\alpha})^2 dR < \infty$ whenever $\alpha > n/2$, it follows that $E_2 \in L^2([1,\infty))$ whenever there is an $\alpha \in (n/2, (n+1)/2]$ with $\int_{\mathbb{R}^n} |\xi|^{-\alpha} d\mu(\xi) < \infty$. Thus,

$$\frac{1}{R^{(n-1)/2}} \int_{\mathrm{bd}\,RK} \widehat{\mu} d\sigma_R - \widehat{\nu}(R) \in L^2([1,\infty))$$

whenever there is an $\alpha > n/2$ with $\int_{\mathbb{R}^n} |\xi|^{-\alpha} d\mu(\xi) < \infty$. Thus,

$$\widehat{\nu} \in L^2([1,\infty)) \Leftrightarrow \int_1^\infty \Big| \int_{\mathrm{bd}\,RK} \widehat{\mu} d\sigma_R \Big|^2 \frac{dR}{R^{n-1}} < \infty$$

[2]Since $\int_{\mathbb{R}^n} |x|^{-\alpha} d\mu(x) < \infty$, we have $\mu(\{0\}) = 0$, so one need not worry about $\kappa(x/|x|)$ when $x = 0$.

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whenever there is an $\alpha > n/2$ with $\int_{\mathbb{R}^n} |\xi|^{-\alpha} d\mu(\xi) < \infty$.

Thus, Theorem 1.2, (4.3), and (4.4) imply that ν is absolutely continuous to Lebesgue measure on $\mathbb{R}$. Since $\operatorname{supp} \nu \subset \Delta_K(\mu) \cup (-\Delta_K(\mu))$ and $\|\nu\| > 0$, this in turn implies that $|\Delta_K(\mu)| > 0$. ■

The argument used in the proof of Theorem 4.1 is very close to an argument used in [7] to prove the following theorem.

Theorem 4.2 (Mattila [6]) *Suppose $\lambda \in M(\mathbb{R}^n)$ is a positive compactly supported measure such that*

$$0 < \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{d\lambda(x) d\lambda(y)}{|x-y|^\alpha} < \infty$$

for some $\alpha > n/2$, and

$$\int_1^\infty \Big(\int_{R\mathbb{S}^{n-1}} |\widehat{\lambda}|^2 d\sigma_R \Big)^2 \frac{dR}{R^{n-1}} < \infty.$$

Then $|\{|x-y| : x, y \in \operatorname{supp} \lambda\}| > 0$.

Notice that if λ satisfies the dimensionality condition

$$\lambda(B(x,r)) \leq Cr^\beta \qquad (x \in \mathbb{R}^n, r > 0)$$

for some positive constants β and C, then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{d\lambda(x) d\lambda(y)}{|x-y|^\alpha} < \infty$$

for all $0 < \alpha < \beta$.

Theorem 4.1 extends Mattila's theorem to the convex setting. To see this, define the measure $\tilde{\lambda} \in M(\mathbb{R}^n)$ by $\tilde{\lambda}(E) = \lambda(-E)$ and apply Theorem 4.1 to the measure $\mu = \lambda * \tilde{\lambda}$. We refer the reader to [1, 5–7] for more information about Mattila's theorem and its important applications to Falconer's distance set conjecture.

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