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COEFFICIENT ESTIMATES OF SOME CLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. We are concerned with coefficient estimates, and other similar problems, of the typically real functions and of the functions with positive real part. Following the stream of ideas in [1], new results and generalizations of others given in [1], [2] and [3] are obtained.

1. Introduction. Let $\mathcal{P}$ be the class of all analytic functions in the unit circle $D=\{z:|z|<1\}$ of the form:

$$f(z)=1+\sum_{n=1}^{\infty} a_n z^n$$

with positive real part and let $\mathcal{T}$ be the class of all typically real functions in D, that is all functions of the form:

$$g(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

analytic in D and which are real for z real and for no other values of z.

In this paper we deal with coefficient-estimates and other similar problems concerning functions in the classes $\mathcal{P}$ and $\mathcal{T}$, and in the spirit of [1], [2], [3]. In general results obtained for the class $\mathcal{P}$ can be reformulated for the class $\mathcal{T}$ in view of the well known fact, (see [3]), that a function g belongs to $\mathcal{T}$ if the a_n are real and, $(g(z)\cdot(1-z^2))/z$ belongs to $\mathcal{P}$.

More precisely, Theorem 1 of this paper serves as a basic tool for results which are obtained later. The use of the Lemma in the proof of Theorem 1 indicates once more what in [3] is suggested, that is the usefullness of the Harmonic Analysis methods in the study of problems concerning coefficient estimates and others of similar nature. Theorem 1 applied to the functions of the class $\mathcal{P}$ gives new results which improve others well known (see Corollary 2, Theorem 4).

Also by reformulating Theorem 1 for functions in the class $\mathcal{T}$ we improve and/or generalize previous results, (see [1], [2], [3]).

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We first state a Lemma which is used in the proof of Theorem 1, and whose proof is omitted as obvious.

LEMMA. *Let* $f \in L^1(\mathbb{R})$ *and* $\hat{f}(t) = \int_{-\infty}^{+\infty} f(x)e^{ixt}\,dx$ *the Fourier transform of* f. *Then, if* $\operatorname{Re} f \geq 0$, *the inequality*

$$|\tilde{f}(t) + \hat{f}(t)| \leq 2 \operatorname{Re} \hat{f}(0)$$

holds for all $t \in \mathbb{R}$.

THEOREM. *Let* $f(z) \in \mathcal{P}$, $(z = re^{it})$.
For each ρ, $k \in N$ *we have:*

$$\left| \sum_{m=-\rho}^{\rho} (a_{k-m} + \bar{a}_{m-k}) \exp[i(k-m)\tau](\rho + 1 - |m|) \right|$$

$$\leq 2 \operatorname{Re}\left[\sum_{m=0}^{\rho} a_m \exp(im\tau) \cdot (\rho+1) - |m|) \right], \quad \text{for all } \tau \in \mathbb{R},$$

where $a_0 = 1$ *and* $a_\rho = 0$ *for* $\rho < 0$.

Proof. Put $\rho + 1 = 2\delta$. For r fixed, set

$$f_\delta(x) = [\sin^2(\delta x) \cdot f(r \exp(-ix))]/x^2.$$

It is clear that the function f_δ satisfies the hypothesis of the Lemma above. Set

$$q_\delta(t) = (\sin^2(\delta x)/x^2)\hat{} = (\pi/2)\sup(0, 2\delta - |t|).$$

We have

$$\hat{f}_\delta(t) = \sum_{n=0}^{\infty} a_n r^n q_\delta(t-n)$$

If we apply the above Lemma for $t = k$ we get:

$$\left| \sum_{m=-\rho}^{\rho} (a_{k-m} r^{k-m} + \bar{a}_{m-k} \cdot r^{m-k})(\rho + 1 - |m|) \right| \leq 2 \operatorname{Re} \sum_{m=0}^{\rho} a_m r^m (\rho + 1 - |m|)$$

since for $|t| \geq \rho + 1$ we have $q_\delta(t) = 0$. We now observe that for each real number the function

$$f_\tau(z) = f(z \cdot \exp(i\tau))$$

belongs to $\mathcal{P}$, and has coefficients

$$a_n \exp(in\tau), \qquad n = 0, 1, 2 \cdots$$

By applying the last inequality to f_τ and letting $r \to 1$ we get the desired result.

2. COROLLARY. *Let*

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$$

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be a function of the class $\mathcal{P}$. *Set* $\theta_q = -\text{Arg}\, a_q + \pi$ $(q = 1, 2, \ldots)$. *Then*

$$|a_q| \le 2 - (1/2) \sup_{k\in N} \left| \sum_{m=-1}^{1} (a_{(m-k)q} + \bar{a}_{(m-k)q}) \cdot \exp[i(k-m)\theta_q] \cdot (2-|m|) \right|.$$

Proof. It is known [4, p. 2] that for $B_n = a_{q\cdot n}$, where q is fixed, the function

$$g(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$$

is a member of $\mathcal{P}$.

If we apply Theorem 1 to g for $\rho = 1$ and $t = \theta_q$ we get the desired result.

3. COROLLARY. *Let* $f \in \mathcal{T}$. *Set* $S_n = 1 + a_2 + \cdots + a_n$ *for* $n \ge 1$ *and* $S_n = 0$ *for* $n < 1$, *and*

$$A_\rho^k = S_{k+\rho+1} - 2S_k - 2S_{k-1} + S_{k-\rho-1} + S_{-k+\rho+2} + S_{-k+\rho+1} + S_{-k+\rho}$$

Then for each ρ, $k \in N$ *we have*

$$|A_\rho^k| \le 2S_\rho + 2S_{\rho+1}$$

Proof. The function

$$g(z) = f(z) \cdot (1-z^2)/z = \sum_{n=0}^{\infty} (a_{n+1} - a_{n-1}) z^n$$

belongs to $\mathcal{P}$. Hence for $t = 0$ and $a_n = S_n - S_{n-1}$ Theorem 1 gives the desired result.

REMARKS. (a) Corollary 2 improves.the well known inequality $|a_q| \le 2$ which holds for functions in $\mathcal{P}$. Also if $a_1 = 2\eta$, where $\eta = \exp(i\theta_0)$, then from Corollary 2 we get by induction $a_n = 2\eta^n$, so that

$$f(z) = 1 + \sum_{n=1}^{\infty} 2\eta^n z^n = (1+\eta z)/(1-\eta z)$$

(see [3]).

(b) Corollary 3 improves the inequality $S_\rho + S_{\rho+1} \ge 0$, which holds for all functions in $\mathcal{T}$ [3].

We have noticed in Remark (b) above, that equation $a_1 = 2\eta$ determines uniquely the extreme function $(1+\eta z)/(1-\eta z)$. Theorem 4 below provides another extreme case.

4. THEOREM. *Let* $f \in \mathcal{P}$, *and suppose there is a number* $\eta = \exp(i\theta_0)$ *such that*

$$\text{Re}(3 + 2\bar{a}_1\eta + \alpha_2\eta^2) = 0$$

then

$$f(z) = (1 - z^2\eta^{-2} + icz\eta^{-1})/(1 + z^2\eta^{-2} + z\eta^{-1})$$

where

$$c=-i\cdot(a_1\eta+1)=i\cdot(a_2\eta^2+1)=\text{real},\quad (|c|\le 3^{1/2})$$

Conversely if f has the above form with $|c|\le 3^{1/2}$ *and* $|\eta|=1$ *then*

$$f\in\mathcal{T}\quad\text{and}\quad \operatorname{Re}(3+2a_1\eta+a_2\eta^2)=0$$

Proof. Let $f\in\mathcal{P}$ and suppose for the moment that $\theta_0=0$ so that

$$\operatorname{Re}(3+2a_1+a_2)=0$$

Then, for $t=0$ and $\rho=2$, we get from Theorem 1 for $k=1$, $k=2$ and $k\ge 3$ respectively

(1) $$a_3+2a_2+3a_1+2a_0+2\bar{a}_0+\bar{a}_1=0$$

(2) $$a_4+2a_3+3a_2+2a_1+a_0+\bar{a}_0=0$$

$$\vdots$$

(k) $$a_{k+1}+2a_{k+1}+3a_k+2a_{k-1}+a_{k-2}=0$$

For $k\ge 3$ subtracting relation (k) from $(k+1)$ we have

$$a_{k+3}+a_{k+2}+a_{k+1}=a_k+a_{k-1}+a_{k-2}$$

The last equality is equivalent to the following

$$a_{3n+3}+a_{3n+2}+a_{3n+1}=a_3+a_2+a_1$$
$$a_{3n+4}+a_{3n+3}+a_{3n+2}=a_4+a_3+a_2$$
$$a_{3n+5}+a_{3n+4}+a_{3n+3}=a_5+a_4+a_3$$
$$n=0,1,3,\ldots.$$

Subtracting the first of these equalities from the second and the second from the third we get

$$a_{3n+4}-a_{3n+1}=a_4-a_1$$
$$a_{3n+5}-a_{3n+2}=a_5-a_2$$

It follows that

$$a_{3n+1}=a_4+(n-1)\cdot(a_4-a_1)$$
$$a_{3n+2}=a_5+(n-1)\cdot(a_5-a_2)$$

Also due to the inequality $|a_n|\le 2$ we must have $a_4-a_1=a_5-a_2=0$. Hence

$$a_{3n+1}=a_4=a_1$$
$$a_{3n+2}=a_5=a_2$$
$$a_{3n+3}=a_3$$

Also, from (2) and (3), we get, since $a_0=1$,

$$a_5+a_4+a_3=a_2+a_1+2$$

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so that

$$a_{3n} = a_3 = 2$$

From (1) and (2) we get

$$6 + 2a_2 + 3a_1 + \bar{a}_1 = 0$$
$$3a_2 + 6 + 3a_1 = 0$$

so that

$$a_2 = \bar{a}_1, \qquad \text{Re } a_1 = \text{Re } a_2 = -1$$

Hence

$$a_1 = -1 + ic, \qquad a_2 = -1 - ic$$

with $|c| \le 3^{1/2}$, since $|a_1| = |a_2| \le 2$.

The function f can now be written as follows:

$$f(z) = 1 + 2 \cdot \sum_{n=1}^{\infty} z^{3n} + (-1 + ic) \cdot \sum_{n=0}^{\infty} z^{3n+1} + (-1 - ic) \cdot \sum_{n=0}^{\infty} z^{3n+2}$$
$$= (1 - z^2 + icz)/(1 + z^2 + z)$$

This form of f corresponds to the case $\theta_0 \neq 0$. If $\theta_0 \neq 0$ the theorem follows if we apply the last formula to the function $f(\eta z)$.

Conversely let

$$f(z) = (1 - z^2\eta^{-2} + icz)/(1 + z^2\eta^{-2} + z\eta^{-1})$$

then

$$f(\eta z) = (1 - z^2 + icz)/(1 + z^2 + z)$$

We have

$$a_1 = (-1 + ic)/\eta, \qquad a_2 = (-1 - ic)/\eta^2$$

We prove that

$$\text{Re } f(\eta z) > 0$$

Set

$$z = r(\cos\theta + i \sin\theta)$$

Then the inequality to prove is equivalent to

$$\cos\theta - c \sin\theta > -(1 + r^2)/r$$

Put $c = \tan\xi$. Since $|c| \le 3^{1/2}$, we have

$$-\pi/3 \le \xi \le \pi/3$$

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and

$$\cos(\theta+\xi) > -\cos(\xi)\cdot(1+r^2)/r$$

since

$$\cos\xi \geq 1/2 \quad \text{and} \quad (1+r^2)/r > 2.$$

The theorem is proved.

5. COROLLARY. *If $f \in \mathcal{T}$ and there is a number $\eta = \exp(i\theta_0)$, $(\theta_0 \in \mathbb{R})$ such that*

$$\mathrm{Re}[3+2a_2\eta+(a_3-1)\eta^2]=0$$

Then f is one of the following functions:

$$(1+z-z^2)/[(1-z^3)\cdot(1-z^2)], \qquad z(1-z-z^2)/[(1-z^3)(1-z^2)], \qquad z/(1-z)^2,\ z/(1+z)^2$$

Proof. Clearly, since $f \in \mathcal{T}$, the a_n are real and the function

$$P(z)=f(z)\cdot(1-z^2)/z=\sum_{n=0}^{\infty}(a_{n+1}-a_{n-1})z^n$$

belongs to $\mathcal{P}$. From Theorem 4 we have

$$a_2=(-1+ic)/\eta, \qquad a_3-1=(-1-ic)/\eta^2$$

It follows that

$$\eta^3=[(1+c^2)/a_2\cdot(a_3-1)]=\text{real number}$$

Hence $\eta^3=1$ or $\eta^3=-1$, so that the possible values of η are the cubic roots of 1 and of -1. It is easily seen that the values

$$-1,\ 1,\ (-1-i3^{1/2})/2,\ (1+i3^{1/2})/2$$

correspond to the values of

$$c=0,\ 0,\ -3^{1/2},\ 3^{1/2}$$

which provide the four functions of the statement and in the order they are written.

To the values

$$\eta=(-1+i3^{1/2})/2, \qquad \eta=(1-i3^{1/2})/2$$

correspond the functions

$$z/(1-z)^2,\ z/(1+z)^2$$

respectively.

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