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ON THE LATTICE OF VARIETIES OF COMPLETELY REGULAR SEMIGROUPS

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Abstract

Several morphisms of this lattice $\mathcal{V}(\mathbf{CR})$ are found, leading to decompositions of it, and various sublattices, into subdirect products of interval sublattices. For example the map $\mathbf{V} \to \mathbf{V} \cap \mathbf{G}$ (where $\mathbf{G}$ is the variety of groups) is shown to be a retraction of $\mathcal{V}(\mathbf{CR})$; from modularity of the lattice $\mathcal{V}(\mathbf{BG})$ of varieties of bands of groups it follows that the map $\mathbf{V} \to (\mathbf{V} \cap \mathbf{G}, \mathbf{V} \vee \mathbf{G})$ is an isomorphism of $\mathcal{V}(\mathbf{BG})$.

In addition, identities are provided for the varieties of central completely regular semigroups and of central bands of groups, answering questions of Petrich.

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1. Introduction

The class $\mathbf{CR}$ of *completely regular* semigroups (also called *unions of groups*) forms a variety of universal algebras when considered as semigroups with the additional unary operation $x \to x^{-1}$. Particular sublattices of the lattice $\mathcal{V}(\mathbf{CR})$ of varieties of completely regular semigroups have been the subject of intense study: for instance the lattice $\mathcal{V}(\mathbf{G})$ of varieties of *groups* (see [10]), the lattice $\mathcal{V}(\mathbf{CS})$ of varieties of *completely simple semigroups* [8, 9, 13-17] and the lattice $\mathcal{V}(\mathbf{B})$ of varieties of *bands* [1, 2, 3].

It is the main aim of this paper to study various morphisms of the lattice $\mathcal{V}(\mathbf{CR})$, of the type used in [6] and [13], to extend the knowledge of these special sublattices to larger sublattices by means of subdirect decompositions. In [6], Hall and the author showed that the map $\mathbf{V} \to (\mathbf{V} \cap \mathbf{B}, \mathbf{V} \vee \mathbf{B})$ is an isomorphism of the lattice $\mathcal{V}(\mathbf{BG})$ of varieties of bands of groups upon a subdirect product of the interval sublattices $\mathcal{V}(\mathbf{B})$ and $[\mathbf{B}, \mathbf{BG}]$; from this it could be deduced that $\mathcal{V}(\mathbf{BG})$ is modular, a powerful result some of whose implications are treated here. In [13] a similar decomposition of $\mathcal{V}(\mathbf{CS})$ using **G** instead of **B** was given. The varieties **B** and **G** are called *neutral* ([4]) (in their respective lattices).

We show (Section 3) that **G** is in fact neutral in $\mathcal{V}(\mathbf{BG})$, and also (Section 4) in the lattice $\mathcal{V}(\mathbf{OCR})$ of *orthodox* completely regular semigroups; a partial result is obtained in $\mathcal{V}(\mathbf{CR})$ itself. Our technique has the advantage that explicit knowledge of the (relatively) free objects in the relevant varieties is not required.

Among the similar results obtained it is shown that **CS** is neutral in $\mathcal{V}(\mathbf{BG})$ and that every variety of normal bands is neutral in the whole lattice $\mathcal{V}(\mathbf{CR})$.

In [16] Petrich and Reilly proved a result of a similar type, to the effect that the variety **CCS** of central completely simple semigroups is neutral in the lattice $\mathcal{V}(\mathbf{CS})$. It would be of interest to know to what extent that result could be generalized.

In the final section two questions posed by Petrich in [12] are answered: identities are provided for the varieties **CCR** and **CBG** of *central* completely regular semigroups and bands of groups respectively.

2. Preliminaries

In general, for semigroup theoretic notation and terminology we follow Howie [7]. However we make the following conventions: the term "completely regular" will be abbreviated to "c.r." throughout; for any element x of a c.r. semigroup, x^{-1} and x^0 will denote respectively the inverse of x in, and the identity of, the maximal subgroup to which it belongs; thus $x^0 = xx^{-1} = x^{-1}x$.

For convenience we present a list of the abbreviations used for various varieties of c.r. semigroups:

CR = c.r. semigroups,
CS = completely simple semigroups,
LZ[**RZ**] = left [right] zero semigroups,
RB = rectangular bands,
B = bands,
G = groups,
BG = bands of groups,
NB = normal band,
NBG = normal bands of groups,

SL = semilattices,
SLG = semilattices of groups,
T = trivial semigroups.

Further, prefixing **O** to any variety **V** will indicate the subvariety of **V** consisting of those *orthodox* members of **V**: thus **OV** = **V** ∩ **OCR**. Prefixing **C** to **V** will indicate the subvariety of **V** consisting of those *central* members of **V**: thus **CV** = **V** ∩ **CCR**. (A c.r. semigroup is *central* if the product of any two of its idempotents lies in the centre of the maximal subgroup to which it belongs.) That **CV** is indeed a variety will follow from Theorem 5.1, where it is shown that **CCR** is itself a variety.

For identities defining these and various other varieties of c.r. semigroups we refer the reader to [12].

For reference we quote here the following important result mentioned in the introduction.

RESULT 2.1 [6, Theorem 3.1]. *The lattice* $\mathcal{V}$(**BG**) *is modular.*

Its importance in the context of this paper stems from the next result, for which some preparation is required. In general, for lattice theoretic notation and terminology we follow Grätzer [4].

An element of a lattice L is called *neutral* ([4], Section III.2) if for all a, b, in L,

(i) $(a \vee b) \wedge d = (a \wedge d) \vee (b \wedge d)$,

(ii) $(a \wedge b) \vee d = (a \vee d) \wedge (b \vee d)$, and

(iii) $a \wedge d = b \wedge d$ and $a \vee d = b \vee d$ together imply $a = b$.

Clearly d satisfies (i) if and only if the map $a \to a \wedge d$ is a retraction of L upon the principal ideal $(a]$ generated by a; a dual statement is valid for (ii). Thus d is neutral if and only if the map $a \to (a \wedge d, a \vee d)$ is an isomorphism upon a subdirect product of $(a]$ and its dual $[a)$.

Specializing Theorem III.2.6 of [4] we obtain

RESULT 2.2. *In a modular lattice any element satisfying either* (i) *or* (ii) *is neutral.*

3. Some morphisms in $\mathcal{V}$(CR)

The first main result of this section, which generalizes Theorem 4.4 of [13], is the following.

THEOREM 3.1. *The mapping* **V** → **V** ∩ **G** *is a retraction of* $\mathcal{V}$(**CR**) *upon* $\mathcal{V}$(**G**).

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PROOF. Let $\mathbf{U}, \mathbf{V} \in \mathcal{V}(\mathbf{CR})$. The inclusion

$$(\mathbf{U} \cap \mathbf{G}) \vee (\mathbf{V} \cap \mathbf{G}) \subseteq (\mathbf{U} \vee \mathbf{V}) \cap \mathbf{G}$$

is clear. To prove this converse let G be a group belonging to $\mathbf{U} \vee \mathbf{V}$. Thus there exist c.r. semigroups $A \in \mathbf{U}$ and $B \in \mathbf{V}$, a (regular) subdirect product T of A and B and a morphism ϕ of T upon G. We will show that every finitely generated subgroup of G belongs to $(\mathbf{U} \cap \mathbf{G}) \vee (\mathbf{V} \cap \mathbf{G})$, so that G itself does.

So let F be such a subgroup, generated by $\{g_1, \ldots, g_n\}$, say. For each i, let $y_i \in T$ be such that $y_i\phi = g_i$, and let $e_i = y_i^0$. Put $e = (e_1 \cdots e_n)^0$. Since T is c.r., each $ey_ie\mathcal{H}e$. Thus $\{ey_1e, \ldots, ey_ne\}$ generates a subgroup H, say, of H_e. Clearly $(ey_ie)\phi = g_i$, since G is a group, so $H\phi = F$.

Now since H is a subgroup of $A \times B$, $e = (a, b)$ for some idempotents $a \in A$, $b \in B$. But for any $(u, v) \in T$, $(u, v)\mathcal{H}(a, b)$ if and only if $u\mathcal{H}a$ (in A) and $v\mathcal{H}b$ (in B), so H_e is isomorphic to a subgroup of $H_a \times H_b$. Since $H_a \in \mathbf{U} \cap \mathbf{G}$ and $H_b \in \mathbf{V} \cap \mathbf{G}$, H_e, H and F in turn belong to $(\mathbf{U} \cap \mathbf{G}) \vee (\mathbf{V} \cap \mathbf{G})$, as required.

Applying Results 2.1 and 2.2 to this theorem immediately yields the following, the final statement of which is Theorem 5.5 of [13].

COROLLARY 3.2. *The variety* $\mathbf{G}$ *is neutral in* $\mathcal{V}(\mathbf{BG})$, *that is, the map* $\mathbf{V} \to (\mathbf{V} \cap \mathbf{G}, \mathbf{V} \vee \mathbf{G})$ *is an isomorphism of* $\mathcal{V}(\mathbf{BG})$ *upon a subdirect product of* $\mathcal{V}(\mathbf{G})$ *with the interval* $[\mathbf{G}, \mathbf{BG}]$. *In particular* $\mathbf{G}$ *is neutral in* $\mathcal{V}(\mathbf{CS})$.

We do not know whether $\mathbf{G}$ is neutral in $\mathcal{V}(\mathbf{CR})$. (See, however, Theorem 4.1.)

The proof of Theorem 3.1 may be easily modified (essentially by replacing $\mathcal{H}$ by $\mathcal{D}$ throughout) to obtain

THEOREM 3.3. *The map* $\mathbf{V} \to \mathbf{V} \cap \mathbf{CS}$ *is a retraction of* $\mathcal{V}(\mathbf{CR})$ *upon* $\mathcal{V}(\mathbf{CS})$.

COROLLARY 3.4. *The variety* $\mathbf{CS}$ *is neutral in* $\mathcal{V}(\mathbf{BG})$.

In [6], Proposition 3.5 it was shown that $\mathbf{SL}$ is neutral in the entire lattice $\mathcal{V}(\mathbf{CR})$. We use a similar approach to prove

THEOREM 3.5. *The variety* $\mathbf{LZ}$ *is neutral in* $\mathcal{V}(\mathbf{CR})$.

PROOF. We show directly that the map

$$\mathbf{V} \to (\mathbf{V} \cap \mathbf{LZ}, \mathbf{V} \vee \mathbf{LZ})$$

is an order isomorphism. It is clearly order preserving. So suppose **U**, **V** in $\mathcal{V}(\mathbf{CR})$ are such that

$$\mathbf{U} \cap \mathbf{LZ} \subseteq \mathbf{V} \cap \mathbf{LZ} \quad \text{and} \quad \mathbf{U} \vee \mathbf{LZ} \subseteq \mathbf{V} \vee \mathbf{LZ};$$

we must show $\mathbf{U} \subseteq \mathbf{V}$.

Note that since **LZ** is an atom of the lattice $\mathcal{V}(\mathbf{CR})$ either $\mathbf{LZ} \subseteq \mathbf{V}$ or $\mathbf{V} \cap \mathbf{LZ} = \mathbf{T}$. In the former case the second inclusion yields $\mathbf{U} \subseteq \mathbf{V}$ immediately, so from now on assume $\mathbf{V} \cap \mathbf{LZ} = \mathbf{T}$. In that case $\mathbf{U} \cap \mathbf{LZ} = \mathbf{T}$ also, so both **U** and **V** consist entirely of semilattices of *right groups*.

Now let $S \in \mathbf{U}$. Thus $S \in \mathbf{V} \vee \mathbf{LZ}$ and there exist $A \in \mathbf{V}$ and $L \in \mathbf{LZ}$, a subdirect product T of A and L and a morphism ϕ of T upon S. For each element a of A define $a\bar{\phi} = (a, l)\phi$, for some $(a, l) \in T$. Suppose (a, l) and $(a, m) \in T$: since L is a left zero semigroup it follows that $l \mathcal{L} m$, whence $(a, l)\mathcal{L}(a, m)$ in T and $(a, l)\phi\mathcal{L}(a, m)\phi$ in S. But the $\mathcal{D}$-class of S containing $(a, l)\phi$ is a right group, so $(a, l)\phi\mathcal{H}(a, m)\phi$. Now (a^0, l) is the identity of the $\mathcal{H}$-class of (a, l) in T, so $(a^0, l)\phi$ is the identity of the $\mathcal{H}$-class of $(a, l)\phi$ in S. Therefore

$$(a, l)\phi = \big((a^0, l)(a, m)\big)\phi = (a^0, l)\phi(a, m)\phi = (a, m)\phi.$$

Thus $\bar{\phi}$ defines a mapping of A into S which is clearly a surjective morphism. Hence $S \in \mathbf{V}$, as required.

From duality it follows that **RZ** is also neutral in $\mathcal{V}(\mathbf{CR})$. From the definition of neutrality it is easily seen that the neutral elements of any lattice form a sublattice. Thus **RB** is neutral and in fact the sublattice of $\mathcal{V}(\mathbf{CR})$ generated by **LZ**, **RZ** and **SL** consists of neutral elements. This sublattice is precisely the lattice $\mathcal{V}(\mathbf{NB})$ (see, for example, [7, page 124]), giving

COROLLARY 3.6. *Every variety of normal bounds is neutral in* $\mathcal{V}(\mathbf{CR})$.

Specializing to $\mathcal{V}(\mathbf{BG})$ and noting, additionally, neutrality of **B** there (implicit in [6], Proposition 3.4) it follows that the sublattice generated by **LZ**, **RZ**, **SL**, **G**, **CS** and **B** consists of neutral elements. For a diagram of the bulk of this sublattice see Diagram 1 of [11] (and for the "missing join" see [6]). In particular, for instance, **NBG** and **OBG** are neutral in $\mathcal{V}(\mathbf{BG})$.

4. Neutrality of G in OCR

Before proving the main result of this section we remind the reader of some facts concerning μ, the greatest idempotent separating congruence on a regular semigroup. We will make use of the fact that on any orthodox semigroup S the

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intersection $\mu \cap \gamma$ is trivial, γ denoting the least inverse semigroup congruence on S. (See, for instance [7], Section VI.4.) Thus S is isomorphic to a subdirect product of S/μ and S/γ. When S is, further, c.r., S/γ is clearly a semilattice of groups. We also make use of the fact that if $\phi: S \to T$ is a surjective morphism of regular semigroups, and if $a\mu b$ in S, then $a\phi\mu b\phi$ in T.

THEOREM 4.1. *The variety* **G** *is neutral in* **OCR**.

PROOF. We again prove directly that the map $\mathbf{U} \to (\mathbf{U} \cap \mathbf{G}, \mathbf{U} \vee \mathbf{G})$ is an order isomorphism. So let $\mathbf{U}, \mathbf{V} \in \mathcal{V}(\mathbf{OCR})$ and suppose

$$\mathbf{U} \cap \mathbf{G} \subseteq \mathbf{V} \cap \mathbf{G} \quad \text{and} \quad \mathbf{U} \vee \mathbf{G} \subseteq \mathbf{V} \vee \mathbf{G}.$$

Observe first that since, by Theorem 3.3, the map $\mathbf{A} \to \mathbf{A} \cap \mathbf{CS}$ is a morphism of $\mathcal{V}(\mathbf{CR})$, the above inequalities yield

$$(\mathbf{U} \cap \mathbf{CS}) \cap \mathbf{G} \subseteq (\mathbf{V} \cap \mathbf{CS}) \cap \mathbf{G} \quad \text{and} \quad (\mathbf{U} \cap \mathbf{CS}) \vee \mathbf{G} \subseteq (\mathbf{V} \cap \mathbf{CS}) \vee \mathbf{G}.$$

By Corollary 3.2, **G** is neutral in $\mathcal{V}(\mathbf{CS})$, so $\mathbf{U} \cap \mathbf{CS} \subseteq \mathbf{V} \cap \mathbf{CS}$. Now if **U** does not contain **SL** then $\mathbf{U} \subseteq \mathbf{CS}$, giving $\mathbf{U} \subseteq \mathbf{V}$. Similarly if **V** does not contain **SL**, $\mathbf{V} \subseteq \mathbf{CS}$ and $\mathbf{U} \subseteq \mathbf{U} \vee \mathbf{G} \subseteq \mathbf{V} \vee \mathbf{G} \subseteq \mathbf{CS}$, giving $\mathbf{U} \subseteq \mathbf{V}$ again.

From now on, then, we assume both **U** and **V** contain **SL**. Let $\mathbf{S} \in \mathbf{U}$. Then $S \in \mathbf{V} \vee \mathbf{G}$ and there exist $A \in \mathbf{V}$ and $G \in \mathbf{G}$, a subdirect product T of A and G and a morphism ϕ of T upon S. Let $a \in A$ and suppose (a, g) and $(a, h) \in T$. Then $(a, g)\mu(a, h)$ and so $(a, g)\phi\mu(a, h)\phi$ in S. Hence $(a, g)\phi\mu^\natural = (a, h)\phi\mu^\natural$ in S/μ ($\mu^\natural$ denoting the natural map). The map $\theta: A \to S/\mu$ given by

$$a\theta = (a, g)\phi\mu^\natural \quad \text{for some } (a, g) \in T,$$

is therefore well defined and is clearly a surjective morphism. Therefore $S/\mu \in \mathbf{V}$.

Clearly $S/\gamma \in \mathbf{U} \cap \mathbf{SLG} = (\mathbf{U} \vee \mathbf{SL}) \cap (\mathbf{G} \vee \mathbf{SL}) = (\mathbf{U} \cap \mathbf{G}) \vee \mathbf{SL}$, since $\mathbf{SL} \subseteq \mathbf{U}$ and **SL** is neutral in $\mathcal{V}(\mathbf{CR})$ (see Section 3). Therefore $S/\gamma \in (\mathbf{V} \cap \mathbf{G}) \vee \mathbf{SL} \subseteq \mathbf{V}$. Since S is isomorphic to a subdirect product of S/μ and S/γ, $S \in \mathbf{V}$ also.

5. Central c.r. semigroups

Petrich posed the following two problems (among others) in Section 7 of [12].

Problem 4. Is **CBG** defined by the identity

(1) $$a^0b^0a = ab^0a^0?$$

Problem 6. Is **CCR** a variety? If so find identities defining it.

In the theorem below we answer each question in the affirmative by providing a single identity for **CCR** which reduces to (1) in bands of groups. We will make use

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of the fact ([14], Proposition 6.2; see also [12], Lemma 3.5) that **CCS** is defined, *within* **CS**, by (1).

THEOREM 5.1. a) *The class* **CCR** *is defined by the identity*

$$(a^0b^0a)(b^0a^0)^0 = (a^0b^0)^0(ab^0a^0), \tag{2}$$

and is therefore a variety. In fact **CCR** *consists precisely of the semilattices of central completely simple semigroups.*

b) *The variety* **CBG** *is defined by the identity* (1).

PROOF. a) First let $S \in \mathbf{CCR}$, and let $a, b \in S$. Using Lemma 1 of [5],

$$\begin{aligned} a^0b^0a &= \left[(a^0b^0a)(a^0b^0a)^{-1}a^0\right]\left[b^0a(a^0b^0a)^{-1}a^0b^0\right]\left[a(a^0b^0a)^{-1}(a^0b^0a)\right] \\ &= \left[(a^0b^0a)^{-1}(a^0b^0a)a^0\right]\left[b^0a(a^0b^0a)^{-1}a^0b^0\right]\left[a(a^0b^0a)^0\right] \\ &= \left[(a^0b^0a)^0\right]\left[b^0a(a^0b^0a)^{-1}a^0b^0\right]\left[(a^0b^0)^0a(a^0b^0a)^0\right] \\ &= xyz, \quad \text{say,} \end{aligned}$$

where each of these terms belongs to the same $\mathcal{D}$-class of S, the middle term y is idempotent, and $z\mathcal{H}x$ (since $z\mathcal{R}a^0b^0\mathcal{R}x$ and $z\mathcal{L}a^0b^0a\mathcal{L}x$). So in fact $a^0b^0a = z^0y^0z$.

From the definition of centrality it is clear that each $\mathcal{D}$-class of S is a central completely simple semigroup and therefore satisfies (1), so that $z^0y^0z = zy^0z^0$. Thus

$$\begin{aligned} a^0b^0a &= \left[(a^0b^0)^0a(a^0b^0a)^0\right]\left[b^0a(a^0b^0a)^{-1}a^0b^0\right]\left[(a^0b^0a)^0\right] \\ &= \left[(a^0b^0)^0a(a^0b^0a)^0\right]\left[(a^0b^0a)(a^0b^0a)^{-1}a^0b^0\right]\left[(a^0b^0a)^0\right] \\ &= (a^0b^0)^0a(a^0b^0a)^0a^0b^0(a^0b^0a)^0 \\ &= (a^0b^0)^0a(a^0b^0a)^0(a^0b^0a)a^{-1}(a^0b^0a)^0 \\ &= (a^0b^0)^0a(a^0b^0a)a^{-1}(a^0b^0a)^0 \\ &= (a^0b^0)^0(ab^0a^0)(a^0b^0a)^0. \end{aligned} \tag{3}$$

On the other hand,

$$\begin{aligned} (a^0b^0a)(b^0a^0)^0 &= (a^0b^0a)(a^0b^0a^0)(b^0a^0)^{-1} \\ &= (a^0b^0a)(a^0b^0a^0)^0(a^0b^0a^0)(b^0a^0)^{-1} \\ &= (a^0b^0a)(a^0b^0a^0)^0(b^0a^0)^0 \\ &= (a^0b^0a)(a^0b^0a^0)^0, \quad \text{since } a^0b^0a^0\mathcal{L}b^0a^0, \end{aligned}$$

and so

$$= (a^0b^0)^0(ab^0a^0)(a^0b^0a)^0(a^0b^0a^0)^0, \quad \text{using (3)},$$
$$= (a^0b^0)^0(ab^0a^0)(a^0b^0a^0)^0, \quad \text{since } a^0b^0a\mathcal{R}\,a^0b^0a^0,$$
$$= (a^0b^0)^0(ab^0a^0), \quad \text{since } ab^0a^0\mathcal{L}\,a^0b^0a^0.$$

So S satisfies (2).

Conversely let S be a c.r. semigroup which satisfies (2), and let D be a $\mathcal{D}$-class of S. Since $\mathcal{H}$ is a congruence on D, for any $a, b \in D$ we have $(b^0a^0)^0 = (b^0a)^0$ and $(a^0b^0)^0 = (ab^0)^0$ so that (2) reduces to (1) in D. Hence each $\mathcal{D}$-class is a central completely semigroup. Now if e and f are idempotents of S, then $ef = [(ef)^0e][f(ef)^0]$, where each of these two terms is an idempotent of D_{ef}. Their product thus lies in the centre of the maximal containing subgroup and S is itself central.

The final statement of a) is now clear from the above proof.

b) Since the bands of groups are *precisely* the c.r. semigroups on which $\mathcal{H}$ is a congruence, this follows as in the proof above.

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