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RESEARCH ARTICLE

Superscars for arithmetic point scatters II

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Abstract

We consider momentum push-forwards of measures arising as quantum limits (semiclassical measures) of eigenfunctions of a point scatterer on the standard flat torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Given any probability measure arising by placing delta masses, with equal weights, on $\mathbb{Z}^2$-lattice points on circles and projecting to the unit circle, we show that the mass of certain subsequences of eigenfunctions, in momentum space, completely localizes on that measure and are completely delocalized in position (i.e., concentration on Lagrangian states). We also show that the mass, in momentum, can fully localize on more exotic measures, for example, singular continuous ones with support on Cantor sets. Further, we can give examples of quantum limits that are certain convex combinations of such measures, in particular showing that the set of quantum limits is richer than the ones arising only from weak limits of lattice points on circles. The proofs exploit features of the half-dimensional sieve and behavior of multiplicative functions in short intervals, enabling precise control of the location of perturbed eigenvalues.

1. Introduction

Let (M, g) be a smooth, compact Riemannian manifold with no boundary, unit mass, and let Δ_g denote the Laplace–Beltrami operator. Also, let $\{\phi_\lambda\}$ be an orthonormal basis of eigenfunctions of Δ_g with eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \dots$. For an observable $f \in C^\infty(\mathbb{S}^*M)$, where $\mathbb{S}^*M$ denotes the unit cotangent bundle of M, let $\mathrm{Op}(f)$ denote its quantization, defined as a pseudo-differential operator (cf. [9] for details.) A central problem in quantum chaos (cf. [52, Problem 3.1]) is to understand the set of possible quantum limits (sometimes called semiclassical measures) describing the distribution of mass of the eigenfunctions $\{\phi_\lambda\}$ within $\mathbb{S}^*M$, in the limit as the eigenvalue λ tends to infinity. A cornerstone result in this direction is the quantum ergodicity theorem of Shnirelman [45], Colin de Verdiére [8] and Zelditch [51] which states that if the geodesic flow on M is ergodic there exists a density one subsequence of eigenfunctions $\{\phi_{\lambda_j}\}$ such that

$$\mu_{\phi_{\lambda_j}}(f) = \langle \mathrm{Op}(f)\phi_{\lambda_j}, \phi_{\lambda_j}\rangle \to \int_{\mathbb{S}^*M} f(x)d\mu_L(x)$$

as $\lambda_j \to \infty$, where $d\mu_L$ is the normalized Liouville measure on $\mathbb{S}^*M$. (Note that any quantum limit, by Egorov's theorem, is invariant under the classical dynamics.)

While the quantum ergodicity theorem implies that the mass of almost all eigenfunctions equidistributes in $\mathbb{S}^*M$ with respect to $d\mu_L$, it does not rule out the existence of sparse subsequences along which the mass of the eigenfunctions localizes. Whether or not this happens crucially depends on the geometry of M, cf. Section 1.3.

In this article, we study quantum limits of 'point scatterers' on $M = \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. These are singular perturbations of the Laplacian on M and were used by Šeba [40] in order to study the transition between integrability and chaos in quantum systems. The perturbation is quite weak and has essentially no effect on the classical dynamics, yet the quantum dynamics 'feels' the effect of the scatterer, and an analog of the quantum ergodicity theorem is known to hold [38, 27] (namely, equidistribution holds for a full density subset of the 'new' eigenfunctions), even though classical ergodicity does not hold.

The model also exhibits scarring along sparse subsequences of the new eigenfunctions [25]. In particular, there exist quantum limits whose momentum push-forwards, which can be viewed as probability measures on the unit circle, are of the form $c\mu_{\text{sing}} + (1-c)\mu_{\text{uniform}}$, for some $c \in [1/2, 1]$. Here, both μ_{uniform} and μ_{sing} are normalized to have mass one, and μ_{sing} can be taken to be a sum of delta measures giving equal mass to the four points $\pm(1,0), \pm(0,1)$. We note that μ_{uniform} is the push-forward of the Liouville measure and hence maximally delocalized, whereas μ_{sing} is maximally localized since any quantum limits in this setting must be invariant under a certain eight fold symmetry (cf. equation (1.7)).

Stronger localization, that is, going beyond $c = 1/2$, is interesting given a number of 'half delocalization' results for quantum limits for some other (strongly chaotic) systems, namely quantized cat maps and geodesic flows on manifolds with constant negative curvature -1. In the former case, Faure and Nonnenmacher showed [12] that if a quantum limit ν is decomposed as $\nu = \nu_{\text{pp}} + \nu_{\text{Liouville}} + \nu_{sc}$, with ν_{pp} denoting the pure point part and ν_{sc} denoting the singular continuous part, then $\nu_{\text{Liouville}}(\mathbb{T}^2) \geq \nu_{\text{pp}}(\mathbb{T}^2)$, and thus $\nu_{\text{pp}}(\mathbb{T}^2) \leq 1/2$. (We emphasize that $\mathbb{T}^2$ is the full phase space in this setting.) In the latter case, it was shown that the Kolmogorov-Sinai (KS) entropy with respect to any measure arising as a quantum limit is at least $1/2$. We remark that for arithmetic point scatterers, the KS entropy is zero with respect to any flow invariant probability measure, in particular for any measure arising as a quantum limit.

The aim of this paper is to exhibit essentially maximal localization for a quantum ergodic system, namely arithmetic toral point scatterers. In particular, we construct quantum limits (in momentum) corresponding to $c = 1$ in the above decomposition; other interesting examples include singular continuous measures with support, say, on Cantor sets. This can be viewed as a step towards a 'measure classification' for quantum limits of quantum ergodic systems.

1.1. Description of the model

Let us now describe the basic properties of the point scatterer. This is discussed in further detail in [38, 39, 27, 25, 40, 42]. To describe the quantum system associated with the point scatterer, consider $-\Delta|_{D_{x_0}}$, where

$$D_{x_0} = \{f \in L^2(\mathbb{T}^2) : f(x) = 0 \text{ in some neighborhood of } x_0\}.$$

By von Neumann's theory of self-adjoint extensions (see Appendix A of [38]) there exists a one parameter family of self-adjoint extension of $-\Delta|_{D_{x_0}}$ parameterized by a phase $\varphi \in (-\pi, \pi]$. Moreover, for $\varphi \neq \pi$ the eigenvalues of these operators may be divided into two categories. The *old* eigenvalues which are eigenvalues of $-\Delta$, with multiplicity decreased by one, along with *new* eigenvalues which are solutions to the spectral equation

$$\sum_{m \geq 1} r(m) \left(\frac{1}{m - \lambda} - \frac{m}{m^2 + 1} \right) = \tan(\varphi/2) \sum_{m \geq 1} \frac{r(m)}{m^2 + 1}, \tag{1.1}$$

where

$$r(m) = \#\{(a, b) \in \mathbb{Z}^2 : a^2 + b^2 = m\}.$$

We will refer to the case when φ is fixed as $\lambda \to \infty$ the *weak coupling* quantization. In this regime work of Shigehara [42] suggests that the level spacing of the eigenvalues should have Poisson spacing statistics, and this is supported by work of Rudnick and Ueberschär [39] along with Freiberg, Kurlberg

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and Rosenzweig [14]. In the hope of exhibiting wave chaos, Shigehara proposes the following *strong coupling* quantization

$$\sum_{|m-\lambda|\leq\lambda^{1/2}} r(m)\left(\frac{1}{m-\lambda}-\frac{m}{m^2+1}\right)=\frac{1}{\alpha}, \tag{1.2}$$

where $\alpha \in \mathbb{R}$ is called the physical coupling constant and reflects the strength of the scatterer. The strong coupling quantization restricts the spectral equation to the physically relevant energy levels. Notably, this forces a renormalization of equation (1.1)

$$\tan(\varphi/2)\sum_{m\geq 1}\frac{r(m)}{m^2+1}\sim -\pi\log\lambda$$

so that φ depends on λ in this case (see [48] equation (3.14)). We note that the weak coupling quantization corresponds to a fixed self-adjoint extension, whereas the strong coupling quantization can be viewed as an energy-dependent, albeit very slowly varying, family of self-adjoint extensions.

From the spectral equation, it follows that new eigenvalues interlace with integers which are representable as the sum of two integer squares. We denote these eigenvalues as follows:

$$0<\lambda_0<1<\lambda_1<2<\lambda_2<4<\lambda_4<5<\lambda_5<\cdots$$

and write Λ_{new} for the set of all such eigenvalues. Also, given $n=a^2+b^2$, let n^+ denote the smallest integer greater than n which is also a sum of two squares. Let

$$s_n=\lambda_n-n>0 \tag{1.3}$$

denote the distance between λ_n and the nearest old eigenvalue n to the left. In addition, given $\lambda\in\Lambda_{new}$ the associated Green's function is given by

$$G_\lambda(x)=-\frac{1}{4\pi^2}\sum_{\xi\in\mathbb{Z}^2}\frac{\exp(-i\xi\cdot x_0)}{|\xi|^2-\lambda}e^{i\xi\cdot x}, \qquad g_\lambda(x)=\frac{1}{\|G_\lambda\|_2}G_\lambda(x) \tag{1.4}$$

(see equation (5.2) of [38]). Also, note that the new eigenvalues interlace between the old eigenvalues; hence, G_λ is well defined for $\lambda\in\Lambda_{new}$. Since the torus is homogeneous, we may without loss of generality assume that $x_0=0$. Our main focus will be the behavior of the matrix coefficients $\{\langle \mathrm{Op}(f)g_\lambda, g_\lambda\rangle\}_{\lambda\in\Lambda_{new}}$ as f ranges over the set of pure momentum observables (i.e., $f\in C^\infty(S^1)\subset C^\infty(\mathbb{S}^*(\mathbb{T}^2))$; for such f the matrix coefficients are explicitly given by (cf. equation (5.3))

$$\langle \mathrm{Op}(f)g_\lambda, g_\lambda\rangle=\frac{1}{\sum_{n\geq 0}\frac{r(n)}{(n-\lambda)^2}}\left(\frac{f(1)}{\lambda^2}+\sum_{n>0}\frac{1}{(n-\lambda)^2}\sum_{a^2+b^2=n}f\left(\frac{a+ib}{|a+ib|}\right)\right). \tag{1.5}$$

1.2. Results

Our first main result shows that along a zero density, yet relatively large, subsequence of new eigenvalues $\{\lambda_j\}$ the mass of g_{λ_j}, in momentum space, localizes on measures arising from $\mathbb{Z}^2$-lattice points on circles (after projecting them to the unit circle). To describe these measures in more detail, consider an integer $n=a^2+b^2$, with $a,b\in\mathbb{Z}$, and the following probability measure on the unit circle $S^1\subset\mathbb{C}$

$$\mu_n=\frac{1}{r(n)}\sum_{a^2+b^2=n}\delta_{(a+ib)/|a+ib|}.$$

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We remark that μ_n can be viewed as the matrix coefficient of the 'flat' (old) Laplace eigenfunction $\psi_n(x) = \frac{1}{2\pi\sqrt{r(n)}} \sum_{\xi\in\mathbb{Z}^2:|\xi|^2=n} e^{-i\xi\cdot x}$, in the sense that, for f a pure momentum observable, we have

$$\langle \mathrm{Op}(f)\psi_n, \psi_n\rangle = \sum_{a^2+b^2=n} f\left(\frac{a+ib}{|a+ib|}\right) = \mu_n(f). \tag{1.6}$$

Following Kurlberg and Wigman [29], we call a measure μ_∞ *attainable* if it is a weak limit point of the set $\{\mu_n\}_{n=a^2+b^2}$. Any such measure is invariant under rotation by $\pi/2$, as well as under reflection in the x-axis; for convenience let

$$\mathrm{Sym}_8 := \left\{\left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle\right\} \subset GL_2(\mathbb{Z}) \tag{1.7}$$

denote the group generated by these transformations.

Theorem 1.1. *Let $m_0 = a^2 + b^2 \in \mathbb{N}$ be odd.*[1] *In each of the weak and strong coupling quantizations, there exists a subset of eigenvalues $\mathcal{E}_{m_0} \subset \Lambda_{new}$ with*

$$\frac{\#\{\lambda \le X : \lambda \in \mathcal{E}_{m_0}\}}{\#\{\lambda \le X : \lambda \in \Lambda_{new}\}} \gg \frac{1}{(\log X)^{1+o(1)}}$$

such that for any pure momentum observable $f \in C^\infty(S^1) \subset C^\infty(\mathbb{S}^*(\mathbb{T}^2))$

$$\langle \mathrm{Op}(f) g_\lambda, g_\lambda\rangle \xrightarrow[\lambda\in\mathcal{E}_{m_0}]{\lambda\to\infty} \frac{1}{r(m_0)} \sum_{a^2+b^2=m_0} f\left(\frac{a+ib}{|a+ib|}\right).$$

The key idea of the proof is to show that some new eigenvalues λ lie very close to certain old eigenvalues n, and this implies that g_λ is very well approximated by the flat eigenfunction ψ_n (cf. equations (1.5) and (1.6)), and consequently, in momentum space, the mass of g_λ completely localizes on the measure μ_{m_0}. Further, for any attainable measure μ_∞ there exists $\{m_{0,\ell}\}_\ell$ such that $\mu_{m_{0,\ell}}$ weakly converges to μ_∞, and this implies the following corollary.

Corollary 1.1. *Let μ_∞ be an attainable measure. Then there exists $\{\lambda_j\}_j \subset \Lambda_{new}$ such that for any pure momentum observable $f \in C^\infty(S^1)$*

$$\langle \mathrm{Op}(f) g_{\lambda_j}, g_{\lambda_j}\rangle \xrightarrow{j\to\infty} \int_{S^1} f\, d\mu_\infty.$$

We note that the set of attainable measures is much smaller than the set of probability measures on S^1 that are Sym_8-invariant, in particular the set of attainable measures is *not convex* (cf. [29, Section 3.2].) In our next result, we show that in the strong coupling quantization there is a subsequence of new eigenvalues along which the entire mass of g_λ localizes on a certain convex combination of two measures arising from lattice points on the circle. In particular, the set of quantum limits, in momentum space, is *strictly richer* than the set of attainable measures.

Theorem 1.2. *Let m_0, m_1 be odd integers which are each representable as a sum of two squares. Then in the strong coupling quantization there exists a subsequence of eigenvalues $\mathcal{E}_{m_0,m_1} \subset \Lambda_{new}$ such that*

[1] As far as possible quantum limits go, m_0 being odd is not a restriction as any μ_n for n even can be approximated by μ_{m_0} for m_0 odd.

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for each $\lambda \in \mathcal{E}_{m_0,m_1}$ *there is an integer* ℓ_λ *with* $r(\ell_\lambda) \neq 0$ *and* $r(\ell_\lambda) \ll 1$ *such that for pure momentum observables* $f \in C^\infty(S^1)$

$$\begin{aligned}\langle \mathrm{Op}(f)g_\lambda, g_\lambda\rangle = c_\lambda \cdot \frac{1}{r(m_0)} \sum_{a^2+b^2=m_0} f\left(\frac{a+ib}{|a+ib|}\right) \\ + (1-c_\lambda)\cdot \frac{1}{r(m_1\ell_\lambda)} \sum_{a^2+b^2=m_1\ell_\lambda} f\left(\frac{a+ib}{|a+ib|}\right) + O\left(\frac{1}{(\log\log\lambda)^{1/11}}\right),\end{aligned} \tag{1.8}$$

where

$$c_\lambda = \frac{1}{1+r(m_0)/r(m_1\ell_\lambda)}.$$

Additionally,

$$\frac{\#\{\lambda \le X : \lambda \in \mathcal{E}_{m_0,m_1}\}}{\#\{\lambda \le X : \lambda \in \Lambda_{new}\}} \gg \frac{1}{(\log X)^{2+o(1)}}.$$

Note that, since $\sum_{p|\ell_\lambda} 1 \ll 1$, the measure $\mu_{m_1\ell_\lambda}$ can be viewed as a fairly small perturbation of μ_{m_1}.

Remark 1. By removing a further 'thin' set of eigenvalues (with spectral counting function of size $O(x^{1-\epsilon})$ for $\epsilon > 0$, we can construct quantum limits that are flat in position (for details, cf. [25, Remark 4]), in addition to the momentum push-forward properties given in Theorems 1.1 and 1.2. In particular, taking say $m_0 = 9$ in Theorem 1.1 and noting that $|z|^2 = 9$ for $z \in \mathbb{Z}[i]$ has the four solutions $\pm 3, \pm 3i$, this then yields quantum limits that are completely localized on the superposition of two Lagrangian states – essentially two plane waves, one in the horizontal and one in the vertical direction. This phenomenon is sometimes called **superscarring** (cf. [6, 25]).

Further, assuming a plausible conjecture on the distribution of the prime numbers, we show that given m_0, m_1 as in Theorem 1.2 the quantum limit of $\langle \mathrm{Op}(f)g_\lambda, g_\lambda\rangle$ can be made to be any given convex combination of μ_{m_0} and μ_{m_1}. The conjecture on the distribution of primes concerns obtaining a lower bound on the number solutions (u, v) in almost primes to the Diophantine equation

$$aX - bY = 4,$$

where $v = p_1p_2$, $u = p_3$ with p_j a prime satisfying $p_j = a_j^2 + b_j^2$ and $b_j = o(a_j)$ for $j = 1, 2, 3$. The precise formulation of this conjecture, which we call Hypothesis 1, is given in Section 5.5.

Theorem 1.3. *Assume Hypothesis 1. Let* $\mu_{\infty_0}, \mu_{\infty_1}$ *be attainable measures and* $0 \le c \le 1$. *Then in the strong coupling quantization there exists* $\{\lambda_j\}_j \subset \Lambda_{new}$ *such that for any* $f \in C^\infty(S^1)$

$$\langle \mathrm{Op}(f)g_{\lambda_j}, g_{\lambda_j}\rangle \xrightarrow{j\to\infty} c\int_{S^1} f\,d\mu_{\infty_0} + (1-c)\int_{S^1} f\,d\mu_{\infty_1}.$$

Further, assuming a variation of the prime k-tuple conjecture that also allows for prescribing Gaussian angles, we can show (cf. Appendix C) that **all** Sym_8-invariant probability measures on S^1 arise as quantum limits in momentum space.

1.3. Discussion

For integrable systems it is often straightforward to construct nonuniform quantum limits, for example, 'whispering gallery modes' for the geodesic flow in the unit ball, and for linear flows on $\mathbb{T}^2$, Lagrangian states with maximal localization (i.e., a single plane wave) are easily constructed. We note that strong localization in position for quantum limits on $\mathbb{T}^2$ was ruled out by Jakobson [20] – in position, any quantum limit is given by trigonometric polynomials whose frequencies lie on at most two circles (hence absolutely continuous with respect to Lebesgue measure.) Further, for the sphere, Jakobson and Zelditch in fact obtained a full classification – *any* flow invariant measure on $S^*(S^2)$ is a quantum limit [21].

The quantum ergodicity theorem holds in great generality as long as the key assumption of ergodic classical dynamics holds, but the existence of exceptional subsequence of nonuniform quantum limits ('scarring') is subtle. For classical systems given by the geodesic flow on compact negatively curved manifolds, the celebrated quantum unique ergodicity (QUE) conjecture [37] by Rudnick and Sarnak asserts that the only possible quantum limit is the Liouville measure. Known results for QUE include Lindenstrauss' breakthrough [30] for Hecke eigenfunctions on arithmetic modular surfaces, together with Soundararajan ruling out 'escape of mass' in the noncompact case [46]. On the other hand, for a generic Bunimovich stadium (with strongly chaotic classical dynamics), Hassell [16] has shown that there exists a subsequence of exceptional eigenstates where the mass localizes on sets of bouncing ball trajectories.

For quantized cat maps, again for Hecke eigenfunctions, QUE is known to hold [26]. However, unlike for arithmetic modular surfaces, where Hecke desymmetrization is believed to be unnecessary, it is essential for quantum cat maps. Namely, Faure, Nonnenmacher and de Bièvre [13] constructed, in the presence of extreme spectral multiplicities and no Hecke desymmetrization, quantum limits of the form $\nu = \frac{1}{2}\nu_{\text{pp}} + \frac{1}{2}\nu_{\text{Liouville}}$; in [12], this was shown to be sharp in the sense that the Liouville component always carries at least as much mass as the pure point one. (We note that, on assuming very weak bounds on spectral multiplicities, Bourgain showed [7] that scarring does not occur.) For higher-dimensional analogs of quantum cat maps, Kelmer has for certain maps shown [23] 'super scarring', even after Hecke desymmetrization, on invariant rational isotropic subspaces. Further, these type of scars persist on adding certain perturbations that destroy the spectral multiplicities [24]. Other models where scarring is known to exist include toral point scatterers with irrational aspect ratios [28, 22, 3] and quantum star graphs [4], though neither model is quantum ergodic [28, 4].

Classifying the set of possible quantum limits, in particular for quantum ergodic settings, is an interesting question. Here, Anantharaman proved very strong results for geodesic flows on negatively curved manifolds [1]: any quantum limit has positive KS entropy with respect to the dynamics of the geodesic flow. In particular, this rules out localization on a finite number of closed geodesics (for compact arithmetic surfaces this was already known due to Rudnick and Sarnak [37].) Moreover, in the case of constant negative curvature, Anantharaman and Nonnenmacher showed [2] that the KS-entropy is at least half of the maximum possible. The measure of maximum entropy is given by the Liouville measure, and thus 'eigenfunctions are at least half delocalized'. Dyatlov and Jin [10] consequently showed that any quantum limit must have *full* support in $S^*(M)$, for compact hyperbolic surfaces M with constant negative curvature; together with Nonnenmacher this was recently strengthened [11] to the include the case of surfaces with variable negative curvature.

1.4. Outline of the proofs

Our arguments use the multiplicative structure of the integers to create an imbalance in the spectral equation (1.2) along a zero density, yet relatively large subsequence of new eigenvalues. Through exploiting this imbalance, we control the location of the new eigenvalues in our subsequence and show that they lie close to integers which are sums of two squares (cf. Section 5.3, in particular equation (5.14) for the argument placing full mass at one nearby eigenspace and 5.4, in particular equation (5.18) for placing mass at two nearby eigenspaces.) This greatly amplifies the amount of mass of the corresponding eigenfunctions in momentum space which lies on the terms which correspond to these integers, so much so that the contribution of the remaining terms is negligible in comparison. Consequently, the mass completely localizes on a convex combination of two measures and moreover our construction allows us to completely control the first measure.

In Section 2, we use sieve methods to produce integers $n = p_1 p_2$, where p_j, $j = 1, 2$, is a prime with $p_j = a^2+b^2 = (a+ib)(a-ib)$, $0 < b \le a$, with $0 \le \arctan(b/a) \le \varepsilon$, where ε is a small parameter, such that $Q_0 p_1 p_2+4$ is also a sum of two squares, $Q_1 | Q_0 p_1 p_2+4$ and $(Q_0 p_1 p_2+4)/Q_1$ has a bounded number

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of prime factors, where Q_0, Q_1 are large integers whose purpose we will describe later. In particular, we exploit special features of the half-dimensional sieve using an ingenious observation of Huxley and Iwaniec [18]. Further, in order to find suitable Gaussian primes in narrow sectors we use a classical result of Hecke together with nontrivial bounds on exponential sums over finite fields to control sums of integral lattice points in narrow sectors with norms lying in arithmetic progressions to large moduli.

The subsequence of almost primes $\{n_\ell\}$ constructed as described above creates the imbalance in the spectral equation (1.2) by boosting the contribution of the terms $m = Q_0 n_\ell, Q_0 n_\ell + 4$, without perturbing the target measure(s). The next step in our argument is to show that this imbalance typically overwhelms the contribution of the remaining terms. To do this, we first show in Section 3 that for all new eigenvalues lying outside a small exceptional set the spectral equation (1.2) can be effectively truncated to integers m with essentially $|m - \lambda| \ll (\log \lambda)^{10}$. This is done by controlling sums of $r(n)$ over short intervals and uses a second moment estimate of the Dedekind zeta-function $\zeta_{\mathbb{Q}(i)}$. In Section 4, we apply this result to new eigenvalues which lie between $Q_0 n_\ell$ and $Q_0 n_\ell + 4$ and show that for almost all such new eigenvalues the remaining terms in the spectral sum (i.e., $|m - \lambda| \ll (\log \lambda)^{10}, m \neq Q_0 n_\ell, Q_0 n_\ell + 4$) is relatively small, provided that we take Q_0, Q_1 sufficiently large thereby boosting the contribution of the closest two terms. This is accomplished by using bounds for sums of multiplicative functions over polynomials due to Henriot [17]. Crucially, we need good estimates for these sums in terms of the discriminant of the polynomials.

Finally, to get complete control on the first measure in Theorem 1.2 we choose Q_0 so that it is the product of a given fixed integer m_0 and large primes $p_k = a^2 + b^2$ with $0 \le \arctan(b_k/a_k) \le p_k^{-1/10}$ so that the probability measure on S^1 associated with $Q_0 n_\ell$ weakly converges to the measure associated with m_0 as $\ell \to \infty$. This last construction uses work of Ricci [35] on Gaussian primes in narrow sectors.

1.5. Notation

We write $f(x) \ll g(x)$ provided that $f(x) = O(g(x))$. Additionally, if for all x under consideration $|f(x)| \ge c g(x)$ we write $f(x) \gg g(x)$. If we have both $f(x) \ll g(x)$ and $f(x) \gg g(x)$, we write $f(x) \asymp g(x)$. For some additional notation related to sieves, see Section 2.1.1.

2. Sieve estimates

Let B_0 be a sufficiently large integer, define $\varepsilon = (\log\log x)^{-1/11}$, and let

$$\begin{aligned} \mathcal{P}_{\varepsilon,x} &= \{p \ge (\log x)^{B_0} : p = a^2 + b^2 \text{ and } 0 < \arctan(b/a) \le \varepsilon\}, \\ \mathcal{P}'_{\varepsilon,x} &= \{p \in \mathcal{P}_\varepsilon : p \le x^{1/9}\}. \end{aligned} \tag{2.1}$$

For brevity, we will write $\mathcal{P}_\varepsilon$ and $\mathcal{P}'_\varepsilon$ for $\mathcal{P}_{\varepsilon,x}$ and $\mathcal{P}'_{\varepsilon,x}$, respectively. Also, given $f, g : \mathbb{N} \to \mathbb{C}$ we define the Dirichlet convolution of f and g by

$$(f * g)(n) = \sum_{ab=n} f(a) g(b).$$

Also, let $Q_0, Q_1 \le (\log x)^{1/10}$ be odd coprime integers whose prime factors are all $\equiv 1 \pmod 4$. Moreover, we assume that $Q_0 = f_0^2 e_0 r_0^{a_0}, Q_1 = f_1^2 e_1 r_1^{a_1}$, where e_0, e_1 are square-free, $f_0, f_1 \ll 1$ and r_0, r_1 are primes congruent to $1 \pmod 4$. Throughout, the arithmetic function $b(n)$ is the indicator function of the set of integers which are representable as a sum of two squares. Also, for $\mathcal{S} \subset \mathbb{N}$ we define

$$1_{\mathcal{S}}(n) = \begin{cases} 1 & \text{if } n \in \mathcal{S}, \\ 0 & \text{otherwise,} \end{cases}$$

and let $\varphi(n) = \#\{m < n : (m, n) = 1\}$.

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Proposition 2.1. *Let $\eta > 0$ be sufficiently small, and let $y = x^\eta$. Suppose $y > Q_0Q_1$. Then*

$$\sum_{\substack{x \le n \le 2x \\ Q_1 | Q_0 n+4 \\ (\frac{Q_0 n+4}{Q_1}, \prod_{p \le y} p)=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) b(Q_0 n + 4) \asymp \frac{\varepsilon^2 Q_0}{\eta^{1/2}\varphi(Q_0)} \cdot \frac{x \log\log x}{\varphi(Q_1)(\log x)^2}.$$

This proposition builds on a result of Friedlander and Iwaniec [15, Ch. 4]. The main novelty here is that we capture almost primes $n = p_1p_2$ such that each prime factor $p = a^2 + b^2$, with $0 \le b \le a$, has the property that $a + ib$ lies within a certain small sector.

We also will require the following result.

Proposition 2.2. *We have that*

$$\sum_{\substack{x \le n \le 2x \\ Q_1 | Q_0 n+4}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) b(Q_0 n + 4) \asymp \varepsilon^2 \frac{x \log\log x}{\varphi(Q_1)(\log x)^{3/2}}.$$

Since Proposition 2.2 follows from a similar, yet simpler argument than the one used to prove Proposition 2.1, we will omit its proof. The rest of this section will be devoted to proving Proposition 2.1.

2.1. The Rosser–Iwaniec sieve

Let us first introduce the Rosser–Iwaniec β-sieve and the classical sieve terminology. We start with a sequence of $\mathcal{A} = \{a_n\}$ of nonnegative real numbers, a set of primes $\mathcal{P}$ and a parameter z. Define

$$P(z) = \prod_{\substack{p \in \mathcal{P} \\ p < z}} p. \tag{2.2}$$

Our goal is to obtain an estimate for the sieved set

$$\mathcal{S}(\mathcal{A}, \mathcal{P}, z) := \sum_{\substack{n \le x \\ (n, P(z))=1}} a_n.$$

This will be accomplished through calculating, for square-free $d \in \mathbb{N}$,

$$A_d(x) := \sum_{\substack{n \le x \\ n \equiv 0 \pmod d}} a_n. \tag{2.3}$$

We now make the hypothesis that our estimate for $A_d(x)$ will be of the form

$$A_d(x) = g(d)X + r_d, \tag{2.4}$$

where $g(d)$ is a multiplicative function with $0 \le g(p) < 1$. The number r_d should be thought of as a remainder term, so X is an approximation to $A_1(x)$, and the function $g(d)$ can be interpreted as a density.

Let

$$V(z) = \prod_{p | P(z)} (1 - g(p)).$$

We further suppose for all $w < z$ that

$$\frac{V(w)}{V(z)} = \prod_{\substack{w \le p < z \\ p \in \mathcal{P}}} (1 - g(p))^{-1} \le \left(\frac{\log z}{\log w}\right)^{\kappa} \left(1 + O\left(\frac{1}{\log w}\right)\right) \tag{2.5}$$

for some $\kappa > 0$. The constant κ is referred to as the *dimension of the sieve*.

Our arguments also require sieve weights. Let $\Lambda = \{\lambda_d\}_d$ be a sequence of real numbers, where d ranges over square-free integers. The sequence Λ is referred to as an upper bound sieve provided that

$$1_{n=1} = \sum_{d|n} \mu(d) \le \sum_{d|n} \lambda_d, \qquad \forall n \in \mathbb{N}, \tag{2.6}$$

where $1_{n=1}$ equals one if $n = 1$ and equals zero otherwise. We call Λ a lower bound sieve if

$$\sum_{d|n} \lambda_d \le 1_{n=1}, \qquad \forall n \in \mathbb{N}. \tag{2.7}$$

For a sieve $\Lambda = \{\lambda_d\}$, we use the notation

$$(\lambda * 1)(n) = \sum_{d|n} \lambda_d. \tag{2.8}$$

(This will be used to show the existence of primes, or almost primes, with desired properties.) Additionally, we say that the sieve Λ has *level D* if $\lambda_d = 0$ for $d > D$.

Given $\kappa > 0$, the β-sieve gives both an upper and lower bound for $\mathcal{S}(\mathcal{A}, \mathcal{P}, z)$ whenever $s = \log D/\log z$ is sufficiently large in terms of κ. The bounds consist of an error term, which is a sum of the remainder terms $|r_d|$ for $d \le D$ and a main term $XV(z)F(s)$, $XV(z)f(s)$ (resp.), where F, f are certain continuous functions with $0 \le f(s) < 1 < F(s)$. For precise definitions, motivation and context, we refer the reader to [15, Chapter 11].

Theorem 2.1 (Cf. [15, Theorem 11.13]). *Let $D \ge z$, and write $s = \frac{\log D}{\log z}$. Then there exists β-sieve weights such that*

$$\mathcal{S}(A, \mathcal{P}, z) \le XV(z)\Big(F(s) + O((\log D)^{-1/6}\Big) + R(D, z)$$
$$\mathcal{S}(A, \mathcal{P}, z) \ge XV(z)\Big(f(s) + O((\log D)^{-1/6}\Big) - R(D, z)$$

for $s \ge \beta(\kappa) - 1$ and $s \ge \beta(\kappa)$ (resp.), where

$$R(D, z) \le \sum_{\substack{d \le D \\ d|P(z)}} |r_d|$$

and $\beta(\kappa)$ denotes the sifting limit of dimension κ (cf. [15, Ch. 6.4].)

In particular, note that for $\kappa = 1/2$ (which is of particular interest to us since we sieve out by the density $1/2$ sequence of primes $\equiv 3 \pmod 4$ to detect sums of two squares), it is well known that $\beta(\kappa) = 1$ (e.g., see [15, Ch. 14.2]), which will be important for us as the 'sifting variable' s (which measures the sifting range relative to the sifting level, for example, smaller s corresponds to a smaller sifting range) only needs to be ≥ 1 to provide a lower bound for $\mathcal{S}(A, \mathcal{P}, z)$, whereas for the linear sieve

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$\beta(1) = 2$ so that one needs $s \geq 2$. In our arguments, we will use β-sieve weights, which are as defined in [15] Sections 6.4–6.5. In particular for these weights, we have $|\lambda_d| \leq 1$. We will sometimes refer to the fundamental lemma of the sieve, by which we mean the following result (see [15, Lemma 6.11].)

Theorem 2.2. *Let $\Lambda^{\pm} = \{\lambda_d^{\pm}\}$ be upper and lower bound (resp.) β-sieves of level D with $\beta \geq 4\kappa + 1$. Also, let $s = \log D/\log z$. Then for any multiplicative function satisfying equation (2.5) and $s \geq \beta + 1$ we have*

$$\sum_{d|P(z)} \lambda_d^{\pm} g(d) = V(z)\Big(1 + O\Big(s^{-s/2}\Big)\Big).$$

We also require the following estimate for the convolution of two sieves (see equation (5.97) and Theorem 5.9 of [15]).

Theorem 2.3. *Let $\Lambda_1 = \{\lambda_d\}$ and $\Lambda_2 = \{\lambda_d'\}$ be upper-bound sieve weights of level D_1, D_2 (resp.). Also, let g_1, g_2 be multiplicative functions satisfying equation (2.5) with $\kappa = 1$. Then*

$$\Bigg| \sum_{\substack{d,e \\ (d,e)=1}} \lambda_d \lambda_e' g_1(d) g_2(e) \Bigg| \leq (4e^{2\gamma} + o(1)) \prod_p (1 + h_1(p)h_2(p)) \prod_{j=1}^{2} \prod_{p<D_j} (1 - g_j(p))$$

as $\min\{D_1, D_2\} \to \infty$, where for $j = 1, 2$, $h_j(n) = g_j(n)(1 - g_j(n))^{-1}$ and γ is Euler's constant.

If in addition $g_1(p), g_2(p) \leq 1/p$ so that $h_1(p)h_2(p) \ll 1/p^2$, which will be the case for us, then

$$\Bigg| \sum_{\substack{d,e \\ (d,e)=1}} \lambda_d \lambda_e' g_1(d) g_2(e) \Bigg| \leq C \prod_{p<D_1} (1 - g_1(p)) \prod_{p<D_2} (1 - g_2(p)), \tag{2.9}$$

where $C > 0$ is an absolute constant.

2.1.1. Notation

We will also use the notation

$$P_3(z_1, z_2) := \prod_{\substack{z_1 \leq p \leq z_2 \\ p \equiv 3 \pmod 4}} p, \qquad \text{and} \qquad P_3(z) := P_3(3, z).$$

Additionally, let $1(n) = 1_{\mathbb{N}}(n) = 1$ denote the identity function and let $\tau(n) = (1 * 1)(n) = \sum_{d|n} 1$. Also, define

$$\mathcal{B}(x; q, a, \varepsilon) := \sum_{\substack{x \leq n \leq 2x \\ n \equiv a \pmod q}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}_\varepsilon'})(n) - \frac{1}{\varphi(q)} \sum_{\substack{x \leq n \leq 2x \\ (n,q)=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}_\varepsilon'})(n). \tag{2.10}$$

Further, $\delta > 0$ will denote a small, but fixed real number.

2.2. Preliminary lemmas

We begin by showing that the difference between the upper and lower bound sieves is 'small'.

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Lemma 2.1. *Let $\Lambda^{\pm} = \{\lambda_d^{\pm}\}$ be upper and lower bound linear sieves (resp.) each of level $w = x^{\sqrt{\eta}}$, where $\eta > 0$ is sufficiently small, whose sieve weights are supported on integers d such that $d|P(y)$, where $y = x^{\eta}$, $y > Q_0Q_1$, and $(d, 2Q_0f_1r_1) = 1$; in particular,*

$$\lambda_d^{\pm} = 0 \text{ if } (d, 2Q_0f_1r_1) > 1. \tag{2.11}$$

Then

$$\sum_{\substack{x\le n\le 2x\\ Q_1|Q_0n+4}} \left((\lambda^+ * 1)\left(\frac{Q_0n+4}{Q_1}\right) - (\lambda^- * 1)\left(\frac{Q_0n+4}{Q_1}\right)\right)(1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)$$
$$\ll \varepsilon^2\eta^{1/(4\eta^{1/2})-1}\frac{Q_0}{\varphi(Q_0)}\frac{x\log\log x}{\varphi(Q_1)(\log x)^2} + \frac{x}{(\log x)^{10}}.$$

Remark 2. We imposed the assumption that $\eta > 0$ is sufficiently small so small that the error $O(\eta^{1/(4\eta^{1/2})})$ in equation (2.16) is less than $1/2$. The requirement $y > Q_0Q_1$ is not essential; in the case $y < Q_0Q_1$ the argument proceeds similarly, but some additional, straightforward estimates are needed to treat the contribution of the primes between y and Q_0Q_1.

Proof. Switching order of summation, it follows that

$$\sum_{\substack{x\le n\le 2x\\ Q_1|Q_0n+4}} \left((\lambda^+ * 1)\left(\frac{Q_0n+4}{Q_1}\right) - (\lambda^- * 1)\left(\frac{Q_0n+4}{Q_1}\right)\right)(1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)$$
$$= \sum_{\pm} \pm \sum_{\substack{d<w\\ d|P(y)\\ (d,2Q_0f_1r_1)=1}} \lambda_d^{\pm} \sum_{\substack{x\le n\le 2x\\ Q_0n+4\equiv 0 \pmod{d)Q_1}}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n). \tag{2.12}$$

The inner sum on the right-hand side (RHS) of equation (2.12) equals

$$\frac{1}{\varphi(dQ_1)}\sum_{\substack{x\le n\le 2x\\ (n,dQ_1)=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) + \mathcal{B}(x; dQ_1, \gamma, \varepsilon), \tag{2.13}$$

where γ is the unique reduced residue $\pmod{dQ_1}$ satisfying $\gamma\cdot Q_0 \equiv -4 \pmod{dQ_1}$ and $\mathcal{B}$ is as defined in equation (2.10). Also,

$$\sum_{\substack{x\le n\le 2x\\ (n,dQ_1)=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) = \sum_{x\le n\le 2x} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) + O\left(\sum_{\substack{p_1p_2\le 2x\\ (p_1p_2,dQ_1)\ne 1}} 1_{\mathcal{P}_\varepsilon}(p_1)1_{\mathcal{P}'_\varepsilon}(p_2)\right). \tag{2.14}$$

Since $dQ_1 \le x^{1/9}$ (as η is small) and $p_2 \le x^{1/9}$ the contribution to the error term from $p_1p_2 \le x$ with $p_1|(p_1p_2, dQ_1)$ is $\ll \sum_{p_2\le x^{1/9}}\sum_{p_1\le x^{1/9}} 1 \ll x^{2/9}$. Also, since $p_2 \ge (\log x)^{B_0}$

$$\sum_{\substack{p_1p_2\le 2x\\ (p_1p_2,dQ_1)=p_2}} 1_{\mathcal{P}_\varepsilon}(p_1)1_{\mathcal{P}'_\varepsilon}(p_2) \le \sum_{\substack{p_2|dQ_1\\ p_2\ge(\log x)^{B_0}}}\sum_{p_1\le 2x/p_2} 1 \ll \frac{x}{\log x}\sum_{\substack{p_2|dQ_1\\ p_2\ge(\log x)^{B_0}}}\frac{1}{p_2} \ll \frac{x(\log\log x)}{(\log x)^{B_0}}. \tag{2.15}$$

Hence, using equations (2.13), (2.14) and (2.15) along with the fundamental lemma of the sieve (see Theorem 2.2 and recall $|\lambda_d| \le 1$) with $g(d) = \varphi(Q_1)/\varphi(Q_1 d)$,[2] and $s = \log w/\log y = \eta^{-1/2}$ we have that

$$\begin{aligned}
&\sum_{\substack{d<w\\ d|P(y)\\ (d,2Q_0)=1}} \lambda_d^{\pm} \sum_{\substack{x\le n\le 2x\\ Q_0 n+4\equiv 0 \pmod{d} Q_1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)\\
&= \frac{1}{\varphi(Q_1)} \sum_{x\le n\le 2x} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) \prod_{\substack{p\le y\\ (p,2Q_0 f_1 r_1)=1}} \left(1-\frac{\varphi(Q_1)}{\varphi(Q_1 p)}\right)(1+O(\eta^{1/(4\eta^{1/2})}))\\
&\quad + O\Bigg(\sum_{\substack{d<w\\ (d,2)=1}} |\mathcal{B}(x; dQ_1, \gamma, \varepsilon)|\Bigg) + O\left(\frac{x\log\log x}{(\log x)^{B_0-1}}\right).
\end{aligned} \tag{2.16}$$

Applying Theorem A.1 from the appendix, since $w = x^{\sqrt{\eta}} < x^{1/2-o(1)}$ we get that

$$\sum_{\substack{d<w\\ (d,2)=1}} |\mathcal{B}(x; dQ_1, \gamma, \varepsilon)| \ll \frac{x}{(\log x)^{10}}.$$

Using the two estimates above in equation (2.12) (note the main terms in equation (2.16) are the same for each of the sieves $\Lambda^{\pm}$ so they cancel in equation (2.12)) and applying equation (A.3) (with $q = 1$) from the appendix to estimate the sum over n completes the proof upon noting that

$$\prod_{\substack{p\le y\\ (p,2Q_0 f_1 r_1)=1}} \left(1-\frac{\varphi(Q_1)}{\varphi(Q_1 p)}\right) \asymp \frac{Q_0}{\varphi(Q_0)\log y} = \frac{Q_0}{\varphi(Q_0)\eta\log x}. \qquad \square$$

We next give a lower bound on the upper bound sieve, which together with Lemma 2.1 is strong enough (given suitable parameter choices) to show the existence of infinitely many integers with *exactly* two prime factors with the desired properties.

Lemma 2.2. *Let $w = x^{\sqrt{\eta}}$, $y = x^{\eta}$ and Λ^+ be as in Lemma 2.1. Let $\delta > 3\sqrt{\eta} > 0$ and $z = x^{\frac12-\delta}$. Then there exists a constant $C_1 > 0$ such that*

$$\sum_{\substack{x\le n\le 2x\\ (Q_0 n+4, P_3(y,z))=1\\ Q_1|Q_0 n+4}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)(\lambda^+ * 1)\left(\frac{Q_0 n+4}{Q_1}\right) \ge C_1 \frac{\varepsilon^2\delta^{1/2}}{\eta^{1/2}} \frac{Q_0}{\varphi(Q_0)} \frac{x\log\log x}{\varphi(Q_1)(\log x)^2}.$$

Proof. We now implement the sieve as discussed in Section 2. We start with the sifting sequence

$$\mathcal{A} = \left\{(1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})\left(\frac{m-4}{Q_0}\right)(\lambda^+ * 1)\left(\frac{m}{Q_1}\right) : Q_1|m, Q_0|m-4\right\}$$

and primes $\mathcal{P} = \{p \ge y : p \equiv 3 \pmod 4\}$. With (2.11) in mind, we may choose

$$\begin{aligned}
X &:= \sum_{\substack{e<w\\ e|P(y)}} \frac{\lambda_e^+}{\varphi(eQ_1)} \sum_{\substack{x\le n\le 2x\\ (n,Q_1 e)=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)\\
&= \sum_{\substack{x\le n\le 2x\\ (n,Q_1)=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) \sum_{\substack{e<w\\ e|P(y)\\ (e,2Q_0 f_1 r_1 n)=1}} \frac{\lambda_e^+}{\varphi(eQ_1)}.
\end{aligned} \tag{2.17}$$

[2]Note that g is multiplicative on the set of square-free d with $(d, f_1 r_1) = 1$.

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Arguing as in the proof of Lemma 2.1, to estimate the inner sum we apply the fundamental lemma of the sieve (see equation (2.16) and take $D = w$, $z = y$ in Theorem 2.2 and note that we then have $s = \eta^{-1/2}$) to get that it is

$$\frac{1}{\varphi(Q_1)} \sum_{\substack{e<w\\ e|P(y)\\ (e,2Q_0f_1r_1n)=1}} \frac{\lambda_e^+}{\varphi(eQ_1)}\varphi(Q_1) = \frac{1}{\varphi(Q_1)} \prod_{\substack{p\le y\\ (p,2Q_0f_1r_1n)=1}} \left(1 - \frac{\varphi(Q_1)}{\varphi(Q_1 p)}\right)(1 + O(\eta^{1/(4\eta^{1/2})})).$$

For $n = p_1p_2$, recalling that $f_1 \ll 1$ and r_1 is prime, the RHS above is

$$\asymp \frac{1}{\varphi(Q_1)} \prod_{\substack{p\le y\\ (p,Q_0)=1}} \left(1 - \frac{1}{p-1}\right) \asymp \frac{1}{\varphi(Q_1)} \cdot \frac{Q_0}{\varphi(Q_0)} \frac{1}{\log y}.$$

Using equations (2.14) and (2.15) along with the prime number theorem for Gaussian primes in sectors (see equations (A.1) and (A.3) in the appendix, with $q = 1$) yields

$$\sum_{\substack{x\le n\le 2x\\ (n,Q_1)=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) \sim 4\varepsilon^2 \cdot \frac{x \log\log x}{\log x}.$$

We conclude that

$$X \asymp \varepsilon^2 \frac{Q_0}{\varphi(Q_0)} \frac{x \log\log x}{\varphi(Q_1)(\log y)(\log x)}. \tag{2.18}$$

For $d|P_3(y, z)$, note that $(d, eQ_0Q_1) = 1$ for e such that $p|e \Rightarrow p < y$, and $(1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) = 0$ if $(d, n) \ne 1$. To apply the sieve, we require an estimate for A_d (cf. equations (2.3) and (2.4) for the definition of A_d) and recalling our choice for X and the definition of $\mathcal{B}$ in equation (2.10) it follows that

$$\begin{aligned} A_d(Q_0x + 4) &:= \sum_{\substack{x\le n\le 2x\\ Q_1|Q_0n+4\\ Q_0n+4\equiv 0 \pmod d}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)(\lambda^+ * 1)\left(\frac{Q_0n + 4}{Q_1}\right) \\ &= \sum_{\substack{e<w\\ e|P(y)}} \lambda_e^+ \sum_{\substack{x\le n\le 2x\\ Q_0n+4\equiv 0 \pmod{eQ_1}\\ Q_0n+4\equiv 0 \pmod d}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) \\ &= \sum_{\substack{e<w\\ e|P(y)}} \frac{\lambda_e^+}{\varphi(deQ_1)} \sum_{\substack{x\le n\le 2x\\ (n,Q_1e)=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) + r_d = \frac{1}{\varphi(d)}X + r_d, \end{aligned} \tag{2.19}$$

where

$$r_d \ll \sum_{\substack{e<w\\ (e,2)=1}} |\mathcal{B}(x; deQ_1, \gamma, \varepsilon)| \tag{2.20}$$

and γ is the unique residue class $\pmod{deQ_1}$ with $Q_0\gamma \equiv -4 \pmod{eQ_1}$ and $Q_0\gamma \equiv -4 \pmod d$; also note that $(d, eQ_1) = 1$ and $\mathcal{B}$ is as in equation (2.10).

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Hence, the half-dimensional Rosser–Iwaniec sieve Theorem 2.1 with set of primes $\mathcal{P} = \{p \equiv 3 \pmod 4 : p > y\}$ (so that equation (2.21) holds with $\kappa = 1/2$), gives for any $D \geq z$ with $s = \log D/\log z$

$$\begin{aligned}\sum_{\substack{x \leq n \leq 2x \\ (Q_0 n+4, P_3(y,z))=1 \\ Q_1 | Q_0 n+4}} &(1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)(\lambda^+ * 1)\left(\frac{Q_0 n + 4}{Q_1}\right) \\ &\geq XV(z)\left(f(s) + O\left(\frac{1}{(\log D)^{1/6}}\right)\right) - \sum_{\substack{d<D \\ d | P_3(y,z)}} |r_d|,\end{aligned} \tag{2.21}$$

where

$$V(z) = \prod_{\substack{y \leq p \leq z \\ p \equiv 3 \pmod 4}} \left(1 - \frac{1}{p-1}\right) \asymp \sqrt{\frac{\log y}{\log z}} \asymp \eta^{1/2}. \tag{2.22}$$

Taking $D = z^{1+\delta}$, so $s = 1+\delta$, we have by Theorem A.1, which is proved in the appendix, that (taking $q = edQ_1$)

$$\sum_{\substack{d<D \\ d | P_3(y,z)}} |r_d| \ll \sum_{\substack{q < DQ_1 w \\ (q,2)=1}} \left(\tau(q) \max_{(a,q)=1} |\mathcal{B}(x; q, a, \varepsilon)|\right) \ll \frac{x}{(\log x)^3}. \tag{2.23}$$

Here, note that $DQ_1 w < x^{\frac{1}{2} - \frac{\delta}{2} + \sqrt{\eta}} < x^{\frac{1}{2} - \frac{\delta}{6}}$, and the contribution of the divisor function is handled by using Cauchy–Schwarz along with the trivial bound $|\mathcal{B}(x; q, a, \varepsilon)| \ll x/q$. Also, note that $f(t) \sim 2\sqrt{\frac{e^\gamma}{\pi}} \cdot \sqrt{t-1}$ as $t \to 1^+$ (see the equation after (14.3) of [15]), so $f(s) = f(1+\delta) \gg \sqrt{\delta}$. Using this along with equations (2.17), (2.22) and (2.23) in equation (2.21) completes the proof. □

Sieving as in the previous lemma, we will now deduce the claimed upper bound in Proposition 2.1.

Lemma 2.3. *Let $\eta > 0$ be sufficiently small and $y = x^\eta$, with $y > Q_0 Q_1$. Then*

$$\sum_{\substack{x \leq n \leq 2x \\ Q_1 | Q_0 n+4 \\ (\frac{Q_0 n+4}{Q_1}, \prod_{p \leq y} p)=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) b(Q_0 n + 4) \ll \frac{\varepsilon^2 Q_0}{\eta^{1/2}\varphi(Q_0)} \cdot \frac{x \log\log x}{\varphi(Q_1)(\log x)^2}.$$

Proof. Write $Q_0 n + 4 = Q_1 f^2 s$, where s is square-free and note that since $((Q_0 n+4)/Q_1, P(y)) = 1$ all the prime divisors of f (and s as well) are $\geq y$, in particular f is coprime to $Q_0 Q_1$. We now note that $b(s) \leq 1_S(s)$ for $S = \{n : (n, P_3(y,z)) = 1\}$ and take Λ^+ to be the upper bound sieve from Lemma 2.1, which we use to bound the condition $((Q_0 n+4)/Q_1, \prod_{p \leq y} p) = 1$ to get that

$$\begin{aligned}&\sum_{\substack{x \leq n \leq 2x \\ Q_1 | Q_0 n+4 \\ (\frac{Q_0 n+4}{Q_1}, \prod_{p \leq y} p)=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) b(Q_0 n + 4) \\ &\leq \sum_{\substack{f \leq (\log x)^{10} \\ p | f \Rightarrow p > y}} \sum_{\substack{x \leq n \leq 2x \\ ((Q_0 n+4)/f^2, P_3(y,z'))=1 \\ f^2 Q_1 | Q_0 n+4}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)(\lambda^+ * 1)\left(\frac{Q_0 n+4}{Q_1}\right) + O(x/(\log x)^{10}),\end{aligned} \tag{2.24}$$

where the error term arises from the contribution of $f > (\log x)^{10}$. The following sieving argument is similar to the previous lemma (in fact we have already handled the case $f = 1$). For f as above let

$$X_f := \sum_{\substack{e<w \\ e|P(y)}} \frac{\lambda_e^+}{\varphi(eQ_1 f^2)} \sum_{\substack{x\le n\le 2x \\ (n, fQ_1e)=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) \asymp \varepsilon^2 \frac{Q_0}{\varphi(Q_0)} \frac{x\log\log x}{\varphi(Q_1 f^2)(\log y)(\log x)}, \tag{2.25}$$

where the last estimate follows from repeating the argument given in equation (2.18). Similarly, arguing as in equation (2.19) we have for each $d|P_3(y,z)$ with d coprime to f that

$$A_d(Qx+4; f) := \sum_{\substack{x\le n\le 2x \\ Q_1 f^2|Q_0 n+4 \\ Q_0 n+4\equiv 0 \pmod d}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)(\lambda^+ * 1)\left(\frac{Q_0 n+4}{Q_1}\right) = \frac{1}{\varphi(d)} X_f + r_{d,f},$$

where

$$r_{d,f} \ll \sum_{\substack{e<w \\ (e,2)=1}} \left|\mathcal{B}(x; def^2 Q_1, \gamma, \varepsilon)\right|.$$

We will now apply an upper bound sieve. For $D = x^{1/50}$ and $z' = x^{1/100}$, Theorem 2.1 with set of primes $\mathcal{P} = \{p \equiv 3 \pmod 4 : (p,f) = 1 \,\&\, p > y\}$ gives for each $f \le (\log x)^{10}$ with prime divisors all $> y$ that the inner sum on the RHS of equation (2.24) is

$$\le X_f V_f(z')\left(F(2) + O\left(\frac{1}{(\log D)^{1/6}}\right)\right) + \sum_{\substack{d<D \\ d|P_3(y,z), (d,f)=1}} |r_{d,f}|, \tag{2.26}$$

where the sum over d is $O(x^{1/25})$ since $D = x^{1/50}$, and

$$V_f(z') = \prod_{\substack{y\le p\le z' \\ p\equiv 3 \pmod 4,\ p\nmid f}} \left(1 - \frac{1}{p-1}\right) \ll \frac{f}{\varphi(f)} \eta^{1/2}, \tag{2.27}$$

where the upper bound follows from equation (2.22). Using equations (2.25) and (2.27) in equation (2.26) then applying the resulting estimate in equation (2.24) and summing over f completes the proof. □

2.3. *The Proof of Proposition 2.1*

We first require a Brun–Titchmarsh type bound for primes in narrow sectors.

Lemma 2.4. *Let $Q, q \le x^{2/3-o(1)}$ be odd. Then*

$$\sum_{\substack{p=a^2+b^2\le x \\ |\arctan(b/a)|\le\varepsilon \\ qp+4=Qp_1,\ p_1 \text{ prime}}} 1 \ll \varepsilon \frac{q}{\varphi(q)} \frac{x}{\varphi(Q)(\log x)^2}.$$

Remark 3. The point of the lemma is that it holds for large moduli $Q > x^{1/2}$. To accomplish this, we use asymptotic estimates for Gaussian integers $\alpha = a + ib$ with $N(\alpha) \le x$ and $N(\alpha) \equiv a \pmod Q$ **and** $|\arg(\alpha)| \le \varepsilon$, where $N(\alpha) = \alpha\overline{\alpha}$ is the norm of α. Details are given in Appendix, cf. section A.2.

The main step in the proof of Proposition 2.1 is the following lemma.

Lemma 2.5. *Let $z = x^{\frac{1}{2}-\delta}$, where $\delta > 0$ is sufficiently small and $y = x^{\eta}$ with $0 < \eta < 1/3$. There exists a constant $C_2 > 0$ such that*

$$\sum_{\substack{x\le n\le 2x\\ Q_1|Q_0n+4\\ (\frac{Q_0n+4}{Q_1},P(y)P_3(y,z))=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) = \sum_{\substack{x\le n\le 2x\\ Q_1|Q_0n+4\\ (\frac{Q_0n+4}{Q_1},P(y))=1\\ p|Q_0n+4\Rightarrow p\equiv 1 \pmod 4}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) + R,$$

where

$$0 \le R \le C_2\cdot\varepsilon^2\cdot\frac{\delta^{3/2}}{\eta^{1/2}}\frac{Q_0}{\varphi(Q_0)}\cdot\frac{x\log\log x}{\varphi(Q_1)(\log x)^2}.$$

Proof. By construction, if $(1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)\ne 0$, then $Q_0n+4\equiv 1 \pmod 4$ and $Q_1\equiv 1 \pmod 4$ so that $(Q_0n+4)/Q_1\equiv 1 \pmod 4$ and must have an even number of prime factors which are congruent to 3 (mod 4). Since $z > x^{1/4}$ the integers which contribute to R must have precisely two such prime factors. Dropping several conditions on the integers n which contribute to R, it follows that R is bounded by the number of integers $n = p_1p_2\le 2x$, $(1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)\ne 0$ such that $(Q_0n+4)/Q_1 = aq_1q_2$, where $b(a) = 1$, $(a, P(y)) = 1$, $q_1\equiv q_2\equiv 3 \pmod 4$ and q_1, q_2 are primes with $z < q_1, q_2\le 4Q_0x/Q_1$ so $a\le 4Q_0x/(Q_1z^2)$. By symmetry, it suffices to consider the terms with $q_1\le q_2$. We get that

$$R\le 2\sum_{p_2\le(2x)^{1/9}} 1_{\mathcal{P}'_\varepsilon}(p_2)\sum_{\substack{a\le\frac{4Q_0x}{Q_1z^2}\\ (a,P(y))=1}} b(a)\sum_{z<q_1\le\sqrt{\frac{4Q_0x}{aQ_1}}}\ \sum_{q_1\le q_2\le 4Q_0x/Q_1}\ \sum_{\substack{p_1\le 2x/p_2\\ Q_0p_1p_2+4=aq_1q_2Q_1}} 1_{\mathcal{P}_\varepsilon}(p_1). \tag{2.28}$$

Applying Lemma 2.4 with $q = Q_0p_2$ and $Q = aq_1Q_1$,

$$\sum_{\substack{p_1\le 2x/p_2\\ Q_0p_1p_2+4=aq_1q_2Q_1}} 1_{\mathcal{P}_\varepsilon}(p_1) \ll \varepsilon\frac{Q_0}{\varphi(Q_0)}\frac{x}{\varphi(aQ_1)q_1p_2(\log x)^2}. \tag{2.29}$$

Note that $x/p_2\ge 2^{-1/9}x^{8/9}$ and $Q_0p_2, aq_1Q_1\le\left(\frac{x}{p_2}\right)^{2/3-o(1)}$, for $\delta > 0$ sufficiently small, so the application of Lemma 2.4 is valid.

We claim that

$$\sum_{\substack{a\le\frac{4Q_0x}{Q_1z^2}\\ (a,P(y))=1}}\frac{b(a)}{\varphi(a)}\ll\sqrt{\frac{\log x/z^2}{\log y}}, \tag{2.30}$$

which we will justify below. Additionally,

$$\sum_{z<q_1\le\sqrt{\frac{4Q_0x}{aQ_1}}}\frac{1}{q_1}\sim\log\frac{\log\sqrt{\frac{2Q_0x}{aQ_1}}}{\log z}\ll\frac{\log\frac{x}{z^2}}{\log z}+\frac{\log Q_0}{\log z}\ll\frac{\log\frac{x}{z^2}}{\log z}\ll\delta. \tag{2.31}$$

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Therefore, using equations (2.29), (2.30) and (2.31) in equation (2.28) we conclude that

$$R \ll_\varepsilon \cdot \frac{Q_0}{\varphi(Q_0)} \cdot \frac{x \log x/z^2}{\varphi(Q_1)(\log x)^2 \log z}\sqrt{\frac{\log x/z^2}{\log y}} \sum_{p_2 \le (2x)^{1/9}} \frac{1_{\mathcal{P}'_\varepsilon}(p_2)}{p_2}$$
$$\ll_{\varepsilon^2} \cdot \frac{\delta^{3/2}}{\eta^{1/2}} \cdot \frac{Q_0}{\varphi(Q_0)} \cdot \frac{x \cdot \log\log x}{\varphi(Q_1)(\log x)^2}$$

as desired.

It remains to justify equation (2.30). Let $F(n)$ be the completely multiplicative function defined by $F(p) = 1$ if $p \ge y$ and zero otherwise. Then for all $t \ge y$, it follows from basic estimates for multiplicative functions (see (1.85) of [19]) that

$$\sum_{\substack{n \le t \\ (n,P(y))=1}} b(n)\frac{n}{\varphi(n)} \le \sum_{n \le t} b(n)\frac{n}{\varphi(n)}F(n)$$
$$\ll \frac{t}{\log t}\prod_{p \le t}\left(1 + \frac{b(p)F(p)}{p-1}\right) \ll \frac{t}{\sqrt{\log t \log y}}.$$

For $1 \le t \le y$, the sum on the left-hand side (LHS) is empty so the bound is true in that case as well. Hence, equation (2.30) follows from this estimate along with partial summation. □

Proof of Proposition 2.1. The upper bound has already been established in Lemma 2.3. It remains to establish the lower bound. Let δ be sufficiently small in terms of C_1 and C_2. Applying the inequality (2.7) for a lower bound sieve (also recall our notation (2.8)) along with Lemmas 2.1 and 2.2, using a lower bound sieve to take care of the condition $(\frac{Q_0 n+4}{Q_1}, P(y)) = 1$ (and recalling that $z = x^{\frac{1}{2}-\delta}$ for $\delta > 0$), we have that

$$\begin{aligned}
\sum_{\substack{x \le n \le 2x \\ Q_1 | Q_0 n+4 \\ (\frac{Q_0 n+4}{Q_1}, P(y)P_3(y,z))=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) &\ge \sum_{\substack{x \le n \le 2x \\ Q_1 | Q_0 n+4 \\ (Q_0 n+4, P_3(y,z))=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)(\lambda^- * 1)\left(\frac{Q_0 n+4}{Q_1}\right) \\
&= \sum_{\substack{x \le n \le 2x \\ Q_1 | Q_0 n+4 \\ (Q_0 n+4, P_3(y,z))=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n)(\lambda^+ * 1)\left(\frac{Q_0 n+4}{Q_1}\right) \\
&\quad + O\left(\varepsilon^2 \eta^{1/(4\eta^{1/2})-1}\frac{Q_0}{\varphi(Q_0)}\frac{x \log\log x}{\varphi(Q_1)(\log x)^2}\right) \\
&\ge C_1 \frac{\varepsilon^2 \delta^{1/2}}{\eta^{1/2}}\frac{Q_0}{\varphi(Q_0)}\frac{x \log\log x}{\varphi(Q_1)(\log x)^2}\left(1 + O\left(\frac{\eta^{\frac{1}{4\eta^{1/2}}-\frac{1}{2}}}{\delta^{1/2}}\right)\right).
\end{aligned} \tag{2.32}$$

Choosing η sufficiently small in terms of δ (which we choose in a way that only depends on C_1, C_2; see below) the O-term above is $\le 1/2$ in absolute value. Therefore, by equation (2.32) along with Lemma 2.5 it follows that

$$\sum_{\substack{x \le n \le 2x \\ Q_1 | Q_0 n+4 \\ (\frac{Q_0 n+4}{Q_1}, P(y)P_3(y,z))=1 \\ p | Q_0 n+4 \Rightarrow p \equiv 1 \pmod 4}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) \ge \left(\frac{C_1}{2}\frac{\varepsilon^2 \delta^{1/2}}{\eta^{1/2}} - \frac{C_2 \varepsilon^2 \delta^{3/2}}{\eta^{1/2}}\right)\frac{Q_0}{\varphi(Q_0)}\frac{x \log\log x}{\varphi(Q_1)(\log x)^2}.$$

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The term $\left(\frac{C_1}{2}\delta^{1/2} - C_2\delta^{3/2}\right)$ is positive for δ sufficiently small in terms of C_1 and C_2. Also, $b(Q_0 n+4) = 1$ for n such that all the prime factors of $Q_0 n+4$ are congruent to $1 \pmod 4$. This completes the proof. $\square$

3. Truncating the spectral equation

In this section, we show that it is possible to achieve a very short truncation of the spectral equation which holds for almost all new eigenvalues.

Theorem 3.1. *Let $A \geq 1$. Then for $B = B(A)$ sufficiently large, we have for every eigenvalue $\lambda_n \in \Lambda_{new} \cap [1, x]$ except those outside an exceptional set of size $O(x/(\log x)^A)$ that*

$$\sum_{m:|m-n|\leq \frac{n}{x}(\log x)^B} \frac{r(m)}{m-\lambda_n} = \begin{cases} \pi \log \lambda_n + O(1) & \text{in the weak coupling quantization,} \\ \frac{1}{\alpha} + O(1) & \text{in the strong coupling quantization.} \end{cases} \tag{3.1}$$

The above theorem is proved by capturing cancellation in the spectral equation even at very small scales for almost all new eigenvalues. This is done by showing that the average behavior of sums of $r(n)$ over even very short intervals is fairly regular.

Lemma 3.1. *Let $x \geq 3$ and $3 \leq L \leq x$. Then*

$$\frac{1}{x}\sum_{\ell\leq x}\left|\sum_{\ell\leq n\leq \ell+\frac{\ell}{L}} r(n) - \pi\frac{\ell}{L}\right|^2 \ll \frac{x}{L}(\log x)^2. \tag{3.2}$$

Proof. We repeat a classical argument, which was used by Selberg [41] to study primes in short intervals. Consider

$$\zeta_{\mathbb{Q}(i)} := \frac{1}{4}\sum_{n\geq 1}\frac{r(n)}{n^s} = L(s,\chi_4)\zeta(s) \qquad \mathrm{Re}(s) > 1,$$

where $L(s,\chi_4)$ is the Dirichlet L-function attached to the nontrivial Dirichlet character $\pmod 4$, and $\zeta(s)$ denotes the Riemann zeta-function. Note $L(1,\chi_4) = \pi/4$. Applying Perron's formula, then shifting contours to $\mathrm{Re}(s) = 1/2$ (which is valid since it is well known that $\zeta_{\mathbb{Q}(i)}(\sigma+it) \ll t^{1-\sigma+o(1)}$, for $0 \leq \sigma \leq 1$) and picking up a simple pole at $s = 1$ we see that for $v, v + v/L \notin \mathbb{Z}$

$$\begin{aligned}\sum_{v\leq n\leq v+\frac{v}{L}} r(n) &= \frac{1}{2\pi i}\int_{(2)} 4\zeta_{\mathbb{Q}(i)}(s)\frac{(v+\frac{v}{L})^s - v^s}{s}\,ds \\ &= 4L(1,\chi_4)\cdot\frac{v}{L} + \frac{v^{1/2}}{2\pi}\int_{\mathbb{R}} 4\zeta_{\mathbb{Q}(i)}(\tfrac{1}{2}+it)\frac{(1+\frac{1}{L})^{\frac{1}{2}+it}-1}{\frac{1}{2}+it}\cdot e^{it\log v}\,dt.\end{aligned}$$

Notice that the integral on the RHS is a Fourier transform. Writing $\nu = \log(1+\frac{1}{L})$, making a change of variables $v = e^{\tau}$ and then applying Plancherel's theorem yields

$$\begin{aligned}\frac{1}{x^2}\int_1^x\left(\sum_{v\leq n\leq v+\frac{v}{L}} r(n) - \pi\cdot\frac{v}{L}\right)^2 dv &\leq \int_{\mathbb{R}}\left(\sum_{e^\tau\leq n\leq e^{\tau+\nu}} r(n) - \pi\cdot\frac{e^\tau}{L}\right)^2\frac{d\tau}{e^\tau} \\ &= \frac{8}{\pi}\int_{\mathbb{R}}|\zeta_{\mathbb{Q}(i)}(\tfrac{1}{2}+it)|^2|w_\nu(\tfrac{1}{2}+it)|^2\,dt,\end{aligned}$$

where $w_\nu(s) = (e^{\nu s}-1)/s \ll \min\{\nu, 1/(1+|t|)\}$ uniformly for $\frac{1}{4} \leq \mathrm{Re}(s) \leq 1$. To estimate the integral on the RHS, we apply the well-known bound

$$\int_0^T |\zeta_{\mathbb{Q}(i)}(\tfrac{1}{2}+it)|^2\,dt \ll T(\log T)^2$$

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(see the introduction of [32]). Hence, we see that

$$\int_{\mathbb{R}} |\zeta_{\mathbb{Q}(i)}(\tfrac{1}{2}+it)|^2 |w_v(\tfrac{1}{2}+it)|^2\,dt \ll v^2 \int_{|t|\le 1/v} |\zeta_{\mathbb{Q}(i)}(\tfrac{1}{2}+it)|^2\,dt + \int_{|t|\ge 1/v} |\zeta_{\mathbb{Q}(i)}(\tfrac{1}{2}+it)|^2 \frac{dt}{t^2}$$
$$\ll v(\log 1/v)^2 \ll \frac{1}{L}(\log L)^2.$$

Combining the estimates above, we conclude that

$$\frac{1}{x}\int_x^{2x} \Big(\sum_{v\le n\le v+\frac{v}{L}} r(n) - \pi\frac{v}{L}\Big)^2 dv \ll \frac{x}{L}(\log x)^2. \tag{3.3}$$

We will now bound the sum over integers $\ell \le x$ on the LHS of equation (3.2) in terms of an integral over $1 \le v \le x$. Let

$$F(v) = \sum_{v\le n\le v+\frac{v}{L}} r(n) - \pi\cdot\frac{v}{L},$$

and let $v_\ell \in [\ell,\ell+1]$ be a point where the minimum of $|F(v)|$ on $[\ell,\ell+1]$ is achieved. Observe that

$$F(\ell) = F(v_\ell) + O(r(\ell) + r(\ell^*) + 1),$$

where $\ell^* = \lfloor \ell + 1 + (\ell+1)/L \rfloor$. Hence,

$$\frac{1}{x}\sum_{\ell\le x} F(\ell)^2 \ll \frac{1}{x}\sum_{\ell\le x} F(v_\ell)^2 + \frac{1}{x}\sum_{\ell\le x}(r^2(\ell) + r^2(\ell^*)) + 1$$
$$\ll \frac{1}{x}\int_1^x F(x)^2\,dx + \log x \ll \frac{x}{L}(\log x)^2,$$

where the last bound follows from equation (3.3) and the bound $\sum_{\ell\le x} r(l)^2 \ll x\log x$ (which in turns follows from a Wirsing type estimate (cf. [49]) or by taking $k = 1$, $R_1(X) = X$ and $F_1(n) = r(n)$ in Lemma 4.1). □

Lemma 3.2. *Let $A \ge 3$ and $x, Y \ge 3$. Then for all but $\ll x/(\log x)^A$ integers $m \in [1,x]$ we have*

$$\Bigg| \sum_{Y\frac{m}{x}<k\le x^{1/2}\frac{m}{x}} \frac{r(m+k)-r(m-k)}{k} \Bigg| \le \frac{(\log x)^{3A}}{\sqrt{Y}}.$$

Proof. Let

$$R_m(t) = \sum_{1\le k\le t} (r(m+k) - r(m-k)).$$

It suffices to consider $m \in [x/(\log x)^A, x]$. Hence, by summation by parts for each integer $m \in [x/(\log x)^A, x]$ we have that

$$\sum_{Y\frac{m}{x}<k\le x^{1/2}\frac{m}{x}} \frac{r(m+k)-r(m-k)}{k} = \frac{R_m(x^{1/2}\frac{m}{x})}{x^{1/2}\frac{m}{x}} - \frac{R_m(Y\frac{m}{x})}{Y\frac{m}{x}} + \int_{Y\frac{m}{x}}^{x^{1/2}\frac{m}{x}} \frac{R_m(t)}{t^2}\,dt.$$

Using this along with Chebyshev's inequality and the elementary inequality $(|a| + |b| + |c|)^2 \le 3^2(a^2 + b^2 + c^2)$, it follows that

$$\#\left\{\frac{x}{(\log x)^A} \le m \le x : \left|\sum_{Y\frac{m}{x} < k \le x^{1/3}\frac{m}{x}} \frac{r(m+k) - r(m-k)}{k}\right| \ge \frac{(\log x)^{3A}}{\sqrt{Y}}\right\}$$
$$\le 9\frac{Y}{(\log x)^{6A}} \sum_{\frac{x}{(\log x)^A} \le m \le x} \left(\frac{R_m\left(x^{1/2}\frac{m}{x}\right)^2 (\log x)^{2A}}{x} + \frac{R_m\left(Y\frac{m}{x}\right)^2 (\log x)^{2A}}{Y^2} + \left(\int_{Y\frac{m}{x}}^{x^{1/2}\frac{m}{x}} \frac{R_m(t)}{t^2}\,dt\right)^2\right). \tag{3.4}$$

In the integral, we make a change of variables and apply the Cauchy–Schwarz inequality to get for each $m \in [x/(\log x)^A, x]$ that

$$\left(\int_{Y\frac{m}{x}}^{x^{1/2}\frac{m}{x}} \frac{R_m(t)}{t^2}\,dt\right)^2 \le \frac{(\log x)^{2A}}{Y} \int_Y^{x^{1/2}} \frac{1}{t^2} R_m\left(t\frac{m}{x}\right)^2 dt. \tag{3.5}$$

Observe that

$$R_m\left(H\frac{m}{x}\right) = \sum_{m \le n \le m + \frac{m}{x}H} r(n) - \sum_{m - \frac{m}{x}H \le n \le m} r(n).$$

Hence, by Lemma 3.1 with $L = x/H$ (along with an analogue of this lemma for the second sum, which is proved in the same way) we get

$$\frac{1}{x}\sum_{m \le x} R_m\left(H\frac{m}{x}\right)^2 \ll H(\log x)^2,$$

for $1 \le H \le x/3$. Using this bound and equation (3.5) in equation (3.4) gives

$$\#\left\{\frac{x}{(\log x)^A} \le m \le x : \left|\sum_{Y\frac{m}{x} < k \le x^{1/2}\frac{m}{x}} \frac{r(m+k) - r(m-k)}{k}\right| \ge \frac{(\log x)^{3A}}{\sqrt{Y}}\right\}$$
$$\ll \frac{Y \cdot x}{(\log x)^{4A}}\left(\frac{(\log x)^2}{x^{1/2}} + \frac{(\log x)^2}{Y} + \frac{(\log x)^3}{Y}\right) \ll \frac{x}{(\log x)^{4A-3}}$$

since we may assume $Y \le x^{1/2}$, otherwise the set on the LHS above is empty. □

Before proving the main result of this section, we require the following technical lemma.

Lemma 3.3. *Let u, v be sufficiently large positive real numbers such that $v^{9/10} \le u \le 2v$. Let $t > 1$ be a real number that is not expressible as a sum of two squares such that $|u - t| \le v^{1/3}$. Then*

$$\sum_{m:|m-u|>v^{\frac{1}{2}}} r(m)\left(\frac{1}{m-t} - \frac{m}{m^2+1}\right) = -\pi \log t + O(1).$$

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Proof. Let $A(x) = \sum_{1\le n\le x} r(n) = \pi x + E(x)$. It is well known that (cf. [43]) that $E(x) \ll x^{\frac{1}{3}}$. Also, let $f_t(x) = \log \frac{|x-t|}{(x^2+1)^{1/2}}$, (so $f_t(x) \to 0$ as $x \to \infty$). Since $|u-t| \le v^{1/3}$, partial summation gives

$$\sum_{m:|m-u|>v^{\frac{1}{2}}} r(m)\left(\frac{1}{m-t} - \frac{m}{m^2+1}\right) = \int_{u+v^{\frac{1}{2}}}^{\infty} f_t'(x)dA(x) + \int_{1^-}^{(u-v^{\frac{1}{2}})^-} f_t'(x)dA(x)$$
$$= \pi\left(f_t(u-v^{\frac{1}{2}}) - f_t(u+v^{\frac{1}{2}}) - \log t\right)$$
$$+ O\left(1 + \max_{\pm} \frac{u^{\frac{1}{3}}}{|u \pm v^{\frac{1}{2}} - t|}\right).$$

The error is $O(1)$ since we assumed $|u-t| \le v^{1/3}$. Also,

$$f_t(u-v^{\frac{1}{2}}) - f_t(u+v^{\frac{1}{2}}) = \log \frac{|u-t-v^{\frac{1}{2}}|}{|u-t+v^{\frac{1}{2}}|} + O(1) \ll 1. \qquad \square$$

We are now ready to prove the main result of this section.

Proof of Theorem 3.1. Let $A \ge 1$. In the weak coupling quantization, it follows from the spectral equation (1.1) along with Lemma 3.3 that

$$\sum_{m:|m-n|\le\frac{n}{x}x^{1/2}} \frac{r(m)}{m-\lambda_n} = \pi \log \lambda_n + O(1) \tag{3.6}$$

for every integer $\frac{x}{(\log x)^A} \le n \le x$, which is a sum of two squares. Note that the application of Lemma 3.3 is justified since it is well known that $\lambda_n - n \le n^+ - n \le 10n^{1/4}$ (see for instance [31] p. 43).

In the strong coupling quantization, applying Lemma 3.3 twice we get for $\frac{x}{(\log x)^A} \le n \le x$ that

$$\left|\sum_{m:|m-n|>\frac{n}{x}x^{1/2}} r(m)\left(\frac{1}{m-\lambda_n} - \frac{m}{m^2+1}\right) - \sum_{m:|m-\lambda_n|>\lambda_n^{1/2}} r(m)\left(\frac{1}{m-\lambda_n} - \frac{m}{m^2+1}\right)\right| \ll 1.$$

Hence, using this along with the spectral equation (1.2) we have

$$\sum_{|m-n|\le\frac{n}{x}x^{1/2}} r(m)\left(\frac{1}{m-\lambda_n} - \frac{m}{m^2+1}\right) = \sum_{|m-\lambda_n|\le\lambda_n^{1/2}} r(m)\left(\frac{1}{m-\lambda_n} - \frac{m}{m^2+1}\right) + O(1)$$
$$= \frac{1}{\alpha} + O(1).$$

Hence, in the strong coupling quantization for each $\frac{x}{(\log x)^A} \le n \le x$

$$\sum_{m:|m-n|\le\frac{n}{x}x^{1/2}} \frac{r(m)}{m-\lambda_n} = \frac{1}{\alpha} + O(1). \tag{3.7}$$

For $\frac{x}{(\log x)^A} \le n \le x$, we now analyze the sum that appears on the LHS of both equations (3.6) and (3.7). Let $B \ge 1$, to be determined later, and consider

$$\sum_{|m-n|\le\frac{n}{x}x^{1/2}} \frac{r(m)}{m-\lambda_n} = \sum_{|m-n|\le\frac{n}{x}(\log x)^B} \frac{r(m)}{m-\lambda_n} + \sum_{\frac{n}{x}(\log x)^B<|k|\le\frac{n}{x}x^{1/2}} \frac{r(n+k)}{k-s_n}, \tag{3.8}$$

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where recall $s_n = \lambda_n - n$. Note that

$$\sum_{\substack{n\le x\\ s_n \ge (\log x)^{B/2}}} b(n) \le \frac{1}{(\log x)^{B/2}} \sum_{n\le x} b(n) s_n \le \frac{1}{(\log x)^{B/2}} \sum_{n\le x} b(n)(n^+ - n) \ll \frac{x}{(\log x)^{B/2}}.$$

Hence, for all but $O(x/(\log x)^{B/2})$ integers $n \le x$ which are representable as a sum of two squares, $s_n < (\log x)^{B/2}$. For these integers, the second sum on the RHS of equation (3.8) equals

$$\sum_{\frac{n}{x}(\log x)^B \le k \le \frac{n}{x} x^{1/2}} \frac{r(n+k) - r(n-k)}{k} + O\left((\log x)^{B/2} \sum_{\frac{n}{x}(\log x)^B \le |k| \le x^{1/2}} \frac{r(n+k)}{k^2}\right). \tag{3.9}$$

Since

$$\#\left\{\frac{x}{(\log x)^A} \le n \le x : (\log x)^{B/2} \sum_{\frac{n}{x}(\log x)^B \le |k| \le x^{1/2}} \frac{r(n+k)}{k^2} \ge 1\right\} \le (\log x)^{B/2} \sum_{(\log x)^{B-A} \le |k| \le x^{1/2}} \frac{1}{k^2} \sum_{n\le x} r(n+k) \ll \frac{x}{(\log x)^{B/2-A}}$$

the O-term in equation (3.9) is $\ll 1$ for all but $O(x/(\log x)^{B/2-A})$ integers $\frac{x}{(\log x)^A} \le n \le x$. The first sum in equation (3.9) is estimated using Lemma 3.2, with $Y = (\log x)^B$; so for $B \ge 6A$, this sum is $\ll 1$ for all but at most $\ll x/(\log x)^A$ integers $n \le x$. Hence, applying the two previous estimates in equation (3.9) and using the resulting bound along with equation (3.8) in equations (3.6) and (3.7) completes the proof upon taking $B \ge 6A$. □

4. Estimates for new eigenvalues nearby almost primes

In this section, we analyze the location of eigenvalues in Λ_{new} nearby certain integers which are almost primes. To state the result, let

$$\begin{aligned} \mathcal{N}_{1,x} &= \{n \in \mathbb{N} : (1_{\mathcal{P}_{\varepsilon,x}} * 1_{\mathcal{P}'_{\varepsilon,x}})(n) \ne 0,\ b(Q_0 n + 4) = 1,\ \&\ Q_1 | Q_0 n + 4\}, \\ \mathcal{N}_{2,x} &= \left\{n \in \mathcal{N}_{1,x} : \left(\frac{Q_0 n + 4}{Q_1}, P(y)\right) = 1\right\}, \end{aligned} \tag{4.1}$$

where $y = x^\eta$ with η as in Proposition 2.1 and $Q_0, Q_1, \varepsilon, 1_{\mathcal{P}_\varepsilon}$ and $b(\cdot)$ are as defined in the beginning of Section 2; we will write $\mathcal{N}_{1,x} = \mathcal{N}_1, \mathcal{N}_{2,x} = \mathcal{N}_2$ for brevity. For $j = 1, 2$, let $\mathcal{N}_j(x) = \mathcal{N}_{j,x} \cap [x, 2x]$. In particular, for each $n \in \mathcal{N}_2(x)$, $Q_0 n + 4 = Q_1 \ell_n$, where ℓ_n is an integer which is a sum of two squares. Moreover, since every prime divisor of ℓ_n is $\ge y = x^\eta$, we have for $n \le x$ that $x^{\eta\cdot\#\{p|\ell_n\}} \le \ell_n \le 2Q_0 x$ and

$$\#\{p|\ell_n\} \le \frac{2}{\eta}. \tag{4.2}$$

Note that by Propositions 2.1 and 2.2

$$\begin{aligned} \#\mathcal{N}_1(x) &\asymp \varepsilon^2 \frac{1}{\varphi(Q_1)} \frac{x \log\log x}{(\log x)^{3/2}}, \\ \#\mathcal{N}_2(x) &\asymp \varepsilon^2 \frac{Q_0}{\varphi(Q_0 Q_1)} \frac{x \log\log x}{(\log x)^2}. \end{aligned} \tag{4.3}$$

The main result of this section is the following proposition.

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Proposition 4.1. *For all $n \in \mathcal{N}_j(x)$, $j = 1, 2$, apart for elements in an exceptional set of size*

$$\ll \frac{\#\mathcal{N}_j(x)}{\varepsilon^2(\log\log x)^{1-o(1)}}$$

we have for $m = Q_0 n$ that $m^+ = m + 4$ and

$$\frac{r(m)}{m-\lambda_m} + \frac{r(m^+)}{m^+-\lambda_m} = \begin{cases} \pi \log \lambda_m + O\big((\log\log x)^5\big) & \text{in the weak coupling quantization,} \\ O\big((\log\log x)^5\big) & \text{in the strong coupling quantization.} \end{cases}$$

We also require a sieve estimate for averages of correlations of multiplicative functions. The following result is due to Henriot [17], which builds on the work of Nair and Tenenbaum [33]. See Corollary 1 of [17] and the subsequent remark therein. Recall that $\tau(n) = \sum_{d|n} 1$ denotes the divisor function, and for a polynomial $R = \sum a_n X^n \in \mathbb{Z}[X]$, let $\|R\|_1 = \sum |a_n|$ denote the norm of R.

Lemma 4.1. *Let $R_1(X), \dots, R_k(X) \in \mathbb{Z}[X]$ be irreducible, pairwise coprime polynomials, for which each polynomial R_j does not have a fixed prime divisor.*[3] *Let D be the discriminant of $R = R_1 \cdots R_k$ and $\varrho_{R_j}(n) = \#\{a \pmod n : R_j(a) \equiv 0 \pmod n\}$. Then there exist $C, c_0 > 0$ such that for any nonnegative multiplicative functions F_j, $j = 1, \dots, k$ with $F_j(n) \le \tau(n)^E$ for some $E \ge 1$, we have for $x \ge c_0\|R\|_1^{1/10}$ and some $A \ge 1$ that*

$$\sum_{n\le x}\prod_{j=1}^{k} F_j(|R_j(n)|) \ll \Delta_D^C\, x \prod_{p\le x}\left(1 - \frac{\varrho_R(p)}{p}\right)\prod_{j=1}^{k}\left(\sum_{n\le x}\frac{F_j(n)\varrho_{R_j}(n)}{n}\right),$$

where

$$\Delta_D := \prod_{p|D}\left(1 + \frac{1}{p}\right)$$

and the implicit constant, C and c_0 depend at most on the degree of R and E.

We first start with a technical lemma.

Lemma 4.2. *Let f be a nonnegative multiplicative function with $f(n) \le \tau(n)$ and $f(mn) \le \max\{1, f(n)\} f(m)$ for $m \in \mathbb{N}$ and n such that $b(n) = 1$. Then for $1 \le |h| \le x^{1/30}$, with $h \ne 4$ and $j = 1, 2$, we have*

$$\sum_{n\in\mathcal{N}_j(x)} f(Q_0 n + h) \ll \frac{1}{\varepsilon^2}\cdot g(h)\prod_{p|Q_0Q_1}\left(1+\frac{1}{p}\right)^C\prod_{p\le x}\left(1 + \frac{f(p)-1}{p}\right)\#\mathcal{N}_j(x), \tag{4.4}$$

where $C > 0$ is an absolute constant and

$$g(h) = \tau(|h|)\tau(|h-4|)\prod_{p|h}\left(1+\frac{1}{p}\right)^C\prod_{p|h-4}\left(1+\frac{1}{p}\right)^C.$$

[3] That is, polynomials $R(X)$ such that $p|R(n)$ for all $n \in \mathbb{Z}$ and p some prime; for example, $R(X) = 2X^2$ and $R(X) = X(X+1)$.

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Additionally, (for $h = 4$) there exists $C > 0$ such that

$$\sum_{n \in \mathcal{N}_1(x)} f(Q_0 n + 4) \ll \frac{1}{\varepsilon^2} \cdot f(Q_1) \prod_{p | Q_0 Q_1} \left(1 + \frac{1}{p}\right)^C \prod_{\substack{p \le x \\ p \equiv 1 \pmod 4}} \left(1 + \frac{f(p) - 1}{p}\right) \# \mathcal{N}_1(x).$$

Remark 4. When applying this lemma, we will take $f(n) = \frac{1}{4} \cdot r(n), b(n)$ or $2^{-\omega_1(n)}$, where $\omega_1(n) = \#\{p | n : p \equiv 1 \pmod 4\}$. The hypotheses of the lemma are satisfied for each of these choices.

Proof. Let $T_j = 2$ if $j = 1$ and $T_j = y = x^\eta$ if $j = 2$. Dropping several of the conditions on $n \in \mathcal{N}_j$, we get that (here $q < p$ denote primes; also recall that $P(z)$ is defined in equation (2.2))

$$\sum_{n \in \mathcal{N}_j(x/2)} f(Q_0 n + h) \le 2 \sum_{\substack{q \le \sqrt{x} \\ q \equiv 1 \pmod 4}} \sum_{\substack{p \le x/q \\ Q_1 | Q_0 pq + 4 \\ (\frac{Q_0 pq + 4}{Q_1}, P(T_j)) = 1}} b(Q_0 qp + 4) f(Q_0 qp + h). \tag{4.5}$$

Let $K = Q_0 q$ and $Y = x/q$. Note that the sum above is empty unless $(K, Q_1) = 1$. Since $(K, Q_1) = 1$, there exist integers $\overline{K}, \overline{Q_1}$ with $1 \le |\overline{K}| < Q_1$ and $1 \le |\overline{Q_1}| < K$ such that $K\overline{K} - Q_1 \overline{Q_1} = 1$. Also, for $Z \ge 1$ let F_Z be the totally multiplicative function given by $F_Z(p) = 1$ if $p \ge Z$ and zero otherwise. The inner sum on the RHS of equation (4.5) is bounded by

$$\begin{aligned} &\ll \sum_{n \le Y, Q_1 | Kn+4} F_{\sqrt{Y}}(n) F_{T_j}\left(\frac{Kn+4}{Q_1}\right) b(Kn+4) f(Kn+h) + Y^{1/2+o(1)} \\ &= \sum_{m \le \frac{Y - 4\overline{K}}{Q_1}} F_{\sqrt{Y}}(Q_1 m - 4\overline{K}) F_{T_j}\left(Km - 4\overline{Q_1}\right) b(KQ_1 m - 4Q_1 \overline{Q_1}) f(KQ_1 m + h - 4K\overline{K}) \\ &\quad + O(Y^{1/2+o(1)}), \end{aligned} \tag{4.6}$$

where the error term $Y^{1/2+o(1)}$ accounts for the primes $p \le \sqrt{Y}$. First, note $b(KQ_1 n - 4Q_1\overline{Q_1}) = b(Kn - 4\overline{Q_1})$. Let $d = (KQ_1, h - 4K\overline{K})$, and suppose that $h \ne 4$. We have

$$f(KQ_1 m + h - 4K\overline{K}) \le \max\{1, f(d)\} f\left(\frac{KQ_1}{d} m + \frac{h - 4K\overline{K}}{d}\right).$$

Let $R_1(X) = Q_1 X - 4\overline{K}$, $R_2(X) = KX - 4\overline{Q_1}$, $R_3(X) = \frac{KQ_1}{d} X + \frac{h - 4K\overline{K}}{d}$ and D denote the discriminant of $R = R_1 R_2 R_3$. The polynomials R_1, R_2, R_3 and multiplicative functions $F_1 = F_{\sqrt{Y}}$, $F_2 = F_{T_j} \cdot b$ and $F_3 = f$ satisfy the assumptions of Lemma 4.1. Also, for $(p, KQ_1) = 1$ we have $\varrho_R(p) = 3$ and $\varrho_{R_j}(p^k) = 1$ for each $j = 1, 2, 3$ and $k \ge 1$, which follows from Hensel's lemma. Hence, the sum in equation (4.6) is bounded by

$$\begin{aligned} &\ll \max\{1, f(d)\} \Delta_D^C \frac{Y}{Q_1} \prod_{p \le Y} \left(1 + \frac{F_{\sqrt{Y}}(p) + F_{T_j}(p) b(p) + f(p) - 3}{p}\right) \prod_{p | KQ_1} \left(1 + \frac{1}{p}\right)^C \\ &\ll \max\{1, f(d)\} \Delta_D^C \prod_{p | KQ_1} \left(1 + \frac{1}{p}\right)^C \frac{Y}{Q_1 (\log Y)^{3/2} (\log T_j)^{1/2}} \prod_{p \le Y} \left(1 + \frac{f(p) - 1}{p}\right). \end{aligned}$$

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Write $d = p_1^{a_1} \cdots p_\ell^{a_\ell}$. For each $j = 1, \ldots, \ell$, we have $p_j^{a_j} | h$ or $p_j^{a_j} | h - 4$ (depending on whether $p_j^{a_j} | K$ or $p_j^{a_j} | Q_1$, respectively); so $f(d) \ll \tau(|h|)\tau(|h-4|)$. Note the discriminant of R equals $D = 16\frac{K^2 Q_1^2}{d^4} h^2 (h-4)^2$ so that

$$\max\{1, f(d)\}\Delta_D^C \ll g(h) \prod_{p | Q_1 K} \left(1 + \frac{1}{p}\right)^C.$$

Also, since $Y = x/q \geq \sqrt{x}$, $\prod_{p \leq Y}\left(1 + \frac{f(p)-1}{p}\right) \ll \prod_{p \leq x}\left(1 + \frac{f(p)-1}{p}\right)$. Hence, applying the estimates above in equation (4.5), summing over q and using equation (4.3) gives the claimed bound for $h \neq 4$.

For $h = 4$, we argue similarly. Only now in order to estimate equation (4.6) we use Lemma 4.1 with R_1, R_2 as before, $R = R_1 R_2$ (so the discriminant is $D = 16$) and $F_1 = F_{\sqrt{Y}}$, $F_2 = b \cdot f$. Also, noting that here $d = Q_1$ we conclude that equation (4.6) is bounded by

$$\begin{aligned} &\ll f(Q_1) \prod_{p | Q_1 K} \left(1 + \frac{1}{p}\right)^C \frac{Y}{Q_1 (\log Y)^2} \prod_{p \leq x} \left(1 + \frac{b(p) f(p)}{p}\right) \\ &\ll f(Q_1) \prod_{p | Q_1 K} \left(1 + \frac{1}{p}\right)^C \frac{Y}{Q_1 (\log x)^{3/2}} \prod_{\substack{p \leq x \\ p \equiv 1 \pmod 4}} \left(1 + \frac{f(p) - 1}{p}\right). \end{aligned}$$

Hence, the claim follows in the same way as before. □

Lemma 4.3. *Let* $(\log\log x)^4 \leq U \leq \frac{1}{10}(\log x)^{1/2}$. *There exists* $C > 0$ *such that for all* $n \in \mathcal{N}_j(x)$, $j = 1, 2$, *outside a set of size*

$$\ll \frac{1}{\varepsilon^2} \cdot \#\mathcal{N}_j(x) \prod_{p | Q_1 Q_0} \left(1 + \frac{1}{p}\right)^C \frac{(\log\log x)^4}{U}$$

the following hold:

$$\sum_{\substack{1 \leq |k| \leq \frac{1}{U}(\log x)^{1/2} \\ k \neq 4}} b(Q_0 n + k) = 0, \tag{4.7}$$

$$\sum_{\substack{1 \leq |k| \leq \frac{n}{2x}(\log x)^B \\ k \neq 4}} \frac{r(Q_0 n + k)}{|k|} \leq U \tag{4.8}$$

and

$$\sum_{|k| \geq U} \frac{r(Q_0 n + k)}{k^2} \leq \frac{1}{\log\log x}. \tag{4.9}$$

Proof. We first establish equation (4.7). By Chebyshev's inequality

$$\#\left\{n \in \mathcal{N}_j(x) : \sum_{\substack{1 \leq |k| \leq \frac{1}{U}(\log x)^{1/2} \\ k \neq 4}} b(Q_0 n + k) \geq 1\right\} \leq \sum_{\substack{1 \leq |k| \leq \frac{1}{U}(\log x)^{1/2} \\ k \neq 4}} \sum_{n \in \mathcal{N}_j(x)} b(Q_0 n + k). \tag{4.10}$$

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Applying Lemma 4.2 to the inner sum and noting that

$$\prod_{p\le x}\left(1+\frac{b(p)-1}{p}\right) \ll \frac{1}{\sqrt{\log x}},$$

we get that the LHS of equation (4.10) is bounded by

$$\begin{aligned} &\ll \prod_{p|Q_1Q_0}\left(1+\frac{1}{p}\right)^C \frac{\#\mathcal{N}_j(x)}{\varepsilon^2\sqrt{\log x}} \sum_{\substack{1\le|k|\le\frac{1}{U}(\log x)^{1/2}\\ k\neq 4}} g(k)\\ &\ll \prod_{p|Q_1Q_0}\left(1+\frac{1}{p}\right)^C \frac{\#\mathcal{N}_j(x)}{\varepsilon^2}\frac{(\log\log x)^2}{U}, \end{aligned} \tag{4.11}$$

where the second step follows upon using Lemma 4.1.

To prove equation (4.8), we argue similarly and apply Lemmas 4.1 and 4.2 to get

$$\begin{aligned} \#\Bigg\{n\in\mathcal{N}_j(x) : &\sum_{\substack{1\le|k|\le\frac{n}{2x}(\log x)^B\\ k\neq 4}} \frac{r(Q_0n+k)}{|k|} > U\Bigg\}\\ &\le \frac{1}{U}\sum_{\substack{1\le|k|\le(\log x)^B\\ k\neq 4}} \frac{1}{|k|}\sum_{n\in\mathcal{N}_j(x)} r(Q_0n+k)\\ &\ll \frac{\#\mathcal{N}_j(x)}{\varepsilon^2U}\prod_{p|Q_0Q_1}\left(1+\frac{1}{p}\right)^C \sum_{\substack{1\le|k|\le(\log x)^B\\ k\neq 4}} \frac{g(k)}{|k|}\\ &\ll \frac{\#\mathcal{N}_j(x)}{\varepsilon^2U}\prod_{p|Q_0Q_1}\left(1+\frac{1}{p}\right)^C (\log\log x)^3. \end{aligned}$$

We will omit the proof of equation (4.9) since it follows similarly. □

For almost all $n\in\mathcal{N}_1(x)$, it is possible to show that $r(Q_0n+4)\asymp(\log n)^{\log 2/2\pm o(1)}$; however, since we do not actually need this estimate we will record the weaker estimate below, which suffices for our purposes and is simpler to prove.

Lemma 4.4. *Let $\nu>0$ be sufficiently small. There exists $C>0$ such that for all $n\in\mathcal{N}_1(x)$ outside a set of size*

$$\ll \frac{r(Q_1)}{\varepsilon^2}\#\mathcal{N}_1(x)\frac{(\log\log x)^C}{(\log x)^\nu}$$

the following holds

$$(\log x)^{1/4-\nu} \le r(Q_0n+4) \le (\log x)^{1/2+\nu}. \tag{4.12}$$

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Proof. We begin with proving the lower bound stated in equation (4.12). Let $\omega_1(n) = \sum_{\substack{p|n \\ p\equiv 1 \pmod 4}} 1$. For n which is a sum of two squares $r(n) \geq 2^{\omega_1(n)}$. Using this with Chebyshev's inequality and Lemma 4.2, the number of $n \in \mathcal{N}_1(x)$ which $r(Q_0 n+4) < (\log x)^{1/4-\nu}$ is bounded by

$$(\log x)^{1/4-\nu} \sum_{n\in\mathcal{N}_1(x)} 2^{-\omega_1(Q_0n+4)} \ll (\log x)^{1/4-\nu} \cdot \frac{(\log\log x)^C}{\sqrt{\log x}} \prod_{\substack{p\leq x \\ p\equiv 1 \pmod 4}} \left(1+\frac{1}{2p}\right) \cdot \frac{1}{\varepsilon^2}\#\mathcal{N}_1(x)$$
$$\ll \frac{1}{\varepsilon^2}\#\mathcal{N}_1(x)\frac{(\log\log x)^C}{(\log x)^\nu}$$

using Lemma 4.2.

The proof of the upper bound is similar: The number of $n \in \mathcal{N}_1(x)$ for which $r(Q_0n+4) > (\log x)^{1/2+\nu}$ is, again by using Lemma 4.2, bounded by

$$\frac{1}{(\log x)^{1/2+\nu}} \sum_{n\in\mathcal{N}_1(x)} r(Q_0n+4) \ll \frac{r(Q_1)(\log\log x)^C \cdot \#\mathcal{N}_1(x)}{\varepsilon^2\cdot(\log x)^{1/2+\nu}} \prod_{\substack{p\leq x \\ p\equiv 1 \pmod 4}} \left(1+\frac{r(p)-1}{p}\right)$$
$$\ll \frac{r(Q_1)(\log\log x)^C}{\varepsilon^2\cdot(\log x)^\nu}\#\mathcal{N}_1(x). \qquad \square$$

Proof of Proposition 4.1. By Theorem 3.1, for say $A = 3$, we get for all but $O(x/(\log x)^A)$ new eigenvalues $\lambda_\ell \leq 2x$ that

$$\sum_{|m-\ell|\leq\frac{\ell}{2x}(\log x)^B} \frac{r(m)}{m-\lambda_\ell} = \begin{cases} \pi\log\lambda_\ell + O(1) & \text{in the weak coupling quantization,} \\ \frac{1}{\alpha}+O(1) & \text{in the strong coupling quantization.} \end{cases}$$

We now consider integers $\ell = Q_0 n$ with $n \in \mathcal{N}_j(x)$, $j = 1,2$ such that the above holds. Writing $m = \ell+k$ and using Lemma 4.3 with $U = (\log\log x)^5$, we find, by equation (4.7), that $r(\ell+k)/(\ell+k-\lambda_l) = 0$ for $k \leq (\log x)^{1/2}/U$ unless $k = 0, 4$. Further, for $k \geq (\log x)^{1/2}/U$, we have $|r(\ell+k)/(\ell+k-\lambda_l)| \ll r(\ell+k)/k$, and it follows (cf. equation (4.8)) that for all but $O(\#\mathcal{N}_j/(\varepsilon^2(\log\log x)^{1-o(1)}))$ of these integers $n \in \mathcal{N}_j(x)$, $j = 1,2$, with $\ell = Q_0 n$ that $\ell^+ = \ell+4$ and

$$\sum_{|m-\ell|\leq\frac{\ell}{x}(\log x)^B} \frac{r(m)}{m-\lambda_\ell} = \frac{r(\ell)}{\ell-\lambda_\ell} + \frac{r(\ell^+)}{\ell^+-\lambda_\ell} + O\Big((\log\log x)^5\Big).$$

Combining the two estimates above completes the proof. $\square$

5. Proofs of the main theorems

5.1. *Quantization of observables*

On the unit cotangent bundle $\mathbb{S}^*M\cong\mathbb{T}^2\times S^1$, a smooth function $f \in C^\infty(T^2\times S^1)$ has the Fourier expansion

$$f(x,e^{i\phi}) = \sum_{\zeta\in\mathbb{Z}^2,k\in\mathbb{Z}} \widehat{f}(\zeta,k)e^{i\langle x,\zeta\rangle+ik\phi}.$$

Following Kurlberg and Ueberschär [27], we quantize our observables as follows. For $g \in L^2(\mathbb{T}^2)$, let

$$(\mathrm{Op}(f)g)(x) = \sum_{\xi\in\mathbb{Z}^2\setminus 0}\sum_{\zeta\in\mathbb{Z}^2,k\in\mathbb{Z}} \widehat{f}(\zeta,k)e^{ik\arg\xi}\widehat{g}(\xi)e^{i\langle\zeta+\xi,x\rangle} + \sum_{\zeta\in\mathbb{Z}^2,k\in\mathbb{Z}} \widehat{f}(\zeta,k)\widehat{g}(0)e^{i\langle\zeta,x\rangle}. \qquad (5.1)$$

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Hence, for pure momentum observables $f : S^1 \to \mathbb{R}$, one has

$$(\mathrm{Op}(f)g)(x) = \sum_{\xi \in \mathbb{Z}^2} f\left(\frac{\xi}{|\xi|}\right)\widehat{g}(\xi)e^{i\langle \xi, x\rangle}, \tag{5.2}$$

where for $\xi = 0$, $f(\frac{\xi}{|\xi|})$ is defined to be $f(1)$. (Another option is defining it as the average $\int_{S^1} f(\theta)\,\frac{d\theta}{2\pi}$; a technical point is that some choice must be made to extend f to a smooth observable on the cotangent bundle; for example, see [50, Section 3]. However, this choice only affects the matrix coefficients in equation (5.3) by $O(1/(\|G_\lambda\|_2 \cdot \lambda^2))$.)

Let g_λ be as given in equation (1.4). Then for f a pure momentum observable it follows from equations (1.4) and (5.2) that

$$\begin{aligned}\langle \mathrm{Op}(f)g_\lambda, g_\lambda\rangle &= \frac{1}{16\pi^4} \cdot \frac{1}{\|G_\lambda\|_2^2} \sum_{n\geq 0} \frac{1}{(n-\lambda)^2} \sum_{a^2+b^2=n} f\left(\frac{a+ib}{|a+ib|}\right) \\ &= \frac{1}{\sum_{n\geq 0} \frac{r(n)}{(n-\lambda)^2}} \sum_{n\geq 0} \frac{1}{(n-\lambda)^2} \sum_{a^2+b^2=n} f\left(\frac{a+ib}{|a+ib|}\right).\end{aligned} \tag{5.3}$$

5.2. *Measures associated to sequences of almost primes in narrow sectors*

Let $\mathcal{N}_1, \mathcal{N}_2$ be as in equation (4.1). Before proceeding to the main result of this section, we will specify our choice of Q_0, Q_1. Consider the set of primes

$$\mathcal{S} = \{p : p = a^2 + b^2, 0 \leq b \leq a \text{ and } 0 < \arctan(b/a) \leq p^{-1/10}\}, \tag{5.4}$$

and let q_j be the jth element of $\mathcal{S}$. It follows from work of Ricci [35, Th'm 2, p. 21–22] that

$$\#\{p \leq x : p \in \mathcal{S}\} \asymp \frac{x^{9/10}}{\log x},$$

so $q_j \asymp (j \log j)^{10/9}$. Let $T = \lfloor \log\log x \rfloor$, $H = \lfloor 100 \log\log\log x \rfloor$ and

$$Q_0' = \prod_{j=T}^{T+H-1} q_j, \qquad Q_1' = \prod_{j=T+H}^{T+2H-1} q_j. \tag{5.5}$$

Also, let $r_0, r_1 \in \mathcal{S}$ with $\frac{1}{4}\log\log x \leq r_0, r_1 \leq \frac{1}{2}\log\log x$ and $a_0, a_1 \in \mathbb{Z}$ with $0 \leq a_0, a_1 \leq \log\log\log x$. Let m_0, m_1 be integers, which are fixed (in terms of x), whose prime factors are all congruent to 1 (mod 4). Write $(m_0, m_1) = p_1^{e_1} \cdots p_s^{e_s}$, and let $g' = \widetilde{p}_1^{e_1} \cdots \widetilde{p}_s^{e_s}$, where $\frac{1}{2}\log\log x < \widetilde{p}_j < \log\log x$, $\widetilde{p}_j = c_j^2 + d_j^2$ with $0 \leq c_j \leq d_j$ and $\arctan(c_j/d_j) = \arctan(b_j/a_j) + O(1/(\log\log x)^{1/10})$, where $a_j^2 + b_j^2 = p_j$ with $0 \leq b_j \leq a_j$, for each $j = 1, \ldots, s$. (Note that such primes exist by Ricci's result on angular equidistribution of Gaussian primes.) We now take

$$Q_0 = Q_0' m_0 r_0^{a_0}, \qquad Q_1 = Q_1' \frac{m_1}{(m_0, m_1)} r_1^{a_1} g'. \tag{5.6}$$

Note that $(Q_0, Q_1) = 1$ and that $Q_0, Q_1 \ll \exp(200(\log\log\log x)^2) \leq (\log x)^{1/10}$ so that this choice of Q_0, Q_1 is consistent with our prior assumption. For $j = 1, 2$, let

$$\mathcal{M}_j(x) = \{x \leq m \leq 2x : m = Q_0 n \text{ and } n \in \mathcal{N}_j\}. \tag{5.7}$$

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It follows from equation (4.3) that

$$\#\mathcal{M}_1(x) \asymp \varepsilon^2 \frac{1}{\varphi(Q_1)} \frac{x \log\log x}{Q_0 (\log x)^{3/2}} \tag{5.8}$$

and

$$\#\mathcal{M}_2(x) \asymp \varepsilon^2 \frac{1}{\varphi(Q_0 Q_1)} \frac{x \log\log x}{(\log x)^2}. \tag{5.9}$$

Recall that we have chosen

$$\varepsilon = (\log\log x)^{-1/11}.$$

Lemma 5.1. *Let Q_0, Q_1 be as in equation (5.6) and $\eta > 0$ be as in Proposition 2.1. Let $m \in \mathcal{M}_j(x)$, $j = 1, 2$, where $\mathcal{M}_j(x)$ is defined as in equation (5.7). Then for $f \in C^1(S^1)$ with $|f'| \ll 1$*

$$\frac{1}{r(m)} \sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right) = \frac{1}{r(m_0)} \sum_{a^2+b^2=m_0} f\left(\frac{a+ib}{|a+ib|}\right) + O(\varepsilon). \tag{5.10}$$

Under the same hypotheses, we have for $m = Q_0 n \in \mathcal{N}_2(x)$ that there exists an integer ℓ_n which is a sum of two squares with $\#\{p | \ell_n\} \le 2/\eta$ such that

$$\frac{1}{r(m^+)} \sum_{a^2+b^2=m^+} f\left(\frac{a+ib}{|a+ib|}\right) = \frac{1}{r(m_1 \ell_n)} \sum_{a^2+b^2=m_1 \ell_n} f\left(\frac{a+ib}{|a+ib|}\right) + O(\varepsilon). \tag{5.11}$$

Proof. First, note that, for a unit, u of $\mathbb{Z}[i]$, that is, $u \in \{\pm 1, \pm i\}$, for any $n \in \mathbb{N}$

$$\sum_{a^2+b^2=n} f\left(\frac{u(a+ib)}{|a+ib|}\right) = \sum_{a^2+b^2=n} f\left(\frac{a+ib}{|a+ib|}\right). \tag{5.12}$$

For $m \in \mathcal{M}_j(x)$ with $j = 1$ or $j = 2$, write $m = Q_0' m_0 r_0^{a_0} n$, where $n \in \mathcal{N}_j(x)$. The factorizations of the ideals $(m) = ((a+ib)(a-ib))$ in $\mathbb{Z}[i]$ are in one-to-one correspondence with factorizations $(Q_0') = ((c+id)(c-id))$, $(m_0) = ((e+if)(e-if))$, $(r_0^{a_0}) = ((g+ih)(g-ih))$ and $(n) = ((k+il)(k-il))$ since Q_0', m_0, n are pairwise coprime. Hence, it follows from this and equation (5.12) that

$$\frac{1}{r(m)} \sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right) = \frac{1}{r(Q_0') r(m_0) r(r_0^{a_0}) r(n)} \sum_{\substack{\alpha \in \mathbb{Z}[i] \\ \alpha\overline{\alpha} = Q_0'}} \sum_{\substack{\beta \in \mathbb{Z}[i] \\ \beta\overline{\beta} = m_0}} \sum_{\substack{\gamma \in \mathbb{Z}[i] \\ \gamma\overline{\gamma} = r_0^{a_0}}} \sum_{\substack{\delta \in \mathbb{Z}[i] \\ \delta\overline{\delta} = n}} f\left(\frac{\alpha\beta\gamma\delta}{|\alpha\beta\gamma\delta|}\right). \tag{5.13}$$

Let $\mathcal{S}$ be as in equation (5.4), and write the jth element of $\mathcal{S}$ as $q_j = a_j^2 + b_j^2$, with $0 \le b_j \le a_j$. By construction, for $\alpha \in \mathbb{Z}[i]$ with $\alpha\overline{\alpha} = Q_0'$ we can write $\alpha = u \prod_{j \in J} (a_j + \epsilon_j i b_j)$ where $J = \{T, T+1, \ldots, T + H_1 - 1\}$, $\epsilon_j \in \{\pm 1\}$ and u is a unit. It follows that

$$\begin{aligned} \frac{\alpha}{|\alpha|} &= u \prod_{j \in J} \frac{a_j + \epsilon_j i b_j}{|a_j + i b_j|} \\ &= u\left(1 + O\left(\sum_{j \in J} |\arctan(b_j / a_j)|\right)\right) = u + O\left(\frac{1}{(\log\log x)^{1/11}}\right), \end{aligned}$$

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where the unit u depends on α. Also, for $\gamma \in \mathbb{Z}[i]$ with $\gamma\overline{\gamma} = r_0^{a_0}$, we have $\frac{\gamma}{|\gamma|} = u + O(1/(\log\log x)^{1/11})$, and for $\delta \in \mathbb{Z}[i]$ with $\delta\overline{\delta} = n$, we have $\frac{\delta}{|\delta|} = u + O(\varepsilon)$. Hence, by this and equation (5.12)

$$\sum_{\substack{\alpha\in\mathbb{Z}[i]\\ \alpha\overline{\alpha}=Q_0'}} \sum_{\substack{\beta\in\mathbb{Z}[i]\\ \beta\overline{\beta}=m_0}} \sum_{\substack{\gamma\in\mathbb{Z}[i]\\ \gamma\overline{\gamma}=r_0^{a_0}}} \sum_{\substack{\delta\in\mathbb{Z}[i]\\ \delta\overline{\delta}=n}} f\left(\frac{\alpha\beta\gamma\delta}{|\alpha\beta\gamma\delta|}\right) = \sum_{\substack{\alpha\in\mathbb{Z}[i]\\ \alpha\overline{\alpha}=Q_0'}} \sum_{\substack{\gamma\in\mathbb{Z}[i]\\ \gamma\overline{\gamma}=r_0^{a_0}}} \sum_{\substack{\delta\in\mathbb{Z}[i]\\ \delta\overline{\delta}=n}} \left(\sum_{\substack{\beta\in\mathbb{Z}[i]\\ \beta\overline{\beta}=m_0}} f\left(\frac{u_{\alpha,\gamma,\delta}\cdot\beta}{|\beta|}\right)\right) + O(\varepsilon r(m))$$
$$= r(Q_0)r(r_0^{a_0})r(n) \sum_{a^2+b^2=m_0} f\left(\frac{a+ib}{|a+ib|}\right) + O(\varepsilon r(m)),$$

thereby proving equation (5.10).

The proof of equation (5.11) follows along the same lines upon noting that for $m = Q_0 n \in \mathcal{M}_2(x)$ we can write $m^+ = Q_1' r_1^{a_1} \frac{m_1}{(m_1,m_0)} g'\ell_n$, where ℓ_n is a sum of two squares. Note that $Q_1', \frac{m_1}{(m_1,m_0)}, r_1^{a_1}, g', \ell_n$ are pairwise coprime by construction since all the prime divisors of ℓ_n are $\geq y$; the latter also implies that $\#\{p|\ell_n\} \leq 2/\eta$. Also, note that since all primes dividing g', Q_1', r_1 have very small Gaussian angles (by construction), the set of Gaussian angles associated with m^+ is very close to the set of angles associated with $m_1\ell_n$, after taking multiplicities into account. □

5.3. *Proof of Theorem 1.1*

Since Gaussian angles associated with inert primes are trivial, we can without loss of generality assume all the prime factors of m_0 are congruent to1 (mod 4). Let Q_0, Q_1 be as in equation (5.6) and $\mathcal{M}_1(x)$ be as in equation (5.7), and recall for $m \in \mathcal{M}_1(x)$ that $m = Q_0 n$, where $n \in \mathcal{N}_1(x)$ and $\mathcal{N}_1$ is as in equation (4.1). By equation (4.7) and Lemma 4.4 it follows that for all but at most $o(\#\mathcal{M}_1(x))$ integers $m \in \mathcal{M}_1(x)$ that $m^+ = m+4$, $(\log x)^{1/4-\nu} \leq r(m^+) \leq (\log x)^{1/2+\nu}$ (for any fixed $\nu > 0$) and $4 \leq r(m) \ll (\log x)^{o(1)}$. Combining this with Proposition 4.1, we get that for all but $o(\#\mathcal{M}_1(x))$ integers $m \in \mathcal{M}_1(x)$ that $\lambda_m - m = o(1)$ and moreover

$$\lambda_m - m \asymp \begin{cases} \dfrac{r(m)}{\log \lambda_m} & \text{in the weak coupling quantization,} \\ \dfrac{r(m)}{r(m^+)} & \text{in the strong coupling quantization.} \end{cases} \tag{5.14}$$

Also, note that, for such m as above, we also have $|\lambda_m - m^+| \geq 3$. Hence, using the above estimate along with equations (4.7) and (4.9) with $U = (\log\log x)^5$ we get for all but at most $o(\#\mathcal{M}_1(x))$ integers $m \in \mathcal{M}_1(x)$ that (in both cases)

$$\begin{aligned} \sum_{\ell\geq 0} \frac{r(\ell)}{(\ell-\lambda_m)^2} &= \frac{r(m)}{(m-\lambda_m)^2} + \frac{r(m^+)}{(m^+-\lambda_m)^2} + o(1) \\ &= \frac{r(m)}{(m-\lambda_m)^2}\left(1 + O\left(\frac{r(m^+)(m-\lambda_m)^2}{r(m)}\right)\right) + o(1) \\ &= \frac{r(m)}{(m-\lambda_m)^2}(1+o(1)). \end{aligned} \tag{5.15}$$

Similarly, for all but at most $o(\#\mathcal{M}_1(x))$ integers $m \in \mathcal{M}_1(x)$

$$\sum_{\ell\geq 0} \frac{1}{(\ell-\lambda_m)^2} \sum_{a^2+b^2=\ell} f\left(\frac{a+ib}{|a+ib|}\right) = \frac{1}{(m-\lambda_m)^2} \sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right) + O(r(m^+)). \tag{5.16}$$

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Therefore, combining equations (5.3), (5.14), (5.15) and (5.16), it follows for all but at most $o(\#\mathcal{M}_1(x))$ integers $m \in \mathcal{M}_1(x)$ we have that

$$\begin{aligned}\langle \mathrm{Op}(f)g_{\lambda_m}, g_{\lambda_m}\rangle =&(1+o(1))\frac{(m-\lambda_m)^2}{r(m)}\cdot\left(\frac{1}{(m-\lambda_m)^2}\sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right)+O(r(m^+))\right)\\ =&(1+o(1))\frac{1}{r(m)}\sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right)+o(1)\\ =&(1+o(1))\frac{1}{r(m_0)}\sum_{a^2+b^2=m_0} f\left(\frac{a+ib}{|a+ib|}\right)+O(\varepsilon),\end{aligned}$$

where the last step follows by equation (5.10). The estimate for the density of this subsequence of eigenvalues follows immediately from equation (5.8), noting that $Q_0, Q_1 \ll (\log x)^{o(1)}$.

5.4. Proof of Theorem 1.2

As before, without loss of generality, we can assume all the prime factors of m_0, m_1 are congruent to 1 (mod 4). For the sake of brevity, let $\mathcal{L}_2 = \log\log x$. Let Q_0, Q_1 be as in equation(5.6) and $\mathcal{M}_2(x)$ be as in equation (5.7), and recall for $m \in \mathcal{M}_2(x)$ that $m = Q_0 n$, where $n \in \mathcal{N}_2(x)$ where $\mathcal{N}_2$ is as in equation (4.1). Note for each $m \in \mathcal{M}_2(x)$ that $r(m) \gg \mathcal{L}_2^{10}$. Also, by construction $r(m)/r(m+4) \asymp \frac{a_0+1}{a_1+1}$, where H, a_0, a_1 are also as in equation (5.6), and note $a_0, a_1 \le \log\mathcal{L}_2$. Applying Proposition 4.1, we get that for all $m \in \mathcal{M}_2(x)$ outside an exceptional set of size $o(\#\mathcal{M}_2(x))$ that $m^+ = m+4$ and

$$\frac{\lambda_m - m}{m^+ - \lambda_m} = \frac{r(m)}{r(m^+)}\left(1+O\left(\frac{\mathcal{L}_2^6}{r(m)}\right)\right) = \frac{r(m)}{r(m^+)}\left(1+O\left(\mathcal{L}_2^{-4}\right)\right). \tag{5.17}$$

In particular, this implies that $\lambda_m - m \gg \mathcal{L}_2^{-1}$ and $m^+ - \lambda_m \gg \mathcal{L}_2^{-1}$. As before, using equations (4.7) and (4.9) with $U = \mathcal{L}_2^5$, we get for all but at most $o(\#\mathcal{M}_2(x))$ integers $m \in \mathcal{M}_2(x)$ that

$$\sum_{\ell\ge 0}\frac{r(\ell)}{(\ell-\lambda_m)^2} = \frac{r(m)}{(m-\lambda_m)^2} + \frac{r(m^+)}{(m^+-\lambda_m)^2} + O(\mathcal{L}_2^{-1}) \tag{5.18}$$

and

$$\begin{aligned}\sum_{\ell\ge 0}\frac{1}{(\ell-\lambda_m)^2}\sum_{a^2+b^2=\ell} f\left(\frac{a+ib}{|a+ib|}\right) =&\frac{1}{(m-\lambda_m)^2}\sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right)\\ &+\frac{1}{(m^+-\lambda_n)^2}\sum_{a^2+b^2=m^+} f\left(\frac{a+ib}{|a+ib|}\right)+O(\mathcal{L}_2^{-1}).\end{aligned} \tag{5.19}$$

Let $C_m = \frac{1}{1+r(m)/r(m^+)}$. Applying equations (5.17),(5.18) and (5.19) equation in (5.3), we get

$$\begin{aligned}\langle \mathrm{Op}(f)g_{\lambda_m}, g_{\lambda_m}\rangle =& (1+O(\mathcal{L}_2^{-1}))\left(\frac{r(m)}{(m-\lambda_m)^2}+\frac{r(m^+)}{(m^+-\lambda_m)^2}\right)^{-1}\\ &\times\left(\frac{1}{(m-\lambda_m)^2}\sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right)+\frac{1}{(m^+-\lambda_m)^2}\sum_{a^2+b^2=m^+} f\left(\frac{a+ib}{|a+ib|}\right)+O(\mathcal{L}_2^{-1})\right)\\ =& \frac{C_m}{r(m)}\sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right)+\frac{1-C_m}{r(m^+)}\sum_{a^2+b^2=m^+} f\left(\frac{a+ib}{|a+ib|}\right)+O(\mathcal{L}_2^{-1}).\end{aligned} \tag{5.20}$$

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Applying equation (5.10) to the first sum above, we get

$$\frac{C_m}{r(m)} \sum_{a^2+b^2=m} f\left(\frac{a+ib}{|a+ib|}\right) = \frac{C_m}{r(m_0)} \sum_{a^2+b^2=m_0} f\left(\frac{a+ib}{|a+ib|}\right) + O(\varepsilon). \tag{5.21}$$

Similarly, applying equation (5.11) to the second sum on the RHS of equation (5.20) we get that

$$\frac{1-C_m}{r(m^+)} \sum_{a^2+b^2=m^+} f\left(\frac{a+ib}{|a+ib|}\right) = \frac{1-C_m}{r(m_1\ell_n)} \sum_{a^2+b^2=m_1\ell_n} f\left(\frac{a+ib}{|a+ib|}\right) + O(\mathcal{L}_2^{-1/11}), \tag{5.22}$$

for some integer ℓ_n with $\#\{p : p|\ell_m\} \le 2/\eta$ by equation (4.2). Using equations (5.21) and (5.22) in equation (5.20) completes the proof upon recalling that $\varepsilon = \mathcal{L}_2^{-1/11}$. The estimate for the density of this subsequence of eigenvalues follows from equation (5.9).

5.5. *Proof of Theorem 1.3*

The proof of Theorem 1.3 relies on the following hypothesis concerning the distribution of primes.

Hypothesis 1. Let Q_1, Q_0 be as in equation (5.6) and $\varepsilon \ge (\log\log x)^{-1/2}$ be sufficiently small. Also, let $y = x^\eta$, where $\eta > 0$ is sufficiently small. Then the number of solutions $(u, v) \in \mathbb{Z}^2$ to

$$Q_1 u - Q_0 v = 4,$$

where $v = p_1 p_2$ and $u = p_3$ are primes satisfying $1_{\mathcal{P}_\varepsilon}(p_1) 1_{\mathcal{P}'_\varepsilon}(p_2) 1_{\mathcal{P}_\varepsilon}(p_3) = 1$, $p_3 > y$ such that $v \le x$ is

$$\gg \varepsilon^3 \frac{Q_0}{\varphi(Q_0 Q_1)} \frac{x \log\log x}{(\log x)^2},$$

where $\mathcal{P}_\varepsilon, \mathcal{P}'_\varepsilon$ are as in equation (2.1).

Proof of Theorem 1.3. Recall the definition of $\mathcal{N}_2$ given in equation (4.1). Let us define

$$\mathcal{N}_3 = \{n \in \mathcal{N}_2 : Q_0 n + 4 = Q_1 p, b(p) = 1, \ \& \ |\theta_p| \le \varepsilon\}.$$

Following equation (5.7), we also define

$$\mathcal{M}_3(x) = \{m \le x : m = Q_0 n \text{ and } n \in \mathcal{N}_3\}.$$

By Hypothesis 1 and equation (5.9), it follows that

$$\#\mathcal{M}_3(x) \asymp \varepsilon \#\mathcal{M}_2(x), \tag{5.23}$$

where we also have used an upper bound sieve to get that $\#\mathcal{M}_3(x) \ll \varepsilon\#\mathcal{M}_2(x)$. Observe that $\mathcal{M}_3(x) \subset \mathcal{M}_2(x)$ and the exceptional set in Proposition 4.1 is $o(\#\mathcal{M}_3(x))$ since we take $\varepsilon = (\log\log x)^{-1/4}$. Hence, we get that equation (5.17) holds for $m \in \mathcal{M}_3(x)$ outside an exceptional set of size $o(\#\mathcal{M}_3(x))$. Similarly, we can conclude that equations (5.18) and (5.19) also hold for all $m \in \mathcal{M}_3(x)$ outside an exceptional set of size $o(\#\mathcal{M}_3(x))$. Therefore, arguing as in equations 5.20–5.22 we conclude that for $m \in \mathcal{M}_3(x)$ outside an exceptional set of size $o(\#\mathcal{M}_3(x))$ we have that

$$\langle \mathrm{Op}(f) g_{\lambda_m}, g_{\lambda_m}\rangle = \frac{C_m}{r(m_0)} \sum_{a^2+b^2=m_0} f\left(\frac{a+ib}{|a+ib|}\right) + \frac{1-C_m}{r(m_1\ell_n)} \sum_{a^2+b^2=m_1\ell_n} f\left(\frac{a+ib}{|a+ib|}\right) + O(\mathcal{L}_2^{-1/11}), \tag{5.24}$$

where m_0, m_1 are arbitrary, fixed integers whose prime factors are all congruent to 1 (mod 4) and $C_m = 1/(1 + r(m)/r(m + 4))$. By our hypothesis, we have that $\ell_n = p$ with $|\theta_p| \le \varepsilon$ and $(m_1, p) = 1$. Hence, repeating the argument used to prove equation (5.10) it follows that

$$\frac{1}{r(m_1\ell_n)} \sum_{a^2+b^2=m_1\ell_n} f\left(\frac{a+ib}{|a+ib|}\right) = \frac{1}{r(m_1)} \sum_{a^2+b^2=m_1} f\left(\frac{a+ib}{|a+ib|}\right) + O(\varepsilon). \tag{5.25}$$

Given $0 < c < 1$ with $c = d/e \in \mathbb{Q}$, we will now specify our choice of a_0, a_1 (from equation (5.5)). Recall we allow a_0, a_1 to grow slowly with x and Q_0', Q_1' have the same number of prime factors. Also, by construction $r(\frac{m_1}{(m_0,m_1)} g') = r(m_1)$. Let $\mathcal{L} = \lfloor (\log\log\log x)^{1/2} \rfloor$. We take

$$a_0 = 2(e - d)r(m_1)\mathcal{L} \qquad \text{and} \qquad a_1 = dr(m_0)\mathcal{L}.$$

Hence,

$$C_m = \frac{1}{1 + \frac{8r(m_0)(a_0+1)}{16r(m_1)(a_1+1)}} = \frac{d}{e} + o(1). \tag{5.26}$$

We are now ready to complete the proof. Given any attainable measures μ_{∞_0}, μ_{∞_1} and $0 \le c \le 1$, we can take $\{m_{0,j}\}_j$ $\{m_{1,j}\}$ such that $\mu_{0,j}$ weakly converges to μ_{∞_0} and $\mu_{1,j}$ weakly converges to μ_{∞_1}, as $j \to \infty$. We also take $\{a_{0,j}\}_j, \{a_{1,j}\}_j$ so that $d_j/e_j \to c$ as $j \to \infty$. Therefore, by equations (5.24), (5.25) and (5.26) we conclude that there exists $\{\lambda_\ell\}_\ell \subset \Lambda_{\text{new}}$ such that

$$\langle \mathrm{Op}(f) g_{\lambda_\ell}, g_{\lambda_\ell} \rangle \xrightarrow{\ell\to\infty} c\int_{S^1} f\, d\mu_{\infty_0} + (1-c)\int_{S^1} f\, d\mu_{\infty_1}. \qquad \square$$

A. Arithmetic over $\mathbb{Q}(i)$

Consider the number field $\mathbb{Q}(i)$ with the corresponding ring of integers $\mathbb{Z}[i]$. For $\mathfrak{b}$ a nonzero integral ideal of $\mathbb{Z}[i]$, the residue classes $\alpha \pmod{\mathfrak{b}}$, where (α) and $\mathfrak{b}$ are relatively prime ideals, form the multiplicative group $(\mathbb{Z}[i]/\mathfrak{b})^*$. We now summarize some well-known facts, which may be found in [34] or [19]. A *Dirichlet character* (mod $\mathfrak{b}$) is a group homomorphism

$$\chi : (\mathbb{Z}[i]/\mathfrak{b})^* \to S^1.$$

We extend χ to all of $\mathbb{Z}[i]$ by setting $\chi(\mathfrak{a}) = 0$ for $\mathfrak{a}$ and $\mathfrak{b}$ which are not relatively prime. Let I denote multiplicative group of nonzero fractional ideals and $I_{\mathfrak{b}} = \{\mathfrak{a} \in I : \mathfrak{a} \text{ and } \mathfrak{b} \text{ are relatively prime}\}$. A *Hecke Großencharakter* (mod $\mathfrak{b}$) is a homomorphism $\psi : I_{\mathfrak{b}} \to \mathbb{C} \setminus \{0\}$ for which there exists a pair of homomorphisms

$$\chi : (\mathbb{Z}[i]/\mathfrak{b})^* \to S^1, \qquad \chi_\infty : \mathbb{C}^* \to S^1$$

such that for an ideal (α) with $\alpha \in \mathbb{Z}[i]$

$$\psi((\alpha)) = \chi(\alpha)\chi_\infty(\alpha).$$

Conversely, given any χ (mod $\mathfrak{b}$) and χ_∞ there exists a Großencharakter ψ (mod $\mathfrak{b}$) such that $\psi = \chi \cdot \chi_\infty$ provided that $\chi(u)\chi_\infty(u) = 1$ for each unit $u \in \mathbb{Z}[i]$.

In particular, for $4|k$ and $\mathfrak{a} = (\alpha)$ a nonnegative integer

$$\psi(\mathfrak{a}) = \left(\frac{\alpha}{|\alpha|}\right)^k$$

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is a Hecke Großenchakter (mod 1) and these Hecke Großencharakteren can be used to detect primes in sectors. Additionally, given a positive rational integer q with $(4, q) = 1$ the homomorphism

$$\chi \circ N : I_q \to S^1$$

given by $(\chi \circ N)(\mathfrak{a}) = \chi(N(\mathfrak{a}))$ is a Dirichlet character (mod q), where χ is a Dirichlet character (mod q) for $\mathbb{Z}$, that is $\chi : (\mathbb{Z}/(q))^* \to S^1$, where $N\mathfrak{a}$ is the norm of $\mathfrak{a}$. Hence, for $4|k$

$$\psi(\mathfrak{a}) = (\chi \circ N)(\alpha)\left(\frac{\alpha}{|\alpha|}\right)^k$$

is a Hecke Großencharakter with modulus q and frequency k, where $\mathfrak{a} = (\alpha)$. (A priori α is only defined up to multiplication by i, but for these characters the choice does not matter.) The L-function attached to the Großencharakter ψ given by

$$L(s, \psi) = \sum_{\mathfrak{a}} \frac{\psi(\mathfrak{a})}{N(\mathfrak{a})^s}$$

has a functional equation and admits an analytic continuation to $\mathbb{C} \setminus \{1\}$.

Moreover, if ψ is not a real character, $L(s, \psi)$ has a standard zero-free region. That is, we have

$$L(\sigma + it, \psi) \neq 0 \qquad \text{for} \qquad \sigma > 1 - \frac{c}{\log(q(|t| + 1)(|k| + 1))}$$

(see [19, Section 5.10]). In particular, if $k \neq 0$,

$$\sum_{N(\pi) \leq x} \chi(N(\pi))\left(\frac{\pi}{|\pi|}\right)^k \ll ((|k| + 1)q) \cdot x \exp\left(-c\sqrt{\log x}\right),$$

where the summation is over prime ideal $\mathfrak{p} = (\pi)$ with norm $\leq x$.

Furthermore, for $k = 0$ the same estimate holds for any complex χ (mod q). However, for $k = 0$ and χ (mod q) a real character, there may be a possible Siegel zero, and in this case we have Siegel's estimate (see Section 5.9 of [19])

$$L(\sigma + it, \chi) \neq 0 \qquad \text{for} \qquad \sigma \geq 1 - \frac{c(\epsilon)}{q^\epsilon}$$

for any $\epsilon > 0$. Consequently, we have the Siegel–Walfisz type prime number theorem for $(a, q) = 1$ and $(q, 2) = 1$

$$\sum_{\substack{N(\pi) \leq x \\ N(\pi) \equiv a \pmod q \\ 0 \leq \arg \pi \leq \varepsilon}} 1 = \frac{1}{\varphi(q)} \sum_{\substack{N(\pi) \leq x \\ (N(\pi), q) = 1 \\ 0 \leq \arg \pi \leq \varepsilon}} 1 + O\left(\frac{x}{(\log x)^A}\right) \tag{A.1}$$

for any $A \geq 1$. (After multiplication by i^l for some l, we can ensure that $\theta = \arg i^l \pi \in [0, \pi/2)$; we will let $\arg \pi$ denote this angle.)

Recall that a prime $p \equiv 3 \pmod 4$ is inert in $\mathbb{Z}[i]$; additionally, a prime $p \equiv 1 \pmod 4$ splits in $\mathbb{Z}[i]$ so that $p = \pi\overline{\pi} = a^2 + b^2$, where π is a prime in $\mathbb{Z}[i]$. Writing

$$\mathcal{B}(x; q, a, \varepsilon) = \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \leq x \\ (n,q)=1}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n),$$

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formula (A.1) gives, for $(a, q) = 1$ and $(q, 2) = 1$, that

$$|\mathcal{B}(x; q, a, \varepsilon)| \ll \frac{x}{(\log x)^A}, \tag{A.2}$$

for $q \le (\log x)^A$. In addition, it is worth noting that equation (A.1) also implies

$$\sum_{\substack{n \le x \\ n \equiv a \pmod q}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) \sim \frac{4\varepsilon^2}{\varphi(q)} \frac{x \log\log x}{\log x}. \tag{A.3}$$

We are now ready to state the following result which is an analog of the Bombieri–Vinogradov theorem.

Theorem A.1. *There exists B_0 sufficiently large so that*

$$\sum_{\substack{q \le Q \\ (q,2)=1}} \max_{(a,q)=1} |\mathcal{B}(x; q, a, \varepsilon)| \ll \frac{x}{(\log x)^{10}}$$

for $Q \le x^{1/2}/(\log x)^{B_0}$.

Let $\mathcal{S} \subset \mathbb{N}$. A sequence of complex numbers $\{\beta_n\}$ with $|\beta_n| \le \tau(n)$ satisfies the *Siegel–Walfisz property for $\mathcal{S}$* provided that for every $q \in \mathcal{S}$ and $A \ge 0$ and $N \ge 2$ we have

$$\sum_{\substack{n \le N \\ n \equiv a \pmod q}} \beta_n = \frac{1}{\varphi(q)} \sum_{\substack{n \le N \\ (n,q)=1}} \beta_n + O\left(\frac{N}{(\log N)^A}\right)$$

for every $a \in \mathbb{Z}$ with $(a, q) = 1$.

A.1. An application of the large sieve

We next recall a consequence of the large sieve, which follows applying a minor modification of Theorem 9.17 of [15].

Lemma A.1. *Let $A \ge 1$ and $Q = x^{1/2}(\log x)^{-B}$, where $B = B(A)$ is sufficiently large. Suppose $\{\beta_n\}$ satisfies the Siegel–Walfisz property for all q with $(q, 2) = 1$. Then for any sequence $\{\alpha_n\}$ of complex numbers such that $|\alpha_n| \le \tau(n)$*

$$\sum_{\substack{q \le Q \\ (q,2)=1}} \max_{(a,q)=1} \left| \sum_{\substack{mn \le x \\ m,n \le \frac{x}{(\log x)^B} \\ mn \equiv a \pmod q}} \beta_m \alpha_n - \frac{1}{\varphi(q)} \sum_{\substack{mn \le x \\ m,n \le \frac{x}{(\log x)^B} \\ (mn,q)=1}} \beta_m \alpha_n \right| \ll \frac{x}{(\log x)^A}.$$

Proof of Theorem A.1. By equation (A.2), the sequence $\beta_n = 1_{\mathcal{P}_\varepsilon}(n)$ satisfies the Siegel–Walfisz condition for all q with $(q, 2) = 1$. Take $\alpha_n = 1_{\mathcal{P}'_\varepsilon}(n)$, and note that (cf. equation (2.1))

$$\sum_{\substack{n \le x \\ n \equiv a \pmod q}} (1_{\mathcal{P}_\varepsilon} * 1_{\mathcal{P}'_\varepsilon})(n) = \sum_{\substack{mn \le x \\ m,n \le \frac{x}{(\log x)^{B_0}} \\ mn \equiv a \pmod q}} 1_{\mathcal{P}_\varepsilon}(m) 1_{\mathcal{P}'_\varepsilon}(n)$$

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and

$$\sum_{\substack{n\le x\\(n,q)=1}}(1_{\mathcal{P}_\varepsilon}*1_{\mathcal{P}'_\varepsilon})(n)=\sum_{\substack{mn\le x\\ m,n\le \frac{x}{(\log x)^{B_0}}\\(mn,q)=1}}1_{\mathcal{P}_\varepsilon}(m)1_{\mathcal{P}'_\varepsilon}(n).$$

Hence, applying Lemma A.1 completes the proof. □

A.2. Gaussian integers in sectors with norms in progressions

The goal of this section is to show that a result of Smith [44] (also cf. [47]) holds for Gaussian integers in sectors. We recall that, for $\alpha\in\mathbb{Z}[i]$, $N(\alpha)=|\alpha|^2$ denotes the norm of α. For $a,q>0$, define

$$\eta_a(q):=|\{\alpha_1,\alpha_2\pmod q:\alpha_1^2+\alpha_2^2\equiv a\pmod q\}|.$$

Proposition A.1. *Let $a,q>0$ be integers, and put $g=(a,q)$. Given an angle θ and $\epsilon\in(0,2\pi)$, let $S=S_{\epsilon,\theta}$ denote the set of lattice points $\alpha\in\mathbb{Z}[i]$ contained in the sector defined by*[4] $|\arg(\alpha)-\theta|<\epsilon/2$. *Then, uniformly for $\epsilon>0$,*

$$|\{\alpha\in S:N(\alpha)\equiv a\pmod q,N(\alpha)\le x\}|$$
$$=\frac{\epsilon x\eta_a(q)}{q^2}+O\left(\frac{x^{1-\delta/3}}{q}\right)$$

provided that $q^3g<x^{2(1-2\delta)}$ for $\delta>0$.

We begin by showing that solutions to $\alpha_1^2+\alpha_2^2\equiv a\pmod q$ is well distributed in fairly small boxes. Given q, let $f:(\mathbb{Z}/q\mathbb{Z})^2\to\mathbb{C}$ denote the characteristic function of the set $\{(\alpha_1,\alpha_2)\in(\mathbb{Z}/q\mathbb{Z})^2:\alpha_1^2+\alpha_2^2\equiv a\pmod q\}$. With the modulo q Fourier transform given by

$$\widehat{f}(\xi_1,\xi_2):=\sum_{\alpha_1,\alpha_2\pmod q}f(\alpha_1,\alpha_2)e^{-2\pi i(\xi_1\alpha_1+\xi_2\alpha_2)/q},\tag{A.4}$$

we recall the following estimate by Tolev [47]:

$$|\widehat{f}(\xi_1,\xi_2)|\ll q^{1/2}\tau(q)^2(q,\xi_1,\xi_2)^{1/2}(q,a,\xi_1^2+\xi_2^2)^{1/2}\le q^{1/2}\tau(q)^2(q,\xi_1,\xi_2)^{1/2}(q,a)^{1/2}.\tag{A.5}$$

By the Chinese remainder theorem, $\eta_a(q)$ is multiplicative in q, and we note that $\widehat{f}(0,0)=\eta_a(q)$.

Let $B\subset[0,q)\times[0,q)$ be a 'box' with side lengths T, and let $g=g_B$ denote the characteristic function of $B\cap(\mathbb{Z}/q\mathbb{Z})^2$. By standard estimates (from summing a geometric series), we have, for $\xi_1,\xi_2\neq0$,

$$\widehat{g}(\xi_1,\xi_2)\ll q^2/|\xi_1\xi_2|,\tag{A.6}$$

for $\xi_1\neq0$,

$$\widehat{g}(\xi_1,0)\ll Tq/|\xi_1|,\tag{A.7}$$

(and similarly for $\xi_2\neq0$) and trivially

$$\widehat{g}(0,0)=T^2.$$

[4]By $\arg(\alpha)$, we denote the complex argument chosen in such a way that it is single valued in an $\epsilon/2$-neighborhood of θ.

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Lemma A.2. *Let $g = (a, q)$. Then*

$$|\{(\alpha_1, \alpha_2) \in B : \alpha_1^2 + \alpha_2^2 \equiv a \pmod q\}| = T^2 \cdot \frac{\eta_a(q)}{q^2} + O(q^{1/2}\tau(q)^3 \log(q)^2 g^{1/2}).$$

Proof. By Fourier analysis on $(\mathbb{Z}/q\mathbb{Z})^2$ (i.e., Plancherel's theorem for finite abelian groups), we have

$$\begin{aligned}|\{(\alpha_1, \alpha_2) \in B : \alpha_1^2 + \alpha_2^2 \equiv a \pmod q\}| &= \sum_{\alpha_1, \alpha_2 \pmod q} f(\alpha_1, \alpha_2) g(\alpha_1, \alpha_2) \\ &= \frac{1}{q^2} \sum_{\xi_1, \xi_2 \pmod q} \widehat{f}(\xi_1, \xi_2)\overline{\widehat{g}(\xi_1, \xi_2)}.\end{aligned}$$

The main term is given by $\xi_1 = \xi_2 = 0$ and equals

$$\frac{\widehat{f}(0,0)\widehat{g}(0,0)}{q^2} = T^2 \frac{\eta_a(q)}{q^2}.$$

Using equations (A.5) and (A.7), the contribution from (say) $\xi_1 = 0$ and $\xi_2 \neq 0$ is

$$\begin{aligned}&\ll \frac{1}{q^2} \sum_{\xi_2=1}^{q-1} \frac{Tq}{\xi_2} q^{1/2}\tau(q)^2 (q, \xi_2)^{1/2} g^{1/2} \ll \frac{Tq^{3/2}\tau(q)^2 g^{1/2}}{q^2} \sum_{d|q} \sum_{0<\xi_2<q/d} \frac{d^{1/2}}{d\xi_2} \\ &\ll \frac{T\tau(q)^3 \log(q) g^{1/2}}{q^{1/2}} = O(q^{1/2}\tau(q)^3 \log(q) g^{1/2}).\end{aligned} \tag{A.8}$$

The contribution from terms $\xi_2 = 0$ and $\xi_1 \neq 0$ is bounded similarly.

As for the terms $\xi_1, \xi_2 \neq 0$, we have by equation (A.5)

$$\begin{aligned}\frac{1}{q^2} \sum_{\xi_1, \xi_2 \neq 0} \widehat{f}(\xi_1, \xi_2)\overline{\widehat{g}(\xi_1, \xi_2)} &\ll \frac{q^{1/2}\tau(q)^2}{q^2} \sum_{\xi_1, \xi_2 \neq 0} \frac{q^2}{\xi_1\xi_2} (q, \xi_1, \xi_2)^{1/2} g^{1/2} \\ &= q^{1/2}\tau(q)^2 \sum_{d|q} \sum_{0<\xi_1, \xi_2 \leq q/d} \frac{d^{1/2} g^{1/2}}{d^2 \xi_1 \xi_2} \ll q^{1/2}\tau(q)^2 \log(q)^2 g^{1/2}.\end{aligned}$$

□

Concluding the proof of Proposition A.1. Take $T = x^{(1-\delta)/2}$. The case $T > q$ is straightforward using a simple tiling argument, and we only give details for $T \leq q$.

By a simple geometry of numbers argument, we may 'tile' the sector S, intersected with a ball of radius $x^{1/2}$, with $\epsilon x/T^2 + O(x^{1/2}/T)$ boxes B (with side lengths T) entirely contained in the sector and with $O(x^{1/2}/T)$ boxes intersecting the boundary. By Lemma A.2, each box B contains

$$T^2 \cdot \frac{\eta_a(q)}{q^2} + O(q^{1/2}\tau(q)^2 \log(q)^2 g^{1/2})$$

points satisfying $\alpha_1^2 + \alpha_2^2 \equiv a \pmod q$.

As $\eta_a(q) < q^{1+o(1)}$ (cf. [5, Lemma 2.8]), we find that the number of lattice points in the sector is

$$\begin{aligned}&(\epsilon x/T^2 + O(x^{1/2}/T))(T^2 \cdot \frac{\eta_a(q)}{q^2} + O(q^{1/2}\tau(q)^3 \log(q)^2 g^{1/2})) \\ &= \frac{\epsilon \eta_a(q) x}{q^2} + O\left(\frac{x^{1-\delta/2}}{q^{1-o(1)}} + \epsilon g^{1/2} q^{1/2+o(1)} x^{\delta}\right).\end{aligned}$$

For $q^3 g < x^{2(1-2\delta)}$, the error term is $\ll \frac{x^{1-\delta/3}}{q}$.

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A.3. *Proof of Lemma 2.4*

We may assume $(Q, q) = 1$; otherwise, the result is trivial. Let $\delta > 0$ be sufficiently small but fixed, and set

$$r_\varepsilon(n) = \sum_{\substack{a^2+b^2=n \\ |\arg(a+ib)|\le\varepsilon}} 1.$$

Also, for $n \in \mathbb{N}$ and $z > 0$ let $\widetilde{P}_n(z) = \prod_{2<p<z} p$. Let $\Lambda_1 = \{\lambda_d\}$, $\Lambda' = \{\lambda'_e\}$ be upper bound sieves of level $D = x^\delta$ with $(d, 2q) = 1$ and $(e, 2Q) = 1$. Then for $z = x^{\delta/2}$, we have

$$\begin{aligned}\sum_{\substack{p=a^2+b^2\le x \\ |\arg(a+ib)|\le\varepsilon \\ qp+4=Qp_1 \text{where } p_1 \text{ is prime}}} 1 &\le \sum_{m\le qx+4} \sum_{\substack{n\le x \\ qn+4=Qm \\ (m,\widetilde{P}_q(z))=1 \\ (n,\widetilde{P}_Q(z))=1}} r_\varepsilon(n) + O(x^{\delta/2}) \\ &\le \sum_{m\le qx+4} \sum_{\substack{n\le x \\ qn+4=Qm}} r_\varepsilon(n)(\lambda' * 1)(n)(\lambda * 1)(m) + O(x^{\delta/2}).\end{aligned}$$

Switching order of summation, we have that the sum on the LHS above is

$$\begin{aligned}&= \sum_{\substack{d,e<D \\ (d,e)=1 \\ (d,2q)=1,(e,2Q)=1}} \lambda_d\lambda'_e \sum_{\substack{n\le x \\ e|n}} r_\varepsilon(n) \sum_{\substack{m\le qx+4 \\ d|m \\ qn+4=Qm}} 1 \\ &= \sum_{\substack{d,e<D \\ (d,e)=1 \\ (d,2q)=1,(e,2Q)=1}} \lambda_d\lambda'_e \sum_{\substack{n\le x \\ n\equiv\gamma \pmod{Qed}}} r_\varepsilon(n)\end{aligned} \tag{A.9}$$

since the inner sum in the first equation above consists of precisely one term provided that $qn + 4 \equiv 0 \pmod{Qd}$ and is empty otherwise. Also, here $\gamma = -4e\overline{eq}$, where $q\overline{q} \equiv 1 \pmod{Qd}$ and $e\overline{e} \equiv 1 \pmod{Qd}$. In particular, $(\gamma, Qed) = e$.

Let us note some properties of the function $\eta_a(q)$. Recall, $\eta_a(\cdot)$ is multiplicative. Moreover, for $p > 2$ and $\ell \ge 1$

$$\eta_a(p^\ell) = p^\ell \sum_{0\le j\le\ell} \frac{\chi_4(p)^j}{p^j} c_{p^j}(a) \tag{A.10}$$

and for any $a, q \ge 1$

$$\eta_q(q) \ll \frac{q^2}{\varphi(q)}\tau((a, q)) \tag{A.11}$$

(see [5, Eqn. (2.20) and Lemma 2.8]), where

$$c_q(a) = \sum_{\substack{b \pmod q \\ (b,q)=1}} e\left(\frac{ab}{q}\right) = \frac{\varphi(q)}{\varphi(q/(q,a))}\mu(q/(q,a)) \tag{A.12}$$

is the Ramanujan sum and χ_4 is the nonprincipal Dirichlet character (mod 4). In particular, note that if $(a, q) = g$ then $\eta_a(q) = \eta_g(q)$ for odd q.

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By Proposition A.1, equations (A.10) and (A.11) and recalling that $(Qed, \gamma) = e$, we get the RHS of equation (A.9) equals

$$2\varepsilon x \sum_{\substack{d,e<D \\ (d,e)=1 \\ (d,2q)=1,(e,2Q)=1}} \frac{\lambda_d \lambda'_e}{(Qed)^2} \eta_\gamma(Qed) + O\left(\frac{x^{1-\delta/4}}{Q}\right)$$
$$= \frac{2\varepsilon x \eta_1(Q)}{Q^2} \sum_{\substack{d,e<D \\ (d,e)=1 \\ (d,2q)=1,(e,2Q)=1}} \frac{\lambda_d \lambda'_e}{(ed)^2} \frac{\eta_1(Qd)\eta_e(e)}{\eta_1(Q)} + O\left(\frac{x^{1-\delta/4}}{Q}\right)$$

provided that $Q^3 D^7 < x^{2(1-2\delta)}$ which we rewrite as $Q < x^{2/3-11\delta/3}$. Using Theorem 2.3 in the form of equation (2.9), and noting that $\eta_1(Qd)/\eta_1(Q)$ is a multiplicative function, we get that the above sum is

$$\ll \frac{\varepsilon x \eta_1(Q)}{Q^2} \prod_{\substack{p<D \\ (p,2q)=1}} \left(1 - \frac{\eta_1(Qp)}{p^2 \eta_1(Q)}\right) \prod_{\substack{p<D \\ (p,2Q)=1}} \left(1 - \frac{\eta_p(p)}{p^2}\right). \tag{A.13}$$

To evaluate the Euler products we use equation (A.10) to get $\eta_p(p) = p(1+\chi_4(p)-\frac{1}{p})$, $\eta_1(Qp)/\eta_1(Q) = p + O(1)$ and $\eta_1(Q) = Q\prod_{p|Q}\left(1 - \frac{\chi_4(p)}{p}\right)$. Hence, by these estimates we get that equation (A.13) is

$$\ll \frac{\varepsilon x \eta_1(Q)}{Q^2} \prod_{p|Q}\left(1 + \frac{\chi_4(p)+1}{p}\right) \prod_{p|q}\left(1 + \frac{1}{p}\right) \cdot \frac{1}{(\log D)^2}$$
$$\ll \frac{q}{\varphi(q)} \cdot \frac{\varepsilon x}{Q\delta^2(\log x)^2} \prod_{p|Q}\left(1 + \frac{1}{p}\right) \ll \frac{q}{\varphi(q)} \cdot \frac{\varepsilon x}{\varphi(Q)\delta^2(\log x)^2}$$

for $Q < x^{2/3-11\delta/3}$ which completes the proof since $\delta > 0$ is arbitrary.

B. Nonattainable quantum limits

Given an integer n such that $r(n) > 0$, define a probability measure μ_n on the unit circle by

$$\mu_n := \frac{1}{r(n)} \sum_{\lambda \in \mathbb{Z}[i]:|\lambda|^2=n} \delta_{\lambda/|\lambda|},$$

that is, μ_n is obtained by projecting the set of $\mathbb{Z}^2$-lattice points on a circle of radius $n^{1/2}$ to the unit circle and δ here denotes the Dirac delta function. A measure μ is said to be *attainable* if μ is a weak* limit of some subsequence of measures μ_{n_i}. A partial classification of the set of attainable measures was given in [29] in terms of their Fourier coefficients. Namely, for $k \in \mathbb{Z}$, let $\widehat{\mu}(k) := \int z^k \, d\mu(z)$ denote the k-th Fourier coefficient of μ. By [29, Theorem 1.3], the inequalities

$$2\widehat{\mu}(4)^2 - 1 \le \widehat{\mu}(8) \le \max(\widehat{\mu}(4)^4, (2|\widehat{\mu}(4)| - 1)^2)$$

hold if μ is attainable. In particular, for $\gamma > 0$ small and $\widehat{\mu}(4) = 1-\gamma$, we must have $\widehat{\mu}(8) = 1-4\gamma+O(\gamma^2)$.

Now, by Theorem 1.2, there exists quantum limits that are convex combinations $c\nu_1 + (1-c)\nu_2$ for $c > 0$ arbitrarily small and where ν_1 is the uniform measure (with $(\widehat{\nu_2}(4), \widehat{\nu_2}(8)) = (0, 0)$), and ν_2 is a Cilleruelo type measure, that is, localized on the four points $\pm 1, \pm i$, and with $(\widehat{\nu_2}(4), \widehat{\nu_2}(8)) = (1, 1)$. Clearly, such convex combinations cannot be attainable for c small.

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C. Convexity assuming a k-prime tuple analog

C.1. Preliminaries

We begin by noting that it is enough to show that any probability measure on the unit circle can be approximated by a convex combination of Dirac measures with uniform weights and similarly for Sym_8-invariant measures. Namely, by the Krein–Milman theorem (cf. [36, §3.21]), any measure on the unit circle is in the closed convex hull of its extreme points. Now, the extreme points are exactly the Dirac deltas: If one tries to decompose the measure $\delta_0 = c\alpha + (1-c)\beta$ (with α, β probability measures and $c \in [0,1]$) it's clear that $\alpha(0) = \beta(0) = 1$ (both of them have mass ≤ 1 at 0, and if < 1, then δ_0 would have too little mass at 0.) On the other hand, if some measure μ puts positive mass on two disjoint subsets A, B whose union is S^1, then as long as we have $\mu(X) = \mu(X \cap A) + \mu(X \cap B)$ we arrive, after renormalizing, at a convex combination of μ in terms of two probability measures. Thus, for Borel probability measures on S^1, the extremal points are exactly the Dirac measures.

We also find that any measure can be approximated by $k^{-1}\sum_{i\leq k}\delta_{\theta_i}$ for any subsequence of integers k tending to infinity since a finite convex combination of delta measures can be approximated by a uniformly weighted sum of delta measures. Further, we can also approximate via delta measures whose angles are restricted to come from Gaussian primes by using Hecke's theorem.

Let $k > 0$ be an even integer. Given collection of angles $\theta_1, \ldots, \theta_k$ define a probability measure $\mu = \frac{1}{k}\sum_{i\leq k}\nu_{\theta_i}$ on the unit circle, where ν_θ denotes the Sym_8-invariant probability measure $\nu_\theta = \frac{1}{8}\sum_{l=1}^{4}\delta_{\pm\theta+l\pi/2}$. As explained above, it is enough to show that any such μ is a quantum limit.

Assuming a plausible analogue of the prime k-tuple, or the Bateman–Horn, conjectures, we show that there exists an infinite subsequence of new eigenvalues λ so that

$$\langle \mathrm{Op}(f) g_\lambda, g_\lambda\rangle = \mu(f) + o(1), \tag{C.1}$$

as $\lambda \longrightarrow \infty$ along said subsequence, in the strong coupling limit.

To state our 'Bateman–Horn type hypothesis' precisely, we define an 'admissibility parameter' $Q_0 = \prod_{p\leq k} k$ and moduli Q_i, $i = 1, \ldots, k$ which are square-free and pairwise coprime. Let

$$Q := \prod_{i=1}^{k} Q_i.$$

We also require admissible shifts $h_i, i = 1, \ldots, k$ with

$$h_i = 1 + l_i Q_0,$$

where l_i are distinct integers such that each prime divisor of Q is larger than $\max_{i,j}|l_j - l_i|$, and $h_0 \pmod{Q}$ is an integer satisfying $h_0 \equiv 0 \pmod{Q_0}$ and $h_0 \equiv h_i \pmod{Q_i}$, $i = 1, \ldots, k$ (these conditions will ensure admissibility; also such an h_0 with $|h_0| \leq Q$ exists by the Chinese remainder theorem by pairwise coprimality of the Q_i). The point of admissibility is to ensure that there are no local obstructions to certain k-tuples being simultaneously prime (e.g., there are only finitely many prime pairs of the form $(n, n+1)$ and finitely many prime triples of the form $(n, n+2, n+4)$; cf. equation (C.3) for the exact formulation in our setting.) Finally, let

$$\mathcal{P}_\varepsilon = \{p \text{ prime} : p = a^2 + b^2,\ a, b > 0,\ \text{ and } 0 < \arctan(b/a) \leq \varepsilon\}.$$

We formulate the following conjecture.

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Conjecture 1. *Let* $\varepsilon > 0$*. Suppose* $Q \le x^{o(1)}$*,* $|h_1|, \ldots, |h_k| \ll_k 1$ *and* $\prod_{p|Q}(1 + 1/p) \ll 1$*. In the notation above, we have*

$$|\{n \le x : \frac{Qn - h_0 + h_1}{Q_1}, \ldots, \frac{Qn - h_0 + h_k}{Q_k} \in \mathcal{P}_\epsilon\}| \gg_{\epsilon,k} \frac{x}{(\log x)^k}.$$

To justify this conjecture, we will show in Section C.3 that the polynomial

$$L(x) = \prod_{i=1}^{k} L_i(x), \quad L_i(x) = A_i x + B_i, \quad A_i = Q/Q_i, B_i = (h_i - h_0)/Q_i \tag{C.2}$$

has no fixed prime divisor (note by construction that $B_i \in \mathbb{Z}$ since $h_i \equiv h_0 \pmod{Q_i}$) so that there are no local obstructions to k-tuples of integers of the form

$$\left(\frac{Qn - h_0 + h_1}{Q_1}, \ldots, \frac{Qn - h_0 + h_k}{Q_k}\right) \tag{C.3}$$

being simultaneously prime.

C.2. *Proof of equation (C.1), assuming Conjecture 1*

C.2.1. The construction and a high-level overview of the argument

Let k be a given *even* integer. Similar to Section 5.2., given a large value of x choose moduli $Q_1, \ldots, Q_k$ as follows: Put $T = \lfloor \log\log x \rfloor$, $H = \lfloor 100 \log\log\log x \rfloor$, and let

$$Q_i = q_i' \cdot \prod_{j=T+1+(i-1)H}^{T+iH-1} q_j,$$

where q_j denotes the j-th element of $\{q \in \mathcal{S} : q \equiv 1 \pmod{Q_0}\}$ (cf. equation (5.4); note that $q_j \asymp_k (j \log j)^{10/9}$ holds), and $q_i' \asymp T$ is a Gaussian prime with associated angle $\theta_i + o(1)$ as x grows. In particular, note that $Q_i \equiv 1 \pmod{Q_0}$ and that

$$\mu_{Q_i} \to \nu_{\theta_i} \tag{C.4}$$

as x grows, where μ_{Q_i} is the probability measure with delta masses placed at the angles of lattice points lying on the circle of radius $\sqrt{Q_i}$. Let $\mathcal{H} = \{h_1, \ldots, h_k\}$, where

$$h_i = 1 + l_i Q_0, \qquad i = 1, \ldots, k$$

and the integers l_i are distinct and chosen so that

$$h_1, \ldots, h_{k/2} \in [-W^3 - W, -W^3], \qquad h_{k/2+1}, \ldots, h_k \in [W^3, W^3 + W], \tag{C.5}$$

where $W = W_0 k Q_0$, and $W_0 \ge 10$ is a (large) parameter. Observe that $\max_{i,j} |l_i - l_j| \le 3W^3/Q_0$, so any prime divisor of Q does not divide $\max_{i,j} |l_i - l_j|$ for x sufficiently large. We note that we may apply Conjecture 1 with this choice of $h_1, \ldots, h_k, Q_1, \ldots, Q_k$. Let

$$\mathcal{N}_3 := \{n \in \mathcal{N} : L_i(n) \in \mathcal{P}_\varepsilon \text{ for } 1 \le i \le k\}.$$

Let us now give an overview of the argument to establish equation (C.1). The basic idea is that for most such values of $n \in \mathcal{N}_3$, if we put $m = Qn - h_0$ there exists a corresponding new eigenvalue $\lambda \in (m + h_{k/2}, m + h_{k/2+1})$ which satisfies $\lambda = m + O(W)$ (which we will prove later). In other words, the new eigenvalue λ 'sits in the middle' of two clusters of k old eigenvalues, where $k/2$ of them lie in

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$[m - W^3 - W, m - W^3]$, and the remaining $k/2$ of them lie in $[m + W^3, m + W^3 + W]$. Moreover, we will see later that, for most such n in a positive density subset of $\mathcal{N}_3 \cap [x/2, x]$, essentially all of the L^2-mass is carried by terms arising from the two clusters, in the sense that for f a pure momentum observable,

$$\begin{aligned}\langle \mathrm{Op}(f)G_\lambda, G_\lambda\rangle &= \frac{1}{k}\sum_{i=1}^{k}\sum_{\xi\in\mathbb{Z}[i]:|\xi|^2=m+h_i}\frac{f(\xi/|\xi|)}{(m+h_i-\lambda)^2} + o(||G_\lambda||^2),\\ &= \frac{2^H}{kW^6}\left(\sum_{i=1}^{k}(\nu_{\theta_i}(f)+o(1))\cdot(1+O(1/W^2))\right) + o(||G_\lambda||^2),\end{aligned}$$

where we have used equation (C.4) in the last step. As this construction also gives $||G_\lambda||_2^2 = \frac{2^H}{W^6}(1+o(1))$, we find that

$$\langle \mathrm{Op}(f)g_\lambda, g_\lambda\rangle = \frac{1}{k}\sum_{i=1}^{k}(\nu_{\theta_i}(f)+o(1))\cdot(1+O(1/W^2)) = (\mu(f)+o(1))\cdot(1+O(1/W^2)),$$

which completes the argument by taking a sequence of W_0's tending to infinity.

C.2.2. Restricting to typical $n \in \mathcal{N}_3$

We also require the following analog of Lemma 4.3, which follows from the techniques used in Section 4 of the paper. A formal proof of this result is given in Section C.4.

Lemma C.1. *Let $U = (\log\log x)^5$. There exists $C > 0$ such that for all $n \in \mathcal{N}_3 \cap [x/2, x]$, outside a set of size*

$$\ll_{k,W} \frac{x}{(\log x)^k}\frac{(\log\log\log\log x)^{2C}\cdot\log\log x}{U},$$

the following hold:

$$\sum_{|h|\le(\log x)^{1/2}/U,\, h\notin\mathcal{H}} b(Qn - h_0 + h) = 0, \tag{C.6}$$

$$\sum_{|h|\le(\log x)^B,\, h\notin\mathcal{H}} \frac{r(Qn-h_0+h)}{|h|} \le U \tag{C.7}$$

and

$$\sum_{|h|\ge U,\, h\notin\mathcal{H}} \frac{r(Qn-h_0+h)}{h^2} \ll \frac{1}{\log\log x}. \tag{C.8}$$

Note that Lemma C.1 and Conjecture 1 imply that equations (C.6), (C.7) and (C.8) hold for a full density subset of $n \in \mathcal{N}_3 \cap [x/2, x]$.

C.2.3. Proof of equation (C.1)

By equations (C.6) and (C.7) and Theorem 3.1, we have for $m = Qn - h_0$ and all $n \in \mathcal{N}_3 \cap [x/2, x]$ that lie outside a subset of size $o(x/(\log x)^k)$, that the new eigenvalue $\lambda_* = \lambda_{m+h_{k/2+1}}$ satisfies the spectral equation

$$\sum_{i\le k}\frac{r(m+h_i)}{m+h_i-\lambda_*} = O((\log\log x)^5). \tag{C.9}$$

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Thus, by Conjecture 1 the above holds for a density one subset of $n \in \mathcal{N}_3 \cap [x/2, x]$. Letting

$$F_m(\lambda) := \sum_{i \le k} \frac{1}{m + h_i - \lambda},$$

we find, as $r(m + h_i) = 2^H$ for $i = 1, \dots, k$, with $H = [100 \log\log\log x]$ and recalling equation (C.9), that

$$F_m(\lambda_*) = o(1).$$

Recalling equation (C.5), we then note that

$$F_m(m) = \sum_{i \le k} 1/h_i = \frac{k/2}{-(W^3 + O(W))} + \frac{k/2}{W^3 + O(W)} = O(k/W^5).$$

Further, for $\lambda \in [m_{h_{k/2}}, m_{h_{k/2+1}}]$, we have

$$F_m'(\lambda) \ge (k/2)\frac{1}{(W^3 + O(W))^2} \gg k/W^6.$$

Thus, combining the previous three assertions and using the mean value theorem gives

$$\lambda_* = m + O(W). \tag{C.10}$$

This tells us that λ_* lies essentially at the center of the two clusters of old eigenvalues (which lie of distance $2W^3 + O(W)$ apart).

Hence, using equations (C.6) and (C.8) we find that

$$||G_\lambda||_2^2 = \frac{k 2^H}{(W^3 + O(W))^2} + O(1/\log\log x)$$

and further, recalling $m + h_i = Qn - h_0 + h_i$, $n \in \mathcal{N}_3$, and using equations (C.5), (C.4) and (C.10), we have that

$$\sum_{i \le k} \sum_{\xi \in \mathbb{Z}[i]:|\xi|^2 = m + h_i} \frac{f(\xi/|\xi|)}{(m + h_i - \lambda_*)^2} = \sum_{i \le k} \frac{2^H(\nu_{\theta_i}(f) + o(1))}{(W^3 + O(W))^2}$$

and consequently

$$\langle \mathrm{Op}(f) g_{\lambda_*}, g_{\lambda_*} \rangle = \frac{2^H(\mu(f) + o(1))(1 + O(1/W^2))}{2^H} = (\mu(f) + o(1)) \cdot (1 + O(1/W^2)).$$

C.3. Admissibility

We need to show that the polynomial $L(x)$ as above has no fixed prime divisor. That is, for each prime p there exists an integer n with $L(n) \not\equiv 0 \pmod{p}$.

We first consider small primes $p \le k$. Since $Q_i B_i = h_i - h_0 \equiv 1 \pmod{Q_0}$, we find that $(B_i, Q_0) = 1$ and thus $L(0) \not\equiv 0 \pmod{p}$ for $p \le k$.

We next treat large primes $p > k$. If p does not divide $\prod_{i \le k} A_i$, $L(x) \pmod{p}$ is a polynomial of degree k and hence can have at most k roots in $\mathbb{Z}/p\mathbb{Z}$, and thus there exists an integer n so that $L(n) \not\equiv 0 \pmod{p}$.

If $p | \prod_{i \le k} A_i$, we must rule out $L(x) \equiv 0 \pmod{p}$ for all $x \in \mathbb{Z}/p\mathbb{Z}$ (i.e., that the reduction of L modulo p is the constant trivial polynomial). Since Q_i are coprime and $p > k$, p can divide at most one

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element in $\{Q_i\}_{i=1}^k$; say $p|Q_i$. As $A_i = Q/Q_i$ is coprime to p, we find that $L_i(x) \bmod p$ is nonconstant and hence has exactly one root. For $j \neq i$, we next show $L_j(x) \bmod p$ has no roots by showing that $p \nmid B_j$. Assume that $p|B_j$. Now, $B_j = (h_j - h_0)/Q_j \equiv (h_j - h_i)/Q_j \pmod{Q_i}$, and since $h_j - h_i = Q_0(l_j - l_i)$ we must have $p|l_j - l_i$; however, since $p|\prod_{i\le k} A_i$, we have $p|Q$, so this contradicts our assumption that all prime divisors of Q are larger than $\max_{i,j}|l_i - l_j|$.

In conclusion, the polynomial $L(x)$ has no fixed prime divisor, and the linear forms $L_1(x), \dots, L_k(x)$ are indeed admissible.

C.4. *Proof of Lemma C.1*

We begin with the following simple consequence of the prime number theorem.

Lemma C.2. *Given an integer $D \ge 2$, we have*

$$\prod_{p|D}(1 + 1/p) \ll \log\log D.$$

Proof. First, note that the product will be maximized if $D = \prod_{p<t} p \asymp e^t$, with t chosen so that $\pi(t) = \omega(D) \le \log D$. In this case, $\prod_{p|D}(1 + 1/p) \ll \exp(\sum_{p<t} 1/p) = \exp(\log\log t) = \log t$, and the result follows. □

Given an integer h with $|h| \le x^{1/4}$ and $h \notin \{h_1, \dots, h_k\}$, define

$$L_0(x) = L_{0,h}(x) := Qx - h_0 + h.$$

We next determine the prime divisors of $D(h)$, the discriminant of

$$L(x) := \prod_{i=0}^{k} L_i(x).$$

With $A = \prod_{i=1}^k A_i$ and $r_i = -B_i/A_i = (h_0 - h_i)/Q$ for $i = 1, \dots, k$ and $r_0 = (h_0 - h)/Q$, the discriminant $D(h)$ of L equals

$$\pm A^{2k} \prod_{0\le i<j\le k} (r_i - r_j)^2.$$

For $0 < i < j \le k$, $r_i - r_j = (h_i - h_j)/Q = Q_0(l_i - l_j)/Q$, whereas $r_0 - r_j = (h_j - h)/Q$. In particular, as $Q|A$, we find that $p|D(h)$ implies that

$$p|Q_0 Q \cdot \prod_{1\le i<j\le k} (l_i - l_j) \cdot \prod_{i=1}^{k}(h_i - h).$$

Lemma C.3. *For each fixed $C > 0$, we have*

$$\prod_{p|D(h)} (1 + 1/p)^C \ll_{k,W} \prod_{p|\prod_{i=1}^k (h-h_i)} (1 + 1/p)^C.$$

Proof. As we have seen, if $p|D(h)$, then $p|Q_0 Q \cdot \prod_{1\le i<j\le k}(l_i - l_j) \cdot \prod_{i=1}^k (h_i - h)$. Since $\prod_{p|Q}(1+1/p) \ll 1$ and $\prod_{p|Q_0}(1 + 1/p) \ll_k 1$, together with $\prod_{1\le i<j\le k}(l_i - l_j) \ll_{k,W} 1$, we find that

$$\prod_{p|D(h)} (1 + 1/p)^C \ll_{k,W} \prod_{p|\prod_{i=1}^k (h_i-h)} (1 + 1/p)^C.$$

□

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We also record a useful estimate involving prime divisors of

$$D_2 := \pm \prod_{0\le i<j\le k} (h_i - h_j)^2,$$

the discriminant of the polynomial $\prod_{i=0}^k (x - h_i)$.

Lemma C.4. *For each fixed $C > 0$, we have*

$$\prod_{p|D_2} (1 + 1/p)^C \ll_{k,W} (\log\log\log\log x)^{2C}.$$

Proof. For $i, j > 0$, we have $h_i - h_j = Q_0(l_i - l_j) \ll_{k,W} 1$, and thus $\prod_{1\le i<j\le k}(h_i - h_j)^2 \ll_{k,W} 1$. Further, as $|h_0| \le Q$ and $h_1, \dots, h_k \ll_{k,W} 1$, we have

$$\prod_{i=1}^k (h_i - h_0)^2 \ll_{k,W} Q^{2k}.$$

Since $Q = \prod_{i=1}^k Q_i$ and each $Q_i \asymp T^H$, with $H \ll \log\log\log x$ and $T \ll \log\log x$ we find that

$$\log D_2 \ll_{k,W} (\log\log\log x)^2.$$

The result now follows from Lemma C.2. □

C.4.1. Applying Henriot's result

Recall that $L_0(x) = Qx - h_0 + h$. Let f_1 be the characteristic function supported on the set of small angle primes $p \in \mathcal{P}_\varepsilon \cap [x/2, x]$; putting $f_1(1) = 1$ we may extend f_1 to a multiplicative function. In what follows, $C = C(k) > 0$ is the constant in Lemma 4.1; note that we allow C to depend on k. Let g denote the multiplicative function

$$g(n) := \prod_{p|n} (1 + 1/p).$$

Lemma C.5. *With $f(n) = b(n)$, or $f(n) = r(n)/4$, we have*

$$\sum_{x/2\le n\le x} f(L_0(n)) \prod_{1\le i\le k} f_1(L_i(n)) \ll f((h - h_0, Q)) g(D(h))^{C+k+1} \frac{x}{(\log x)^{k+1}} \prod_{p\le x} (1 + f(p)/p),$$

where $D(h)$ is the discriminant of the polynomial

$$L(x) = \prod_{i=0}^k L_i(x).$$

Further, for $H > (\log x)^{1/4}$, we have

$$\sum_{|h|\le H, h\notin \mathcal{H}} f((h - h_0, Q)) g(D(h))^{C+k+1} \ll_{k,W} H(\log\log\log\log x)^{2C}.$$

Proof. We first assume that $(h - h_0, Q) = 1$. Now, for $p \nmid Q$, the linear forms L_i are nondegenerate modulo p for $0 \le i \le k$, and we have $\rho_{L_i}(p) = 1$ for $0 \le i \le k$. Further, we have $\rho_L(p) = k+1$ provided p does not divide $Q_0 Q \prod_{1\le i<j\le k}(l_i - l_j) \cdot \prod_{i=1}^k (h_i - h)$.

If $p|Q_i$ for some $i \in [1, k]$, since $p \nmid A_i$, we have $\rho_{L_i}(p) = 1$. For $j \ne i$ and $1 \le j \le k$, as $p|A_j$ and $p \nmid B_j$, we have $\rho_{L_j}(p) = 0$. Further, as we assume that $(h - h_0, Q) = 1$, we also have $\rho_{L_0}(p) = 0$.

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Since $\rho_L(p) = k+1$ for $p \nmid D(h)$, we find that

$$\prod_{p\le x}(1-\rho_L(p)/p) \ll_k \frac{g(D(h))^{k+1}}{(\log x)^{k+1}}$$

since

$$\prod_{p\le x}(1-(k+1)/p) \ll 1/(\log x)^{k+1}$$

and the contribution from primes $p|D(h)$ is of size $g(D(h))^{k+1}$.

Finally, noting that $\sum_{n\le x}\frac{f_1(n)}{n} = 1+O(\sum_{p\in[x/2,x]}1/p) = 1+O(1/\log x)$ and

$$\sum_{n\le x} f(n)/n \ll \prod_{p\le x}(1+f(p)/p)$$

the result follows for $(h-h_0, Q)=1$.

The case $(h-h_0,Q)>1$ is then easily deduced as follows. First, note that $f(mn)=f(m)f(n)$ if $mn|Q$ since $p\equiv 1 \pmod 4$ for any prime divisor $p|Q$, as in our construction we assumed that each prime divisor of Q_i is $\equiv 1 \pmod 4$, together with Q being square-free, as well as the estimate $f(p^k) \le f(p)\cdot f(p^{k-1})$. Letting $\tilde{L_0}(x) = (Q/(h-h_0,Q))\cdot x + (h-h_0)/(h-h_0,Q)$, we have $f(L_0(n)) \le f((h-h_0,Q))f(\tilde{L_0}(n))$ and we may apply the previous argument to the polynomials $\tilde{L_0}, L_1,\dots,L_k$ (note that the two linear polynomials L_0 and $\tilde{L_0}$ have the same roots).

To bound the h-sum, define $f_Q(n) = \prod_{p|(Q,n)} f(p)$, and note, again using Lemma 4.1, that

$$\begin{aligned}\sum_{|h|<H} f((h-h_0,Q))g(D(h))^{C+k+1} &\ll_{k,W} \sum_{|h|<H} f_Q(h-h_0)\prod_{i=1}^{k} g(h-h_i)^{C+k+1}\\ &\ll \frac{H\cdot g(D_2)^C}{(\log H)^{k+1}}\prod_{p<H}(1+f_Q(p)/p)\Big(\prod_{p<H}(1+g(p)^{C+k+1}/p)\Big)^k\\ &\ll_k g(D_2)^C H \ll_{k,W} H(\log\log\log\log x)^{2C}.\end{aligned}$$

□

C.4.2. Completing the proof of Lemma C.1

We are now ready to prove Lemma C.1.

Proof of Lemma C.1. With $g(n) := \prod_{p|n}(1+1/p)$, note that $\Delta_{D(h)} = g(D(h))^C$. Using Lemma C.5, we find that

$$\sum_{n\in\mathcal{N}_3(x)} f(Qn-h_0+h) \ll f((h-h_0,Q))g(D(h))^{C+k+1}\frac{x}{(\log x)^{k+1}}\prod_{p\le x}(1+f(p)/p)$$

and, for $H \ge (\log x)^{1/4}$,

$$\sum_{|h|<H} f((h-h_0,Q))g(D(h))^{C+k+1} \ll H(\log\log\log\log x)^{2C}.$$

Taking $f=b$ and $H=\sqrt{\log x}/U$, together with Chebyshev's inequality, gives the desired bound for the first sum.

Taking $f=r/4$ and using that apart from n in a small subset of $\mathcal{N}_3$, $b(Qn-h_0+h)=0$ for $|h|\le(\log x)^{1/2}/U$ and $h\notin\mathcal{H}$ (from the condition in the first sum), the second bound follows from Lemma C.5 if we use a dyadic decomposition of the h-sum, say for intervals of the form $[2^i(\log x)^{1/2}/U, 2^{i+i}(\log x)^{1/2}/U]$, and then using Chebyshev's inequality.

For the third sum, again we use that $b(Qn - h_0 + h) = 0$ for $|h| \le (\log x)^{1/2}/U$ and $h \notin \mathcal{H}$ which gives, for n outside a small exceptional set, that

$$\sum_{|h|\ge U, h\notin\mathcal{H}} \frac{r(Qn-h_0+h)}{h^2} = \sum_{|h|\ge(\log x)^{1/2}/U, h\notin\mathcal{H}} \frac{r(Qn-h_0+h)}{h^2}.$$

Summing over n outside this small subset, the result follows from Lemma C.5 by taking $f = r/4$, together with Chebyshev's inequality. (The sum over h is easily treated by splitting into dyadic intervals.) □

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