Canad. J. Math. 2024, pp. 1–19
http://dx.doi.org/10.4153/S0008414X2400066X

Convolution of periodic multiplicative functions and the divisor problem

Marco Aymone, Gopal Maiti, Olivier Ramaré, and
Priyamvad Srivastav

Abstract. We study a certain class of arithmetic functions that appeared in Klurman's classification of ± 1 multiplicative functions with bounded partial sums; c.f., Comp. Math. 153(2017), 2017, no. 8, 1622–1657. These functions are periodic and 1-pretentious. We prove that if f_1 and f_2 belong to this class, then $\sum_{n\leq x}(f_1 * f_2)(n) = \Omega(x^{1/4})$. This confirms a conjecture by the first author. As a byproduct of our proof, we studied the correlation between $\Delta(x)$ and $\Delta(\theta x)$, where θ is a fixed real number. We prove that there is a nontrivial correlation when θ is rational, and a decorrelation when θ is irrational. Moreover, if θ has a finite irrationality measure, then we can make it quantitative this decorrelation in terms of this measure.

1 Introduction

1.1 Main result and background

A question posed by Erdős in [7], known as the Erdős discrepancy problem, states that whether for all arithmetic functions $f : \mathbb{N} \to \{-1, 1\}$, we have that the discrepancy

$$\sup_{x,d} \left| \sum_{n\leq x} f(nd) \right| = \infty. \tag{1.1}$$

When, in addition, f is assumed to be completely multiplicative, then this reduces to whether f has unbounded partial sums.

In 2015, Tao [17] proved that (1.1) holds for all $f : \mathbb{N} \to \{-1, 1\}$, and a key point of its proof is that it is sufficient to establish (1.1) only in the class of completely multiplicative functions f taking values in the unit (complex) circle.

When $f : \mathbb{N} \to \{-1, 1\}$ is assumed to be only multiplicative, then not necessarily f has unbounded partial sums. For example, $f(n) = (-1)^{n+1}$ is multiplicative and clearly has bounded partial sums. In this case, $f(2^k) = -1$ for all positive integers k. It was observed by Coons [6] that, for bounded partial sums, this rigidity on powers of 2 is actually necessary under suitable conditions on the values that a multiplicative

Received by the editors January 6, 2024; revised June 27, 2024; accepted July 8, 2024.

The visit of the first author to the Aix-Marseille Université was funded by CNPq grant PDE no. 400010/2022-4 (200121/2022-7). His research also is supported by FAPEMIG, grant Universal no. APQ-00256-23 and by CNPq grant Universal no. 403037/2021-2. The second and third author are supported by the joint FWF-ANR project Arithrand: FWF: I 4945-N and ANR-20-CE91-0006.

AMS subject classification: **11N37**.

Keywords: Dirichlet divisor problem, correlations in the divisor problem, omega bounds, periodic multiplicative functions.

function f takes at the remaining primes. Later, in the same paper [17], Tao gave a partial classification of multiplicative functions taking values ±1 with bounded partial sums: They must satisfy the previous rigidity condition on powers of 2, and they must be 1-pretentious (for more on pretentious Number Theory, we refer the reader to [8] by Granville and Soundararajan); that is,

$$\sum_{p} \frac{1-f(p)}{p} < \infty.$$

Later, Klurman [12] proved that the only multiplicative functions f taking ±1 values and with bounded partial sums are the periodic multiplicative functions with sum 0 inside each period, thus closing this problem for ±1 multiplicative functions.

Building upon the referred work of Klurman, the first author proved in [1] that if we allow values outside the unit disk, a M-periodic multiplicative function f with bounded partial sums such that $f(M) \neq 0$ satisfies

i. For some prime $q|M$, $\sum_{k=0}^{\infty} \frac{f(q^k)}{q^k} = 0$.
ii. For each $p^a \| M$, $f(p^k) = f(p^a)$ for all $k \geq a$.
iii. For each $\gcd(p, M) = 1$, $f(p^k) = 1$, for all $k \geq 1$.

Conversely, if $f : \mathbb{N} \to \mathbb{C}$ is multiplicative and the three conditions above are satisfied, then f has period M and has bounded partial sums. Therefore, these three conditions above give examples of multiplicative functions with values outside the unit disk with bounded partial sums, despite the fact that $f(M)$ is zero or not.

Remark 1.1 It is interesting to observe that when it is assumed that $|f| \leq 1$, the only way to achieve condition i. is with $q = 2$ and $f(2^k) = -1$ for all $k \geq 1$.

Remark 1.2 What makes the difference between a multiplicative function f satisfying i-ii-iii from a nonprincipal Dirichlet character χ is that χ neither satisfies i. nor iii.

Here, we are interested in the convolution $f_1 * f_2(n) := \sum_{d|n} f_1(d) f_2(n/d)$ for f_1 and f_2 satisfying i-ii-iii above. It was proved in [1] that

$$\sum_{n \leq x} (f_1 * f_2)(n) \ll x^{\alpha+\varepsilon},$$

where α is the infimum over the exponents $a > 0$ such that $\Delta(x) \ll x^a$, where $\Delta(x)$ is the classical error term in the Dirichlet divisor problem defined by

$$\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + \Delta(x).$$

It is widely believed that $\alpha = 1/4$, and many results were proven in this direction. The best upper bound up to date is due to Huxley [9]: $\alpha \leq 131/416 \approx 0.315$. Regarding Ω bounds, Soundararajan [16] proved that

$$\Delta(x) = \Omega\left((x \log x)^{1/4} \frac{(\log\log x)^{3/4(2^{4/3}-1)}}{(\log\log\log x)^{5/8}} \right).$$

It was conjectured in [1] that the partial sums of $f_1 * f_2$ obey a similar Ω-bound for $\Delta(x)$; that is, $\sum_{n \leq x} (f_1 * f_2)(n) = \Omega(x^{1/4})$. Here, we establish this conjecture.

https://doi.org/10.4153/S0008414X2400066X Published online by Cambridge University Press

Theorem 1.3 *Let f_1 and f_2 be periodic multiplicative functions satisfying i-ii-iii above. Then $\sum_{n\le x}(f_1 * f_2)(n) = \Omega(x^{1/4})$.*

Example 1.4 The results from [1] give that for each prime q, there exists a unique q-periodic multiplicative function f with bounded partial sums and such that $f(q) \neq 0$. In the case $q = 2$, the corresponding function is $f(n) = (-1)^{n+1}$. Therefore, in this particular case, we have that $\sum_{n\le x}(f * f)(n) = \Omega(x^{1/4})$. In particular, this establishes the conjecture in an uncovered case by Proposition 3.1 of [1].

Remark 1.5 Another class of periodic multiplicative functions with bounded partial sums is that of the nonprincipal Dirichlet characters. In a forthcoming work, the first author is finishing a study where he shows a similar Ω-bound for the partial sums of the convolution between these Dirichlet characters.

Our proof relies on two ingredients. The second one is a study of a family of quadratic forms and is explained in Section 5. The first ingredient is a generalization of a result of Tong [18] and proves the next theorem.

Theorem 1.6 *When a and b are nonnegative integers, $\lambda = \gcd(a,b)$, $c = a/\lambda$ and $d = b/\lambda$, we have*

$$\lim_{X\to\infty} \frac{1}{X^{3/2}} \int_1^X \Delta(x/a)\Delta(x/b)dx = \frac{\tau(cd)}{6\pi^2\sqrt{\lambda cd}} \frac{\zeta(3/2)^4}{\zeta(3)} \prod_{p^k \| cd} \frac{1 - \frac{k-1}{(k+1)p^{3/2}}}{1 + 1/p^{3/2}}.$$

Furthermore, when $\theta > 0$ is irrational, we have

$$\lim_{X\to\infty} \frac{1}{X^{3/2}} \int_1^X \Delta(x)\Delta(\theta x)dx = 0.$$

1.2 The proof in the large

To prove Theorem 1.3, our starting point is the following formula from [1]: If M_1 and M_2 are the periods of f_1 and f_2, respectively, then

$$\sum_{n\le x}(f_1 * f_2)(n) = \sum_{n|M_1M_2}(f_1 * f_2 * \mu * \mu)(n)\Delta(x/n), \tag{1.2}$$

where μ is the Möbius function. Therefore, the partial sums of $f_1 * f_2$ can be written as a finite linear combination of the quantities $(\Delta(x/n))_n$. Apart from the fact that $\Delta(x) = \Omega(x^{1/4})$, we cannot, at least by a direct argument, prevent a conspiracy among the large values of $(\Delta(x/n))_n$ that would yield a cancellation among a linear combination of them.

To circumvent this, our approach is inspired by an elegant result of Tong [18]:

$$\int_1^X \Delta(x)^2 dx = \frac{(1+o(1))}{6\pi^2} \sum_{n=1}^{\infty} \frac{\tau(n)^2}{n^{3/2}} X^{3/2}. \tag{1.3}$$

By (1.2), the limit

$$\lim_{X\to\infty} \frac{1}{X^{3/2}} \int_1^X \left| \sum_{n\le x}(f_1 * f_2)(n) \right|^2 dx$$

can be expressed as a quadratic form with matrix $(c_{a,b})_{a,b|M_1M_2}$, where $c_{a,b}$ is the correlation

$$c_{a,b} := \lim_{X\to\infty} \frac{1}{X^{3/2}} \int_1^X \Delta(x/a)\Delta(x/b)dx.$$

As it turns out, these correlations do not vanish and are computed in Theorem 1.6. With that in hand, the matrix correlation-term $c_{a,b}$ can be expressed as

$$\frac{C}{\sqrt{\gcd(a,b)}}\varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right), \tag{1.4}$$

for some constant $C > 0$ and multiplicative function φ.

This matrix entanglement is hard to analyze directly. In Section 5, we explore sufficient conditions for a matrix of the form (1.4) to be positive definite. When this happens, this ensures the referred Ω-bound. Thanks to the Selberg diagonalization process, we show that when φ is completely multiplicative and satisfies other conditions, then this matrix is positive definite. The main proof somehow reduces to this case; we indeed find a way to conjugate our original matrix to reach a matrix related to a completely multiplicative function. With standard linear algebra of Hermitian matrices, we conclude that our matrix $(c_{a,b})_{a,b|M_1M_2}$ is positive definite. We ended up with the following result.

Theorem 1.7 *Let f_1 and f_2 be two periodic multiplicative functions satisfying i-ii-iii above with periods M_1 and M_2, respectively. Let $g = f_1 * f_2 * \mu * \mu$ and $\gamma(n)$ the multiplicative function defined by*

$$\gamma(n) = \prod_{p^k \| n} \frac{1 - \frac{(k-1)}{(k+1)}p^{-3/2}}{1 + p^{-3/2}}.$$

Then the following limit

$$\lim_{X\to\infty} \frac{1}{X^{3/2}} \int_1^X \left|\sum_{n\le x} (f_1 * f_2)(n)\right|^2 dx$$

is positive and equals to

$$\frac{\zeta(3/2)^4}{\zeta(3)} \sum_{n,m|M_1M_2} g(n)\overline{g(m)}\frac{\gcd(n,m)^{3/2}}{nm}\tau\left(\frac{nm}{\gcd(n,m)^2}\right)\gamma\left(\frac{nm}{\gcd(n,m)^2}\right).$$

1.3 Byproduct study

Motivated by Nyman's reformulation of the Riemann hypothesis [15], in recent papers [2, 3, 4] by Balazard, Duarte, and Martin, the correlation

$$A(\theta) := \int_0^\infty \{x\}\{\theta x\}\frac{dx}{x^2}$$

has been thoroughly studied. Here, $\theta > 0$ is any real number and $\{x\}$ stands for the fractional part of x. Several analytic properties for the function $A(\theta)$ have been shown.

https://doi.org/10.4153/S0008414X2400066X Published online by Cambridge University Press

Motivated by this, we studied the "*divisor*" analogue

$$I(\theta) = \lim_{X\to\infty} \frac{1}{X^{3/2}} \int_1^X \Delta(x)\Delta(\theta x)dx.$$

As stated in Theorem 1.6, when $\theta = p/q$ is a rational number, the limit above is described by a positive multiplicative function depending on p and q. However, and somewhat surprisingly, when θ is irrational, this correlation vanishes. The next proposition establishes that this vanishing is indeed very strong, except maybe at points θ that are well approximated by rationals.

Proposition 1.8 *Let $\theta > 0$ be an irrational number with irrationality measure $\eta + 1$; that is, for each $\varepsilon > 0$, there is a constant $C > 0$ such that the inequality*

$$|n - m\theta| \geq \frac{C}{m^{\eta+\varepsilon}}$$

is violated only for a finite number of positive integers n and m. Then, for every positive ε, we have

$$\int_1^X \Delta(x)\Delta(\theta x)dx = O(X^{3/2-1/(18\eta)+\varepsilon}).$$

In the other cases of irrationals θ, the integral above is $o(X^{3/2})$.

This shows that we have decorrelation among the values $\Delta(x)$ and $\Delta(\theta x)$ when θ is irrational, and moreover, this gives that the function $I(\theta)$ is continuous at the irrational numbers and discontinuous at the rationals. Another interesting remark is a result due to Khintchine [11] that states that almost all irrational numbers, with respect to (w.r.t.) Lebesgue measure, have irrationality measure equals to 2.

Therefore, this result of Khintchine allow us to state the following Corollary from Proposition 1.8.

Corollary 1.9 *For almost all irrational numbers θ w.r.t. Lebesgue measure, for all small fixed $\varepsilon > 0$,*

$$\int_1^X \Delta(x)\Delta(\theta x)dx = O(X^{3/2-1/18+\varepsilon}).$$

We mention that a similar decorrelation also has been obtained by Ivić and Zhai in [10]. In this paper, they show decorrelation between $\Delta(x)$ and $\Delta_k(x)$, where $\Delta_k(x)$ is the error term related to the k-fold divisor function, and $k = 3$ or 4.

2 Notation

2.1 Asymptotic notation

We employ both Vinogradov's notation $f \ll g$ or $f = O(g)$ whenever there exists a constant $C > 0$ such that $|f(x)| \leq C|g(x)|$, for all x in a set of parameters. When not specified, this set of parameters is $x \in (a, \infty)$ for sufficiently large $a > 0$. We employ $f = o(g)$ when $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$. In this case, a can be a complex number or $\pm\infty$. Finally, $f = \Omega(g)$ when $\limsup_{x\to a} \frac{|f(x)|}{g(x)} > 0$, where a is as in the previous notation.

https://doi.org/10.4153/S0008414X2400066X Published online by Cambridge University Press

2.2 Number-theoretic notation

Here, p stands for a generic prime number. We sometimes denote the least common multiple between a, b as $\mathrm{lcm}(a, b)$. The greatest common divisor is denoted by $\gcd(a, b)$. The symbol $*$ stands for Dirichlet convolution between two arithmetic functions: $(f * g)(n) = \sum_{d|n} f(d)g(n/d)$.

3 Multiplicative auxiliaries

Our first task is to evaluate $\sum_{n\geq 1} \tau(cn)\tau(dn)/n^{3/2}$ for coprime positive integers c and d.

Lemma 3.1 *Let c be fixed positive number and $f(n)$ be a multiplicative function with $f(c) \neq 0$. Then $n \mapsto \frac{f(cn)}{f(c)}$ is multiplicative.*

Proof For positive integers u, v, we have

$$f(u)f(v) = f(\gcd(u, v))f(\mathrm{lcm}(u, v)).$$

Let $u = cn$, $v = cm$ with $\gcd(n, m) = 1$. Then $f(cn)f(cm) = f(c)f(cnm)$. Therefore, we obtain

$$\frac{f(cm)}{f(c)}\frac{f(cn)}{f(c)} = \frac{f(cnm)}{f(c)}. \qquad \blacksquare$$

Lemma 3.2 *Let c, d be two fixed positive integers with $\gcd(c, d) = 1$. Then*

$$\sum_{n=1}^{\infty} \frac{\tau(cn)\tau(dn)}{n^s} = \tau(cd)\frac{\zeta(s)^4}{\zeta(2s)} \prod_{p^k \| cd} (1 + p^{-s})^{-1} \left(1 - \tfrac{(k-1)}{(k+1)} p^{-s}\right).$$

The quantity we compute appears in several places – for instance, in [14] by Lee and Lee and in [5] by Borda, Munsch, and Shparlinski.

Proof Note that $\frac{\tau(cn)}{\tau(c)}$ is a multiplicative function in the variable n by Lemma 3.1, and so is $\frac{\tau(cn)\tau(dn)}{\tau(c)\tau(d)}$. Therefore, for $\Re(s) > 1$, we have the following Euler factorization:

$$\sum_{n=1}^{\infty} \frac{\tau(cn)\tau(dn)}{\tau(c)\tau(d)n^s} = \prod_{p\nmid cd} \left(1 + \sum_{\ell=1}^{\infty} \frac{\tau(p^\ell)^2}{p^{\ell s}}\right) \prod_{p|cd} \left(1 + \sum_{\ell=1}^{\infty} \frac{\tau(cp^\ell)\tau(dp^\ell)}{\tau(c)\tau(d)p^{\ell s}}\right).$$

For $|x| < 1$, we know that

$$\sum_{\ell=0}^{\infty} (\ell+1)x^\ell = \frac{1}{(1-x)^2}, \qquad \sum_{\ell=0}^{\infty} (\ell+1)^2 x^\ell = \frac{(1+x)}{(1-x)^3},$$

from which we also derive that

$$\sum_{\ell=0}^{\infty} \ell(\ell+1)x^\ell = \frac{2x}{(1-x)^3}.$$

https://doi.org/10.4153/S0008414X2400066X Published online by Cambridge University Press

Now,

$$\prod_{p\nmid cd}\left(1+\sum_{\ell=1}^{\infty}\frac{\tau(p^\ell)^2}{p^{\ell s}}\right)=\prod_{p}\left(1+\sum_{\ell=1}^{\infty}\frac{(\ell+1)^2}{p^{\ell s}}\right)\prod_{p\mid cd}\left(1+\sum_{\ell=1}^{\infty}\frac{(\ell+1)^2}{p^{\ell s}}\right)^{-1}$$
$$=\prod_{p}\frac{(1+p^{-s})}{(1-p^{-s})^3}\prod_{p\mid cd}\frac{(1-p^{-s})^3}{(1+p^{-s})}$$
$$=\frac{\zeta(s)^4}{\zeta(2s)}\prod_{p\mid cd}\frac{(1-p^{-s})^3}{(1+p^{-s})}.$$

If $\gcd(c,d)=1$,

$$\prod_{p\mid cd}\left(1+\sum_{\ell=1}^{\infty}\frac{\tau(cp^\ell)\tau(dp^\ell)}{\tau(c)\tau(d)p^{\ell s}}\right)=\prod_{p^k\parallel cd}\left(1+\sum_{\ell=1}^{\infty}\frac{(k+1+\ell)(\ell+1)}{(k+1)p^{\ell s}}\right)$$
$$=\prod_{p^k\parallel cd}\left(1+\sum_{\ell=1}^{\infty}\frac{(\ell+1)}{p^{\ell s}}+\frac{1}{k+1}\sum_{\ell=1}^{\infty}\frac{\ell(\ell+1)}{p^{\ell s}}\right)$$
$$=\prod_{p^k\parallel cd}(1-p^{-s})^{-3}\left(1-\tfrac{(k-1)}{(k+1)}p^{-s}\right).\qquad\blacksquare$$

4 Correlations of the Δ function

We continue with the proof with the following Lemma.

Lemma 4.1 *Let $a>0$. Then*

$$\int x^2\cos(ax)dx=x^2\frac{\sin(ax)}{a}+2x\frac{\cos(ax)}{a^2}-2\frac{\sin(ax)}{a^3}.$$

Moreover, for any $X>1$,

$$\int_1^X x^2\cos(ax)dx\ll\frac{X^2}{a},\quad\int_1^X x^2\sin(ax)dx\ll\frac{X^2}{a}.$$

Proof We do integration by parts:

$$\int x^2\cos(ax)dx=x^2\frac{\sin(ax)}{a}-\int\frac{2x\sin(ax)}{a}dx.$$

By making the trivial bound $|\sin(ax)|\le 1$ in the right-hand side of the equation above, we reach to the second claim of the proposed Lemma. By making integration by parts, the last integral of the equation above gives the first claim of the Lemma. Similar arguments give similar results for sin in place of cos. ■

Lemma 4.2 *Let a,b be positive integers, $\lambda=\gcd(a,b)$, $c=a/\lambda$ and $d=b/\lambda$. Then*

$$\lim_{X\to\infty}\frac{1}{X^{3/2}}\int_1^X\Delta(x/a)\Delta(x/b)dx=\frac{1}{6\pi^2\sqrt{\lambda}cd}\sum_{n=1}^{\infty}\frac{\tau(cn)\tau(dn)}{n^{3/2}}.$$

https://doi.org/10.4153/S0008414X2400066X Published online by Cambridge University Press

Proof Let $N > 0$ and $\varepsilon > 0$ be a small number that may change from line after line. We proceed with Voronoï's formula for $\Delta(x)$ in the following form (see [13]):

$$\Delta(x) = \frac{x^{1/4}}{\pi\sqrt{2}} \sum_{n \le N} \frac{\tau(n)}{n^{3/4}} \cos(4\pi\sqrt{nx} - \pi/4) + R_N(x),$$

where, for every positive ε, we have

$$R_N(x) \ll x^{\varepsilon} + \frac{x^{1/2+\varepsilon}}{N^{1/2}}.$$

We select N at the end. With this formula, we have that in the range $1 \le x \le X$,

$$\begin{aligned}\Delta(x/a) &= \frac{(x/a)^{1/4}}{\pi\sqrt{2}} \sum_{n \le N} \frac{\tau(n)}{n^{3/4}} \cos(4\pi\sqrt{nx/a} - \pi/4) + R_N(x/a)\\ &= U_N(x/a) + R_N(x/a)\end{aligned}$$

say.

Now,

$$\begin{aligned}\int_1^X \Delta(x/a)\Delta(x/b)dx &= \int_1^X U_N(x/a)U_N(x/b)dx\\ &\quad + \int_1^X U_N(x/a)R_N(x/b)dx\\ &\quad + \int_1^X U_N(x/b)R_N(x/a)dx\\ &\quad + \int_1^X R_N(x/a)R_N(x/b)dx\\ &= \int_1^X U_N(x/a)U_N(x/b)dx\\ &\quad + O\left(X^{1+1/4+\varepsilon} + \frac{X^{1+3/4+\varepsilon}}{\sqrt{N}} + \frac{X^{2+\varepsilon}}{N}\right),\end{aligned}$$

where we used the Cauchy-Schwarz inequality and (1.3) in the last equality. By making the change of variable $u = x/\lambda$, we reach

$$\begin{aligned}\int_1^X & U_N(x/a)U_N(x/b)dx = \lambda \int_1^{X/\lambda} U_N(x/c)U_N(x/d)dx\\ &= \frac{\lambda}{2\pi^2(cd)^{1/4}} \sum_{n,m \le N} \frac{\tau(n)\tau(m)}{(nm)^{3/4}}\cdot\\ &\quad \cdot \int_1^{X/\lambda} x^{1/2} \cos(4\pi\sqrt{nx/c} - \pi/4)\cos(4\pi\sqrt{mx/d} - \pi/4)dx\\ &= \frac{\lambda}{\pi^2(cd)^{1/4}} \sum_{n,m \le N} \frac{\tau(n)\tau(m)}{(nm)^{3/4}}\cdot\\ &\quad \cdot \int_1^{(X/\lambda)^{1/2}} u^2 \cos(4\pi u\sqrt{n/c} - \pi/4)\cos(4\pi u\sqrt{m/d} - \pi/4)du,\end{aligned}$$

where in the last equality above we made a change of variable $u = \sqrt{x}$. We claim now that the main contribution comes when $n/c = m/d$. Since c and d are coprime, this implies that $m = dk$ and $n = ck$. Therefore, the sum over these n and m can be written as

$$\frac{\lambda}{\pi^2 cd}\sum_{k=1}^{\infty}\frac{\tau(ck)\tau(dk)}{k^{3/2}}\int_1^{(X/\lambda)^{1/2}} u^2\cos^2(4\pi u\sqrt{k}-\pi/4)du + O\left(\frac{X^{3/2+\varepsilon}}{\sqrt{N}}\right). \tag{4.1}$$

We recall now that $\cos^2(v) = \frac{1+\cos(2v)}{2}$; hence, by Lemma 4.1, the integral above is

$$\int_1^{X^{1/2}/\lambda^{1/2}} x^2\cos^2(4\pi\sqrt{n}x-\pi/4)dx = \frac{X^{3/2}}{6\lambda^{3/2}} + O(X), \tag{4.2}$$

where the big-oh term is uniform in n. Now we will show that the sum over those n and m such that $n/c \neq m/d$ will be $o(X^{3/2})$. With this, the proof is complete by combining (4.1) and (4.2).

We recall the identity $2\cos(u)\cos(v) = \cos(u-v)+\cos(u+v)$. Thus, for $\sqrt{n/c} \neq \sqrt{m/d}$, by Lemma 4.1, we find that

$$\begin{aligned}
&\int_1^{X^{1/2}/\lambda^{1/2}} x^2\cos(4\pi\sqrt{n/c}x-\pi/4)\cos(4\pi\sqrt{m/d}x-\pi/4)dx\\
&= \frac{1}{2}\int_1^{X^{1/2}/\lambda^{1/2}} x^2\cos(4\pi(\sqrt{n/c}-\sqrt{m/d})x)dx\\
&\quad + \frac{1}{2}\int_1^{X^{1/2}/\lambda^{1/2}} x^2\sin(4\pi(\sqrt{n/c}+\sqrt{m/d})x)dx\\
&\ll \frac{X}{\left|\sqrt{n/c}-\sqrt{m/d}\right|} + \frac{X}{\sqrt{n/c}+\sqrt{m/d}}\\
&\ll \frac{\sqrt{n/c}+\sqrt{m/d}}{|nd-mc|}X.
\end{aligned}$$

Let $\mathbb{1}_P(n)$ be the indicator that n has property P. We find that

$$\begin{aligned}
&\sum_{\substack{n,m\le N\\ nd-mc\neq 0}}\frac{\tau(n)\tau(m)}{(nm)^{3/4}}\int_1^{X/\lambda} x^{1/2}\cos(4\pi\sqrt{nx/c}-\pi/4)\cos(4\pi\sqrt{mx/d}-\pi/4)dx\\
&\ll XN^{\varepsilon}\sum_{\substack{n,m\le N\\ nd-mc\neq 0}}\frac{\sqrt{n/c}+\sqrt{m/d}}{(nm)^{3/4}|nd-mc|}\\
&= XN^{\varepsilon}\sum_{\substack{n,m\le N\\ nd-mc\neq 0}}\frac{\sqrt{n/c}+\sqrt{m/d}}{(nm)^{3/4}|nd-mc|}\sum_{\substack{k=-N\max(c,d)\\ k\neq 0}}^{N\max(c,d)}\mathbb{1}_{nd-mc=k}.
\end{aligned}$$

https://doi.org/10.4153/S0008414X2400066X Published online by Cambridge University Press

On calling this sum S, we readily continue with

$$\begin{aligned} S &\ll XN^{\varepsilon}\sum_{k=1}^{N\max(c,d)}\frac{1}{k}\sum_{m\le N}\frac{\sqrt{m}+\sqrt{k}}{((k+mc)m)^{3/4}} \\ &\ll XN^{\varepsilon}\left(O(\log N)^2+\sum_{k\le N}\frac{1}{\sqrt{k}}\sum_{m\le N}\frac{1}{(m^2+mk)^{3/4}}\right) \\ &\ll XN^{\varepsilon}\left(O(\log N)^2+\sum_{k\le N}\frac{1}{\sqrt{k}}\left(\sum_{k\le m\le N}\frac{1}{m^{3/2}}+\frac{1}{k^{3/4}}\sum_{m\le k}\frac{1}{m^{3/4}}\right)\right) \\ &\ll XN^{\varepsilon}(\log N)^2. \end{aligned}$$

Finally, by selecting $N = X^2$, we arrive at

$$\int_1^X \Delta(x/a)\Delta(x/b)dx = \frac{1}{6\pi^2\sqrt{\lambda}cd}\left(\sum_{n=1}^{\infty}\frac{\tau(cn)\tau(dn)}{n^{3/2}}\right)X^{3/2}+O(X^{3/2-1/4+\varepsilon}),$$

where the main contribution in the O-term above comes from the usage of Cauchy-Schwarz in the beginning of the proof.

The proof is complete. ∎

Now we deviate from the main line and prove Proposition 1.8.

Proof of Proposition 1.8 By the proof of Lemma 4.2 we have that

$$\begin{aligned} I_\theta(X) &:= \int_1^X \Delta(x)\Delta(\theta x)dx \\ &= \frac{1}{\pi^2}\sum_{n,m\le N}\frac{\tau(n)\tau(m)}{(nm)^{3/4}}\int_1^{X^{1/2}} x^2\cos(4\pi x\sqrt{n}-\pi/4)\cos(4\pi x\sqrt{m\theta}-\pi/4)dx \\ &\quad + O\left(X^{1+1/4+\varepsilon}+\frac{X^{1+3/4+\varepsilon}}{\sqrt{N}}+\frac{X^{2+\varepsilon}}{N}\right). \end{aligned}$$

Now, by appealing to the identity $2\cos(u)\cos(v) = \cos(u-v)+\cos(u+v)$, we reach at

$$\begin{aligned} I_\theta(X) &= \frac{1}{2\pi^2}\sum_{n,m\le N}\frac{\tau(n)\tau(m)}{(nm)^{3/4}}\int_1^{X^{1/2}} x^2\cos(4\pi x(\sqrt{n}-\sqrt{m\theta}))dx \\ &\quad + \frac{1}{2\pi^2}\sum_{n,m\le N}\frac{\tau(n)\tau(m)}{(nm)^{3/4}}\int_1^{X^{1/2}} x^2\sin(4\pi x(\sqrt{n}+\sqrt{m\theta}))dx \\ &\quad + O\left(X^{1+1/4}+\frac{X^{1+3/4+\varepsilon}}{\sqrt{N}}+\frac{X^{2+\varepsilon}}{N}\right). \end{aligned}$$

We have that, by Lemma 4.1,

$$\begin{aligned} &\sum_{n,m\le N}\frac{\tau(n)\tau(m)}{(nm)^{3/4}}\int_1^{X^{1/2}} x^2\sin(4\pi x(\sqrt{n}+\sqrt{m\theta}))dx \\ &\qquad \ll XN^{\varepsilon}\sum_{n,m\le N}\frac{1}{m^{3/4}n^{5/4}+n^{3/4}m^{5/4}} \end{aligned}$$

$$\ll XN^{\varepsilon}\sum_{n,m\le N}\frac{1}{n^{3/4}m^{5/4}}$$
$$\ll XN^{1/4+\varepsilon}.$$

Thus, we reach at

$$I_\theta(X)=\frac{1}{2\pi^2}\sum_{n,m\le N}\frac{\tau(n)\tau(m)}{(nm)^{3/4}}\int_1^{X^{1/2}}x^2\cos(4\pi x(\sqrt{n}-\sqrt{m\theta}))dx$$
$$+O\left(X^{1+1/4}+\frac{X^{1+3/4+\varepsilon}}{\sqrt{N}}+\frac{X^{2+\varepsilon}}{N}+XN^{1/4+\varepsilon}\right).$$

On calling the last sum above $S_\theta(X)$, $a_{n,m}:=4\pi(\sqrt{n}-\sqrt{m\theta})$, we obtain that

$$S_\theta(X)=X^{3/2}\sum_{n,m\le N}\frac{\tau(n)\tau(m)}{(nm)^{3/4}}\Lambda(a_{n,m}\sqrt{X}),$$

where, by Lemma 4.1, $\Lambda(0):=1/3$ and for $u\neq 0$,

$$\Lambda(u):=\frac{\sin(u)}{u}+2\frac{\cos(u)}{u^2}-2\frac{\sin(u)}{u^3}.$$

A careful inspection shows that Λ is continuous and for large $|u|$, $\Lambda(u)\ll|u|^{-1}$.

Now, for a large parameter T (to be chosen), we split

$$S_\theta(X)=X^{3/2}\sum_{\substack{n,m\le N\\ |a_{n,m}\sqrt{X}|\le T}}\frac{\tau(n)\tau(m)}{(nm)^{3/4}}\Lambda(a_{n,m}\sqrt{X})$$
$$+X^{3/2}\sum_{\substack{n,m\le N\\ |a_{n,m}\sqrt{X}|>T}}\frac{\tau(n)\tau(m)}{(nm)^{3/4}}\Lambda(a_{n,m}\sqrt{X}).$$

We call the first sum in the right-hand side above *diagonal* contribution and the second sum the *nondiagonal* contribution. We select $T=X^{1/2-\delta}$ and $N=X^{1/2+\delta}$, for some small $\delta>0$.

The diagonal contribution. We have that

$$D(X)=X^{3/2}\sum_{\substack{n,m\le N\\ |a_{n,m}\sqrt{X}|\le T}}\frac{\tau(n)\tau(m)}{(nm)^{3/4}}\Lambda(a_{n,m}\sqrt{X}) \tag{4.3}$$

$$\ll X^{3/2}N^{\varepsilon}\sum_{m\le N}\frac{1}{m^{3/4}}\sum_{\substack{n;\\ |n-m\theta|\le\frac{2\sqrt{m\theta}}{X^{\delta}}+\frac{1}{X^{2\delta}}}}\frac{|\Lambda(a_{n,m}\sqrt{X})|}{n^{3/4}}. \tag{4.4}$$

The inner sum above we split accordingly $\frac{2\sqrt{m\theta}}{X^{\delta}}+\frac{1}{X^{2\delta}}$ is below and above 1. In the case that this quantity is greater or equal to 1, we have that $m\ge(4\theta)^{-1}X^{2\delta}$, and hence,

https://doi.org/10.4153/S0008414X2400066X Published online by Cambridge University Press

$$\begin{aligned} D(X) &\ll X^{3/2}N^{\varepsilon} \sum_{(4\theta)^{-1}X^{2\delta}\le m\le N} \frac{1}{m^{3/4}} \sum_{\substack{n;\\ |n-m\theta|\le \frac{2\sqrt{m\theta}}{X^{\delta}}+\frac{1}{X^{2\delta}}}} \frac{|\Lambda(a_{n,m}\sqrt{X})|}{n^{3/4}} \\ &\ll X^{3/2}N^{\varepsilon} \sum_{(4\theta)^{-1}X^{2\delta}\le m\le N} \frac{1}{m^{3/4}} \cdot \frac{1}{m^{3/4}} \frac{\sqrt{m}}{X^{\delta}} \\ &\ll X^{3/2-\delta}N^{\varepsilon}. \end{aligned}$$

In the case that $\frac{2\sqrt{m\theta}}{X^{\delta}} + \frac{1}{X^{2\delta}} \le 1$, we have that $m \le (4\theta)^{-1}X^{2\delta}$, and now the Diophantine properties of θ come into play. If the irrationality measure of θ is $\eta + 1$, we have that for each ε, there is a constant $C > 0$ such that the inequality

$$|n - m\theta| \ge \frac{C}{m^{\eta+\varepsilon}}$$

is violated only for a finite number of positive integers n and m. In our case, this allows us to lower bound $|a_{n,m}\sqrt{X}|$ for all but a finite number of n and m such that $1 \le m \ll X^{2\delta}$ and $1/2 \le \sqrt{n}/\sqrt{m\theta} \le 2$:

$$\begin{aligned} |a_{n,m}\sqrt{X}| \cdot \frac{\sqrt{n}+\sqrt{m\theta}}{\sqrt{n}+\sqrt{m\theta}} &= \sqrt{X}\frac{|n-m\theta|}{\sqrt{n}+\sqrt{m\theta}} \\ &\ge \frac{\sqrt{X}}{m^{\eta+\varepsilon}(\sqrt{n}+\sqrt{m\theta})} \\ &\gg X^{1/2-(2\eta+1)\delta-\varepsilon}. \end{aligned}$$

Observe that the diagonal contribution from those exceptional n and m will be at most $O(X)$. With these estimates on hand and recalling that $\Lambda(u) \ll |u|^{-1}$, we obtain

$$\begin{aligned} &X^{3/2}N^{\varepsilon} \sum_{m\le(4\theta)^{-1}X^{2\delta}} \frac{1}{m^{3/4}} \sum_{\substack{n;\\ |n-m\theta|\le \frac{2\sqrt{m\theta}}{X^{\delta}}+\frac{1}{X^{2\delta}}}} \frac{|\Lambda(a_{n,m}\sqrt{X})|}{n^{3/4}} \\ &\ll X^{3/2}N^{\varepsilon} \sum_{m\le(4\theta)^{-1}X^{2\delta}} \frac{1}{m^{3/2}} \cdot \frac{1}{X^{1/2-(2\eta+1)\delta-\varepsilon}} + O(X) \\ &\ll X^{1+(2\eta+1)\delta+\varepsilon}. \end{aligned}$$

Therefore, the diagonal contribution is at most

$$D(X) \ll X^{1+(2\eta+1)\delta+\varepsilon} + X^{3/2-\delta+\varepsilon}.$$

The nondiagonal contribution. Now, we reach

$$\begin{aligned} X^{3/2} \sum_{\substack{n,m\le N\\ |a_{n,m}\sqrt{X}|>T}} \frac{\tau(n)\tau(m)}{(nm)^{3/4}} \Lambda(a_{n,m}\sqrt{X}) &\ll \frac{X^{3/2}N^{1/2+\varepsilon}}{T} \\ &= X^{3/2+1/4+(\delta+\varepsilon)/2+\varepsilon\delta-1/2+\delta} \\ &= X^{1+1/4+3\delta/2+\varepsilon/2+\varepsilon\delta}. \end{aligned}$$

https://doi.org/10.4153/S0008414X2400066X Published online by Cambridge University Press

We choose $\delta = \frac{1}{3(2\eta+1)}$ and obtain

$$I_\theta(X) = O(X^{3/2-1/(18\eta)}).$$

The proof of the first part of Proposition 1.8 is complete.

Now we assume that θ is a Liouville number (i.e., θ does not have finite irrationality measure). We see that the nondiagonal argument does not depend on the Diophantine properties of θ. Let $\eta > 0$ be a large fixed number, $t > 0$ a small number that will tend to 0. For $D(X)$ as in (4.3), by repeating verbatim the estimates above, we have that

$$D(X) \ll X^{3/2} \sum_{m \le (4\theta)^{-1} X^{2\delta}} \frac{\tau(m)}{m^{3/4}} \sum_{\substack{n; \\ |n - m\theta| \le \frac{2\sqrt{m\theta}}{X^\delta} + \frac{1}{X^{2\delta}}}} \frac{\tau(n)|\Lambda(a_{n,m}\sqrt{X})|}{n^{3/4}} + O(X^{3/2-\delta} N^\varepsilon).$$

Let $\|x\|$ be the distance from x to the nearest integer. We split the sum over m above into two sums: One over those m such that $\|m\theta\| > tm^{-\eta}$ and the other over m such that $\|m\theta\| \le tm^{-\eta}$.

Repeating the argument above for non-Liouville numbers, we have that the contribution over those m such that $\|m\theta\| > tm^{-\eta}$ is $O(t^{-1}X^{1+\delta(2\eta+1)})$. Therefore,

$$D(X) \ll X^{3/2} \sum_{\substack{m=1 \\ \|m\theta\| \le tm^{-\eta}}}^{\infty} \frac{1}{m^{3/2-\varepsilon}} + O(t^{-1}X^{1+\delta(2\eta+1)} + X^{3/2-\delta+\varepsilon}).$$

Combining all these estimates, we see that

$$\limsup_{X\to\infty} \frac{1}{X^{3/2}} \left| \int_1^X \Delta(x)\Delta(\theta x)\,dx \right| \ll \sum_{\substack{m=1 \\ \|m\theta\| \le tm^{-\eta}}}^{\infty} \frac{1}{m^{3/2-\varepsilon}}.$$

Since the upper bound above holds for all $t > 0$, we have that as $t \to 0^+$, the sum above converges to 0 and thus implies that the lim sup is 0. The proof is complete. ■

Proof of Theorem 1.6 On combining Lemma 4.2 together with Lemma 3.2, we get the first part of Theorem 1.6. The second part is a trivial consequence of Proposition 1.8. ■

5 Quadratic forms auxiliaries

The main proof will lead to considering the quadratic form attached to a matrix of the form

$$M_{S,\varphi} = \left(\frac{1}{\sqrt{\gcd(a,b)}} \varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)} \right) \right)_{a,b \in S}, \tag{5.1}$$

where S is some finite set of integers, while φ is a *nonnegative multiplicative function such that* $\varphi(p^k) \le 1$. So we stray somewhat from the main line and investigate this situation. Our initial aim is to find conditions under which the associated quadratic form is positive definite, but we shall finally restrict our scope. GCD-matrices have

https://doi.org/10.4153/S0008414X2400066X Published online by Cambridge University Press

received quite some attention, but it seems the matrices occuring in (5.1) have not been explored. We obtain results in two specific contexts.

Completely multiplicative case

Here is our first result.

Lemma 5.1 *When φ is completely multiplicative and $p^{1/4}\varphi(p) \in (0,1]$, the matrix $M_{S,\varphi}$ is nonnegative. When in addition we assume that $p^{1/4}\varphi(p) \in (0,1)$ and S is divisor closed, this matrix is positive definite. The determinant in that case is given by the formula*

$$\det\left(\frac{1}{\sqrt{\gcd(a,b)}}\varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right)\right)_{a,b\in S} = \prod_{d\in S}\varphi(d)^2(\mu\star\psi)(d),$$

where ψ is the completely multiplicative function given by $\psi(p) = 1/(\sqrt{p}\varphi(p)^2)$.

By *divisor closed*, we mean that every divisor of an element of S also belongs to S.

Proof We write

$$\frac{1}{\sqrt{\gcd(a,b)}}\varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right) = \varphi(a)\varphi(b)\psi(\gcd(a,b)),$$

where $\psi(n) = 1/(\varphi(n)^2\sqrt{n})$ is another nonnegative multiplicative function. We introduce the auxiliary function $h = \mu \star \psi$. Notice that this function is multiplicative and nonnegative, as $\psi(p) \geq 1$. We use Selberg's diagonalization process to write

$$\begin{aligned}\sum_{a,b\in S}\frac{1}{\sqrt{\gcd(a,b)}}\varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right)x_a x_b &= \sum_{a,b\in S}\psi(\gcd(a,b))\varphi(a)x_a\varphi(b)x_b\\ &= \sum_{a,b\in S}\sum_{d\mid\gcd(a,b)}h(d)\varphi(a)x_a\varphi(b)x_b\\ &= \sum_d h(d)\Bigg(\sum_{\substack{a\in S\\ d\mid a}}\varphi(a)x_a\Bigg)^2\end{aligned}$$

from which the nonnegativity follows readily. When φ verifies the more stringent condition that $p^{1/4}\varphi(p) \in (0,1)$, we know that both φ and h are strictly positive. Let us define $y_d = \sum_{\substack{a\in S\\ d\mid a}}\varphi(a)x_a$. The variable d varies in the set D of divisors of S. We assume that S is divisor closed, so that $D = S$. We can readily invert the triangular system giving the y_d's as functions of the x_a's into

$$\varphi(a)x_a = \sum_{a\mid b}\mu(b/a)y_b.$$

Indeed, the fact that the mentioned system is triangular ensures that a solution y is unique if it exists. We next verify that the proposed expression is indeed a solution by

$$\sum_{\substack{a\in S\\ d\mid a}}\varphi(a)x_a = \sum_{\substack{a\in S\\ d\mid a}}\sum_{a\mid b}\mu(b/a)y_b = \sum_{\substack{b\in S\\ d\mid b}}y_b\sum_{d\mid a\mid b}\mu(b/a) = y_d$$

https://doi.org/10.4153/S0008414X2400066X Published online by Cambridge University Press

as the last inner sum vanishes when $d \neq b$. We thus have a writing as a linear combination of squares of independant linear forms. In a more pedestrian manner, if our quadratic form vanishes, then all y_d's do vanish, and hence, so do the x_a's. ∎

Here is a corollary.

Lemma 5.2 *When the set S contains solely squarefree integers, the matrix $M_{S,\varphi}$ is nonnegative.*

Proof Simply apply Lemma 5.1 to the completely multiplicative function φ' that has the same values on primes as φ. ∎

Now we recall the Sylvester's criterion.

Lemma 5.3 *A hermitian complex valued matrix $M = (m_{i,j})_{i,j\leq K}$ defines a positive definite form if and only if all its principal minors $\det(m_{i,j})_{i,j\leq k}$ for $k \leq K$ are positive.*

A tensor product-like situation

Lemma 5.2 is enough to solve our main problem when M_1 and M_2 are coprime squarefree integers. We need to go somewhat further. Let S be a divisor closed set. We consider the quadratic form

$$\sum_{a,b\in S} \varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right) x_a x_b, \tag{5.2}$$

where the variables x_a's are also multiplicatively split; that is,

$$x_a = \prod_{p^k \| a} x_{p^k}. \tag{5.3}$$

Let $S(p)$ the subset of S made only of 1 and of prime powers. We extend S so that it contains every products of integers from any collection of distinct $S(p)$.[1] We then find that

$$\sum_{a,b\in S} \frac{1}{\sqrt{\gcd(a,b)}} \varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right) x_a x_b = \prod_{p\in S}\left(\sum_{p^k,p^\ell\in S(p)} \frac{\varphi\left(p^{\max(k,\ell)-\min(k,\ell)}\right)}{p^{\min(k,\ell)/2}} x_{p^k} x_{p^\ell}\right). \tag{5.4}$$

We check this identity simply by opening the right-hand side and seeing that every summand from the left-hand side appears one and only one time.

6 Proof of the main result

Proof By [1, Theorem 1.4], we have

$$S(x) = \sum_{n\leq x} (f_1 * f_2)(n) = \sum_{a|M_1M_2} g(a)\Delta(x/a),$$

[1] This is not automatically the case, as the example $S = \{1, 2, 3, 5, 6, 10\}$ shows, since 30 does not belong to S.

https://doi.org/10.4153/S0008414X2400066X Published online by Cambridge University Press

where $g = f_1 * f_2 * \mu * \mu$. We infer from this formula that

$$\int_1^X |S(x)|^2 dx = \sum_{a,b|M_1M_2} g(a)g(b) \int_1^X \Delta(x/a)\Delta(x/b)dx$$
$$= \frac{(1+o(1))}{6\pi^2} X^{3/2} \sum_{a,b|M_1M_2} g(a)g(b) \frac{\gcd(a,b)^{3/2}}{ab} \sum_{n=1}^{\infty} \frac{\tau\left(\frac{an}{\gcd(a,b)}\right)\tau\left(\frac{bn}{\gcd(a,b)}\right)}{n^{3/2}}$$

by Lemma 4.2. We next use Lemma 3.2 to infer that

$$\lim_{X\to\infty} \frac{1}{X^{3/2}} \int_1^X |S(x)|^2 dx = \frac{\zeta(3/2)^4}{6\pi^2\zeta(3)} \sum_{a,b|M_1M_2} g(a)g(b) \frac{1}{\sqrt{\gcd(a,b)}} \varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right),$$

where φ is multiplicative and at prime powers:

$$\begin{aligned} \varphi(p^k) &= \frac{(k+1)}{p^k} \frac{1}{1+p^{-3/2}} \left(1 - \frac{(k-1)}{(k+1)p^{3/2}}\right) \\ &= \frac{1}{p^k(1+p^{-3/2})} \left((k+1) - (k-1)p^{-3/2}\right) \\ &= \frac{1}{p^k(1+p^{-3/2})} \left(k(1-p^{-3/2}) + (1+p^{-3/2})\right) \\ &= \frac{k\beta(p)+1}{p^k}, \end{aligned} \tag{6.1}$$

where

$$\beta(p) = \frac{1-p^{-3/2}}{1+p^{-3/2}}.$$

Now, we can write

$$\frac{1}{\sqrt{\gcd(a,b)}} \varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right) = \frac{1}{(ab)^{1/4}} \left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right)^{1/4} \varphi\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right).$$

Since the terms $a^{-1/4}$ and $b^{-1/4}$ can be absorbed into the variables $g(a)$ and $g(b)$ of the quadratic form, it is enough to consider the quantity

$$\varphi^*\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right), \quad \text{where } \varphi^*(n) = n^{1/4}\varphi(n).$$

We note that, at the prime power p^k, we have

$$\varphi^*(p^k) = p^{k/4}\varphi(p^k) = \frac{k\beta(p)+1}{p^{3k/4}}. \tag{6.2}$$

Due to (5.4) and the discussion before it, we now restrict to the prime power case; that is, we look to matrices of the form

$$\mathcal{M}_K = \left(\varphi^*(p^{|i-j|})\right)_{i,j\le K}.$$

As we are dealing with a given prime p, we shorten $\beta(p)$ in β.

https://doi.org/10.4153/S0008414X2400066X Published online by Cambridge University Press

Since φ^* is not completely multiplicative, it is not clear how to handle the matrix $\mathcal{M}_K$ directly. So, our aim will be to transform this into another matrix which in some way associates with a completely multiplicative function. So, let us consider

$$\mathcal{A}_K = \mathcal{U}_K^\top \mathcal{M}_K \mathcal{U}_K,$$

where,

$$\mathcal{U}_K(i,j) = \begin{cases} \frac{\mu(p^{|i-j|})}{p^{3(|i-j|)/4}}, & \text{when } i \geq j \text{ or } (i,j) = (K-1,K), \\ 0, & \text{otherwise.} \end{cases} \tag{6.3}$$

Simply speaking, $\mathcal{U}_K$ is 1 on the diagonal and $-p^{-3/4}$ on all $(i+1,i)$ as well as $(K-1,K)$. Also,

$$\det(\mathcal{U}_K) = 1 - p^{-3/2}.$$

We now calculate the entries of the matrix $\mathcal{A}_K$. We have the following:

Proposition 6.1 *The matrix $\mathcal{A}_K$ above is given by*

$$\mathcal{A}_K(i,j) = \beta(1-p^{-3/2}) \cdot \begin{cases} p^{-3|i-j|/4}, & \textit{when } 1 \leq i,j \leq K-1 \textit{ or } i=j=K, \\ 0, & \textit{otherwise.} \end{cases} \qquad \blacksquare$$

We begin with the following lemma:

Lemma 6.2 *We have*

$$\varphi^*(p^m) - p^{-3/4}\varphi^*(p^{|m-1|}) = p^{-3m/4}\beta, \quad \textit{for all } m \geq 0.$$

Proof First, assume $m \geq 1$. We have

$$\varphi^*(p^m) - p^{-3/4}\varphi^*(p^{m-1}) = \frac{m\beta+1}{p^{3m/4}} - p^{-3/4}\frac{(m-1)\beta+1}{p^{3(m-1)/4}} = p^{-3m/4}\beta.$$

When $m = 0$, we have

$$1 - p^{-3/4}\varphi^*(p) = 1 - p^{-3/2}(\beta+1) = 1 - \frac{2p^{-3/2}}{1+p^{-3/2}} = \beta. \qquad \blacksquare$$

Now, we shall proceed with the proof of the Proposition 6.1.

Proof of Proposition 6.1 Let us first assume that $1 \leq i,j \leq K-1$. We have

$$\begin{aligned} \mathcal{A}_K(i,j) &= \sum_{k_1,k_2} \mathcal{U}_K^\top(i,k_1)\mathcal{M}_K(k_1,k_2)\mathcal{U}_K(k_2,j) \\ &= \sum_{\substack{k_1-i\in\{0,1\}\\ k_2-j\in\{0,1\}}} \frac{\mu(p^{k_1-i})}{p^{3(k_1-i)/4}}\frac{\mu(p^{k_2-j})}{p^{3(k_2-j)/4}}\varphi^*(p^{|k_1-k_2|}) \qquad (6.4) \\ &= \left(\varphi^*(p^{|i-j|})(1+p^{-3/2}) - \frac{\varphi^*(p^{|i-j+1|}) + \varphi^*(p^{|i-j-1|})}{p^{3/4}}\right). \end{aligned}$$

Here, we do not have the contribution coming from $\mathcal{U}_K(K-1,K)$ or $\mathcal{U}_K^\top(K, K-1)$ as we have assumed $i,j \leq K-1$. This assumption is necessary because we are considering the values $k_1 = i+1$ and $k_2 = j+1$ (both of which should remain $\leq K$).

https://doi.org/10.4153/S0008414X2400066X Published online by Cambridge University Press

First, let us consider the case $i \geq j$. Letting $i - j = m \geq 0$, (6.4) becomes

$$\begin{aligned}\mathcal{A}_K(i+m,i) &= \varphi^*(p^m) - p^{-3/4}\varphi^*(p^{|m-1|}) - p^{-3/4}\big(\varphi^*(p^{m+1}) - p^{-3/4}\varphi^*(p^m)\big)\\ &= p^{-3m/4}\beta - p^{-3/4}p^{-3(m+1)/4}\beta\\ &= \beta(1-p^{-3/2})p^{-3m/4}.\end{aligned}$$

Similarly, for $j \geq i$, we will obtain the same expression in terms of $m = j - i$. This proves Proposition 6.1 for $1 \leq i, j \leq K-1$.

Next, we consider the case when one of i or j equals K.

Claim $\mathcal{A}_K(i,K) = \mathcal{A}_K(K,j) = 0$, *for all* $1 \leq i, j \leq K-1$. ∎

We revert to the first line of the expression (6.4). Letting $m = K - i \geq 1$, we obtain

$$\begin{aligned}\mathcal{A}_K(i,K) &= \sum_{\substack{k_1\in\{i,i+1\}\\ k_2\in\{K-1,K\}}} \frac{\mu(p^{k_1-i})}{p^{3(k_1-i)/4}}\frac{\mu(p^{K-k_2})}{p^{3(K-k_2)/4}}\varphi^*(p^{|k_1-k_2|})\\ &= -p^{-3/4}\varphi^*(p^{m-1}) + p^{-3/2}\varphi^*(p^{|m-2|}) + \varphi^*(p^m) - p^{-3/4}\varphi^*(p^{m-1})\\ &= -p^{-3/4}\big(\varphi^*(p^{m-1}) - p^{-3/4}\varphi^*(p^{|m-2|})\big) + \varphi^*(p^m) - p^{-3/4}\varphi^*(p^{m-1})\\ &= -p^{-3/4}p^{-3(m-1)/4}\beta + p^{-3m/4}\beta = 0.\end{aligned}$$

It similarly follows that $\mathcal{A}_K(K,j) = 0$ for $1 \leq j \leq K-1$, proving the claim.

Next, we see that

$$\begin{aligned}\mathcal{A}_K(K,K) &= \sum_{k_1,k_2\in\{K-1,K\}} \frac{\mu(p^{K-k_1})}{p^{3(K-k_1)/4}}\frac{\mu(p^{K-k_2})}{p^{3(K-k_2)/4}}\varphi^*(p^{|k_1-k_2|})\\ &= 1 - p^{-3/4}\varphi^*(p) - p^{-3/4}\big(\varphi^*(p) - p^{-3/4}\big)\\ &= \beta - p^{-3/4}\big(p^{-3/4}\beta\big) = \beta(1-p^{-3/2}).\end{aligned}$$

This completes the proof of Proposition 6.1.

Now since $n \mapsto n^{-3/4}$ is completely multiplicative, by the proof of Lemma 5.1, the matrix

$$\mathcal{B}_K = \left(\left(\frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}\right)^{-3/4}\right)_{a,b\in\{1,\dots,p^K\}}$$

is positive definite for all K. Since the entries (i,j) with $1 \leq i, j \leq K-1$ of $\mathcal{A}_K$ coincide with the ones of the matrix $c\mathcal{B}_K$ for some positive constant c, and that the entries (i,K) and (K,j) of $\mathcal{A}_K$ are all zero with a single exception at the entry (K,K), by Sylvester's criterion (Lemma 5.3), we conclude that the matrix $\mathcal{A}_K$ is positive definite for all K.

Since $\mathcal{A}_K = \mathcal{U}_K^{\top}\mathcal{M}_K\mathcal{U}_K$, we have

$$\det(\mathcal{A}_K) = \det(\mathcal{U}_K)^2\,\det(\mathcal{M}_K) = (1-p^{-3/2})^2\,\det(\mathcal{M}_K).$$

This proves that $\det(\mathcal{M}_K) > 0$, and by induction over K in Lemma 5.3, $\mathcal{M}_K$ is positive definite for all K.

The factorization (5.4) completes the proof of Theorem 1.3.

Acknowledgements We would like to thank the referee for a careful revision and for his/her comments, suggestions, and corrections that improved this paper. This project started while the first author was a visiting Professor at Aix-Marseille Université, and he is thankful for their kind hospitality. The main bulk of this paper was built when the fourth author was a guest of the Aix-Marseille Université I2M laboratory. We thank this place for its support.

References

[1] M. Aymone, *Complex valued multiplicative functions with bounded partial sums*. Bull. Braz. Math. Soc. (N.S.) **53**(2022), 1317–1329.
[2] L. Báez-Duarte, M. Balazard, B. Landreau and E. Saias, *Étude de l'autocorrélation multiplicative de la fonction 'partie fractionnaire'*. Ramanujan J. **9**(2005), 215–240.
[3] M. Balazard and B. Martin, *Sur l'autocorrélation multiplicative de la fonction partie fractionnaire et une fonction définie par J. R. Wilton*. https://hal.archives-ouvertes.fr/hal-00823899v1, (2013).
[4] M. Balazard and B. Martin, *Sur une équation fonctionnelle approchée due à J. R. Wilton*. Mosc. Math. J. **15**(2015), 629–652.
[5] B. Borda, M. Munsch and I. E. Shparlinski, *Pointwise and correlation bounds on Dedekind sums over small subgroups*. Res. Number Theory **10**(2024), Paper No. 28, 12.
[6] M. Coons, *On the multiplicative Erdös discrepancy problem*. Preprint, 2010, arXiv:1003.5388.
[7] P. Erdős, *Some unsolved problems*. Michigan Math. J. **4**(1957), 291–300.
[8] A. Granville and K. Soundararajan, *Pretentious multiplicative functions and an inequality for the zeta-function*. In Anatomy of integers, CRM Proc. Lecture Notes, Vol. 46, Amer. Math. Soc., Providence, RI, 2008, 191–197.
[9] M. N. Huxley, *Exponential sums and lattice points. III*. Proc. London Math. Soc. (3) **87**(2003), 591–609.
[10] A. Ivić and W. Zhai, *On certain integrals involving the Dirichlet divisor problem*. Funct. Approx. Comment. Math. **62**(2020), 247–267.
[11] A. Khintchine, *Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen*. Math. Ann. **92**(1924), 115–125.
[12] O. Klurman, *Correlations of multiplicative functions and applications*. Compos. Math. **153**(2017), 1622–1657.
[13] Y.-K. Lau and K.-M. Tsang, *Mean square of the remainder term in the Dirichlet divisor problem*. In Les Dix-huitièmes Journées Arithmétiques (Bordeaux, 1993), Vol. 7, 1995, 75–92.
[14] S. H. Lee and S. Lee, *The twisted moments and distribution of values of Dirichlet L-functions at 1*. J. Number Theory **197**(2019), 168–184.
[15] B. Nyman, *On the one-dimensional translation group and semi-group in certain function spaces*, Thesis, University of Uppsala, Uppsala, 1950.
[16] K. Soundararajan, *Omega results for the divisor and circle problems*. Int. Math. Res. Not. (2003), 1987–1998.
[17] T. Tao, *The Erdös discrepancy problem*. Discrete Anal. (2016), Paper No. 1, 29.
[18] K.-C. Tong, *On divisor problems. II, III*. Acta Math. Sinica **6**(1956), 139–152, 515–541.

Universidade Federal de Minas Gerais (UFMG), Belo Horizonte, Brazil
e-mail: aymone.marco@gmail.com

Max Planck Institute for Mathematics (MPIM), Bonn, Germany
e-mail: maiti@mpim-bonn.mpg.de

CNRS / Aix-Marseille Université, Marseille, France
e-mail: olivier.ramare@univ-amu.fr

Mumbai, India
e-mail: priyamvads@gmail.com

https://doi.org/10.4153/S0008414X2400066X Published online by Cambridge University Press