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FINITELY STABLE ADDITIVE BASES

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Abstract

An additive basis A is finitely stable when the order of A is equal to the order of $A \cup F$ for all finite subsets $F \subseteq \mathbb{N}$. We give a sufficient condition for an additive basis to be finitely stable. In particular, we prove that $\mathbb{N}^2$ is finitely stable.

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1. Introduction

An additive basis is a subset $A \subseteq \mathbb{N} = \{0, 1, 2, 3, \ldots\}$ with the property that there exists $h \in \mathbb{Z}_+ = \mathbb{N} - \{0\}$ such that every $n \in \mathbb{N}$ is the sum of h elements of A. The minimum h satisfying this definition is called the order of A and is denoted by $h = o(A)$. Examples of additive bases are:

(a) the squares $\mathbb{N}^2 = \{0, 1, 4, 9, 16, \ldots\}$, which has order 4 (Lagrange's theorem);
(b) the cubes $\mathbb{N}^3 = \{0, 1, 8, 27, 64, \ldots\}$, which has order 9 (Wieferich's theorem);
(c) the triangular numbers $\mathbb{N}_3 = \{0, 1, 3, 6, 10, \ldots\}$, which has order 3 (Gauss's theorem).

For more information on additive bases, see [2].

An additive basis is called *finitely stable* when $o(A) = o(A \cup F)$ for all finite subsets $F \subseteq \mathbb{N}$. It is obvious from these definitions that $\mathbb{N}$ is finitely stable. Moreover, it is easy to see that an additive basis A of order 2 is finitely stable if and only if $\mathbb{N} - A$ is infinite. So, the study of finitely stable additive bases is nontrivial only for bases whose order is greater then 2. This article aims to present a sufficient condition for an additive basis of order greater than 2 to be finitely stable. As an application of the result, we prove that $\mathbb{N}^2$ is finitely stable.

2. The results

We first set the notation. If $A, B \subseteq \mathbb{N}$, then

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

If $t \in \mathbb{Z}_+$ and $A \subseteq \mathbb{N}$, then $tA = \underbrace{A + \cdots + A}_{t \text{ times}}$. Also, we define $0A = \{0\}$. Finally, if $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, then $A(n) = |\{a \in A : 1 \leq a \leq n\}|$.

Lemma 2.1 (Binomial theorem for additive bases). *If* $\{0\} \subseteq A, B \subseteq \mathbb{N}$ *and* $t \in \mathbb{Z}_+$, *then*

$$t(A \cup B) = \bigcup_{i=0}^{t} [(t-i)A + iB].$$

Proof. Note that $n \in t(A \cup B)$ if and only if there exists i, $0 \leq i \leq t$, such that

$$n = a_1 + a_2 + \cdots + a_{t-i} + b_1 + b_2 + \cdots + b_i,$$

where $a_j \in A$ for $j = 1, \ldots, t-i$ and $b_j \in B$ for $j = 1, \ldots, i$. That is, $n \in t(A \cup B)$ if and only if $n \in (t-i)A + iB$ for some i. The lemma follows. □

Theorem 2.2. *Let* A *be an additive basis and suppose that* $o(A) = h \geq 3$. *If*

$$\lim_{n\to\infty} \frac{((h-2)A)(n)}{n} = 0$$

and

$$\limsup \frac{((h-1)A)(n)}{n} < 1,$$

then A *is finitely stable.*

Proof. Note first that since $tA \subseteq (t+1)A$ for all $t \in \mathbb{Z}_+$, then $\lim_{n\to\infty}(tA)(n)/n = 0$ for all $t \in \{1, \ldots, h-2\}$. Now suppose that the statement is false. Then there exists a finite subset $F \subseteq \mathbb{N}$ such that $o(A \cup F) < o(A)$. Suppose without loss of generality that $F \cap A = \emptyset$. Since $o(A \cup F) < h$, then $(h-1)(A \cup F) = \mathbb{N}$. So, if $n \in \mathbb{Z}_+$, then

$$\begin{aligned} n = ((h-1)(A \cup F))(n) &= \Big(\bigcup_{i=0}^{h-1} (h-1-i)A + iF\Big)(n) \\ &\leq \sum_{i=0}^{h-1} ((h-1-i)A + iF)(n) \leq \sum_{i=0}^{h-1} |iF| \cdot ((h-1-i)A)(n). \end{aligned}$$

Dividing by n and taking lim sup,

$$\begin{aligned} 1 &\leq \limsup \sum_{i=0}^{h-1} \frac{|iF| \cdot ((h-1-i)A)(n)}{n} \leq \sum_{i=0}^{h-1} |iF| \limsup \frac{((h-1-i)A)(n)}{n} \\ &= \limsup \frac{((h-1)A)(n)}{n} + \sum_{i=1}^{h-1} |iF| \limsup \frac{((h-1-i)A)(n)}{n} \\ &= \limsup \frac{((h-1)A)(n)}{n} < 1, \end{aligned}$$

which is a contradiction. Hence, A is finitely stable. □

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As an application of the previous theorem, we will prove that $\mathbb{N}^2$ is finitely stable. For this, we will need the following results of Landau [1].

THEOREM 2.3 (Landau).

$$\lim_{n\to\infty} \frac{(2\mathbb{N}^2)(n)}{n(\log n)^{-1/2}} = \Big(2\prod_p (1-p^{-2})\Big)^{-1/2},$$

the product being taken over all primes p such that $p \equiv 3 \bmod 4$.

THEOREM 2.4 (Landau).

$$\lim_{n\to\infty} \frac{(3\mathbb{N}^2)(n)}{n} = \frac{5}{6}.$$

COROLLARY 2.5. $\mathbb{N}^2$ *is finitely stable.*

PROOF. Since $o(\mathbb{N}^2) = 4$, Theorems 2.3 and 2.4 show that the hypotheses of Theorem 2.2 are satisfied. Thus, $\mathbb{N}^2$ is finitely stable. □

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