ISOMORPHIC SUBGRAPHS HAVING MINIMAL INTERSECTIONS

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Abstract

Given a finite graph H and G, a subgraph of it, we define $\sigma(G, H)$ to be the largest integer such that every pair of subgraphs of H, both isomorphic to G, has at least $\sigma(G, H)$ edges in common; furthermore, $R(G, H)$ is defined to be the maximum number of subgraphs of H, all isomorphic to G, such that any two of them have $\sigma(G, H)$ edges common between them. We are interested in the values of $\sigma(G, H)$ and $R(G, H)$ for general H and G. A number of combinatorial problems can be considered as special cases of this question; for example, the classical set-packing problem is equivalent to evaluating $R(G, H)$ where G is a complete subgraph of the complete graph H and $\sigma(G, H) = 0$, and the decomposition of H into subgraphs isomorphic to G is equivalent to showing that $\sigma(G, H) = 0$ and $R(G, H) = \varepsilon(H)/\varepsilon(G)$ where $\varepsilon(H), \varepsilon(G)$ are the number of edges in H, G respectively.

A result of S. M. Johnson (1962) gives an upper bound for $R(G, H)$ in terms of $\sigma(G, H)$. As a corollary of Johnson's result, we obtain the upper bound of McCarthy and van Rees (1977) for the Cordes problem. The remainder of the paper is a study of $\sigma(G, H)$ and $R(G, H)$ for special classes of graphs; in particular, H is a complete graph and G is, in most instances, a union of disjoint complete subgraphs.

1. Introduction

Let H be any finite graph and let G be any subgraph of H. We are interested in two integer-valued functions of G and H, $\sigma(G, H)$ and $R(G, H)$. The function $\sigma(G, H)$ is defined to be the largest integer such that every pair of subgraphs of H, say G_1 and G_2, both isomorphic to G, has at least $\sigma(G, H)$ edges in common. This means that each pair of such subgraphs has at least $\sigma(G, H)$ common edges and some pair of them has precisely $\sigma(G, H)$ common edges. If the graph H

under consideration is clear from the discussion, we will refer to $\sigma(G, H)$ simply as $\sigma(G)$ or, still more simply, as σ, in the event that there is no ambiguity about the subgraph G either. The function $R(G, H)$ is defined to be the largest integer r such that there exist r subgraphs of H, say $G_1, G_2, \ldots, G_r$, each isomorphic to G, with the property that each pair of them has σ common edges. When appropriate, we will refer to $R(G, H)$ as $R(G)$ or R.

It is interesting to note that the classical set-packing problem is a special case of this problem. The object in the packing problem is to determine the maximum number of k-subsets of an n-set such that any pair of elements from the n-set is in at most one of the k-subsets. This is equivalent to the determination of $R(G, H)$ where H is the complete graph K_n, G is K_k and $\sigma = 0$.

Cordes (1978)[*] considered the problem of determining $R(G, H)$ and $\sigma(G, H)$ in the special case where H is the graph K_{mn} (the complete graph on mn vertices) and G is the union of n disjoint complete graphs, each on m vertices. Since then, others have considered this problem, including Mullin and Stanton (1978), McCarthy and van Rees (1977), and Nemeth (1976). Nemeth was the first to consider the more general problem described above.

For any graph H, let $\varepsilon(H)$ denote the number of edges in H. In Section 2, we observe that a result of Johnson (1962) gives

$$R(G, H) \leqslant \frac{E(e - \sigma(G, H))}{e^2 - E \cdot \sigma(G, H)}$$

where $E = \varepsilon(H)$ and $e = \varepsilon(G)$. This bound gives the upper bound for $R(G, H)$ obtained by McCarthy and van Rees (1977) in the special case where $H = K_{mn}$ and G is the disjoint union of n K_m's.

In Section 3, we consider lower bounds for $R(G, K_{2n})$ where G is the disjoint union of 2 K_n's; in Section 4, we study $R(G, K_n)$ where G is the disjoint union of two complete graphs K_m and K_{n-m}; and in Section 5, we consider $R(G, K_n)$ where G is a double-cone (see Section 5) on n vertices. For G a double-cone on n vertices, where $n \geqslant 8$, it is shown that $\sigma(G, K_n) = 8$ and $R(G, K_n) = 2$.

We conclude the introduction by describing some notation to be used in later sections. For any graph G, $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges; furthermore, $\nu(G) = |V(G)|$ and $\varepsilon(G) = |E(G)|$.

2. The Cordes problem

Though Cordes (1978) originally phrases this problem in the language of combinatorial designs, we will use the graph theoretical language employed in

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[*] Though the paper by Cordes was published in 1978, he introduced the problem in 1976.

Section 1. As mentioned in the introduction, Cordes investigated $\sigma(G, H)$ and $R(G, H)$ in the special case where H is a complete graph on mn vertices and G is the union of n disjoint complete subgraphs, each on m vertices. In this special case, we denote $\sigma(G, H)$ by $\sigma(m^n)$ and $R(G, H)$ by $R(m^n)$.

Let us consider the more general situation where G is the union of disjoint complete subgraphs of H, but the complete subgraphs need not all have the same cardinality. We begin by looking at $\sigma(G, H)$. The following result (see Hartman, Mullin and Stinson (1980)) plays a significant role in several of the proofs given below.

LEMMA 2.1. *For any b non-negative integers* $k_1, k_2, \ldots, k_b$, *let*

$$\sum_{i=1}^{b} k_i = qb + r, \quad \text{where } 0 \leqslant r < b.$$

If $q + 1 \geqslant u$, *then*

$$\sum_{i=1}^{b} \binom{k_i}{u} \geqslant r\binom{q+1}{u} + (b - r)\binom{q}{u}$$

with equality if and only if r of the k_i's *are equal to* $q + 1$ *and the remaining* $b - r$ *are equal to q. If* $q + 1 < u$, *then*

$$\sum_{i=1}^{b} \binom{k_i}{u} \geqslant 0$$

with equality if and only if each $k_i < u$.

We are now in a position to prove

THEOREM 2.2. *For m, n and s positive integers such that* $m, n \geqslant 1$, $0 \leqslant s < n$, *let* $H = K_t$ *where* $t = mn + s$; *that is, let H be a complete graph on t vertices. Let G be the union of n disjoint complete subgraphs of H, s of which have* $m + 1$ *vertices and* $n - s$ *of which have m vertices. Let* $m = kn + l$, *where* $0 \leqslant l < n$. *Then*

$$\sigma(G, H) = (ln + s)\binom{k+1}{2} + (n^2 - ln - s)\binom{k}{2};$$

furthermore, whenever two subgraphs isomorphic to G, say G_1 *and* G_2, *intersect in* σ *edges, then each connected component of* G_1 *intersects each connected component of* G_2 *in k or* $k + 1$ *vertices.*

PROOF. Label the connected components of G_i, $i \in \{1, 2\}$, by $C_i^1, C_i^2, \ldots, C_i^n$. Let $x_{ij} = \nu(C_1^i \cap C_2^j)$. Clearly

$$\sum_{i,j} x_{ij} = nm + s = kn^2 + ln + s.$$

The number of edges common to both G_1 and G_2 is given by

$$\sum_{i,j} \binom{x_{ij}}{2}.$$

We can apply Lemma 2.1 to the x_{ij}'s with $b = n^2$, $q = k$, $r = ln + s$ and $u = 2$. Thus

$$\sum_{i,j} \binom{x_{ij}}{2} \geq (ln + s)\binom{k+1}{2} + (n^2 - ln - s)\binom{k}{2}$$

with equality if and only if each x_{ij} is either $k + 1$ or k. We conclude that

$$\sigma(G, H) = (ln + s)\binom{k+1}{2} + (n^2 - ln - s)\binom{k}{2}$$

and G_1, G_2 have σ edges common if and only if each connected component of G_1 intersects each connected component of G_2 in k or $k + 1$ vertices.

If $s = 0$, we have

$$\sigma(G, H) = n\left[l\binom{k+1}{2} + (n-l)\binom{k}{2}\right]$$

which was first established by Cordes (1978).

We now turn our attention to the function $R(G, H)$. The more general graph-theoretic formulation of Cordes' problem permits us to employ an elegant result of S. M. Johnson (1962) to obtain an upper bound for $R(G, H)$. For the sake of completeness, we include the proof of Johnson's Theorem.

As usual, we let $|X|$ denote the cardinality of the set X.

THEOREM 2.3 (1962). *Let H be a finite set of cardinality E and let $G_1, G_2, \ldots, G_\delta$ be subsets of H such that*
(i) $|G_i| = e$ *for* $1 \leq i \leq \delta$, *and*
(ii) $|G_i \cap G_j| \leq \theta$ *for* $1 \leq i < j \leq \delta$.
Then, for $e^2 > E\theta$,

$$\delta \leq \frac{E(e - \theta)}{e^2 - E\theta}.$$

PROOF. *Let A be the multi-set of set intersections*

$$(G_i \cap G_j \mid 1 \leq i < j \leq \delta).$$

Define a bipartite graph, with bipartition (A, H), as follows: a vertex $h \in H$ is joined to a vertex $G_i \cap G_j \in A$ if and only if the element $h \in H$ is in the intersection $G_i \cap G_j$. In order to count the edges of this graph, for each $h \in H$,

we define k_h to be the number of G_i's which contain the element h. Clearly, the number of edges incident with the vertex set H is

$$\sum_{h \in H} \binom{k_h}{2}.$$

The number of edges incident with the vertex set A is at most $\binom{\delta}{2}\theta$ since $|G_i \cap G_j| \leqslant \theta$. Hence

(I) $$\sum_{h \in H} \binom{k_h}{2} \leqslant \binom{\delta}{2}\theta.$$

It is immediate from the definition of the k_h's that

(II) $$\sum_{h \in H} k_h = \delta e.$$

Combining (I) and (II), we have

(III) $$\sum_{h \in H} k_h^2 \leqslant \delta(\delta - 1)\theta + \delta e.$$

By the Cauchy-Schwarz inequality,

$$\left(\sum a_i b_i \right)^2 \leqslant \left(\sum a_i^2 \right)\left(\sum b_i^2 \right),$$

applied with $a_i = k_h$ and $b_i = 1$, we obtain

$$\delta^2 e^2 \leqslant (\delta(\delta - 1)\theta + \delta e)E,$$

which simplifies to give

$$\delta \leqslant \frac{E(e - \theta)}{e^2 - E\theta},$$

under the hypothesis that $e^2 > E\theta$.

Johnson also observed that given any upper bound for δ, one can readily obtain a second inequality which must be satisfied. Let $q = [\delta e/E]$ and let $r = \delta e - qE$. Applying Lemma 2.1 with $q = q$, $r = r$, $b = E$ and $u = 2$, we obtain that

$$\sum_{h \in H} \binom{k_h}{2} \geqslant r\binom{q + 1}{2} + (E - r)\binom{q}{2}.$$

Combining this with inequality (I) of the above theorem, we obtain

$$\binom{\delta}{2}\theta \geqslant rq + E\binom{q}{2}.$$

Thus, we have established

LEMMA 2.4. *Under the hypothesis of Theorem* 2.3,

$$\binom{\delta}{2}\theta \geqslant rq + E\binom{q}{2}$$

where $q = [\delta e/E]$ *and* $r = \delta e - qE$.

This result will occasionally permit us to obtain an improved value for δ; for example, if we have an upper bound for δ and if the inequality of Lemma 2.4 is not satisfied, then $\delta - 1$ is a better upper bound.

In the generalized graph-theoretical formulation of the Cordes problem, the graphs G and H are completely determined by their edge-sets $E(G)$ and $E(H)$, respectively. Thus, if we apply Johnson's Theorem to these two edge-sets, we immediately obtain

COROLLARY 2.5. *Let* H *be a graph with* E *edges and let* G *be a subgraph with* e *edges. Then for* $\sigma = \sigma(G, H)$,

$$R(G, H) \leqslant \frac{E(e - \sigma)}{e^2 - E\sigma}.$$

Combining Theorem 2.2 in the special case when $s = 0$ and Corollary 2.5, we obtain the following result of McCarthy and van Rees.

LEMMA 2.6. *For the classical Cordes problem*,

$$R(m^n) \leqslant \frac{(mn - 1)[m(m - 1) - k(m + l - n)]}{m(m - 1)^2 - k(mn - 1)(m + l - n)}$$

where $m = kn + l$ *and* $0 \leqslant l < n$.

PROOF. This follows immediately from Corollary 2.5 when we set

$$E = \binom{nm}{2}, \quad e = n\binom{m}{2},$$

and from Theorem 2.2, with $s = 0$,

$$\sigma(m^n) = n\left[l\binom{k + 1}{2} + (n - l)\binom{k}{2}\right].$$

One can also find an upper bound for $R(G, H)$, with G and H as defined in Theorem 2.2, since

$$E = \binom{n^2k + nl + s}{2}, \quad e = s\binom{nk + l + 1}{2} + (n - s)\binom{nk + l}{2},$$

and $\sigma(G, H)$ is as given in Theorem 2.2.

We now use Lemma 2.4 to establish the following improved upper bound of McCarthy and van Rees (1977).

THEOREM 2.7. $R((2m + 1)^2) \leqslant 4m + 4$, *provided* $m \geqslant 2$.

PROOF. From Lemma 2.6, we get $R((2m + 1)^2) \leqslant 4m + 5$. Now let $\delta = 4m + 5$. For this case,

$$E = \binom{4m + 2}{2} = 8m^2 + 6m + 1, \quad e = 2 \cdot \binom{2m + 1}{2} = 4m^2 + 2m,$$

and

$$\theta = 2 \cdot \left[\binom{m}{2} + \binom{m + 1}{2} \right] = 2m^2.$$

Then $q = 2m + 1$, $r = 8m^2 + 2m - 1$, and hence,

$$rq + E \cdot \binom{q}{2} = 16m^4 + 36m^3 + 20m^2 + m - 1,$$

$$\binom{\delta}{2} \theta = 16m^4 + 36m^3 + 20m^2.$$

Hence $\delta \leqslant 4m + 4$.

When $H = K_{4m}$ and G is the union of two disjoint K_{2m}'s, Cordes (1978) established

THEOREM 2.8. $R((2m)^2) \leqslant 4m - 1$ *with equality if and only if there exists a Hadamard matrix of order* $4m$.

3. Some lower bounds for $R\big((2m + 1)^2\big)$

McCarthy and van Rees (1977) establish a number of results on $R((2m + 1)^2)$.

LEMMA 3.1. $R((2m + 1)^2) \geqslant R((2m)^2)$.

LEMMA 3.2. $R((4n + 1)^2) \geqslant 2 \min[R((2n)^2), R((2n + 1)^2)] + 1$.

LEMMA 3.3. $R((4n - 1)^2) \geqslant 2 \min[R((2n - 1)^2), R((2n)^2)] + 1$.

LEMMA 3.4. $2^{n+1} - 1 \leqslant R((2^n - 1)^2) \leqslant 2^{n+1}$ *for* $n \geqslant 3$.

LEMMA 3.5. *If* $p^\alpha = 4n + 1$, *where* p *is a prime, then* $4n + 1 \leqslant R((2n + 1)^2) \leqslant 4n + 4$.

McCarthy and van Rees also demonstrate that $R(5^2) = 12$ and determine upper and lower bounds for $R((2m + 1)^2)$ for $2m + 1 \in \{7, 9, 11, 13\}$. Cordes (1978) shows that $R(3^2) = 10$.

In this section, we show that $R((4m + 3)^2) \geqslant 8m + 7$ where $4m + 3$ is a prime or a power of a prime. Combining this result with Lemma 3.3, we show that $8m + 7 \leqslant R((4m + 3)^2) \leqslant 8m + 8$ for many non-prime powers $4m + 3$.

We proceed, now, using the language of combinatorial designs rather than graphs. Let V be a set of cardinality mn. An (m^n)-round, $\mathcal{R}$, is a partition of V into n sets, called *blocks*, each of cardinality m. The *pairs of an* (m^n)-round $\mathcal{R}$ are precisely those pairs $\{a, b\} \subseteq V$, $a \neq b$, such that, for some block $B \in \mathcal{R}$, $\{a, b\} \subseteq B$. Then $\sigma(m^n)$ is the largest integer such that any two (m^n)-rounds of B have at least $\sigma(m^n)$ common pairs. Also, $R(m^n)$ is the largest integer r such that there exist r (m^n)-rounds of V, say $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_r$, with the property that each pair of these rounds have $\sigma(m^n)$ common pairs. Throughout the remainder of this section, we investigate $R((2m + 1)^2)$.

We now consider the problem of constructing two $((2m + 1)^2)$-rounds having $\sigma((2m + 1)^2)$ common pairs.

LEMMA 3.6. *Let V be a set of cardinality $4m + 2$. Constructing two $((2m + 1)^2)$-rounds of V having $\sigma((2m + 1)^2)$ common pairs is equivalent to constructing two blocks (subsets) of V, say A and B, both of cardinality $2m + 1$, such that*

$$|A \cap B| = m \quad or \quad m + 1.$$

PROOF. Clearly, if $\mathcal{R}_1$ and $\mathcal{R}_2$ are $((2m + 1)^2)$-rounds containing $\sigma((2m + 1)^2) = 2m^2$ common pairs, then by Theorem 2.2, each block of $\mathcal{R}_1$ intersects each block of $\mathcal{R}_2$ in m or $m + 1$ elements. Hence, we can take either block of $\mathcal{R}_1$ to be A and either block of $\mathcal{R}_2$ to be B.

Conversely, let A and B be blocks of V, each of cardinality $2m + 1$, such that $|A \cap B| = m$ or $m + 1$. Let $\bar{A} = V \setminus A$ and $\bar{B} = V \setminus B$. It is an easy matter to check that $\mathcal{R}_1 = \{A, \bar{A}\}$ and $\mathcal{R}_2 = \{B, \bar{B}\}$ are $((2m + 1)^2)$-rounds such that each block of $\mathcal{R}_1$ intersects each block of $\mathcal{R}_2$ in m or $m + 1$ elements. Hence by Lemma 2.2, $\mathcal{R}_1$ and $\mathcal{R}_2$ have $\sigma((2m + 1)^2)$ common pairs.

We are now in a position to prove

THEOREM 3.7. *If $p^\alpha = 4m + 3$, where p is a prime and α is a positive integer, then $R((4m + 3)^2) \geqslant 8m + 7$.*

PROOF. Bose (1947) has established that the set of quadratic residues, say Q of the Galois field of order $4m + 3$, $G = GF(4m + 3)$, is a difference set (M. Hall,

Jr. (1967)) of index m, as is the set of quadratic nonresidues, say N. That is, the multi-set

$$(a - b | a, b \in Q, a \neq b)$$

contains each element $g \in G \backslash \{0\}$ precisely m times, as does the multi-set $(a - b | a, b \in N,\ a \neq b)$. Furthermore, $Q' = Q \cup \{0\}$ and $N' = N \cup \{0\}$ are both difference sets of index $m + 1$.

For any $g \in G$ and for any subset $S \subseteq G$, define

$$S + g = \{s + g | s \in S\}.$$

Now, for $a \in Q$ and $g \in G \backslash \{0\}$, $a \in Q \cap (Q + g)$ if and only if there is an element $b \in Q$ such that $a = b + g$ or, equivalently, $a - b = g$. Since Q is a difference set of index m, it follows that, for any $g \in G \backslash \{0\}, |Q \cap (Q + g)| = m$ and hence, for any $\{g, h\} \subseteq G, g \neq h$,

$$|(Q + g) \cap (Q + h)| = m.$$

Similarly, for any $\{g, h\} \subseteq G,\ g \neq h,\ |(N + g) \cap (N + h)| = m,\ |(Q' + g) \cap (Q' + h)| = m + 1$, and $|(N' + g) \cap (N' + h)| = m + 1$.

We make use of the following notation. For $i \in \{1, 2\}$ and for any subset $S \subseteq G$, let $S_i = S \times \{i\}$: furthermore, for any $g \in G$, let $g_i = (g, i) \in G \times \{i\}$. For any $g_i \in G_i$ and any $h \in G$, define

$$g_i + h = (g + h)_i$$

and, for any subset $S \subseteq G_1 \cup G_2$, define $S + h = \{s + h | s \in S\}$.

Now, let $E = Q_1' \cup Q_2$ and let $F = N_1' \cup Q_2$. Then, for $\{g, h\} \subset G, g \neq h$,

$$|(E + g) \cap (E + h)| = |(Q_1' + g) \cap (Q_1' + h)| + |(Q_2 + g) \cap (Q_2 + h)|$$
$$= (m + 1) + m = 2m + 1.$$

Similarly, $|(F + g) \cap (F + h)| = 2m + 1$. Also,

$$|(E + g) \cap (F + h)| = |(Q_1' + g) \cap (N_1' + h)| + |(Q_2 + g) \cap (Q_2 + h)|$$
$$= |(Q_1' + g) \backslash (Q_1 + h)| + m$$
$$= |\{g_1\} \backslash (Q_1 + h)| + |(Q_1 + g) \backslash (Q_1 + h)| + m$$
$$= |\{g_1\} \backslash (Q_1 + h)| + 2m + 1.$$

Now $g_1 \in Q_1 + h$ if and only if $g - h \in Q$. Hence,

$$|(E + g) \cap (F + h)| = \begin{cases} 2m + 1 & \text{if } g - h \in Q, \\ 2m + 2 & \text{if } g - h \in N. \end{cases}$$

For any $g \in G$,

$$|(E + g) \cap (F + g)| = |(Q'_1 + g) \cap (N'_1 + g)| + |(Q_2 + g) \cap (Q_2 + g)|$$
$$= (1) + (2m + 1) = 2m + 2.$$

Finally,

$$|G_1 \cap (E + g)| = |G_1 \cap (F + g)| = 2m + 2.$$

Thus, we have shown that

$$\mathcal{B} = \bigcup_{g \in G} \{E + g, F + g\} \cup \{G_1\}$$

is a set of $8m + 7$ blocks of $G_1 \cup G_2$ such that any two of these blocks intersect in $2m + 1$ or $2m + 2$ elements. By Lemma 3.6, there exist $8m + 7$ $((4m + 3)^2)$-rounds of $G_1 \cup G_2$ having $\sigma((4m + 3)^2)$ common pairs and hence $\mathcal{R}((4m + 3)^2) \geq 8m + 7$, as required.

We conclude this section by observing that Lemma 3.3 and Theorem 3.7 imply

$$R\big((4m + 3)^2\big) \geq 8m + 7$$

for some integers $4m + 3$ which are not primes nor powers of primes, for example, when $4m + 3 \in \{15, 39, 55, 63\}$.

4. $\sigma(mn)$ and $R(mn)$

In this section, we continue our investigation of $\sigma(G, H)$ and $R(G, H)$ using the language of combinatorial designs.

Let $k_1, k_2, \ldots, k_s$ be *distinct* positive integers and let $i_1, i_2, \ldots, i_s$ be positive integers. Let

$$\sum_{j=1}^{s} i_j = t \quad \text{and} \quad \sum_{j=1}^{s} i_j k_j = v.$$

For

$$r = k_1^{i_1} k_2^{i_2} \cdots k_s^{i_s},$$

a $k_1^{i_1} k_2^{i_2} \cdots k_s^{i_s}$-round, or more briefly, an r-round, say $\mathcal{R} = \{B_1, B_2, \ldots, B_t\}$, is a partition of some v-set, say V, into t blocks such that i_1 of the blocks have cardinality k_1, i_2 have cardinality k_2, and so on until i_s have cardinality k_s. As in Section 3, the pairs of $\mathcal{R}$ are all the pairs $\{a, b\} \subseteq V$, $a \neq b$, such that, for some block $B \in \mathcal{R}$, $\{a, b\} \subseteq B$. We define $\sigma(k_1^{i_1} k_2^{i_2} \cdots k_s^{i_s}) = \sigma(r)$ to be the largest integer such that any two r-rounds of V contain at least $\sigma(r)$ common pairs and

define $R(r)$ to be the largest integer l such that there are l r-rounds, say $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_l$, any two of which have $\sigma(r)$ common pairs. If no such largest integer l exists, then we say $R(r) = \infty$.

In this section we study $R(r)$ in the special case where $r = m^1 n^1$ or more simply, $r = mn$. Since we have considered $r = m^2$ in Section 3, we further assume that $m > n$.

LEMMA 4.1. (i) *For $m \geqslant 3n - 1$, two mn-rounds have $\sigma(mn)$ common pairs if and only if the two blocks of size m intersect in $p = m - n$ elements.*

For $n < m < 3n - 1$, let $p = [(3m - n + 1)/4]$, the integer part of $(3m - n + 1)/4$. Two mn-rounds have $\sigma(mn)$ common pairs if and only if the two blocks of size m

(ii) *intersect in p elements when $m + n \not\equiv 2 \pmod 4$ and*
(iii) *intersect in p or $p + 1$ elements when $m + n \equiv 2 \pmod 4$.*

PROOF. Let one round consist of blocks A and $\overline{A}$ and another of B and $\overline{B}$, where $|A| = |B| = m$ and $|\overline{A}| = |\overline{B}| = n$; furthermore, let $|A \cap B| = x$. Then $|A \cap \overline{B}| = m - x$, $|B \cap \overline{A}| = m - x$ and $|\overline{A} \cap \overline{B}| = n - (m - x) = n - m + x$. Clearly, $n - m + x \geqslant 0$ or $x \geqslant m - n$: hence, $m - n \leqslant x \leqslant m$.

The number of pairs common between these two rounds is given by

$$N(x) = \binom{x}{2} + \binom{m - x}{2} + \binom{m - x}{2} + \binom{n - m + x}{2}$$

$$= 2x^2 + (n - 3m)x + \frac{3m^2 - 2mn + n^2 - m - n}{2}.$$

It can readily be shown that $N(x)$ has a minimum at $x = (3m - n)/4$ and that it is symmetric about the line $x = (3m - n)/4$.

If $m - n \geqslant (3m - n - 1)/4$, or equivalently, if $m \geqslant 3n - 1$, then $N(x)$ is strictly increasing for integers x in the admissible interval $m - n \leqslant x \leqslant m$; hence, $\sigma(mn) = N(m - n)$. This establishes (i).

Otherwise, $n < m < 3n - 1$. Since $x = |A \cap B|$ is an integer, it follows that $\sigma(mn) = N(p)$ where $p = [(3m - n + 1)/4]$, the integer part of $(3m - n + 1)/4$. For $m + n \not\equiv 2 \pmod 4$, $N(p) < N(q)$ for any integer $q \neq p$, and, for $m + n \equiv 2 \pmod 4$, $N(p) = N(p + 1)$ and, for any integer $q \notin \{p, p + 1\}$, $N(p) < N(q)$. Hence, we have parts (ii) and (iii).

Before stating a useful corollary of this lemma we require a definition. An (r, λ)-*design* is a pair $(X, \mathcal{C})$ where X is a set and $\mathcal{C}$ is a collection of (not necessarily distinct) subsets of X, called *blocks*, with the following two properties: (i) any element of X is contained in precisely r blocks of $\mathcal{C}$ and (ii) any pair of

distinct elements of X is contained in precisely λ blocks of $\mathcal{C}$. Note that blocks of cardinality one are permitted.

COROLLARY 4.2. *If* $m + n \not\equiv 2(\bmod 4)$, *then the existence of* t *mn-rounds, each pair of them having* $\sigma(mn)$ *common pairs, is equivalent to the existence of an* (r, λ)*-design,* $(X, \mathcal{C})$, *with* $(r, \lambda) = (m, p)$, $|X| = t$ *and* $|\mathcal{C}| \leqslant m + n$.

PROOF. Let there be t mn-rounds of the $(m + n)$-set such that any two of these mn-rounds have $\sigma(mn)$ common pairs. Let $B_1, B_2, \ldots, B_t$ be the blocks of cardinality m from these t mn-rounds and let $V = B_1 \cup B_2 \cup \cdots \cup B_t$. The dual (see, for example, M. Hall, Jr. (1967)) of the design $(V, \{B_1, B_2, \ldots, B_t\})$ is an (r, λ)-design, say $(X, \mathcal{C})$, having $(r, \lambda) = (m, p)$, $|X| = t$ and $|\mathcal{C}| \leqslant m + n$.
It follows readily that the converse also holds.

Since Fisher's inequality (M. Hall, Jr. (1967)) holds for (r, λ)-designs, we immediately have

LEMMA 4.3. $R(mn) \leqslant m + n$, *provided* $m + n \not\equiv 2(\bmod 4)$.

For certain values of the parameters m and n, we can obtain a sharper upper bound.

LEMMA 4.4. *If* $p(m + n - 1) \neq m(m - 1)$, *and* $m + \not\equiv 2(\bmod 4)$ *then*

$$R(mn) < m + n.$$

PROOF. By hypothesis, $p(m + n - 1) \neq m(m - 1)$. To obtain a contradiction, assume $R(mn) = m + n$. By Lemma 4.2, there exists an (r, λ)-design, with $(r, \lambda) = (m, p)$, say $(X, \mathcal{C})$, having $|X| = m + n = |\mathcal{C}|$. Ryser (1950) has shown that such an (r, λ)-design is, in fact, a symmetric balanced incomplete block design with parameters $(v, b, r, k, \lambda) = (m + n, m + n, m, m, p)$ and hence $\lambda(v - 1) = r(k - 1)$ or $p(m + n - 1) = m(m - 1)$. From this contradiction, we conclude that $R(mn) \neq m + n$, which, by Lemma 4.3, implies $R(mn) < m + n$.

Obviously, if A and B are two m-subsets of an $(m + n)$-set which intersect in p elements, then their complements, $\overline{A}$ and $\overline{B}$, are two n-subsets which intersect in $n - m + p$ elements. Hence, it is easy to determine $R(mn)$ when the m-subsets intersect in $m - n$ elements: in this case, $R(mn)$ is the maximum number of disjoint n-subsets that can be selected from an $(m + n)$-set. Now by Lemma 4.1(i), the m-subsets of two mn-rounds having $\sigma(mn)$ common pairs intersect in $m - n$ elements whenever $m \geqslant 3n - 1$. Thus we have

LEMMA 4.5. *If $m \geqslant 3n - 1$, then*

$$R(mn) = [(m + n)/n],$$

the integer part of $(m + n)/n$.

It now follows from the above lemma that $R(m1) = m + 1$.

We now consider the case where $n = m - 1$. For results on Hadamard matrices, the reader is referred to Wallis, Street, Wallis (1972). Note that in this case $m + n \not\equiv 2 \pmod 4$.

THEOREM 4.6. *If there is a Hadamard matrix of order $4k$, then*

$$R((2k)(2k - 1)) = 4k - 1.$$

PROOF. The existence of a Hadamard matrix of order $4n$ implies the existence of a symmetric balanced incomplete block design with parameters (v, b, r, k, λ) $= (4k - 1, 4k - 1, 2k, 2k, k)$ (M. Hall, Jr. (1967)). Hence, by Corollary 4.2, $R(r) \geqslant 4k - 1$ and, consequently, $R = 4k - 1$, as required.

We next establish a lower bound for $R((2k + 1)(2k))$.

LEMMA 4.7. *If there is an Hadamard matrix of order $4k$, then*

$$R((2k + 1)(2k)) \geqslant 4k - 1.$$

PROOF. The existence of a Hadamard matrix of order $4k$ is equivalent to the existence of a symmetric balanced, incomplete block design with parameters $(v, b, r, k, \lambda) = (4k - 1, 4k - 1, 2k - 1, 2k - 1, k - 1)$ (M. Hall, Jr. (1967)). Adjoin two addditional blocks, both containing all $4k - 1$ elements. We obtain an (r, λ)-design $(X, \mathcal{C})$ having $(r, \lambda) = (2k + 1, k + 1), |X| = 4k - 1$ and $|\mathcal{C}| = 4k + 1$. Hence, by Corollary 4.2, $R \geqslant 4k - 1$.

For $p = p((2k + 1)(2k))$, $p(m + n - 1) = (k + 1)4k \neq (2k + 1)2k = m(m - 1)$ and, consequently, by Lemma 4.4, $R((2k + 1)(2k)) < 4k + 1$. Thus we have

THEOREM 4.8. *If there is a Hadamard matrix of order $4k$ then,*

$$4k - 1 \leqslant R((2k + 1)(2k)) \leqslant 4k.$$

The following four $3^1 2^1$-rounds demonstrate that $R(3^1 2^1) = 4$:

$$
\begin{array}{ll}
1\,2\,3 & 4\,5 \\
1\,2\,4 & 3\,5 \\
2\,3\,4 & 1\,5 \\
1\,3\,4 & 2\,5
\end{array}
$$

Also, $R(4^1 2^1) = 15$ by Lemma 4.13 below. These results, together with Lemma 4.5, immediately determine $R(m^1 2^1)$. $R(3^1 2^1) = 4$ and $R(4^1 2^1) = 15$: for $m > 4$, $m \geqslant 3n - 1 = 4$ which implies $R = [(m + 2)/2]$, the integer part of $(m + 2)/2$.

We conclude this section by considering $R(m(m - 2))$. We investigate the cases $R((2k + 1)(2k - 1))$ and $R((2k + 2)(2k))$ separately.

LEMMA 4.9. $R((2k + 1)(2k - 1)) \geqslant R((2k)(2k - 1))$.

PROOF. Let $R((2k)(2k - 1)) = t$. By Lemma 4.1, $p = k$, and by Corollary 4.2, there is a $(2k, k)$-design $(X, \mathcal{C})$, having $|X| = t$ and $|\mathcal{C}| \leqslant 4k - 1$. Let $\mathcal{C}' = \mathcal{C} \cup \{X\}$. Then $(X, \mathcal{C}')$ is a $(2k + 1, k + 1)$-design having $|X| = t$ and $|\mathcal{C}'| = 4k$ which, by Corollary 4.2, implies that

$$R((2k + 1)(2k - 1)) \geqslant t = R((2k)(2k - 1)).$$

THEOREM 4.10. *If there is a Hadamard matrix of order* $4k$, *then*

$$R((2k + 1)(2k - 1)) = 4k - 1.$$

PROOF. By Lemma 4.4, $R((2k + 1)(2k - 1)) < 4k$, and by Lemma 4.6 and 4.9, $R \geqslant 4k - 1$.

We now consider $R((2k + 2)(2k))$. In this case, $m + n = 4k + 2 \equiv 2 \pmod 4$ and Lemma 4.1(iii) implies that two $(2k + 2)(2k)$-rounds have σ common pairs if and only if blocks of size $2k + 2$ intersect in $k + 1$ or $k + 2$ elements, or equivalently, if and only if the blocks of size $2k$ intersect in k or $k - 1$ elements. Thus we have

LEMMA 4.11. $R((2k + 2)(2k))$ *is equal to the number of* $2k$-*subsets which may be selected from a* $4k + 2$-*set such that any two of the* $2k$-*subsets intersect in* k *or* $k - 1$ *elements.*

LEMMA 4.12. $R((2k + 2)(2k)) \leqslant 4k + 5$ *provided* $k \geqslant 4$ *and*
$$R(4^1 2^1) \leqslant 15, \quad R(6^1 4^1) \leqslant 15, \quad R(8^1 6^1) \leqslant 18.$$

PROOF. This follows directly from Theorem 2.3 when we assume

$$E = \binom{4k + 2}{2}, \qquad e = \binom{2k + 2}{2} + \binom{2k}{2},$$

$$\theta = \binom{k + 2}{2} + 3\binom{k}{2} = 3\binom{k + 1}{2} + \binom{k - 1}{2}.$$

THEOREM 4.13. $R(4^1 2^1) = 15$.

PROOF. By virtue of Lemma 4.12, it is enough to construct 15 blocks of size 2 such that any two of them intersect in 0 or 1 element. This is easily done by taking all $\binom{6}{2} = 15$ pairs from the 6 elements.

Now we look to get some lower bounds for the $R((2k + 2)(2k))$ value. In this context we use the following result of Bose.

LEMMA 4.14 (1947). *Let* $p^\alpha = 4k + 1$ *where p is a prime. Then among the totality of differences of the quadratic residues of GF(p^α), every quadratic residue occurs* $(k - 1)$ *times and every quadratic nonresidue occurs k times.*

LEMMA 4.15. *If* $4k + 1$ *is a prime power, then* $R((2k + 2)(2k)) \geqslant 4k + 1$.

PROOF. Let the element-set be $GF(4k + 1) \cup \{\infty\}$ and let x be a primitive element of $GF(4k + 1)$. The set of quadratic residues is

$$B = \{x^0, x^2, \ldots, x^{4k-2}\}.$$

Consider the set of blocks $\{B + y: y \in GF(4k + 1)\}$. It follows from Lemma 4.14 that

$$|(B + y) \cap (B + w)| = \begin{cases} k - 1 & \text{if } y - w \text{ is a quadratic residue,} \\ k & \text{if } y - w \text{ is a non-residue.} \end{cases}$$

Hence, Lemma 4.11 implies $R((2k + 2)(2k)) \geqslant 4k + 1$.

LEMMA 4.16. $R[(4k + 4)(4k + 2)] \geqslant 8k + 7$ *provided* $4k + 3$ *is a prime power.*

PROOF. We apply the same technique that we use in proving Theorem 3.7. Let G be the Galois field with $4k + 3$ elements and let

$$G_i = G \times \{i\}, \qquad i = 1, 2.$$

Our element set is $G_1 \cup G_2$ so that $|G_1 \cup G_2| = 8k + 6$.

Q is the set of quadratic residues in G and N is the set of quadratic non-residues in G. Let $E = Q_1 \cup Q_2$ and $F = N_1 \cup Q_2$. Then, as in Theorem 3.7, one can show that

$$B = \bigcup_{g \in G} \{E + g, F + g\} \cup (G_1 \setminus \{0_1\})$$

is a set of $8k + 7$ blocks of $G_1 \cup G_2$ such that any two of these blocks intersect in $2k$ or $2k + 1$ elements. Hence Lemma 4.11 implies

$$R[(4k + 4)(4k + 2)] \geqslant 8k + 7.$$

LEMMA 4.17. $R((4k + 2)(4k)) \geqslant 2 \min\{R((2k + 2)(2k)), R((2k)^2)\} + 1$.

PROOF. Let $\min\{R((2k + 2)(2k)), R((2k)^2)\} = t$. Let X be a set with $4k + 2$ elements. Then there are t subsets of X say, $B_1, B_2, \ldots, B_t$, such that $|B_i| = 2k$ and $|B_i \cap B_j| = k$ or $k - 1$ for all $i, j = 1, \ldots, t$. Also let Y be a set with $4k$ elements such that $Y \cap X = \varnothing$. Then there are t subsets of Y, say, $C_1, C_2, \ldots, C_t$, such that $|C_i| = 2k$ and $|C_i \cap C_j| = k$ for all $i, j = 1, 2, \ldots, t$. Now let

$$D_i = B_i \cup C_i, \qquad i = 1, \ldots, t,$$
$$E_i = B_i \cup \overline{C_i}, \qquad i = 1, \ldots, t \quad \text{where } \overline{C_i} = Y \backslash C_i.$$

Then

$$\bigcup_{i=1}^{t} \{D_i, E_i\} \cup Y$$

is a set of $2t + 1$ blocks of $X \cup Y$ such that each has cardinality $4k$ and any two of these blocks intersect in $2k$ or $2k - 1$ elements.

LEMMA 4.18. $R[(4k + 4)(4k + 2)] \geqslant 2 \min\{R((2k + 2)(2k)), R((2k + 2)^2)\} + 1$.

PROOF. Let $\min\{R((2k + 2)(2k)), R((2k + 2)^2)\} = t$. Let X be a set having $4k + 2$ elements. Then, there are t subsets of X, say, $B_1, B_2, \ldots, B_t$, such that $|B_i| = 2k$ and $|B_i \cap B_j| = k$ or $k - 1$ for all $i, j = 1, 2, \ldots, t$. Also, let Y be a set having $4k + 4$ elements, $Y \cap X = \varnothing$. Then, there are t subsets of Y, say, $C_1, C_2, \ldots, C_t$ such that $|C_i| = 2k + 2$ and $|C_i \cap C_j| = k + 1$ for all $i, j = 1, 2, \ldots, t$. Now let

$$D_i = B_i \cup C_i, \qquad i = 1, 2, \ldots, t,$$
$$E_i = B_i \cup \overline{C_i}, \qquad i = 1, 2, \ldots, t \quad \text{where } \overline{C_i} = Y \backslash C_i.$$

Then

$$\bigcup_{i=1}^{t} \{D_i, E_i\} \cup Y$$

is a set of $2t + 1$ blocks of $X \cup Y$ such that each has cardinality $4k + 2$ and any two of them intersect in $2k$ or $2k + 1$ elements.

Hence, the result follows by Lemma 4.11.

5. Double-cones

In this section we consider $\sigma(G, H)$ and $R(G, H)$ where H is a complete graph and G is a graph having $\nu(G) = \nu(H)$. In the preceding sections, we have considered the case where G is the union of disjoint complete subgraphs of H. Nemeth (1976) has considered the cases where G is a star, a cycle, a wheel, a rim-deleted wheel, and a spoke-deleted wheel. We now consider the case where G is a double-cone.

A *double-cone* on $n + 2$ vertices consists of a circuit of length n, together with two independent vertices, say s and t, which are adjacent to each vertex of the circuit. Observe that s and t each have valence n and all the other vertices have valence 4. We call the circuit of length n the *rim* of the double-cone and, for $n > 4$, the rim is uniquely determined by the vertices of valence 4. We will represent a double-cone as shown below.
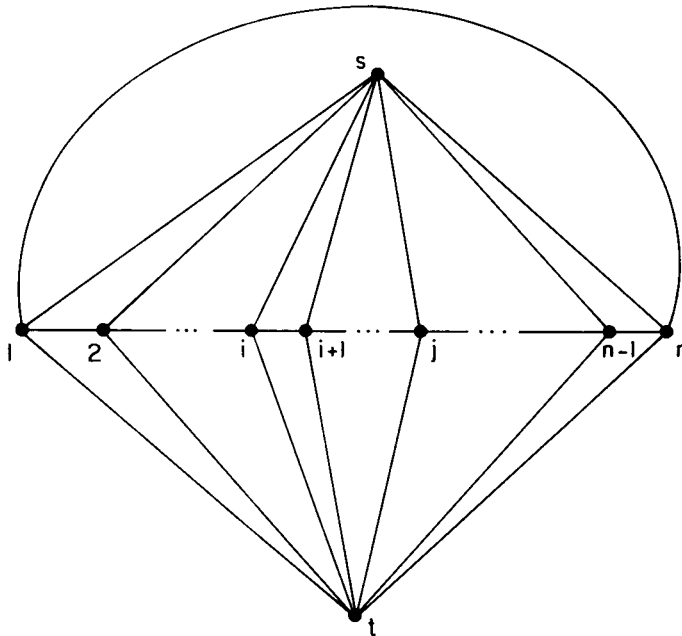
THEOREM 5.1. *Let H be a complete graph and let G be a double-cone (having $\nu(G) = \nu(H)$). For $\nu(H) \geqslant 8$, $\sigma = \sigma(G, H) = 8$ and $R = R(G, H) = 2$.*

PROOF. We begin by showing $\sigma = 8$.

Let G be a subgraph of H which is a double-cone and let G' be a subgraph of H isomorphic to G. Label the vertices of H with the symbols of $\{1, 2, \ldots, n\} \cup \{s, t\}$ so that the rim of G has $1, 2, \ldots, n$ as its vertices.

If the isomorphism $\theta: G \to G'$ maps either of s or t into $\{s, t\}$, then it can be shown that $\varepsilon(G \cap G') \geq n + 3 \geq 9$. Hence, we consider an isomorphism $\theta: G \to G'$ such that

$$\theta(s) = k, \quad \theta(t) = l, \quad \theta^{-1}(s) = i, \quad \theta^{-1}(t) = j$$

where $\{i, j, k, l\} \subseteq \{1, 2, \ldots, n\}$.

Case 1. Vertices i and j are not adjacent in G.
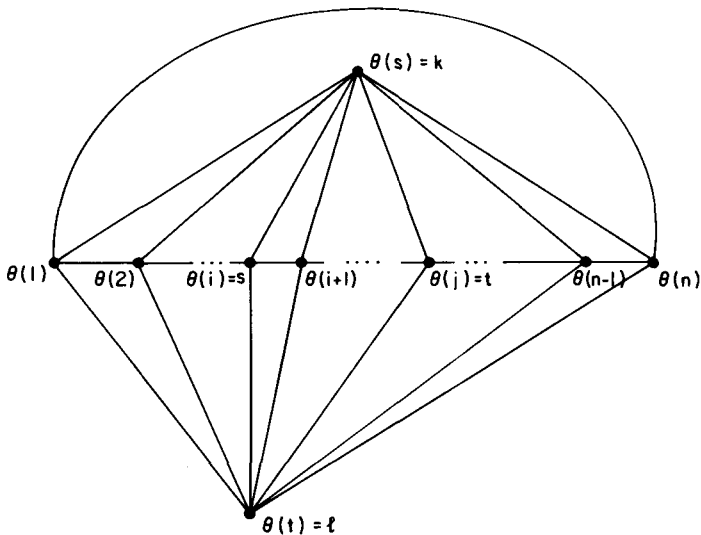
Clearly the 4 edges on s and the 4 edges on t in G' are also in $E(G)$; furthermore, at least one additional edge on k in G' and at least one additional edge on l in G' are in $E(G)$. Therefore $\varepsilon(G \cap G') \geq 10$.

Case 2. Vertices k and l are not adjacent in G. As in Case 1, it can be shown that $\varepsilon(G \cap G') \geq 10$.

Case 3. Vertices i and j are adjacent in G and vertices k and l are adjacent in G.
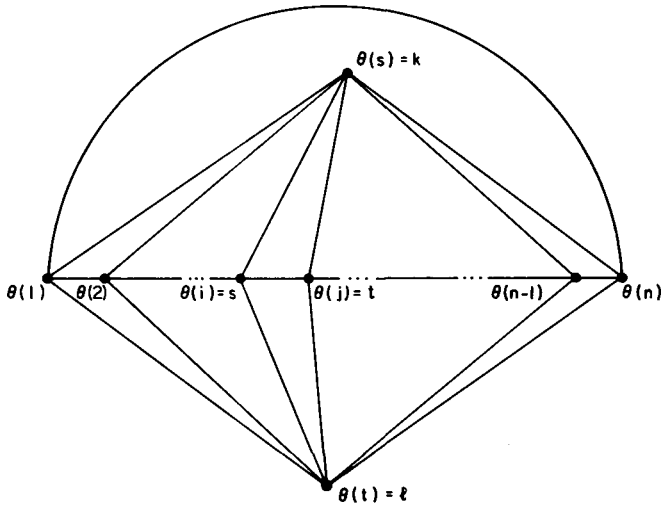
Then three of the edges incident with s in G' are in $E(G)$, as are three of the edges incident with t. Precisely one additional edge incident with k in G' is in $E(G)$, as is one additional edge on l in G. Hence, $\varepsilon(G \cap G') \geqslant 8$. In addition, if the rims of G and G' are edge disjoint, then it follows that $\varepsilon(G \cap G') = 8$. If $\nu(H) \geqslant 8$, then it is possible to obtain G and G' such that their rims are edge-disjoint. This can be shown by Dirac's Theorem on Hamiltonian circuits (Bondy & Murty (1976)). Thus $\sigma = 8$.

To obtain a contradiction, suppose G'' is a third subgraph of H isomorphic to G such that any pair of G, G', G'' have precisely 8 edges in common. By the above argument, s and t must be adjacent on the rims of G' and G''. Since the rims of G' and G'' are not edge-disjoint, $\varepsilon(G' \cap G'') > 8$, a contradiction. Hence, $R = 2$.

LEMMA 5.2. *If D_n represents a double-cone on n vertices, then*

$$\sigma(D_5, K_5) = 8 \quad and \quad R(D_5, K_5) = 10,$$
$$\sigma(D_6, K_6) = 9 \quad and \quad R(D_6, K_6) = 5,$$
$$\sigma(D_7, K_7) = 9 \quad and \quad R(D_7, K_7) = 3.$$

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