Ergod. Th. & Dynam. Sys. (1987), **7**, 351–374
Printed in Great Britain

Dynamics of Markov chains and stable manifolds for random diffeomorphisms

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(*Received* 26 *June* 1986 *and revised* 20 *October* 1986)

Abstract. We consider the Markov chain on a compact manifold M generated by a sequence of random diffeomorphisms, i.e. a sequence of independent $\mathrm{Diff}^2(M)$-valued random variables with common distribution. Random diffeomorphisms appear for instance when diffusion processes are considered as solutions of stochastic differential equations. We discuss the global dynamics of Markov chains with continuous transition densities and construct non-random stable foliations for random diffeomorphisms.

1. *Introduction*

Let μ be a probability measure on the space $\mathrm{Diff}^2(M)$ of C^2-diffeomorphisms of a compact Riemannian manifold M. By a sequence of random diffeomorphisms we mean a sequence $F_1, F_2, \dots$ of independent $\mathrm{Diff}^2(M)$-valued random variables with common distribution μ. Random diffeomorphisms appear for instance when diffusion processes are considered as solutions of stochastic differential equations, see [**Ku**] and [**Ki**, chapter V]. Probability measures

$$P(x, \Gamma) = \mu\{f \in \mathrm{Diff}^2(M): fx \in \Gamma\}, \qquad \Gamma \subset M,$$

give rise to a random walk $X_n = F_n X_{n-1}$. Moreover, since F_n is a random diffeomorphism, we obtain a Markov chain $v_n = DF_n v_{n-1}$ in the tangent bundle TM. By the Oseledec multiplicative ergodic theorem (see Theorem 3.1 below, [**Os**], [**Ru**], [**L**]), if x does not belong to an exceptional set, then for almost every sequence $\omega = (f_1, f_2, \dots)$ the characteristic exponent

$$\chi(x, \omega, v) = \lim_{n\to\infty} \frac{1}{n} \ln \|Df_n \circ \cdots \circ Df_1 v\|$$

exists for every vector $v \in T_xM$. Moreover, there is a filtration

$$\{0\} = V^0_{(x,\omega)} \subset V^1_{(x,\omega)} \subset \cdots \subset V^{\kappa}_{(x,\omega)} = T_xM$$

such that the characteristic exponent is constant on $V^i_{(x,\omega)} \setminus V^{i-1}_{(x,\omega)}$. In general $\chi(x, \omega, v)$ depends non-trivially on ω. However, under certain natural assumptions (see Theorem 3.3 below and [**Ki**, chapter III]), if x does not belong to an exceptional set, then $\chi(x, \omega, v)$ almost surely does not depend on ω for all $v \in T_xM$ and there exists a non-random DF_n-invariant filtration of subspaces

$$\{0\} = L^0_x \subset L^1_x \subset \cdots \subset L^l_x = T_xM$$

such that $\chi(x, \omega, v) = \lambda_j(x)$ is constant on $L_x^j \backslash L_x^{j-1}$ with probability 1. In the ergodic case the λ_j's do not depend on x.

The main purpose of this paper is to construct non-random foliations invariant under random diffeomorphisms. Denote by F^n the composition $F_n \circ \cdots \circ F_1$. According to a version of the stable manifold theorem (see Theorem 4.1 below, [**Ru**], [**P**], [**FHY**]), if $\chi(x, \omega, v) = \lambda < 0$ for $v \in V^i_{(x,\omega)} \backslash V^{i-1}_{(x,\omega)}$, then the set

$$\{y \in M: \limsup_{n\to\infty} \frac{1}{n} \ln (\operatorname{dist}(F^n(\omega)x, F^n(\omega)y)) \le \lambda\}$$

is a submanifold tangent to $V^i_{(x,\omega)}$ at x. The main result of this paper is Theorem 4.2 below. It states that, under certain integrability conditions, if the transition probabilities $P(x, \cdot)$ have continuous densities, then the subbundle L^j is integrable for any j with $\lambda_j < 0$. Its integral manifolds W^j_x foliate the supports of probability measures on M that are invariant under the random walk X_n and ergodic. The corresponding foliation is invariant under any $f \in \operatorname{supp}(\mu)$. Although any vector tangent to a leaf W^j_x is exponentially contracted by $DF^n(\omega)$ with probability 1, the entire leaf W^j_x or any non-random part of it may not be contracted by $F^n(\omega)$ (see Example 4.7). However, for any x, if ω does not belong to a set of measure 0, then there is a neighbourhood of x in W^j_x that is exponentially contracted by $F^n(\omega)$.

Note that if a non-degenerate diffusion process is represented by a sequence of random diffeomorphisms (or in this case a stochastic flow), then the transition probabilities do have continuous densities. The main results and auxiliary statements of this paper hold true if the continuity of the transitional densities is replaced by the weaker assumption that they are bounded. Moreover, it is sufficient to assume that the transition probabilities have bounded densities with respect to a continuous (non-atomic) measure. The continuity assumption allows us to simplify many proofs and to make the probabilistic arguments in § 2 self-contained. In § 2 we discuss the global dynamics of Markov chains with continuous transition densities. Some of the results of this section are well known for more general Markov chains, e.g. Harris or recurrent chains (see [**Or**], [**Re**, chapter 6]). In § 3 we study the invariant filtrations in the tangent bundle and prove, in particular, the Hölder continuity of the non-random invariant subbundles L^j. § 4 contains our main result, the integrability of the subbundles L^j for which $\lambda_j < 0$. In § 5 we prove a general theorem on the Hölder continuity of invariant subbundles for partially non-uniformly hyperbolic dynamical systems.

This paper was written during the visits of the first author to the Hebrew University of Jerusalem in June 1985 and 1986 and the visits of the second author to the University of Maryland in September 1985 and 1986. Both authors were supported by BSF Israel-USA Grant # 84-00028, and the second author was supported by NSF Grant # DMS-82-04024 and by DARPA under NIMMP.

2. *Markov chains with continuous transition densities*

In this section we review the necessary properties of Markov chains with continuous transition densities. The arguments we give below are very similar to those usually applied in the classical case of finite Markov chains.

Let $\{X_n\}$ be a Markov chain on a compact Riemannian manifold M with transition probabilities $P(x, \Gamma)$, i.e. if $X_n = x$ then $X_{n+1} \in \Gamma$ with probability $P(x, \Gamma)$ (cf. [**Re**]). From now on we assume that transition probabilities have continuous transition densities, i.e. there exists a non-negative function $p(x, y)$ continuous in both variables such that for any measurable subset $\Gamma \subset M$ and any $x \in M$

$$P(x, \Gamma) = \int_\Gamma p(x, y)\, dm(y),$$

where m is the normalized Riemannian volume in M. Let $P^k(x, \Gamma)$ be the probability of reaching Γ from x in k steps and let $p^k(x, y)$ denote the continuous density of $P^k(x, \Gamma)$,

$$P^k(x, \Gamma) = \int_\Gamma p^k(x, y)\, dm(y).$$

2.1. *Definition.* A Borel probability measure Q on M is called invariant under $\{X_n\}$, or simply, *invariant* if

$$Q(\Gamma) = \int_M P(x, \Gamma)\, dQ(x)$$

for any measurable $\Gamma \subset M$.

The existence of invariant measures for Markov chains with continuous transition densities is well known. Actually a weaker condition that the measure $P(x, \cdot)$ depends continuously on x in the weak topology implies the existence of invariant measures.

2.2. LEMMA. *If the family of Borel measures $P(x, \cdot)$ depends continuously on x in the weak topology, then there exists at least one invariant Borel measure.*

Proof. Denote by P^* the operator in the space of measures which acts by the formula

$$(P^*\eta)(\Gamma) = \int_M d\eta(x)\, P(x, \Gamma).$$

Let η be an arbitrary Borel probability measure on M and let

$$\eta_n = \frac{1}{n} \sum_{k=0}^{n-1} (P^*)^k \eta.$$

Since the space of probability measures on a compact M is compact in the weak topology (see [**Ro**, p. 100]), there is a subsequence n_i such that $\eta_{n_i} \to^w \rho$ and also

$$P^* \eta_{n_i} = \frac{1}{n_i} \sum_{k=1}^{n_i} (P^*)^k \eta \to^w \rho.$$

Let P denote the induced action in the space of functions:

$$(Pg)(x) = \int_M g(y)\, P(x, dy).$$

Then for a continuous function g the function Pg is also continuous and

$$\int_M g\, dP^* \eta_{n_i} = \int_M Pg\, d\eta_{n_i} \xrightarrow[i \to \infty]{} \int_M Pg\, d\rho = \int_M g\, dP^* \rho.$$

Therefore, $P^* \eta_{n_i} \to^w P^* \rho$ and hence $P^* \rho = \rho$, i.e. ρ is invariant. □

The continuity of $p(x, y)$ easily implies that any invariant measure Q has continuous density $q(x)$. Indeed

$$Q(\Gamma) = \int_M P(x, \Gamma)\, dQ(x) = \int_\Gamma dm(y) \int_M p(x, y)\, dQ(x),$$

so that

$$q(y) = \int_M p(x, y)\, dQ(x).$$

We shall call q an *invariant density.*

2.3. *Definition.* A measurable subset $\Gamma \subset M$ is called *invariant* under $\{X_n\}$, or simply, *invariant* if

$$P(x, \Gamma) = 1$$

for m-a.e. (m-almost every) $x \in \Gamma$.

2.4. *Definition.* An invariant measure Q is *ergodic* if for any invariant set Γ, either $Q(\Gamma) = 0$ or $Q(\Gamma) = 1$. We will refer to the density of Q as an *invariant ergodic density.*

For an invariant measure Q with density q we write

$$A_q = \{x \in M : q(x) > 0\}.$$

The open set A_q is invariant since

$$0 = Q(M \backslash A_q) = \int_{M \backslash A_q} dm(y) \int_M q(x) p(x, y)\, dm(x)$$
$$= \int_{M \backslash A_q} dm(y) \int_{A_q} q(x) p(x, y)\, dm(x),$$

and therefore $p(x, y) = 0$ if $x \in A_q$ and $y \in M \backslash A_q$. Hence $P(x, A_q) = 1$ when $x \in A_q$. Moreover, by the continuity of $p(x, y)$,

$$P(x, A_q) = 1 \qquad \text{for any } x \in \bar{A}_q, \tag{2.1}$$

where $\bar{A}_q$ is the closure of A_q.

2.5. PROPOSITION. *Let q be an invariant ergodic density. Then there exists an integer T called the period of the Markov chain for which*:

(i) *A_q is the union of disjoint open sets A_q^i, $0 \le i \le T-1$, such that if $x \in A_q^i$ and $p(x, y) > 0$, then $y \in A_q^j$, where $j = (i+1) \bmod T$, and*

$$Q(A_q^i) = \int_{A_q^i} q(x)\, dm(x) = \frac{1}{T};$$

(ii) *for every compact subset $K \subset A_q^i$ there are $N = N(K)$ and $\delta = \delta(K) > 0$ such that $p^{NT}(x, y) > \delta$ for any $x \in \bar{A}_q^i$ and $y \in K$.*

We shall need the following auxiliary facts.

2.6. LEMMA. *Let $p^i(x, y) > 0$ and $p^j(y, z) > 0$ for some $x, y, z \in M$ and integers i, j. Then $p^{i+j}(x, z) > 0$.*

Proof. Recall that the functions $p^k(x, y)$ are continuous and that

$$p^{i+j}(x, z) = \int_M p^i(x, v)p^j(v, z)\, dm(v),$$

the integrand being positive in a neighborhood of $v = y$. □

2.7. LEMMA. *For any* $x \in \bar{A}_q$ *and any* $y \in A_q$

$$n(x, y) \overset{\text{def}}{=\!=} \inf\{n: p^n(x, y) > 0\}$$

is finite and the function $n(x, y)$ *is upper semicontinuous in* (x, y).

Proof. For $x \in \bar{A}_q$ let

$$U(x) = \{y: n(x, y) < \infty\}.$$

The set $U(x)$ is clearly open, non-empty and contained in A_q. By Lemma 2.6, $U(x)$ is invariant. Hence, by the ergodicity of q, $U(x) = A_q$ and $n(x, y)$ is finite for any $x \in \bar{A}_q$, $y \in A_q$. If $p^k(x, y) > 0$ then, by the continuity of p, $p^k(\tilde{x}, \tilde{y}) > 0$ for any $(\tilde{x}, \tilde{y})$ close enough to (x, y), and hence $n(\tilde{x}, \tilde{y}) \le n(x, y)$. □

Proof of Proposition 2.5. For $x \in A_q$ let

$$I(x) = \{i: p^i(x, x) > 0\}.$$

By Lemma 2.6, $i + j \in I(x)$ for any $i, j \in I(x)$, i.e. $I(x)$ is an additive semigroup and a simple number theory argument shows that for some big enough i_0 the elements of $I(x)$ greater than i_0 form an arithmetic progression with difference $T(x)$ equal to the greatest common divisor of $I(x)$. Let $y \in A_q$. By Lemma 2.7, there exist k and n such that $p^k(x, y) > 0$ and $p^n(y, x) > 0$, and hence $k + n \in I(y)$. It follows from Lemma 2.6 that $I(x) \supset k + n + I(y)$. Therefore, $T(x)$ divides $T(y)$. By the symmetry, $T(x) = T(y)$. Hence $T(x)$ is a constant which we denote by T.

Fix $x_0 \in A_q$ and set $A_q^i = \{y: p^{kT+i}(x_0, y) > 0$ for some $k\}$, $i = 0, 1, \ldots, T-1$. By Lemma 2.7, $\bigcup_{i=0}^{T-1} A_q^i = A_q$. Every A_q^i is open since it is the union of open sets. Furthermore, the sets A_q^i are disjoint. Indeed, let $y \in A_q^i \cap A_q^j$. Then $p^{kT+i}(x_0, y) > 0$ and $p^{lT+j}(x_0, y) > 0$ for some k and l. By Lemma 2.7, there is an n such that $p^n(y, x_0) > 0$. It follows from Lemma 2.6 that both $n + kT + i$ and $n + lT + j$ are divisible by T, and hence $i - j$ is divisible by T. Thus $i = j$. If $p^{kT+i}(x_0, x) > 0$ and $p(x, y) > 0$ then, by Lemma 2.6, $p^{kT+i+1}(x_0, y) > 0$. It follows from above that the sets A_q^i are cyclically permuted by the Markov chain and so $Q(A_q^i) = 1/T$ which concludes the proof of (i).

By Lemma 2.7, for any $x, y \in A_q^i$ there is an n such that $p^n(x, y) > 0$ and, by (i), all such n's are multiples of T. Recall that $I(x_0)$ contains an arithmetic progression $\{T(i(x_0) + k)\}_{k=0}^{\infty}$. Fix a compact subset $K \subset A_q^i$. By Lemma 2.7, there exists $N_0(K)$ such that if $x \in \bar{A}_q^i$ and $y \in K$ then $p^k(x, x_0) > 0$ and $p^l(x_0, y) > 0$ for some $k, l < N_0(K)$. Therefore, the set $\{n: p^n(x, y) > 0\}$ contains an arithmetic progression $\{T(i(x, y) + k)\}_{k=0}^{\infty}$, where $i(x, y) \le i(x_0) + 2N_0(K)/T + 1$. Let N be the integral part of $i(x_0) + 2N_0(K)/T + 1$ and $\delta = \min_{x \in \bar{A}_q^i, y \in K} p^{NT}(x, y)$. Since $p^{NT}(x, y)$ is continuous in (x, y), it follows that $\delta > 0$, which proves (ii). □

https://doi.org/10.1017/S0143385700004107 Published online by Cambridge University Press

As is well known, Proposition 2.5(ii) implies the following convergence of transition densities (cf. [**D**, p. 197]).

2.8. PROPOSITION. *Under the assumptions of Proposition* 2.5, *for any compact subset* $K \subset A_q^i$ *with* $m(K) > 0$, *there exist* $\gamma(K), 0 < \gamma(K) < 1$, *and* $C(K) > 0$ *such that for any* $x \in \bar{A}_q^i$, $y \in K$ *and any positive integer* k

$$|p^{kN(K)T}(x, y) - Tq(y)| \leq C(K)(\gamma(K))^k.$$

Proof. To simplify the notation set $\tau = N(K)T$. Define

$$\bar{M}_k(y) = \max_{x \in \bar{A}_q^i} p^{k\tau}(x, y), \qquad \underline{M}_k(y) = \min_{x \in \bar{A}_q^i} p^{k\tau}(x, y).$$

We have

$$\begin{aligned} \bar{M}_{k+1}(y) &= \max_{x \in \bar{A}_q^i} \int_{A_q^i} p^\tau(x, z) p^{k\tau}(z, y)\, dm(z) \\ &\leq \max_{x \in \bar{A}_q^i} \int_{A_q^i} p^\tau(x, z) \bar{M}_k(y)\, dm(z) \leq \bar{M}_k(y). \end{aligned}$$

Similarly

$$\underline{M}_{k+1}(y) \geq \underline{M}_k(y).$$

Therefore there exist the following limits

$$\bar{M}(y) = \lim_{k \to \infty} \bar{M}_k(y) \geq \underline{M}(y) = \lim_{k \to \infty} \underline{M}_k(y).$$

For $x, z \in \bar{A}_q^i$ let

$$B_+ = \{v \in A_q^i : p^\tau(x, v) - p^\tau(z, v) > 0\}$$

and

$$B_- = \{v \in A_q^i : p^\tau(x, v) - p^\tau(z, v) < 0\}.$$

Since

$$\int_{A_q^i} p^\tau(x, v)\, dm(v) = \int_{A_q^i} p^\tau(z, v)\, dm(v) = 1,$$

we have

$$\int_{B_+} (p^\tau(x, v) - p^\tau(z, v))\, dm(v) = -\int_{B_-} (p^\tau(x, v) - p^\tau(z, v))\, dm(v). \tag{2.2}$$

Thus

$$\begin{aligned} p^{(k+1)\tau}(x, y) - p^{(k+1)\tau}(z, y) &= \int_{A_q^i} (p^\tau(x, v) - p^\tau(z, v)) p^{k\tau}(v, y)\, dm(v) \\ &\leq \bar{M}_k(y) \int_{B_+} (p^\tau(x, v) - p^\tau(z, v))\, dm(v) \\ &\quad + \underline{M}_k(y) \int_{B_-} (p^\tau(x, v) - p^\tau(z, v))\, dm(v) \\ &= (\bar{M}_k(y) - \underline{M}_k(y)) \int_{B_+} (p^\tau(x, v) - p^\tau(z, v))\, dm(v). \end{aligned} \tag{2.3}$$

By Proposition 2.5(ii),

$$\int_{B_+} p^\tau(x, v)\, dm(v) \le 1 - \delta(K) \cdot m((A_q^i \backslash B_+) \cap K)$$

and

$$\int_{B_+} p^\tau(z, v)\, dm(v) \ge \delta(K) \cdot m(B_+ \cap K).$$

Therefore, the last integral in (2.3) does not exceed $1 - \delta(K)m(K)$. Since x and z were arbitrary, we see that

$$\bar{M}_{k+1}(y) - \underline{M}_{k+1}(y) \le (1 - \delta(K)m(K))(\bar{M}_k(y) - \underline{M}_k(y)).$$

Hence

$$\bar{M}(y) = \underline{M}(y) = \lim_{k\to\infty} p^{k\tau}(x, y)$$

and

$$|\bar{M}(y) - p^{k\tau}(x, y)| \le C(K) \cdot (\gamma(K))^k,$$

where $C(K) = \max p^\tau(x, y)$ and $\gamma(K) = 1 - \delta(K) \cdot m(K)$. By the invariance of q and by Proposition 2.5(i),

$$q(y) = \int_{A_q} q(z) p^{k\tau}(z, y)\, dm(z) = \int_{A_q^i} q(z) p^{kz}(z, y)\, dm(z)$$

$$\xrightarrow[k\to\infty]{} \int_{A_q^i} q(z) \bar{M}(y)\, dm(z) = \frac{1}{T} \bar{M}(y).$$

□

2.9. Theorem. *There exist only finitely many different probability invariant ergodic measures $Q_1, Q_2, \ldots, Q_r$ with densities $q_1, q_2, \ldots, q_r$, the corresponding sets $\bar{A}_{q_1}, \ldots, \bar{A}_{q_r}$ being disjoint. For any invariant probability measure Q*

$$Q = \sum_{i=1}^{r} Q(A_{q_i}) Q_i.$$

We will need the following lemmas.

2.10. Lemma. *For any invariant density q*

$$m(A_q) \ge (\max_{x,y\in M} p(x, y))^{-1}.$$

Proof. By (2.1), if $x \in A_q$ then

$$\int_{A_q} p(x, y)\, dm(y) = 1,$$

and the lemma follows. □

2.11. Lemma. *Let Q be an invariant probability measure with density q and $B \subset A_q$ be an invariant subset with $0 < Q(B) < 1$.*

Then there exist normalized invariant measures Q_1 and Q_2 with densities q_1 and q_2 such that $A_{q_1} \cap A_{q_2} = \varnothing$, $A_{q_1} \cup A_{q_2} = A_q$ and

$$Q = Q(A_{q_1}) Q_1 + Q(A_{q_2}) Q_2.$$

Proof. By the invariance of Q and B,

$$Q(B)=\int_M dQ(x)\,P(x,B)=\int_{A_q} dQ(x)\,P(x,B)$$
$$=\int_B dQ(x)\,P(x,B)+\int_{A_q\setminus B} dQ(x)\,P(x,B)$$
$$=Q(B)+\int_{A_q\setminus B} dQ(x)\,P(x,B).$$

Hence $P(x,B)=0$ for Q-a.e. $x\in A_q\setminus B$, and since Q and m are equivalent on A_q, $P(x,B)=0$ for m-a.e. $x\in A_q\setminus B$. Therefore, by the invariance of A_q, $P(x,A_q\setminus B)=1$ for m-a.e. $x\in A_q\setminus B$, i.e. the set $A_q\setminus B$ is invariant. Now let Q_1 be the normalized restriction of Q to B and Q_2 be the normalized restriction of Q to $A_q\setminus B$. It is easy to see that both measures Q_1 and Q_2 are invariant and mutually singular. Being invariant, Q_1 and Q_2 have continuous densities q_1 and q_2. The corresponding sets A_{q_1} and A_{q_2} obviously have the desired properties. □

Proof of Theorem 2.9. Let Q_1 and Q_2 be two invariant ergodic measures with densities q_1 and q_2 and let $x\in\bar A_{q_1}\cap\bar A_{q_2}$. Then $P(x,A_{q_1}\cap A_{q_2})=1$ by (2.1), and $A_{q_1}\cap A_{q_2}$ is an open non-empty invariant set. Since $Q_i(A_{q_1}\cap A_{q_2})>0$, then, by the ergodicity, $Q_i(A_{q_1}\cap A_{q_2})=1$, $i=1,2$. Therefore, $q_1=q_2$ by Proposition 2.8. Now the first statement of the theorem follows from Lemma 2.10. The second statement follows immediately from Lemmas 2.10 and 2.11. □

The following result implies that the union of the supports of ergodic invariant measures $A=\bigcup_{i=1}^r A_{q_i}$ (see Theorem 2.9) attracts the Markov chain X_n.

2.12. THEOREM. *Let $q_1,\dots,q_r$ be the densities of the ergodic invariant measures and $A_{q_1},A_{q_2},\dots,A_{q_r}$ be the corresponding sets. Then*

$$N(x,\omega)\overset{\text{def}}{=\!=}\inf\left\{n\colon X_n(\omega)\in\bigcup_{i=1}^r A_{q_i}=A,\ X_0(\omega)=x\right\}<\infty$$

with probability 1. *Moreover $Ee^{\delta N(x,\omega)}<\infty$ for some $\delta>0$.*

Proof. If $x\in A$ then $N(x,\omega)=0$. If $x\in\bar A\setminus A$ then $N(x,\omega)=1$. Let now $x\in M\setminus\bar A$ and set

$$U(x)=\{y\colon p^k(x,y)>0\text{ for some }k>0\}.$$

The set $U(x)$ is open and invariant. Assume that $U(x)\cap A=\varnothing$. Then $U(x)\cap\bar A=\varnothing$ and hence $\overline{U(x)}\cap A=\varnothing$. Since $\overline{U(x)}$ is also invariant, Lemma 2.2 implies that there exists an invariant measure supported by $\overline{U(x)}$. This contradicts Theorem 2.9. Therefore $U(x)\cap A\neq\varnothing$ and for each $x\in M$

$$k(x)\overset{\text{def}}{=\!=}\inf\{k\colon p^k(x,y)>0\text{ for some }y\in A\}<\infty.$$

Clearly $k(x)$ is upper semicontinuous and

$$K=\sup_{x\in M}k(x)<\infty.$$

Since A is invariant, $P^K(x,\bar A)>\delta$ for some $\delta>0$ and every $x\in M$. It follows that

$P^{K+1}(x, A) > \delta$. We have

$$P\{N(x,\omega) > n(K+1)\} = P\{X_{n(K+1)} \in M \backslash A\}$$
$$= \int_{M\backslash A} P^{K+1}(x, dz_1) \int_{M\backslash A} \cdots \int_{M\backslash A} P^{K+1}(x, dz_n)$$
$$\leq (1-\delta)^n.$$

This last inequality together with the Borel–Cantelli lemma implies the statement of the theorem. □

3. *Invariant filtrations for random diffeomorphisms*

For the convenience of the reader we formulate here a version of the Oseledec multiplicative ergodic theorem for random diffeomorphisms.

Let M be a compact Riemannian manifold, $F_1, F_2, \ldots$ be a sequence of independent random C^2-diffeomorphisms with a common probability distribution μ in $\mathrm{Diff}^2(M)$ and X_0 be a random variable independent of all F_n with values in M and distribution Q. Then $X_n = F_n \circ F_{n-1} \circ \cdots \circ F_1 X_0$ is a Markov chain on M with transition probabilities

$$P(x, \Gamma) = \mu(\{f \in \mathrm{Diff}^2(M) : fx \in \Gamma\})$$

for any $x \in M$ and any Borel $\Gamma \subset M$. Set

$$(\Omega, \nu) = (\mathrm{Diff}^2(M)^{\mathbb{N}}, \mu^{\mathbb{N}})$$

and let θ be the shift transformation in Ω

$$\theta(f_1, f_2, \ldots) = (f_2, f_3, \ldots).$$

For $x \in M$ and $\omega = (f_1, f_2, \ldots) \in \Omega$ define $\tau : (M \times \Omega) \to (M \times \Omega)$ by the formula

$$\tau(x, \omega) = (f_1 x, \theta\omega).$$

Suppose Q is invariant under $\{X_n\}$ in the sense of Definition 2.1, then the measure $(Q \times \nu)$ is τ-invariant. Indeed, for any measurable $A \subset M$ and $\Phi \subset \Omega$ we have

$$Q \times \nu(\tau^{-1}(A \times \Phi))$$
$$= Q \times \nu(\{(x,\omega) = (x, (f_1, \omega')) : f_1 x \in A, \omega' \in \Phi\})$$
$$= \nu(\Phi) \cdot \int_{\mathrm{Diff}^2(M)} Q(f_1^{-1}A)\, d\mu(f_1) = \nu(\Phi) \int_M \int_{\mathrm{Diff}^2(M)} \mathbb{1}_{f_1^{-1}A}\, d\mu(f_1)\, dQ(x)$$
$$= \nu(\Phi) \int_M \mu(\{f_1 : f_1 x \in A\})\, dQ(x) = \nu(\Phi) \int_M P(x, A)\, dQ(x) = \nu(\Phi) Q(A).$$

We shall use the following abbreviation

$$F^n(\omega) = F_n(\omega) \circ F_{n-1}(\omega) \circ \cdots \circ F_1(\omega).$$

3.1. Theorem (see **[Os]**, **[Ru]**, **[Le]**). *Suppose that*

$$\iint \ln^+ \|D_x f\|\, d\mu(f) dQ(x) < \infty, \tag{3.1}$$

where $\|D_x f\|$ *is the norm of the differential at* x *and* $\ln^+ a = \max(\ln a, 0)$. *Then there is a subset* Λ *of full* $Q \times \nu$ *measure in* $M \times \Omega$ *such that for every* $(x, \omega) \in \Lambda$ *there exists*

https://doi.org/10.1017/S0143385700004107 Published online by Cambridge University Press

a filtration of linear subspaces

$$0=V^0_{(x,\omega)}\subset V^1_{(x,\omega)}\subset\cdots\subset V^\kappa_{(x,\omega)}=T_xM$$

and a sequence of real numbers called characteristic exponents

$$-\infty\le\chi_1(x,\omega)<\chi_2(x,\omega)<\cdots<\chi_\kappa(x,\omega)<\infty$$

with the properties

(i) $$\lim_{n\to\infty}\frac{1}{n}\ln\|D_xF^n(\omega)\|=\chi_\kappa(x,\omega);$$

(ii) *if* $v\in V^i_{(x,\omega)}\setminus V^{i-1}_{(x,\omega)}$ *then*

$$\lim_{n\to\infty}\frac{1}{n}\ln\|D_xF^n(\omega)v\|=\chi_i(x,\omega);$$

(iii) *the functions* $\kappa=\kappa(x,\omega)$, $d_i=d_i(x,\omega)=\dim V^i_{(x,\omega)}-\dim V^{i-1}_{(x,\omega)}$ *and* $\chi_i=\chi_i(x,\omega)$, $i=1,2,\ldots,\kappa$, *are* τ*-invariant;*

(iv) *the subspaces* $V^i_{(x,\omega)}$ *depend measurably on* (x,ω) *and are DF-invariant*

$$D_xF(\omega)V^i_{(x,\omega)}=V^i_{\tau(x,\omega)};$$

(v) *the asymptotic behaviour of the exterior power is described by the following formula:*

$$\lim_{n\to\infty}\frac{1}{n}\ln\|(D_xF^n(\omega)^{\wedge r}\|=\sum_{i=j+1}^{\kappa}d_i(x,\omega)\chi_i(x,\omega)+l\cdot\chi_j(x,\omega),$$

where $l+\sum_{i=j+1}^{\kappa}d_i(x,\omega)=r$ *and* $0<l\le d_j(x,\omega)$.

3.2. Corollary (see c.f. [**FHY**, Proposition 1], [**P**, Theorem 1.1.1]). *Let* $(x,\omega)\in\Lambda$ *and set* $x_n=F^n(\omega)x$, $\omega_n=\theta^n\omega$, $\lambda=\chi_i(x,\omega)$, $\mu=\chi_{i+1}(x,\omega)$, $\lambda<\mu$. *Then for any* $a>0$ *there is a function* $C(n)=C(x,\omega,a,n)\ge 1$ *such that for all* $n,k\ge 0$

(i) $\|D_{x_n}F^k(\omega_n)v\|\le C(n)\,e^{(\lambda+a)k}\|v\|$ *for any* $v\in V^i_{(x_n,\omega_n)}$;

(ii) $\|D_{x_n}F^k(\omega_n)w\|\ge C^{-1}(n)\,e^{(\mu-a)k}\|w\|$ *for any* w *from the orthogonal complement* $(V^i_{(x_n,\omega_n)})^\perp$ *of* $V^i_{(x_n,\omega_n)}$ *in* $T_{x_n}M$;

(iii) *the angle between* $D_{x_n}F^k(\omega_n)V^i_{(x_n,\omega_n)}$ *and* $D_{x_n}F^k(\omega_n)(V^i_{(x_n,\omega_n)})^\perp$ *is greater than* $C^{-1}(n)\,e^{-ak}$;

(iv) *the function* $C(x,\omega,a,n)$ *can be chosen to depend measurably on* (x,ω,a) *and to satisfy* $C(n)\le C(0)\,e^{an}$.

The following result produces a non-random DF-invariant filtration.

3.3. Theorem [**Ki**]. *Suppose* Q *is ergodic (see Definition* 2.4) *and*

$$\iint(\ln^+\|D_xf\|+\ln^+\|D_xf^{-1}\|)\,dQ(x)\,d\mu(f)<\infty. \tag{3.2}$$

Then for Q*-a.e.* $x\in M$ *there exists a measurable filtration of (non-random) subspaces*

$$0=L^0_x\subset L^1_x\subset\cdots\subset L^l_x=T_xM$$

and a sequence of (non-random) numbers

$$-\infty<\lambda_1<\cdots<\lambda_l<\infty$$

such that

(i) $$\lim_{n\to\infty} \frac{1}{n} \ln \|D_x F^n(\omega)\| = \lambda_l \qquad Q \times \nu\text{-a.e.};$$

(ii) *if* $v \in L_x^j \setminus L_x^{j-1}$ *then*

$$\lim_{n\to\infty} \frac{1}{n} \ln \|D_x F^n(\omega) v\| = \lambda_j \qquad Q \times \nu\text{-a.e.}$$

(iii) *the subspaces* L_x^j *depend measurably on* x *and are DF-invariant*

$$D_x F(\omega) L_x^j = L^j_{F(\omega)x} \qquad Q \times \nu\text{-a.e.}$$

In [**C**] a similar result was proved under stronger assumptions.

The next result establishes a relation between the random filtration given by Theorem 3.1 and the non-random filtration given by Theorem 3.3. The functions χ_i are clearly τ-invariant, and hence constant for an ergodic Q. Therefore, there exist $1 \le i_1 < i_2 < \cdots < i_l \le \kappa$ such that $\lambda_j = \chi_{i_j}(x, \omega)$ $Q \times \nu$-a.e. The next statement follows immediately from Theorems 3.1 and 3.3.

3.4. Corollary [**Ki**]. (i) $L_x^j \subseteq V^{i_j}_{(x,\omega)}$ $Q \times \nu$*-a.e. and filtration L is the maximal non-random subfiltration of filtration V in the sense that for Q-a.e.* $x \in M$ *and any* $v \in T_x M$

$$\nu(\{\omega \in \Omega : v \in V^{i_{j+1}-1}_{(x,\omega)} \setminus L_x^j\}) = 0$$

(ii) *Let* (Ω^m, ν^m) *be the direct product of* $m = \dim M$ *copies of* (Ω, ν), *then*

$$\bigcap_{n=1}^{m} V^{i_j}_{(x,\omega_n)} = L_x^j$$

for $Q \times \nu^m$*-almost every* $(x, \omega_1, \omega_2, \ldots, \omega_m)$.

Let $r > 0$ be the injectivity radius of M. If $\operatorname{dist}(x, y) < r$, denote by $P(x, y)\colon T_x M \to T_y M$ the parallel translation from x to y along the unique shortest geodesic connecting x and y. For any points $x, y \in M$ and tangent vectors $v \in T_x M$, $w \in T_y M$ define

$$\operatorname{dist}(v, w) = \begin{cases} \|v\| + \|w\| & \text{if } \operatorname{dist}(x, y) \ge r \\ \|v - P(y, x) w\| & \text{if } \operatorname{dist}(x, y) < r. \end{cases}$$

For $E_x \subset T_x M$ and $E_y \subset T_y M$ define

$$\operatorname{dist}(E_x, E_y) = \max_{\substack{v \in E_x \\ \|v\|=1}} \min_{\substack{w \in E_y \\ \|w\|=1}} \operatorname{dist}(v, w). \tag{3.3}$$

If $D_x : T_x M \to T_{\tilde{x}} M$ and $D_y : T_y M \to T_{\tilde{y}} M$ are linear maps, define

$$\operatorname{dist}(D_x, D_y) = \begin{cases} \|D_x\| + \|D_y\| & \text{if } \max(\operatorname{dist}(x, y), \operatorname{dist}(\tilde{x}, \tilde{y})) \ge r \\ \|D_x - P(\tilde{y}, \tilde{x}) \circ D_y \circ P(x, y)\| & \text{otherwise.} \end{cases} \tag{3.4}$$

3.5. *Definition.* Let $\Lambda \subset M$. A family $\{E_x\}$, $x \in \Lambda$, of subspaces $E_x \subset T_x M$ is called *Hölder continuous in x with exponent* α, $0 < \alpha \le 1$, and constant $C_h > 0$, if for any $x, y \in \Lambda$

$$\operatorname{dist}(E_x, E_y) \le C_h (\operatorname{dist}(x, y))^\alpha.$$

For a diffeomorphism $f : M \to M$ and for a number σ, $0 < \sigma < 1$ define

$$\|Df\|_\sigma = \sup_x \|D_x f\| + \sup_{x,y} \frac{\operatorname{dist}(D_x f, D_y f)}{(\operatorname{dist}(x, y))^\sigma}. \tag{3.5}$$

3.6. Lemma. *Let*

$$\int \ln^+\|Df\|_\sigma \, d\mu(f) \stackrel{\text{def}}{=\!=\!=} b < \infty. \tag{3.6}$$

Then

(i) *for any* $\delta > 0$ *and* ν*-a.e.* $\omega \in \Omega$ *there exists* $C_\delta(\omega)$ *such that* $\|DF_n(\omega)\|_\sigma \le C_\delta(\omega)\, e^{\delta n}$, $n = 1, 2, \ldots$;

(ii) *for* ν*-a.e.* $\omega \in \Omega$ *there exists* $\hat{C}(\omega)$ *such that*

$$\prod_{i=1}^{n} \|DF_i(\omega)\|_\sigma \le \hat{C}(\omega)\, e^{2bn}, \qquad n = 1, 2, \ldots. \tag{3.7}$$

Proof. Let $\mu_n = \mu\{f: \ln^+ \|Df\|_\sigma > \delta n\}$. Then

$$\infty > \int \ln^+ \|Df\|_\sigma \, d\mu(f) \ge \sum_{n=0}^{\infty} \delta n(\mu_n - \mu_{n+1}) = \delta \sum_{n=1}^{\infty} \mu_n$$

and (i) follows from the Borel–Cantelli lemma.

To prove (ii) note that, by the ergodic theorem, for ν-a.e. $\omega \in \Omega$

$$\frac{1}{n} \ln \prod_{i=1}^{n} \|DF_i(\omega)\|_\sigma \xrightarrow[n\to\infty]{} \int \ln \|Df\|_\sigma \, d\mu(f) \le b. \qquad \square$$

3.7. Theorem. *Let* μ *satisfy* (3.2) *and* (3.6), *let* Q *be an ergodic invariant measure with the corresponding set* A_q, *and let the transition probabilities* $P(x, \Gamma)$ *have continuous densities* $p(x, y)$. *Then the subbundles* L_x^j, $j = 1, 2, \ldots, l$, *from Theorem* 3.3 *are defined and Hölder continuous on* $\bar{A}_q$.

Proof. By ergodicity, the characteristic exponents $\chi_1, \ldots, \chi_\kappa$ are constants on a subset $\tilde{\Lambda} \subset \Lambda$ of full $Q \times \nu$-measure in $M \times \Omega$. Fix $i < \kappa$ and set $a = (\chi_{i+1} - \chi_i)/3$. Consider the following sets

$$\Lambda_N = \{(x, \omega) \in \tilde{\Lambda}: C(x, \omega, a, 0) \le N, \hat{C}(\omega) \le N\},$$

$$\Lambda_N(\omega) = \{x \in M: (x, \omega) \in \Lambda_N\},$$

where C is the function introduced in Corollary 3.2. Note that $(Q \times \nu)(\Lambda_N) \to 1$ as $N \to \infty$.

3.8. Lemma. *For any* $N > 0$ *there exist* $C_N, \alpha_N > 0$ *such that*

$$\operatorname{dist}(V^i_{(x,\omega)}, V^i_{(y,\omega)}) \le C_N (\operatorname{dist}(x, y))^{\alpha_N}$$

for any $(x, \omega), (y, \omega) \in \Lambda_N$, *i.e.* $V^i_{(x,\omega)}$ *is Hölder continuous in* x *on* Λ_N.

Proof. It follows from Corollary 3.2 and Lemma 3.6(ii) that the assumptions of Corollary 5.3 are satisfied for ν-a.e. sequence of diffeomorphisms $\omega = \{f_1, f_2, \ldots\} \in \Omega$, $f_i \in \operatorname{supp}(\mu)$. Therefore the subspace $V^i_{(x,\omega)} = E^\lambda_x$ is Hölder continuous in x on the set $\Lambda_N(\omega)$ with exponent and constant depending only on $\chi_i + a = \lambda$, $\chi_{i+1} - a = \mu$, $a = \frac{1}{3}(\chi_{i+1} - \chi_i)$, and N. $\square$

We will now show that L^j_x is Hölder continuous in x on a subset of M of large Q-measure. To do that we fix $\varepsilon > 0$ and choose N so that

$$(Q \times \nu)(\Lambda_N) \ge 1 - \varepsilon.$$

Then

$$(Q\times\nu^m)\{(x,\omega_1,\ldots,\omega_m): (x,\omega_n)\notin\Lambda_N \text{ for some } n\}\le m\varepsilon,$$

where $m=\dim M$. Therefore

$$(Q\times\nu^m)\{(x,\omega_1,\ldots,\omega_m): (x,\omega_n)\in\Lambda_N,\ n=1,2,\ldots,m\}\ge 1-m\varepsilon.$$

Hence, by Corollary 3.4(ii) and Lemma 3.8, there is a set $\Lambda_\varepsilon\subset M\times\Omega^m$ such that

(i) $\bigcap_{n=1}^{m} V^{i_j}_{(x,\omega_n)}=L^j_x$ for any $(x,\omega_1,\ldots,\omega_m)\in\Lambda_\varepsilon$;

(ii) the subspaces $V^{i_j}_{(x,\omega_n)}$, $n=1,2,\ldots,m$, are uniformly Hölder continuous in x;

(iii) $(Q\times\nu^m)(\Lambda_\varepsilon)\ge 1-m\varepsilon$.

By ergodicity, $\dim L^j_x$ is constant Q-a.e., denote it by d. By the Fubini theorem, there is an m-tuple $(\omega^0_1,\omega^0_2,\ldots,\omega^0_m)$ and a subset $\Gamma_\varepsilon\subset M$, $Q(\Gamma_\varepsilon)\ge 1-m\varepsilon$, such that the subspaces $V_{x,n}=V^{i_j}_{(x,\omega^0_n)}$, $n=1,\ldots,m$, are uniformly Hölder continuous in x on Γ_ε, their intersection is L^j_x and $\dim L^j_x=d$ for any $x\in\Gamma_\varepsilon$.

3.9. LEMMA. *For any $x\in\Gamma_\varepsilon$ there exists $r=r(x)>0$ such that L^j is Hölder continuous with some constant and exponent on $\Gamma_\varepsilon\cap B(r,x)$, where $B(r,x)\subset M$ is the ball of radius r centred at x.*

Proof. We first use a coordinate chart at x to identify a small neighbourhood U of x in M with a small neighbourhood in $\mathbb{R}^m$. This allows us to identify any subspace of T_yM, $y\in U$, with the corresponding subspace of T_xM. Denote by $V^\perp_{y,n}$ the orthogonal complement of

$$V_{y,n}=V^{i_j}_{(y,\omega^0_n)},$$

$y\in\Gamma_\varepsilon\cap U$, $n=1,2,\ldots,m$. It is easy to see that the subspace $V^\perp_{y,n}$ depends Hölder continuously on $y\in\Gamma_\varepsilon$ since $V_{y,n}$ does. For every $n=1,2,\ldots,m$ fix a basis $e_{ni}(x)$, $i=1,2,\ldots,k_n$, of $V^\perp_{x,n}$. Denote by $e_{ni}(y)$ the orthogonal projection of $e_{ni}(x)$ onto $V^\perp_{y,n}$, $y\in\Gamma_\varepsilon\cap U$. Then the subspace $V_{y,n}$ is the space of solutions of the following linear system:

$$\langle v, e_{ni}(y)\rangle=0,\qquad i=1,\ldots,k_n,$$

where $\langle\,,\rangle$ denotes the standard inner product in $\mathbb{R}^m$. Recall that $L^j_y=\bigcap_{n=1}^m V_{y,n}$ and hence the subspace L^j_y is the set of solutions of the system of linear equations

$$\langle v, e_{ni}(y)\rangle=0,\qquad i=1,\ldots,k_n,\ n=1,\ldots,m. \tag{3.7}$$

The coefficients of this system are Hölder continuous functions of $y\in\Gamma_\varepsilon\cap U$. Since $\dim L^j_y=d$, the dimension of the space of solutions of (3.7) is d. In particular, there is a subset of $m-d$ equations which is equivalent to the whole system (3.7) at x and a non-zero $(m-d)$-minor of that subsystem at x. If $r(x)$ is small enough, the corresponding minor is uniformly bounded away from 0 for $y\in B(r,x)\cap\Gamma_\varepsilon$ and the corresponding subsystem is equivalent to (3.7). Any scheme of solving $m-d$ linear equations applied consistently to different $y\in B(r,x)\cap\Gamma_\varepsilon$ will produce a basis of L^j_y depending Hölder continuously on y. This implies the Hölder continuity of L^j on $B(r,x)\cap\Gamma_\varepsilon$. □

Let $\Gamma \subset \Gamma_\varepsilon$ be a closed subset with

$$Q(\Gamma) \geq 1 - 2m\varepsilon.$$

Cover Γ by finitely many balls $B(r(x_i)/2, x_i)$, $i = 1, 2, \ldots, N_1$, where $r(x_i)$ are from Lemma 3.9. By Lemma 3.9, the subspace L_x^j is Hölder continuous on each intersection $\Gamma_\varepsilon \cap B(r(x_i), x_i)$. Let C_1 be the worst constant and α be the worst exponent. If $x, y \in \Gamma$ and $\operatorname{dist}(x, y) < \frac{1}{2} \min_i r(x_i)$, then x and y lie in one of the balls, and therefore

$$\operatorname{dist}(L_x^j, L_y^j) \leq C_1(\operatorname{dist}(x, y))^\alpha.$$

For a bigger constant C_2 we have

$$\operatorname{dist}(L_x^j, L_y^j) \leq C_2(\operatorname{dist}(x, y))^\alpha \qquad \text{for any } x, y \in \Gamma.$$

We will now use the invariance of L^j under the random diffeomorphisms to show that L^j is Hölder continuous on $\bar{A}_q$. Since the invariant density q is positive on A_q and since $Q(\Gamma) \geq 1 - 2m\varepsilon$, we can choose ε so small that the volume of Γ is big enough:

$$m(\Gamma) > m(A_q)(1 - (3\bar{p}m(A_q))^{-1}),$$

where $\bar{p} = \max p(x, y)$. Then for any $x \in \bar{A}_q$

$$P(x, A_q \backslash \Gamma) = \int_{A_q \backslash \Gamma} p(x, y)\, dm(y) \leq \bar{p} \cdot m(A_q \backslash \Gamma) < \tfrac{1}{3}.$$

On the other hand, by (2.1), $P(x, A_q) = 1$ and therefore $P(x, \Gamma) > \frac{2}{3}$. Denote by $\mathscr{F}(C)$ the set of diffeomorphisms f such that $\sup_x \|D_x f\| \leq C$ and for any $z, w \in M$ and any subspaces $S_z \subset T_zM$, $S_w \subset T_wM$

$$\operatorname{dist}(D_z f^{-1} S_z, D_w f^{-1} S_w) \leq C(\operatorname{dist}(S_z, S_w) + \operatorname{dist}(z, w)).$$

Choose C_3 so that $\mu(\mathscr{F}(C_3)) > \frac{2}{3}$. Then for any $x, y \in \bar{A}_q$ there is a diffeomorphism $f_{xy} = f \in \mathscr{F}(C_3)$ such that $fx, fy \in \Gamma$. Therefore if L_x^j and L_y^j are defined, then for $f = f_{xy}$ we have

$$\begin{aligned}\operatorname{dist}(L_x^j, L_y^j) &\leq C_3(\operatorname{dist}(L_{fx}^j, L_{fy}^j) + \operatorname{dist}(fx, fy)) \\ &\leq C_3(C_2+1)(\operatorname{dist}(fx, fy))^\alpha \leq C_3(C_2+1)C_3^\alpha(\operatorname{dist}(x, y))^\alpha,\end{aligned}$$

Hence the distribution L^j is Hölder continuous in x with constant $C_h = C_3^{1+\alpha}(C_2+1)$ and exponent α on the set of full Q-measure in $\bar{A}_q$ where it is defined. By the uniform continuity, L_x^j can be defined for any $x \in \bar{A}_q$ so that L^j is Hölder continuous and invariant under any $f \in \operatorname{supp}(\mu)$. □

We extend by continuity the subbundles L^j, $1 \leq j \leq l$, to the set $\bar{A}_q$ and in what follows use the same notations for the extended subbundles. Since L is DF-invariant, the statement of Corollary 3.4 holds for any $x \in \bar{A}_q$.

3.10. Corollary. *Using the notations of Corollary* 3.4, *if* $x \in \bar{A}_q$ *then* $L_x^j \subseteq V_{(x,\omega)}^{i_j}$ *ν-a.e. and there exists a subset* Ω_x^m *of full* ν^m*-measure in* Ω^m *such that* $\bigcap_{n=1}^m V_{(x,\omega_n)}^{i_j} = L_x^j$ *for any* $(\omega_1, \omega_2, \ldots, \omega_m) \in \Omega_x^m$.

4. *Invariant foliations for random diffeomorphisms*

In this section we prove that if $\lambda_j = \chi_{i_j} < 0$, then the subbundle L^j is integrable, see Theorem 4.2 below. Its integral manifolds W_x^j are contracted in a certain probabilistic

https://doi.org/10.1017/S0143385700004107 Published online by Cambridge University Press

sense and form a foliation W^j invariant under any diffeomorphism from the support of μ. Throughout this section we assume that M is compact.

The version of the stable manifold theorem for a sequence of random diffeomorphisms $F_1, F_2, \ldots, F_n, \ldots$ formulated below follows immediately from Theorem 5.1 in [**Ru**] and Theorem 5.1 in [**Ru 1**]. We keep the notations of Theorem 3.1 introduced at the beginning of § 3.

4.1. THE STABLE MANIFOLD THEOREM (see also [**P**], [**FHY**]). *Let* (3.6) *be satisfied and let* $(x, \omega) \in \Lambda$ *and* $\xi < 0$ *satisfy* $\chi_i(x, \omega) < \xi < \chi_{i+1}(x, \omega)$. *Then there exist* $\delta = \delta(x, \omega) > 0$ *and* $R = R(x, \omega, \xi) > 0$ *such that*

(i) $W^{\xi}_{(x,\omega)} = \{y\colon \operatorname{dist}(x, y) < \delta$ *and* $\operatorname{dist}(F^n(\omega)x, F^n(\omega)y) \le Re^{\xi n} \operatorname{dist}(x, y)$ *for all* $n \ge 0\}$ *is a* C^1*-submanifold tangent to* $V^i_{(x,\omega)}$ *at* x *and diffeomorphic with the unit ball in* $V^i_{(x,\omega)}$;

(ii) $(y, \omega) \in \Lambda$ *for any* $y \in W^{\xi}_{(x,\omega)}$ *and the tangent space* $T_y W^{\xi}_{(x,\omega)}$ *is* $V^i_{(y,\omega)}$;

(iii) *for any* $y, z \in W^{\xi}_{(x,\omega)}$, $\operatorname{dist}(F^n(\omega)y, F^n(\omega)z) \le Re^{\xi n} \operatorname{dist}(y, z)$ *for all* $n \ge 0$;

(iv) *the functions* $R^{-1}(F^n(\omega)x, \theta^n\omega, \xi)$ *and* $\delta(F^n(\omega)x, \theta^n\omega)$ *may decrease only subexponentially in* n;

(v) *denote by* W *the connected component of* $F_1(\omega)x$ *in the intersection of* $F_1(\omega)W^{\xi}_{(x,\omega)}$ *with the ball of radius* $\delta(F_1(\omega)x, \theta\omega)$ *centred at* $F_1(\omega)x$; *then* $W \subset W^{\xi}_{(F_1(\omega)x, \theta\omega)}$.

The main result of this section is the following theorem.

4.2. THEOREM. *Let* μ *satisfy* (3.2) *and* (3.6), *let* Q *be an ergodic invariant measure with the corresponding set* A_q, *and let the transition probabilities* $P(x, \Gamma)$ *have continuous densities* $p(x, y)$. *Suppose* $\lambda_j = \chi_{i_j} < 0$.

Then the subbundle $\{L^j_x\}$ *is integrable in the following sense. There exists a foliation* W^j *of* A_q *into* C^1 *complete submanifolds* W^j_x *without boundary such that the tangent space* $T_x W^j_x$ *is* L^j_x *for any point* $x \in A_q$. *The foliation* W^j *is invariant under any diffeomorphism* f *from the support of* μ. *For any* $x \in A_q$ *and* ν*-almost every* ω *there exists a neighbourhood* $W^j_{x,\varepsilon(\omega)}$ *of* x *in* W^j_x *such that*

$$\limsup_{n\to\infty} \frac{1}{n} \ln \operatorname{diam}\,(F^n(\omega) W^j_{x,\varepsilon(\omega)}) \le \lambda_j.$$

Proof. We first give an outline of the argument. For an $x \in A_q$ and a typical sequence ω of diffeomorphisms $f_n \in \operatorname{supp}(\mu)$ we will construct a special sequence of norms $|||\ \ |||_n$. The non-uniformly hyperbolic sequence of matrices $D_{x_n} f_{n+1}$, $x_n = f_n x_{n-1}$, $x_0 = x$, will be uniformly hyperbolic if the norms $|||\ \ |||_n$ are used. Then, after establishing certain properties of the norms $|||\ \ |||_n$, we will use them to prove that any piecewise smooth curve, tangent to L^j and passing through x, is exponentially contracted by the sequence of diffeomorphisms $\{f_n\}$, see Lemma 4.5 below. This geometric property will imply the integrability of L^j.

Fix $x \in A_q$ and $\omega \in \Omega = (\mathrm{Diff}^2(M))^{\mathbb{N}}$, $\omega = \{f_n\}$, such that $L^j_x \subseteq V^{i_j}_{(x,\omega)}$, cf. Corollary 3.10, $(x, \omega) \in \Lambda$ (see Theorem 3.1), and the numbers $\chi_{i_j}(x, \omega) < \chi_{i_j+1}(x, \omega)$ are typical values (recall that, by ergodicity, the exponents are constant $Q \times \nu$-a.e.). Let $\varphi_n : \mathcal{N} \to M$ be the exponential map at $x_n = f^n x$, $f^n = f_n \circ \cdots \circ f_1$, restricted to a neighbourhood

$\mathcal{N} \subset \mathbb{R}^m = T_{x_n}M$ of the origin, $m = \dim M$. By the compactness of M, we may choose $\mathcal{N}$ so small that all φ_n's are one-to-one with uniformly bounded first and second derivatives of φ_n and φ_n^{-1}. Set $\tilde{f}_n = \varphi_n^{-1} \circ f_n \circ \varphi_{n-1}$: $\mathcal{N} \to \mathbb{R}^m$ and introduce the following notations:

$$T_n = D_0 \tilde{f}_{n+1}, \quad E_0^s = D\varphi_0^{-1}(V^{i_j}_{(x,\omega)}), \quad E_0^u = (E_0^s)^\perp,$$
$$E_{n+1}^s = T_n E_n^s = D\varphi_{n+1}^{-1}(V^{i_j}_{(f^{n+1}x,\, \theta^{n+1}\omega)}), \quad E_{n+1}^u = T_n E_n^u,$$
$$T_n^k = T_{n+k-1} \cdots T_n, \quad S_n^k = T_n^k | E_n^s, \quad U_n^k = T_n^k | E_n^u,$$

where $n, k \geq 0$, $T_n^0 = \mathrm{id}$, $S_n^0 = \mathrm{id}$, $U_n^0 = \mathrm{id}$. Set

$$\lambda = \chi_{i_j}, \qquad \mu = \chi_{i_j+1}$$

and fix $a > 0$ such that

$$a < \min\left(\frac{-\alpha\lambda}{2\alpha+5}, \frac{\mu-\lambda}{10}\right), \tag{4.1}$$

where α is the Hölder exponent of L^j. By Corollary 3.2 there is a function $C(n)$ such that for all $n, k \geq 0$

$$\|S_n^k v^s\| \leq C(n)\, e^{(\lambda+a)k} \|v^s\|, \qquad v^s \in E_n^s, \tag{4.2}$$
$$\|U_n^k v^u\| \leq C^{-1}(n)\, e^{(\mu-a)k} \|v^u\|, \qquad v^u \in E_n^u, \tag{4.3}$$
$$\|v^s\| \leq C(n)\|v\|, \quad \|v^u\| \leq C(n)\|v\|, \tag{4.4}$$

where $v = v^s + v^u$, $v^s \in E_n^s$, $v^u \in E_n^u$,

$$1 \leq C(n) \leq C(0)\, e^{an}. \tag{4.5}$$

The following sequence of norms is usually called the *Lyapunov metric* or the *metric adjusted to the dynamical system*:

$$|||v^s|||_n = \sum_{k=0}^{\infty} e^{-(\lambda+2a)k} \|S_n^k v^s\|, \qquad v^s \in E_n^s, \qquad n \geq 0, \tag{4.6}$$
$$|||v^u|||_n = \sum_{k=0}^{n} e^{(\mu-2a)k} \|(U_{n-k}^k)^{-1} v^u\|, \qquad v^u \in E_n^u, \qquad n \geq 0, \tag{4.7}$$
$$|||v|||_n^2 = |||v^s|||_n^2 + |||v^u|||_n^2 \qquad \text{for } v = v^s + v^u, \qquad v^s \in E_n^s, \qquad v^u \in E_n^u. \tag{4.8}$$

4.3. LEMMA [**BN**]. *The sequence of norms* $\{|||\ \ |||_n\}$ *satisfies*

(i) $|||S_n v^s|||_{n+1} \leq e^{\lambda+2a} |||v^s|||_n, \quad v^s \in E_n^s,$

(ii) $|||U_n v^u|||_{n+1} \geq e^{\mu-2a} |||v^u|||_n, \quad v^u \in E_n^u,$

(iii) $\frac{1}{2}\|v\| \leq |||v|||_n \leq C_1\, e^{2an} \|v\|$, *where* $C_1 = 2C^2(0)/(1-e^{-a})$.

Proof. We have

$$|||S_n v^s|||_{n+1} = \sum_{k=0}^{\infty} e^{-(\mu+2a)k} \|S_{n+1}^k S_n v^s\| \leq e^{\lambda+2a} \sum_{k=0}^{\infty} e^{-(\lambda+2a)(k+1)} \|S_n^{k+1} v^s\| \leq e^{\lambda+2a} |||v|||_n,$$

and hence (i) holds. Similarly

$$\begin{aligned} |||U_x v^u|||_{n+1} &= \sum_{k=0}^{n+1} e^{(\mu-2a)k} \|(U_{n+1-k}^k)^{-1} U_n v^u\| \\ &= \|U_n v^u\| + e^{\mu-2a} \sum_{k=1}^{n+1} e^{(\mu-2a)(k-1)} \|(U_{n+1-k}^{k-1})^{-1} v^u\| \\ &\geq e^{-2a} |||v^u|||_n, \end{aligned}$$

and hence (ii) holds.

https://doi.org/10.1017/S0143385700004107 Published online by Cambridge University Press

To prove (iii) we first note that the zero terms in the right-hand sides of (4.6) and (4.7) are $\|v^s\|$ and $\|v^u\|$. Therefore $|||v^s|||_n \geq \|v^s\|$ and $|||v^u|||_n \geq \|v^u\|$. This implies the left inequality in (iii) since

$$\|v\| \leq \|v^s\| + \|v^u\| \leq 2 \max (|||v^s|||_n, |||v^u|||_n).$$

To prove the right inequality observe that, by (4.2) and (4.5),

$$|||v^s|||_n \leq \sum_{k=0}^{\infty} e^{-(\lambda+2a)k} C(n)\, e^{(\lambda+a)k} \|v^s\| \leq \frac{C(0)}{1-e^{-a}}\, e^{an} \|v^u\|.$$

Similarly, by (4.3) and (4.5),

$$|||v^u|||_n \leq \sum_{k=0}^{n} e^{(\mu-2a)k} C(n-k)\, e^{-(\mu-a)k} \|v^u\| \leq \frac{C(0)}{1-e^{-2a}}\, e^{an} \|v^u\|.$$

By the last two estimates, (4.6) and (4.7),

$$\begin{aligned} |||v|||_n^2 &\leq C^2(0)\, e^{2an} \left(\frac{\|v^s\|^2}{(1-e^{-a})^2} + \frac{\|v^u\|^2}{(1-e^{-2a})^2} \right) \\ &\leq 2C^2(0)\, e^{2an} \frac{C^2(n)}{(1-e^{-a})^2} \|v\|^2 \leq 2 \frac{C^4(0)}{(1-e^{-a})^2} e^{4an} \|v\|^2. \end{aligned}$$ □

Denote by dist_n and length_n the distance in $\mathbb{R}^m$ and the length of curves induced by the norm $|||\ \ |||_n$. Then, by Lemma 4.3(iii)

$$\tfrac{1}{2} \text{dist}(y, z) \leq \text{dist}_n(y, z) \leq C_1\, e^{2an} \text{dist}(y, z) \tag{4.9}$$

for any points $y, z \in \mathbb{R}^m$, and

$$\tfrac{1}{2} \text{length}(\sigma) \leq \text{length}_n(\sigma) \leq C_1\, e^{2an} \text{length}(\sigma) \tag{4.10}$$

for any piecewise smooth curve σ. For any matrix T and any vector v we have, by Lemma 4.3(iii)

$$|||Tv|||_{n+1} \leq C_1\, e^{2a(n+1)} \|Tv\| \leq C_1\, e^{2a(n+1)} \|T\| \cdot \|v\| \leq 2C_1\, e^{2a(n+1)} \|T\| \cdot |||v|||_n$$

and hence

$$|||Tv|||_{n+1} \leq 2C_1\, e^{2a(n+1)} \|T\| \cdot |||v|||_n. \tag{4.11}$$

Let $y \in \mathcal{N}$ be such that $\varphi_n(y) \in A_q$. Set

$$\tilde{L}^j_n(y) = (D\varphi_n)^{-1}(L^j_{\varphi_n(y)}).$$

Since L^j is Hölder continuous with exponent α and constant C_h, the distributions $\tilde{L}^j_n$ are uniformly Hölder continuous with the same exponent α and constant $\tilde{C}_h$.

4.4. Lemma. *Let $\varphi_n(y) \in A_q$ and let $v \in \tilde{L}^j_n(y)$ and denote by v^s and v^u the components of v that are parallel to E^s_n and E^u_n respectively, $v = v^s + v^u$. Then*

$$\frac{|||v^u|||_n}{|||v|||_n} \leq C_4\, e^{3an} (\text{dist}_n(y, 0))^\alpha$$

for some constant C_4.

Proof. Denote by $v^\perp$ the component of v perpendicular to E^s_n. Since $\tilde{L}^j_n(0) \subset E^s_n$ and $\tilde{L}^j_n$ is Hölder continuous, by (4.9)

$$|||v^\perp||| \leq C_2 (\text{dist}(y, 0))^\alpha \|v\| \leq 2C_2 (\text{dist}_n(y, 0))^\alpha \|v\|$$

for a constant C_2. Therefore by (4.4) and (4.5)

$$\|v^u\| \le C(n)\|v^\perp\| \le 2C(0)C_2\, e^{an}(\mathrm{dist}_n\,(y,0))^\alpha \|v\|.$$

Hence, by Lemma 4.3(iii),

$$\frac{|||v^u|||_n}{|||v|||_n} \le \frac{C_1\, e^{2an}\|v^u\|}{\frac{1}{2}\|v\|} \le 4C(0)C_2\, e^{3an}(\mathrm{dist}_n(y,0))^\alpha. \qquad \square$$

Denote by $\tilde{f}^n$ the composition $\tilde{f}_n \circ \tilde{f}_{n-1} \circ \cdots \circ \tilde{f}_1$.

4.5. LEMMA. *Let ω in addition satisfy Lemma* 3.6(i) *and let β be such that*

$$\lambda + 2a < \beta < \min(-6a/\alpha, -3a/\sigma), \tag{4.12}$$

where α is the Hölder exponent of L^j and a satisfies (4.1). *Then there exists $\varepsilon > 0$ such that for any piecewise smooth curve $\gamma:[0,1]\to\mathcal{N}$ with* $\gamma(0)=0$, length $(\gamma)\le\varepsilon$ *and* $\dot\gamma(t)\in\tilde{L}_0^j(\gamma(t))$, $t\in[0,1]$, *the following holds*:

(i) $\tilde{f}^n(\gamma(t))\in\mathcal{N}$ *for* $t\in[0,1]$, $\quad n\ge 0$,

(ii) $\mathrm{length}_n\,(\tilde{f}^n\circ\gamma)\le e^{\beta n}\,\mathrm{length}_0\,(\gamma)$, $\quad n\ge 0$,

(iii) $\mathrm{length}\,(\tilde{f}^n\circ\gamma)\le 2C_1\, e^{\beta n}\,\mathrm{length}\,(\gamma)$, $\quad n\ge 0$.

Proof. We first prove (i) and (ii) by induction in n. For $n=0$, (i) and (ii) are obviously true. Suppose (i) and (ii) hold for some positive integer n. Set $\gamma_n(t)=\tilde{f}^n(\gamma(t))$, $t\in[0,1]$, and denote by $T_n(t)$ the differential $D_{\gamma_n}(t)\tilde{f}_{n+1}$. Let C_4 be an upper bound for the first and second derivatives of $\tilde{f}_n$. We have for any $t\in[0,1]$

$$|||T_n(t)\dot\gamma_n(t)|||_{n+1} \le |||(T_n(t)-T_n)\dot\gamma_n(t)|||_{n+1} + |||T_n\dot\gamma_n(t)|||_{n+1}. \tag{4.13}$$

By (4.11) and (4.10)

$$\begin{aligned} &|||(T_n(t)-T_n)\dot\gamma_n(t)|||_{n+1}\\ &\quad\le 2C_1\, e^{2a(n+1)}\cdot \|T_n(t)-T_n\|\cdot |||\dot\gamma_n(t)|||_n\\ &\quad\le 2C_1\, e^{2a(n+1)}\|D\tilde{f}_{n+1}\|_\sigma(2\,\mathrm{length}_n\,(\dot\gamma_n))^\sigma\cdot |||\dot\gamma_n(t)|||_n. \end{aligned} \tag{4.14}$$

Denote by $\dot\gamma_n^s(t)$ and $\dot\gamma_n^u(t)$ the components of $\dot\gamma_n(t)$ parallel to E_n^s and E_n^u respectively. Then

$$|||T_n\dot\gamma_n(t)|||_{n+1} \le |||T_n\dot\gamma_n^s(t)|||_{n+1} + |||T_n\dot\gamma_n^u(t)|||_{n+1}. \tag{4.15}$$

It follows from (4.11) and Lemma 4.4 that

$$\begin{aligned} |||T_n\dot\gamma_n^u(t)||| &\le 2C_1\, e^{2a(n+1)}\|T_n\|\cdot|||\dot\gamma_n^u(t)|||_n\\ &\le 2C_1\, e^{2a(n+1)}\|D\tilde{f}_{n+1}\|_\sigma|||\dot\gamma_n^u(t)|||_n\\ &\le 2C_1\, e^{2a(n+1)}C_4\, e^{3an}\|D\tilde{f}_{n+1}\|_\sigma(\mathrm{length}_n\,(\gamma_n))^\alpha|||\dot\gamma_n(t)|||_n. \end{aligned} \tag{4.16}$$

By Lemma 4.3(i),

$$|||T_n\dot\gamma_n^s(t)|||_{n+1} \le e^{\lambda+2a}|||\dot\gamma_n^s(t)|||_n \le e^{\lambda+2a}|||\dot\gamma_n(t)|||_n.$$

Set $\delta=a$ in Lemma 3.6(i). We now put together (4.13)-(4.16) and use the inductive assumption to obtain

$$\begin{aligned} |||T_n(t)\dot\gamma_n(t)|||_{n+1} &\le [C_5\|D\tilde{f}_{n+1}\|_\sigma(\exp((2a+\sigma\beta)n)\cdot\varepsilon^\sigma\\ &\qquad +\exp((5a+\alpha\beta)n)\varepsilon^\alpha)+e^{\lambda+2a}]\cdot|||\dot\gamma_n(t)|||_n\\ &\le [C_5C_a(\omega)(\exp((3a+\sigma\beta)n)\varepsilon^\sigma\\ &\qquad +\exp((6a+\alpha\beta)n)\varepsilon^\alpha)+e^{\lambda+2a}]\cdot|||\dot\gamma_n(t)|||_n, \end{aligned}$$

where C_5 is a constant and $C_a(\omega)$ is from Lemma 3.6. By our choice of a and β (cf. (4.1) and (4.12)), it follows from the last estimate that for small enough $\varepsilon > 0$

$$|||T_n(t)\dot{\gamma}_n(t)|||_{n+1} \le e^{\beta}|||\dot{\gamma}_n(t)|||_n.$$

This implies (ii) with n replaced by $n+1$ since

$$\text{length}_{n+1}(\gamma_{n+1}) = \int_0^1 |||T_n(t)\dot{\gamma}_n(t)|||_{n+1}\,dt.$$

Statement (i) follows from (ii) if ε is less than half the radius of the biggest ball centred at the origin and contained in $\mathcal{N}$. Statement (iii) follows immediately from (ii) and (4.10). □

We are ready now to prove the integrability of L^j. Let $x \in A_q$. Then, by Corollary 3.10, there is a subset Ω_x^m of full ν^m-measure in Ω^m such that

$$\bigcap_{n=1}^m V^{i_j}_{(x,\,\omega_n)} = L^j_x \qquad \text{for any } (\omega_1, \omega_2, \dots, \omega_m) \in \Omega^m_x. \tag{4.17}$$

Since Q is ergodic we may assume that the characteristics exponents are constant for all ω_i's. Fix such a collection $(\omega_1, \dots, \omega_m)$ and a number $a > 0$ satisfying (4.1) with $\lambda = \chi_{i_j}$, $\mu = \chi_{i_j} + 1$. Choose $\beta < \lambda + 3a$ satisfying (4.12) and set $\xi = \lambda + 4a < \mu$. By Theorem 4.1, there are local stable manifolds $W^{\xi}_{(x,\,\omega_n)}$, $n = 1, \dots, m$. Denote by $W^j_{x,\varepsilon}$ the set of points lying on piecewise C^1-curves $\gamma: [0, 1] \to M$ such that $\gamma(0) = x$, $\dot{\gamma}(t) \in L^j_{\gamma(t)}$, $0 \le t \le 1$, and length $(\gamma) \le \varepsilon$. If ε is small enough, we can lift such a curve γ to the neighbourhood $\mathcal{N} \subset \mathbb{R}^m = T_xM$, apply Lemma 4.5 and obtain that

$$\text{length}\,(F^n(\omega) \circ \gamma) \le 2C_1C_6^2\, e^{\beta n}\, \text{length}\,(\gamma), \tag{4.18}$$

where C_6 is an upper bound for the norm of the differential of the exponential map restricted to $\mathcal{N}$ and for the inverse. Assume now that ε is less than the sizes $\delta(x, \omega_n)$ of the local stable manifolds $W^{\xi}_{(x,\,\omega_n)}$, $n = 1, 2, \dots, m$. Then by Theorem 4.1 and (4.18)

$$W^j_{x,\,\varepsilon} \subset \bigcap_{n=1}^m W^{\xi}_{(x,\,\omega_n)} \stackrel{\text{def}}{=\!=} W.$$

We use a coordinate chart at x to identify a small neighbourhood of the origin in $\mathbb{R}^m$. Recall that $\dim L^j_x = d$. Let $y, z \in W^j_{x,\varepsilon} \subset W$, $y \ne z$. Then the unit vector $\eta = (z-y)/\|z-y\|$ is close to $V^{i_j}_{(x,\,\omega_n)}$ for every n. Therefore η is close to L^j_x and does not belong to the orthogonal complement $L^{j\perp}_x$ of L^j_x. Let P_y denote the $(m-d)$-plane parallel to $L^{j\perp}_x$ and passing through $y \in U$, $P_x = L^{j\perp}_x$. For a tangent vector $w \in L^j_x$ at x let $P_{x,w}$ be the $(m-d+1)$-plane $\{tw + z\colon t \in \mathbb{R}, z \in P_x\}$. If U is small enough, then for any $y \in P_{x,w} \cap U$ the intersection $L^j_y \cap P_{x,w}$ is one-dimensional and there is a unique vector $v_w(y) \in L^j_y \cap P_{x,w}$ whose projection onto the t-axis in $P_{x,w}$ is w. Clearly $v_w(\cdot)$ is a continuous vector field on $U \cap P_{x,w}$ which depends continuously on w. Consider now the differential equation $\dot{y} = v_w(y)$ in $U \cap P_{x,w}$. We claim that for a small enough U there is a unique integral curve $y = g_w(t)$ passing through x. To see that, note that any such curve is tangent to L^j and therefore lies in W. Suppose there are two different integral curves $y = g_i(t)$, $i = 1, 2$, $g_i(0) = 0$, $g_1(t_0) \ne g_2(t_0)$. Let π denote the orthogonal projection onto L^j_x. Then $\pi(g_1(t)) = \pi(g_2(t)) = tw$; i.e. $g_1(t) - g_2(t) \in L^{j\perp}_x$, which is a contradiction.

Since v_w depends continuously on w, the uniqueness of g_w implies that it depends continuously on w. If w_1 and w_2 are not collinear, then the integral curves g_{w_1} and g_{w_2} lie in different planes and hence $g_{w_1}(t) \neq g_{w_2}(t)$ for any $t > 0$. Therefore, if U is small enough, then the restriction ψ of the map $w \to g_w(1)$ to $L^j_x \cap U$ is a homeomorphism. Since no two points from $W^j_{x,\varepsilon}$ have the same projection on L^j_x, for a small ε and properly chosen U the image of ψ is $W^j_{x,\varepsilon}$ and the plane L^j_x is clearly tangent to the image at x. Let $y \in W^j_{x,\tilde{\varepsilon}}$, $\tilde{\varepsilon} < \varepsilon$. Then the above argument shows that for $\delta < \varepsilon - \tilde{\varepsilon}$ the set $W^j_{y,\delta}$ is a C^0-submanifold that coincides locally near y with $W^j_{x,\varepsilon}$ and is tangent to L^j_y at y. It follows that $W^j_{x,\varepsilon}$ is a d-dimensional C^1-submanifold tangent to L^j.

Let W^j_x denote the set of points lying on piecewise C^1-curves passing through x and tangent to L^j. It follows from above that each set W^j_x is a complete submanifold of A_q which depends continuously on x in the C^1-topology. Clearly the sets W^j_x partition A_q. Since L^j is invariant under any diffeomorphism f from the support of measure μ, we have that $f(W^j_x) = W^j_{fx}$, and hence the foliation W^j with leaves W^j_x is also invariant under any such diffeomorphism. This finishes the proof of integrability of L^j.

Let $x \in A_q$. Then the set $\{\omega: L^j_x \subseteq V^{i_j}_{(x,\omega)}\} = \Omega_x$ has full ν-measure by Corollary 3.10. If $\omega \in \Omega_x$ and $\varepsilon(x, \omega) = \varepsilon(\omega)$ is small enough, then, by Lemma 4.5, the set $W^j_{x,\varepsilon(\omega)}$ is contained in the local stable manifold $W^{\xi}_{(x,\omega)}$. Hence, by Theorem 4.1

$$\limsup_{n\to\infty} \frac{1}{n} \ln \operatorname{diam} (F^n(\omega) W^j_{x,\varepsilon(\omega)}) \le \lambda_j.$$

This finishes the proof of Theorem 4.2. □

Since $\lambda_j < 0$ and therefore, by Corollary 3.4, any vector tangent to L^j is exponentially contracted by $DF^n(\omega)$ with probability 1, one may wonder whether the size $\varepsilon(x, \omega)$ of the 'stable' manifold $W^j_{x,\varepsilon(\omega)}$ can be chosen independently of ω and whether the global integral manifolds W^j_x are exponentially contracted by $F^n(\omega)$ with probability 1. An obvious sufficient condition for this to be true is given in Remark 4.6. However, as Example 4.7 shows, in general the answer to both questions is negative.

4.6. *Remark.* Let $\lambda_j < 0$ and assume that

$$\lim_{n\to\infty} \frac{1}{n} \ln \|DF^n(\omega)|L^j\| \le \lambda \le 0 \qquad \nu\text{-a.e.},$$

where $\lambda < \lambda_{j+1}$. Then for a.e. ω all vectors are uniformly contracted by $DF^n(\omega)$ and hence any global integral manifold W^j_x is exponentially contracted by $F^n(\omega)$ with asymptotic rate λ.

4.7. *Example.* Let f be a diffeomorphism of the circle S^1 which we view as the interval $[0, 1]$ with the ends identified. Then $1 = \text{length}\,(f([0, 1])) = \int_0^1 |f'(x)|\,dx$. By the Jensen inequality,

$$0 = \ln \int_0^1 |f'(x)|\,dx \ge \int_0^1 \ln |f'(x)|\,dx,$$

with equality if and only if $|f'| \equiv \text{const}$. Assume now that f does not preserve the

Lebesgue measure, i.e. $|f'|$ is not constant. Then

$$\int_0^1 \ln|f'(x)|\, dx < 0. \tag{4.19}$$

Let θ_n, $n = 1, 2, \ldots$, be a sequence of independent random variables uniformly distributed on $[0, 1]$ and consider the sequence of independent random diffeomorphisms $F_n x = f(x) + \theta_n (\text{mod } 1)$, $n = 1, 2, \ldots$. Then for each x the transition probability $P(x, \cdot)$ of the Markov chain $X_n = F_n X_{n-1}$ is the Lebesgue measure m, and hence m is invariant under X_n. Set $F^n = F_n \circ \cdots \circ F_1$ and let $\chi = \chi(x, \omega)$ denote the characteristic exponent. Since m is obviously ergodic, the ergodic theorem implies that

$$\chi(x, \omega) = \lim_{n\to\infty} \frac{1}{n} \ln|(F^n(\omega))'(x)| = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \ln|F_k'(F^{k-1}x)|$$
$$= \int_0^1 \ln|f'(y)|\, dy$$

for a.e. ω. The last integral is negative by (4.19), and hence $\chi < 0$. It follows from Theorem 4.2 that for any $x \in S^1$ and a.e. sequence of diffeomorphisms ω there is a neighbourhood $W_{x,\varepsilon}$ of size $\varepsilon = \varepsilon(x, \omega)$ that is exponentially contracted by $F^n(\omega)$. Suppose that $\varepsilon(x, \omega) \geq \varepsilon_0 > 0$ for some x and ν-a.e. ω. Then $\varepsilon(F_1 x, \theta\omega) \geq \alpha\varepsilon_0$ for a.e. ω, where $\alpha = \min|f'|$. Therefore, by the definition of F_1, $\varepsilon(x, \omega) \geq \alpha\varepsilon_0$ for a.e. (x, ω). Hence there exist finitely many points $x_1, \ldots, x_k$ such that their $\alpha\varepsilon_0$ neighbourhoods cover S^1 and $\varepsilon(x_i, \omega) \geq \alpha\varepsilon_0$ for ν-a.e. ω. It follows that there is an ω such that $F^n(\omega)$ contracts the whole circle exponentially. This is a contradiction. Hence the size of the local stable manifold is really random in this case. □

4.8. *Remark.* Recall that stochastic flows generated by stochastic differential equations can be represented as compositions of independent identically distributed random diffeomorphisms. In this case the integrability condition (3.6), which was used in the proofs of Theorems 3.7 and 4.2, follows from Sobolev's embedding theorems provided the coefficients of stochastic differential equations are smooth enough. Moreover, even if the coefficients are not smooth enough, the main auxiliary statement, Corollary 5.3, can be obtained by using Gronwall's inequality and arguments similar to Totoki's proof of the multidimensional-time generalization of Kolmogorov's criteria for the path continuity of stochastic processes (see [**E**, p. 302]).

5. *Appendix*: *Hölder continuity of invariant subbundles*

5.1. *Definition.* Let Λ be a (not necessarily complete) metric space, H be a (separable, complete) Hilbert space and $\{E_x\}$, $x \in \Lambda$, be a family of subspaces, $E_x \subset H$. The family $\{E_x\}$ is called ***Hölder continuous*** in x with exponent α, $0 < \alpha \leq 1$, and constant L, $L > 0$, if for any $x, y \in \Lambda$

$$\operatorname{dist}(E_x, E_y) \overset{\text{def}}{=\!=\!=} \sup_{\substack{v \in E_x \\ \|v\|=1}} \inf_{\substack{w \in E_y \\ \|w\|=1}} \operatorname{dist}(v, w) \leq L \cdot (\operatorname{dist}(x, y))^\alpha.$$

Let X be a metric space, $\operatorname{diam}(X) \leq 1$, H be a Hilbert space and $\{T_i(x)\}$, $i = 0, 1, \ldots$, $x \in X$, be a sequence of families of bounded linear operators $T_i(x): H \to H$.

Set

$$T^n(x) = T_n(x) \circ \cdots \circ T_1(x).$$

5.2. THEOREM. *For $C > 1$ and $\lambda < \mu$ let $\Lambda_{C,\lambda,\mu} \subset X$ be the (maybe empty) set of points x for which there exists a splitting*

$$H = E_x^\lambda \oplus E_x^{\lambda\perp}$$

such that for any positive integer n

$$\|T^n(x)v\| \le Ce^{\lambda n}\|v\| \qquad \text{if } v \in E_x^\lambda,$$
$$\|T^n(x)w\| \ge C^{-1}e^{\mu n}\|w\| \qquad \text{if } w \in E_x^{\lambda\perp}.$$

Suppose there is $a > \max(\lambda, 0)$ and $0 < \beta \le 1$ such that

$$\|T^n(x) - T^n(y)\| \le e^{an}(\text{dist}(x, y))^\beta$$

for any positive integer n and any $x, y \in X$.

Then the family $\{E_x^\lambda\}$ is Hölder continuous in x on $\Lambda_{C,\lambda,\mu}$ with exponent $((\lambda-\mu)/(\lambda-a))\beta$ and constant $3C^2e^{\mu-\lambda}$.

Proof. Set

$$K_x^n = \{v \in H : \|T^n(x)v\| \le 2Ce^{\lambda n}\|v\|\}.$$

Let $v \in K_x^n$, $v = v^\lambda + v^\perp$, where $v^\lambda \in E_x^\lambda$, $v^\perp \in E_x^{\lambda\perp}$. By the triangle inequality

$$\|T^n(x)v\| = \|T^n(x)(v^\lambda + v^\perp)\| \ge \|T^n(x)v^\perp\| - \|T^n(x)v^\lambda\| \ge C^{-1}e^{\mu n}\|v^\perp\| - Ce^{\lambda n}\|v\|.$$

Therefore

$$\|v^\perp\| \le Ce^{-\mu n}(\|T^n(x)v\| + Ce^{\lambda n}\|v\|) \le 3C^2e^{(\lambda-\mu)n}\|v\|$$

and

$$\text{dist}(v, E_x^\lambda) \le 3C^2e^{(\lambda-\mu)n}\|v\|. \tag{5.1}$$

Fix $a_1 > \max(a, \lambda)$ and set

$$\gamma = (\lambda - a_1)/\beta.$$

Let $x, y \in \Lambda_{C,\lambda,\mu}$. Since $\gamma < 0$, there is a unique non-negative integer $n = n(x, y)$ such that

$$e^{\gamma(n+1)} < \text{dist}(x, y) \le e^{\gamma n}. \tag{5.2}$$

For any $w \in E_y^\lambda$

$$\begin{aligned}\|T^n(x)w\| &\le \|T^n(y)w\| + \|T^n(x) - T^n(y)\|\,\|w\| \\ &\le Ce^{\lambda n}\|w\| + e^{an}(\text{dist}(x, y))^\beta\|w\| \\ &\le (Ce^{\lambda n} + e^{an}e^{\beta\gamma n})\|w\| \le 2Ce^{\lambda n}\|w\|.\end{aligned}$$

Hence $w \in K_x^n$ and $E_y^\lambda \subset K_x^n$. By symmetry, $E_x^\lambda \subset K_y^n$. It follows from (5.1) and (5.2) that

$$\begin{aligned}\text{dist}(E_x^\lambda, E_y^\lambda) &\le 3C^2e^{(\lambda-\mu)n} \le 3C^2e^{\mu-\lambda}(\text{dist}(x, y))(\lambda-\mu)/\gamma \\ &= 3C^2e^{\mu-\lambda}(\text{dist}(x, y))^{((\lambda-\mu)/(\lambda-a_1))\cdot\beta}.\end{aligned}$$ □

5.3. COROLLARY. *Let M be a Riemannian manifold with injectivity radius $r > 0$ and let $\{f_i\}$, $i = 1, 2, \ldots$, be a sequence of differentiable maps $f_i : M \to M$ such that the differentials Df_i satisfy*

$$\prod_{i=1}^n \|Df_i\|_\sigma \le C_1e^{an}, \qquad n = 1, 2, \ldots, \tag{5.3}$$

where $a, C_1 > 0$, $0 < \sigma \le 1$. Set $f^n = f_n \circ f_{n-1} \circ \cdots \circ f_1$. Fix $C > 0$ and $\lambda < \mu$ and let

$\Lambda_{C,\lambda,\mu}$ *be the (maybe empty) set of points x for which there exists a splitting*

$$T_xM = E_x^{\lambda} \oplus E_x^{\lambda\perp}$$

such that for any positive integer n

$$\|D_xf^nv\| \le Ce^{\lambda n}\|v\| \qquad \text{if } v \in E_x^{\lambda},$$
$$\|D_xf^nw\| \ge C^{-1}e^{\mu n}\|w\| \qquad \text{if } w \in E_x^{\lambda\perp}.$$

Then the family $\{E_x^{\lambda}\}$ *is Hölder continuous in x on* $\Lambda_{C,\lambda,\mu}$ *with constant* $3C^2e^{\mu-\lambda}$ *and exponent* $\alpha = ((\lambda-\mu)/(\lambda-b))\sigma$, *where* $b = \ln(2C_1^2) + 2a - \sigma \ln r + |\lambda|$.

Proof. We will need the following lemma.

5.4. Lemma. *For* $n \ge 1$

$$\operatorname{dist}(D_xf^n, D_yf^n) \le e^{bn}(\operatorname{dist}(x, y))^{\sigma}. \tag{5.4}$$

Proof. Assume that $\operatorname{dist}(f^nx, f^ny) \ge r$ for some $n \ge 0$. Then, by (3.5),

$$r \le \operatorname{dist}(f^nx, f^ny) \le \prod_{i=1}^{n} \|Df_i\|_{\sigma} \cdot \operatorname{dist}(x, y)$$

and hence, by (5.3), $\operatorname{dist}(x, y) \ge rC_1^{-1}e^{-an}$. Therefore, by (3.4), (3.5), (5.3) and by the choice of b, for any $m \ge n$ we have

$$\operatorname{dist}(D_xf^m, D_yf^m) \le 2\prod_{i=1}^{m} \|Df_i\|_{\sigma} \le 2C_1e^{am} \le e^{mb}(\operatorname{dist}(x \cdot y))^{\sigma},$$

where we used the inequality $\operatorname{dist}(D_xf^m, D_yf^m) \le \|D_xf^m\| + \|D_yf^m\|$.

Hence it suffices to prove (5.4) when $\operatorname{dist}(f^ix, f^iy) < r$ for all $i = 0, 1, \ldots, n$. Note that (5.4) follows from (5.3) and the inequality

$$\operatorname{dist}(D_xf^n, D_yf^n) \le \left(\prod_{i=1}^{n} \|Df_i\|_{\sigma}\right)^2 \cdot (\operatorname{dist}(x, y))^{\sigma} \tag{5.5}$$

which we now prove by induction. For $n = 1$ (5.5) follows from (3.5). Assume now that (5.5) holds true for $n = k$. For $n = k+1$ using (3.4), (3.5) and the inductive assumption we have:

$$\begin{aligned}
&\operatorname{dist}(D_xf^{k+1}, D_yf^{k+1}) \\
&\quad = \|D_xf^{k+1} - P(f^{k+1}y, f^{k+1}x) \circ P(x, y)\| \\
&\quad = \|D_{f^kx}f_{k+1} \circ D_xf^k \\
&\qquad - P(f^{k+1}y, f^{k+1}x)D_{f^ky}f_{k+1} \circ P(f^kx, f^ky) \circ P(f^ky, f^kx) \circ D_yf^k \circ P(x, y)\| \\
&\quad \le \|D_{f^kx}f_{k+1} \circ D_xf^k - P(f^{k+1}y, f^{k+1}x) \circ D_{f^ky}f_{k+1} \circ P(f^kx, f^ky) \circ D_xf^k\| \\
&\qquad + \|P(f^{k+1}y, f^{k+1}x) \circ D_{f^ky}f_{k+1} \circ P(f^kx, f^ky) \circ D_xf^k \\
&\qquad - P(f^{k+1}y, f^{k+1}x) \circ D_{f^ky}f_{k+1} \circ P(f^kx, f^ky) \circ P(f^ky, f^kx) \circ D_yf^k \circ P(x, y)\| \\
&\quad \le \operatorname{dist}(D_{f^kx}f_{k+1}, D_{f^ky}f_{k+1}) \cdot \|D_xf^k\| + \|D_{f^ky}f_{n+1}\| \cdot \operatorname{dist}(D_xf^k, D_yf^k) \\
&\quad \le (\|Df_{k+1}\|_{\sigma} - \sup_z \|D_zf_{k+1}\|) \cdot (\operatorname{dist}(f^kx, f^ky))^{\sigma} \cdot \|D_xf^k\| \\
&\qquad + \sup_z \|D_zf_{k+1}\| \cdot \left(\prod_{i=1}^{k} \|Df_i\|_{\sigma}\right)^2 \cdot (\operatorname{dist}(x, y))^{\sigma} \\
&\quad \le \left(\prod_{i=1}^{k+1} \|Df_i\|_{\sigma}\right)^2 \cdot (\operatorname{dist}(x, y))^{\sigma}
\end{aligned}$$

since

$$\text{dist}\,(f^k x, f^k y) \le \prod_{i=1}^{k} \|Df_i\|_\sigma. \qquad \square$$

Fix any points $x, y \in \Lambda_{C,\lambda,\mu}$ and set

$$T_i(x) = D_x f^i, \qquad T_i(y) = P(f^i y, f^i x) \circ D_y f^i \circ P(x, y).$$

Identify the tangent spaces $T_{f^i x}M$ by any sequence of isometries with $H = T_x M$ and consider the discrete set $X = \bigcup_{n=0}^{\infty} (f^n x \cup f^n y)$ with the following distance function: $\text{dist}'\,(f^i x, f^i y) = \text{dist}\,(f^i x, f^i y)$ if this distance is not greater than r, and the distance is r in all other cases. By Theorem 5.2 and Lemma 5.4,

$$\text{dist}\,(E_x^\lambda, E_y^\lambda) \le 3C^2\, e^{\mu-\lambda} (\text{dist}\,(x, y))^\alpha. \qquad \square$$

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https://doi.org/10.1017/S0143385700004107 Published online by Cambridge University Press