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Remark on ordered abelian groups

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Let N be a subgroup of the torsion-free abelian group G. Then a partial order for N is contained in one, two or uncountably many full orders for G, and a full order for nonzero N is contained in one or uncountably many full orders for G.

Fuchs and Sąsiada [2, Theorem 2] exhibit a group G with a proper subgroup N such that every full order for N can be extended to exactly two full orders for G. This fails in abelian G. (Henceforth $N \subseteq G$ are torsion-free abelian groups; thus G is an O^*-group - *cf.* [1, p. 39, Corollary 13] - and so every partial order for N, being a partial order for G, extends to some full order for G.) In fact, this comprehensive result holds:

THEOREM.

(a) *A partial order* $P(N)$ *for* N *is contained in one, two or uncountably many full orders* $L(G)$ *for* G.

(b) *A full order* $L(N)$ *for nonzero* N *is contained in one or uncountably many such* $L(G)$.

Proof *(a)*. Let

$$S \equiv \{g \in G \mid mg \in P(N) \text{ for integer } m \text{ implies } m = 0\}$$

If $S = \emptyset$, then every g in G has a nonzero multiple in $P(N)$, and, trivially, exactly one $L(G)$ extends $P(N)$. If the set S has 'rank 1', that is, the largest independent subset (language of abelian groups) of G contained in S has one element, it is easily checked that

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exactly two $L(G)$ extend $P(N)$. Now suppose S has rank > 1 , and without loss of generality let G be divisible; we test two cases:

Case 1: $G = R \oplus R$, where R is the additive rationals. Consider this simple geometric argument. Any $L(G)$ consists, in usual Cartesian 2-space, of all rational pairs (x, y) in a half-plane T bounded by a line through the origin (and including one of the two rays from the origin which comprise the boundary); conversely, each such T induces a full order for G and distinct T induce distinct orders. Similarly, $P(N)$ is a subset of a 'smallest wedge' W with vertex at the origin, where W has some angle α in $[0, \pi]$. If $\alpha < \pi$, then clearly $P(N)$ extends to continuously many T . But S has rank > 1 , so $\alpha < \pi$.

Case 2: G has (finite or infinite) rank > 2 . If $\{s, t\}$ is an independent subset of G in S , take $G = R_1 \oplus R_2 \oplus \ldots$ where every R_i is R , and R_1 (respectively R_2) consists of all rational multiples of s (respectively t) . Now $P \equiv P(N) \cap (R_1 \oplus R_2)$ is a partial order for $R_1 \oplus R_2$ which by Case 1 extends to uncountably many full orders L_j for $R_1 \oplus R_2$. For each such j , $P_j \equiv P(N) + L_j$ is a partial order for G containing $P(N)$, and the P_j are distinct and extend to distinct $L(G)$.

(b). If $S = \emptyset$, then $L(N)$ extends to exactly one $L(G)$ as above (*cf.* Neumann and Shepperd [3, Lemma 2.9]). If $S \neq \emptyset$, let s be in S and h in $L(N) - \{0\}$. Now G contains $(s) \oplus (h)$, where $(\ )$ is 'subgroup generated by'. Let $(h) \subseteq (s) \oplus (h)$ bear the full order $L((h))$ induced by $L(N)$ on (h) . There are uncountably many full orders L_j on $(s) \oplus (h)$ which induce $L((h))$; for each such j let a partial order P_j for G be the subsemigroup $L_j + (0, L(N))$ of $(s) \oplus N \subseteq G$. Each P_j contains $L(N)$, and the P_j are distinct and extend to distinct $L(G)$.

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