J. Austral. Math. Soc. (*Series A*) 31 (1981), 175–188

POSITIVE VALUES OF INHOMOGENEOUS QUINARY QUADRATIC FORMS OF TYPE (4, 1)

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(Received 17 December 1979, revised 14 October 1980)

Communicated by A. J. van der Poorten

Abstract

Here it is proved that if $Q(x, y, z, t, u)$ is a real indefinite quinary quadratic form of type (4, 1) and determinant D, then given any real numbers x_0, y_0, z_0, t_0, u_0 there exist integers x, y, z, t, u such that

$$0 < Q(x + x_0, y + y_0, z + z_0, t + t_0, u + u_0) \leqslant (8|D|)^{1/5}.$$

All critical forms are also obtained.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 10 E 20.

1. Introduction

Let $Q(x_1, x_2, \ldots, x_n)$ be a real indefinite quadratic form in n variables with signature $(r, n - r)$, $0 < r < n$ and determinant $D \neq 0$. It is known (see Blaney (1948)) that there exists a real number κ, depending upon n and r only, such that given any real numbers $c_1, c_2, \ldots, c_n$ the inequality

$$0 < Q(x_1 + c_1, x_2 + c_2, \ldots, x_n + c_n) \leqslant (\kappa|D|)^{1/n}$$

has a solution in integers $x_1, x_2, \ldots, x_n$. Let $\Gamma_{r,n-r}$ denote the infimum of all such numbers κ. Davenport and Heilbronn (1947) proved that $\Gamma_{1,1} = 4$. $\Gamma_{2,1} = 4$ was proved by Barnes (1961) and $\Gamma_{1,2} = 8$ was obtained by Dumir (1967). Dumir (1968a, b) has also shown that $\Gamma_{3,1} = 16/3$ and $\Gamma_{2,2} = 16$. The authors (1980) proved that $\Gamma_{3,2} = 16$. In this paper we prove that $\Gamma_{4,1} = 8$. All the critical forms are also obtained. More precisely we prove:

https://doi.org/10.1017/S1446788700033450 Published online by Cambridge University Press

THEOREM. *Let $Q(x, y, z, t, u)$ be a real indefinite quinary quadratic form of type* (4, 1) *and determinant D (< 0) then given any real numbers x_0, y_0, z_0, t_0, u_0, there exist integers x, y, z, t, u such that*

$$(1.1) \qquad 0 < Q(x + x_0, y + y_0, z + z_0, t + t_0, u + u_0) \leq (8|D|)^{1/5}.$$

The sign of equality in (1.1) *is necessary if and only if either*

$$(1.2) \qquad Q(x, y, z, t, u) \sim \rho Q_1 = \rho(xy + z^2 + t^2 + u^2 + zt + tu + uz)$$

or

$$(1.3) \qquad Q(x, y, z, t, u) \sim \rho Q_2 = \rho(x^2 + y^2 + z^2 + t^2 - 4u^2),$$

where $\rho > 0$.

For Q_1, the sign of equality in (1.1) *is necessary if and only if* $(x_0, y_0, z_0, t_0, u_0) \equiv (0, 0, 0, 0, 0)$ (mod 1) *while for Q_2 it is needed if and only if* $(x_0, y_0, z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (mod 1).

2. Some lemmas

In the course of the proof we shall use the following lemmas:

LEMMA 1. *If Q is as in the theorem, there exist integers x_1, y_1, z_1, t_1, u_1 such that*

$$(2.1) \qquad 0 < Q(x_1, y_1, z_1, t_1, u_1) \leq (8|D|)^{1/5}.$$

The sign of equality in (2.1) *is necessary if and only if $Q \sim \rho Q_1$, $\rho > 0$.*

This follows from some results of Watson (1968), Jackson (1969) and Oppenheim (1953a). Also see Watson (1958).

Let $\varphi(y, z, t, u)$ be a real indefinite quaternary quadratic form of type (3, 1) and determinant D (< 0). We need the following results:

LEMMA 2. *Given any real numbers y_0, z_0, t_0, u_0, there exist* $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0)$ (mod 1) *such that*

$$(2.2) \qquad |\varphi(y, z, t, u)| \leq (|D|/3)^{1/4}.$$

This is a theorem due to Dumir (1967).

LEMMA 3. *There exist integers y_2, z_2, t_2, u_2 such that*

$$(2.3) \qquad 0 < \varphi(y_2, z_2, t_2, u_2) < (4|D|)^{1/4}$$

except when $\varphi(y, z, t, u) \sim \rho\varphi_1 = \rho(y^2 + yz + z^2 + tu)$ and $\varphi(y, z, t, u) \sim \rho\varphi_2 = \rho(y^2 + z^2 + tu)$, $\rho > 0$.

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This is Theorem 2 of Oppenheim (1953b).

LEMMA 4. *There exist* $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0) \pmod 1$ *such that*

$$0 < -\varphi(y, z, t, u) < (22|D|)^{1/4}. \tag{2.4}$$

This follows from Theorem 1 of the authors (1980).

LEMMA 5. *Let* $\psi(z, t, u)$ *be a real indefinite ternary quadratic form of type* (2, 1) *and determinant* D (< 0). *Then given any real numbers* z_0, t_0, u_0 *there exist* $(z, t, u) = (z_0, t_0, u_0) \pmod 1$ *such that*

$$|\psi(z, t, u)| \leqslant (27|D|/100)^{1/3}. \tag{2.5}$$

This is a theorem due to Davenport (1948).

LEMMA 6. *Let* $\psi(z, t, u)$ *be as in Lemma* 5. *Let* $c = \frac{9}{8}, \frac{1}{2}$ *or* $\frac{1}{3}$. *Then given any real numbers* z_0, t_0, u_0 *there exist* $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ *such that*

$$-c(f(c)|D|)^{1/3} < \psi(z, t, u) \leqslant (f(c)|D|)^{1/3}, \tag{2.6}$$

where $f(\frac{9}{8}) = \frac{512}{2187}$, $f(\frac{1}{2}) = \frac{256}{429}$ *and* $f(\frac{1}{3}) = \frac{27}{32}$. *The sign of equality in* (2.6) *is necessary if and only if* $c = \frac{1}{3}$ *and* $\psi \sim \rho\psi_1$, $\rho > 0$ *where* $\psi_1 = z^2 + t^2 - 4u^2$. *For* ψ_1 *the equality is needed if and only if* $(z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pmod 1$.

For $c = \frac{9}{8}$ and $\frac{1}{2}$, the result follows from a theorem of Dumir (1969). For $c = \frac{1}{3}$, it is due to the authors (1979).

LEMMA 7. *Let* α, β, *and* d *be real numbers with* $d \geqslant 1$. *Then given any real number* x_0, *there exists* $x \equiv x_0 \pmod 1$ *such that*

$$0 < (x + \alpha)^2 - \beta^2 \leqslant d \tag{2.7}$$

provided

$$\beta^2 \begin{cases} \leqslant (d-1)^2/4 & \textit{if } d \textit{ is an integer,} \\ < [d]^2/4 & \textit{if } d \textit{ is not an integer.} \end{cases} \tag{2.8}$$

Further strict inequality in (2.8) *implies strict inequality in* (2.7).

This is Lemma 6 of Dumir (1968a).

LEMMA 8. *Let* n *be an integer* $\geqslant 1$. *If* $f(d)$ *is an increasing function of* d *for* $d > n$ *and if*

$$f(d) < (d-1)^2/4 \quad \textit{for } d \geqslant n + 1, \tag{2.9}$$

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then for $n < d < n + 1$,

(2.10) $$f(d) < [d]^2/4.$$

This obvious lemma is useful in many calculations.

3. Proof of the theorem

Let

(3.1) $$m = \inf_{\substack{x, y, z, t, u \in \mathbb{Z}, \\ Q(x, y, z, t, u) > 0}} Q(x, y, z, t, u).$$

By Lemma 1,

$$0 \leqslant m \leqslant (8|D|)^{1/5}.$$

If $m = 0$, the result follows from a result of Watson (1960). So we can suppose that $m > 0$ in the rest of the paper. Let $0 < \varepsilon_0 < \frac{1}{16}$ be a sufficiently small number. Then we can find integers x_1, y_1, z_1, t_1, u_1 such that

$$Q(x_1, y_1, z_1, t_1, u_1) = \frac{m}{1 - \varepsilon} \leqslant (8|D|)^{1/5},$$

where $0 \leqslant \varepsilon < \varepsilon_0$ and g.c.d. $(x_1, y_1, z_1, t_1, u_1) = 1$. By a suitable unimodular transformation we can suppose that

$$Q(1, 0, 0, 0, 0) = m/(1 - \varepsilon)$$

and write

$$Q(x, y, z, t, u) = m(1 - \varepsilon)^{-1}\{(x + hy + gz + h't + g'u)^2 + \varphi(y, z, t, u)\},$$

where $|h| \leqslant \frac{1}{2}$, $|g| \leqslant \frac{1}{2}$, $|h'| \leqslant \frac{1}{2}$, $|g'| \leqslant \frac{1}{2}$ and $\varphi(y, z, t, u)$ is a real indefinite quadratic form of type (3, 1) with determinant

(3.2) $$D(m/(1 - \varepsilon))^{-5} \leqslant -\frac{1}{8}.$$

Equality in (3.2) occurs if and only if $Q \sim \rho Q_1$ (by Lemma 1). Also by definition of m, we have, for any integers x, y, z, t, u either $Q(x, y, z, t, u) \leqslant 0$ or $Q(x, y, z, t, u) \geqslant m$. Because of homogeneity it suffices to prove:

THEOREM A. *Let* $Q(x, y, z, t, u) = (x + hy + gz + h't + g'u)^2 + \varphi(y, z, t, u)$, *where* $\varphi(y, z, t, u)$ *is a real indefinite quaternary quadratic form of type* (3, 1) *and determinant* D *such that*

(3.3) $$D \leqslant -\frac{1}{8} \qquad \left(D = -\frac{1}{8} \text{ if and only if } Q \sim Q_1\right)$$

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and

$$(3.4) \qquad |h| < \frac{1}{2}, \quad |g| < \frac{1}{2}, \quad |g'| < \frac{1}{2}, \quad |h'| < \frac{1}{2}.$$

Suppose further that for integers x, y, z, t, u *we have*

$$(3.5) \qquad \text{either} \quad Q(x, y, z, t, u) \leq 0 \quad \text{or} \quad Q(x, y, z, t, u) \geq 1 - \varepsilon,$$

where ε (> 0) *is sufficiently small. Let*

$$(3.6) \qquad d = (8|D|)^{1/5}.$$

Then given any real numbers x_0, y_0, z_0, t_0, u_0 *there exist* $(x, y, z, t, u) \equiv (x_0, y_0, z_0, t_0, u_0)$ (mod 1) *satisfying*

$$(3.7) \qquad 0 < Q(x, y, z, t, u) \leq d.$$

The sign of equality in (3.7) *is necessary if and only if* $Q \sim Q_1$ *or* Q_2. *For* Q_1, *equality occurs if and only if* $(x_0, y_0, z_0, t_0, u_0) \equiv (0, 0, 0, 0, 0)$ (mod 1) *while for* Q_2 *it occurs if and only if* $(x_0, y_0, z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (mod 1).

3.1. Proof of Theorem A.

LEMMA 9. *If* $Q(x, y, z, t, u)$ *is as defined in Theorem* A, *then for integers* y, z, t, u *we have*

$$(3.8) \qquad \text{either} \quad \varphi(y, z, t, u) \leq 0 \quad \text{or} \quad \varphi(y, z, t, u) \geq \frac{3}{4} - \varepsilon.$$

This result and its proof is similar to Lemma 4.1 of Dumir (1969).

LEMMA 10. *If* $Q = Q_1$, *then* (3.7) *is true with strict inequality unless* $(x_0, y_0, z_0, t_0, u_0) \equiv (0, 0, 0, 0, 0)$ (mod 1), *in which case equality is necessary.*

PROOF. Here $|D| = \frac{1}{8}$, so that $d = 1$.

Case (i). $(x_0, y_0) \not\equiv (0, 0)$ (mod 1). Suppose without loss of generality that $x_0 \not\equiv 0$ (mod 1). Choose $(z, t, u) \equiv (z_0, t_0, u_0)$ (mod 1) arbitrarily, $x \equiv x_0$ (mod 1) such that $0 < |x| \leq \frac{1}{2}$ and then choose $y \equiv y_0$ (mod 1) to satisfy

$$0 < xy + z^2 + t^2 + u^2 + zt + tu + uz \leq |x| \leq \frac{1}{2} < d = 1.$$

Case (ii). $(x_0, y_0) \equiv (0, 0)$ (mod 1). First we deal with the case when $(z_0, t_0, u_0) \not\equiv (0, 0, 0)$ (mod 1). Without loss of generality we can suppose that $z_0 \not\equiv 0$ (mod 1). Choose $z \equiv z_0$ (mod 1) such that $0 < |z| \leq \frac{1}{2}$. Now choose $t \equiv$

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t_0 (mod 1) such that $0 \leqslant |t + z/3| \leqslant \frac{1}{2}$ and $u \equiv u_0$ (mod 1) such that $0 \leqslant |u + t/2 + z/2| \leqslant \frac{1}{2}$. Take $x = y = 0$. So that

$$\begin{aligned} 0 &< xy + z^2 + t^2 + u^2 + zt + tu + uz \\ &= xy + (z/2 + t/2 + u)^2 + 3(t + z/3)^2/4 + 2z^2/3 \\ &\leqslant \frac{1}{4} + \frac{3}{4}\cdot\frac{1}{4} + \frac{2}{3}\cdot\frac{1}{4} = \frac{29}{48} < 1. \end{aligned}$$

Now let $(x_0, y_0, z_0, t_0, u_0) \equiv (0, 0, 0, 0, 0)$ (mod 1). Then equality is needed in (3.7) because $xy + z^2 + t^2 + u^2 + zt + tu + uz$ takes integral values only.

Since from (3.3), $d = (8|D|)^{1/5} \geqslant 1$ and $d = 1$ if and only if $Q \sim Q_1$, we can suppose that $d > 1$ in the rest of the paper.

LEMMA 11. *Let $\nu_1 = d - \frac{1}{4}$ and $\nu_2 > 0$ be a real number satisfying*

$$\nu_2 \begin{cases} \leqslant (d-1)^2/4 & \text{if } d \text{ is an integer,} \\ < [d]^2/4 & \text{if } d \text{ is not an integer.} \end{cases} \tag{3.9}$$

Suppose that we can find $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0)$ (mod 1) such that

$$-\nu_2 < \varphi(y, z, t, u) \leqslant \nu_1 \tag{3.10}$$

then for any x_0, there exists $x \equiv x_0$ (mod 1) satisfying (3.7). Further strict inequality in (3.10) implies strict inequality in (3.7).

PROOF. If $0 < \varphi(y, z, t, u) \leqslant \nu_1$, choose $x \equiv x_0$ (mod 1) such that

$$|x + hy + gz + h't + g'u| \leqslant \frac{1}{2},$$

so that

$$0 < Q(x, y, z, t, u) \leqslant \nu_1 + \frac{1}{4} = d.$$

Strict inequality holds if we have strict inequality in (3.10). If $-\nu_2 < \varphi(y, z, t, u) \leqslant 0$, then the result follows from Lemma 7 with $\alpha = hy + gz + h't + g'u$ and $\beta^2 = -\varphi(y, z, t, u)$.

LEMMA 12. *If $d > 11$, then (3.7) is true with strict inequality.*

PROOF. By Lemma 4, there exist $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0)$ (mod 1) such that

$$0 < -\varphi(y, z, t, u) < (22|D|)^{1/4}$$

that is

$$-(11d^5/4)^{1/4} < \varphi(y, z, t, u) < 0.$$

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Then the result will follow from Lemma 11, if we have

$$(11d^5/4)^{1/4} < \begin{cases} (d-1)^2/4 & \text{if } d \geqslant 12, \\ [d]^2/4 & \text{if } 11 < d < 12. \end{cases}$$

$f(d) = (11d^5/4)^{1/4}$ is an increasing function of d for $d > 1$. By Lemma 8, it is enough to verify the above inequality for $d \geqslant 12$. This verification is easy and we omit the proof.

LEMMA 13. *If* $4 < d \leqslant 11$, *then again* (3.7) *is true with strict inequality.*

PROOF. By Lemma 2, there exist $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0)$ with

$$|\varphi(y, z, t, u)| \leqslant (|D|/3)^{1/4} = (d^5/24)^{1/4}.$$

The result will follow from Lemma 11, if we have

$$(d^5/24)^{1/4} < d - \frac{1}{4} \tag{3.11}$$

and

$$(d^5/24)^{1/4} < \begin{cases} (d-1)^2/4 & \text{if } 5 \leqslant d \leqslant 11, \\ [d]^2/4 & \text{if } 4 < d \leqslant 5. \end{cases} \tag{3.12}$$

We observe that by Lemma 8, it is enough to verify (3.12) for $5 \leqslant d \leqslant 11$. Verification of these inequalities is easy and is left to the reader.

REMARK. For $1 < d \leqslant 4$, we shall repeat the procedure of reduction described in Section 3. We shall use Lemma 3 on the homogeneous minimum of positive values of quaternary forms of type (3, 1). So we first dispose of the exceptional forms.

LEMMA 14. *If* $\varphi(y, z, t, u) \sim \rho\varphi_1$ *or* $\rho\varphi_2$, $1 < d \leqslant 4$, $\rho > 0$, *then again* (3.7) *is true with strict inequality.*

PROOF. *Case* (i). $\varphi \sim \rho\varphi_1$. It is enough to consider

$$\varphi = \rho\varphi_1 = \rho(y^2 + yz + z^2 + tu).$$

So that

$$Q(x, y, z, t, u) = (x + hy + gz + h't + g'u)^2 + \rho(y^2 + yz + z^2 + tu).$$

If $g' \neq 0$, then by (3.4) we get $0 < Q(0, 0, 0, 0, 1) = g'^2 \leqslant \frac{1}{4} < 1 - \varepsilon$. This contradicts (3.5). Therefore $g' = 0$. Similarly $h' = 0$. Consideration of the values of Q at the points $(0, 0, 1, -1, 1)$ and $(0, 1, 0, -1, 1)$ gives $g = h = 0$. Therefore

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$Q(x, y, z, t, u) = x^2 + \rho(y^2 + yz + z^2 + tu)$ and $|D| = 3\rho^4/16$. Here $\rho = (16|D|/3)^{1/4} = (2d^5/3)^{1/4} < 2d$ for $d \leqslant 4$.

Subcase (i). $(t_0, u_0) \not\equiv (0, 0) \pmod 1$. Without loss of generality we can suppose that $t_0 \not\equiv 0 \pmod 1$. Choose $(x, y, z) \equiv (x_0, y_0, z_0) \pmod 1$ arbitrarily, $t \equiv t_0 \pmod 1$ such that $0 < |t| \leqslant \frac{1}{2}$ and then choose $u \equiv u_0 \pmod 1$ satisfying

$$0 < x^2 + \rho(y^2 + yz + z^2 + tu) \leqslant \rho|t| \leqslant \rho/2 < d.$$

Subcase (ii). $(t_0, u_0) \equiv (0, 0) \pmod 1$. Take $t = u = 0$. Choose $x \equiv x_0 \pmod 1$ such that $|x| \leqslant \frac{1}{2}$, $z \equiv z_0 \pmod 1$ such that $|z| \leqslant \frac{1}{2}$ and $y \equiv y_0 \pmod 1$ such that $|y + z/2| \leqslant \frac{1}{2}$. So that

$$\begin{aligned} 0 &\leqslant x^2 + \rho(y^2 + yz + z^2 + tu) \\ &= x^2 + \rho(y + z/2)^2 + 3 \cdot \rho z^2/4 + \rho tu \leqslant 7\rho/16 + \frac{1}{4} < d. \end{aligned}$$

(It can be easily checked that $7\rho/4 < 4d - 1$, for $d \leqslant 4$.) Therefore (3.7) is satisfied with strict inequality unless $x = 0$, $y + z/2 = 0$, $z = 0$. In this case change x to 1, then (3.7) is satisfied with strict inequality.

Case (ii). $\varphi \sim \rho\varphi_2$, $\rho > 0$ is similar and is left to the reader.

3.2. Proof of Theorem A continued

From now on we can suppose that $1 < d \leqslant 4$ and $\varphi(y, z, t, u) \nsim \rho\varphi_1$ or $\rho\varphi_2$, $\rho > 0$. By Lemma 3, there exist integers y_2, z_2, t_2, u_2 with g.c.d. $(y_2, z_2, t_2, u_2) = 1$ such that

(3.13) $$0 < a = \varphi(y_2, z_2, t_2, u_2) < (4|D|)^{1/4} = (d^5/2)^{1/4}.$$

Also from (3.8) we have $a \geqslant \frac{3}{4} - \varepsilon$. By a suitable unimodular transformation we can suppose that $\varphi(1, 0, 0, 0) = a$. So we can write

$$\varphi(y, z, t, u) = a\{(y + fz + f't + f''u)^2 + \psi(z, t, u)\},$$

where

(3.14) $$\frac{3}{4} - \varepsilon \leqslant a < (d^5/2)^{1/4}$$

and $\psi(z, t, u)$ is a real indefinite ternary quadratic form of type (2, 1) and determinant D/a^4.

In view of Lemma 11, it is enough to prove that there exist $(y, z, t, u) \equiv (y_0, z_0, t_0, u_0) \pmod 1$ such that

(3.15) $$-\nu/a < (y + fz + f't + f''u)^2 + \psi(z, t, u) \leqslant (4d - 1)/4a,$$

where

$$(3.16)\qquad \nu = \begin{cases} \dfrac{9}{4} & \text{if } 3 < d \leqslant 4, \\ 1 & \text{if } 2 < d \leqslant 3, \\ \dfrac{1}{4} & \text{if } 1 < d \leqslant 2. \end{cases}$$

Let $\mu_1 = (4d - 1 - a)/4a$ and $\lambda = (4d - 1 + 4\nu)/4a$. Using (3.14) one can easily verify that $\mu_1 > 0$ and $\lambda > 1$.

LEMMA 15. *Let $\mu_2 > 0$ be a real number satisfying*

$$\mu_2 \leqslant \begin{cases} (\lambda - 1)^2/4 + \nu/a & \text{if } \lambda \text{ is an integer}, \\ [\lambda]^2/4 + \nu/a & \text{if } \lambda \text{ is not an integer}. \end{cases}$$

Suppose that there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ *such that*

$$(3.17)\qquad -\mu_2 < \psi(z, t, u) \leqslant \mu_1.$$

Then we can find $y \equiv y_0 \pmod 1$ *satisfying* (3.15). *Further strict inequality in* (3.17) *implies strict inequality in* (3.15).

The proof is similar to that of Lemma 11, so we omit it.

LEMMA 16. *If* $3 < d \leqslant 4$, *then* (3.17) *and hence* (3.15) *holds with strict inequality.*

PROOF. In this case $\nu = \frac{9}{4}$, so that $\lambda = (d + 2)/a$.

By Lemma 6, we can find $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$-\left(|D|/3a^4\right)^{1/3} < \psi(z, t, u) \leqslant \frac{8}{9}\left(|D|/3a^4\right)^{1/3} = \frac{8}{9}\left(d^5/24a^4\right)^{1/3}.$$

Then (3.17) will hold with strict inequality if we have

$$(3.18)\qquad \frac{8}{9}\left(d^5/24a^4\right)^{1/3} < (4d - 1 - a)/4a$$

and

$$(3.19)\qquad \left(d^5/24a^4\right)^{1/3} < \begin{cases} (\lambda - 1)^2/4 + \dfrac{9}{4}a & \text{if } \lambda \text{ is an integer}, \\ [\lambda]^2/4 + \dfrac{9}{4}a & \text{if } \lambda \text{ is not an integer}. \end{cases}$$

A simple calculation yields the inequality (3.18). So we proceed to verify (3.19). Let $n < \lambda = (d + 2)/a \leqslant n + 1$, $n = 1, 2, 3, \ldots$. Then (3.19) will be satisfied if we have

(3.20) $$d^5/24 < a\left(\frac{9}{4} + n^2a/4\right)^3 = g(a) \quad \text{(say)}.$$

Since $a \geqslant (d+2)/n + 1$, we have

$$g(a) \geqslant g((d+2)/(n+1)) = (d+2)(n+1)^{-4}4^{-3}\{9(n+1) + n^2(d+2)\}^3.$$

We shall have (3.20) if

$$h(d) = d^5 4^3 g((d+2)/(n+1)) > \frac{8}{3}.$$

For fixed n, $h(d)$ is a decreasing function of d and $d \leqslant 4$, therefore

$$h(d) \geqslant h(4) = 6 \cdot 27 \cdot 4^{-5}(n+1)^{-4}\{3(n+1) + 2n^2\}^3 \geqslant \frac{81}{16}$$

because $n \geqslant 1$. This proves (3.20) and hence (3.19).

LEMMA 17. *If* $2 < d \leqslant 3$, *then again* (3.17) *and hence* (3.15) *is satisfied with strict inequality.*

PROOF. In this case $\nu = 1$, so that $\lambda = (3 + 4d)/4a$. Let $n < (3+4d)/4a \leqslant n + 1$, $n = 1, 2, \ldots$. In view of Lemma 15, it is enough to prove that there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

(3.21) $$-(n^2/4 + 1/a) < \psi(z, t, z) < (4d - 1 - a)/4a.$$

Case (I). $n \geqslant 2$. By Lemma 5, there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$|\psi(z, t, u)| \leqslant (27|D|/100a^4)^{1/3} = (27d^5/800a^4)^{1/3}.$$

Then (3.21) will hold if we have

(3.22) $$(27d^5/800a^4)^{1/3} < (4d - 1 - a)/4a$$

and

(3.23) $$(27d^5/800a^4)^{1/3} < n^2/4 + 1/a.$$

We omit the straightforward verification of these inequalities.

Case (II). $n = 1$ that is $1 < (3 + 4d)/4a = \lambda \leqslant 2$. By Lemma 6, with $c = \frac{1}{2}$, we can find $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$-\frac{1}{2}(32d^5/429a^4)^{1/3} < \psi(z, t, u) \leqslant (32d^5/429a^4)^{1/3}.$$

Then (3.21) will hold if we have

(3.24) $$(32d^5/429a^4)^{1/3} < (4d - 1 - a)/4a$$

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and

(3.25) $$(32d^5/429a^4)^{1/3} < 2(n^2/4 + 1/a) = (a+4)/2a.$$

Since $(4d-1-a)/4a < (a+4)/2a$ for $a \geqslant (3+4d)/8$ and $d \leqslant 4$, it is enough to verify (3.24), which can be easily done.

LEMMA 18. *If* $1 < d \leqslant 2$, *then again* (3.17) *and hence* (3.15) *is true.*

PROOF. In this case $\nu = \frac{1}{4}$, so that $\lambda = d/a$. Also from (3.13), $\lambda = d/a \leqslant 8/(3-4\varepsilon) < 3$, on taking ε sufficiently small. We distinguish two cases:

Case (i). $2 < \lambda < 3$. In this case $[\lambda]^2/4 + \nu/a = (1+4a)/4a$. So we have to prove that there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

(3.26) $$-(1+4a)/4a < \psi(z,t,u) \leqslant (4d-1-a)/4a.$$

By Lemma 6, with $c = \frac{1}{2}$, there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$-\frac{1}{2}(32d^5/429a^4)^{1/3} < \psi(z,t,u) \leqslant (32d^5/429a^4)^{1/3}.$$

Then (3.26) will hold with strict inequality if

$$(32d^5/429a^4)^{1/3} < \min\left(\frac{4d-1-a}{4a}, \frac{1+4a}{2a}\right) = (4d-1-a)/4a.$$

This will be so if and only if

$$g(a) = a(d-(1+a)/4)^3 > 32d^5/429.$$

$g(a)$ is an increasing function of a for $d/3 < a < d/2$, therefore

$$g(a) \geqslant g(d/3) = \frac{1}{3}d\{d-(1+d/3)/4\}^3 > 32d^5/429$$

if $h(d) = (11d-3)^3 d^{-4} > 12^3 \cdot 32/143$, which is true for $1 < d \leqslant 2$.

Case (ii). $1 < \lambda \leqslant 2$. In this case $[\lambda]^2/4 + \nu/a = (1+a)/4a$. By Lemma 15, it is enough to prove that there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

(3.27) $$-(1+a)/4a < \psi(z,t,u) \leqslant (4d-1-a)/4a.$$

By Lemma 6, with $c = \frac{1}{3}$, there exist $(z, t, u) \equiv (z_0, t_0, u_0) \pmod 1$ such that

$$-\frac{1}{3}(27|D|/32a^4)^{1/3} < \psi(z,t,u) \leqslant (27|D|/32a^4)^{1/3}.$$

Then (3.27) will follow if we have

$$(27|D|/32a^4)^{1/3} = (27d^5/256a^4)^{1/3} \leqslant \min((4d-1-a)/4a, 3(1+a)/4a).$$

Now $(4d-1-a)/4a \leqslant 3(1+a)/4a$ if and only if $d \leqslant 1+a$, which is true. (Strict inequality holds unless $d = 2$, $a = d/2 = 1$.) So it is enough to verify that

(3.28) $$(27d^5/256a^4)^{1/3} \leqslant (4d-1-a)/4a.$$

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We shall have (3.28) if and only if

(3.29) $$g(a) = a(d - (1 + a)/4)^3 > 27d^5/256.$$

$g(a)$ increases or decreases according as $a < d - \frac{1}{4}$ or $a > d - \frac{1}{4}$ and since $d/2 < d - \frac{1}{4}$, $d/2 \leqslant a < (\frac{1}{2}d^5)^{1/4}$, (3.29) will be true if

$$\min\left\{ g(d/2), g\left((d^5/2)^{1/4}\right)\right\} > 27d^5/256.$$

Now $g(d/2) = d(7d-2)^3/1024 \geqslant 27d^5/256$ if $f(d) = (7d-2)^3d^{-4} > 108$. $f(d)$ increases or decreases according as $d < \frac{8}{7}$ or $d > \frac{8}{7}$. Therefore

$$f(d) \geqslant \min(f(1), f(2)) = f(2) = 108,$$

and strict inequality holds unless $d = 2$. The inequality $g((d^5/2)^{1/4}) > 27d^5/256$ can be easily verified.

Therefore (3.29) is satisfied with strict inequality unless $d = 2$, $a = d/2 = 1$. Hence (3.27) is true. Equality holds in (3.27) only if $d = 2$, $a = 1$, and ψ, z_0, t_0, u_0 are such that equality is needed in (2.6).

This completes the proof of Lemma 18.

4. The case of equality

LEMMA 19. *For $d > 1$, the sign of equality in* (3.7) *is necessary if and only if $Q \sim Q_2$. For Q_2, it is so if and only if $(x_0, y_0, z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$* (mod 1).

PROOF. Equality can hold in (3.7) only if it holds in (3.15). This happens only if $d = 2$, $a = 1$ and ψ, z_0, t_0, u_0 are such that equality is necessary in (2.6) (see Lemma 18). Thus we must have $\psi \sim \rho\psi_1 = \rho(z^2 + t^2 - 4u^2)$, $\rho > 0$ and $(z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (mod 1). Then $4\rho^3 = |D|/a^4 = d^5/8a^4 = 4$ so that $\rho = 1$. Therefore $\varphi(y, z, t, u) = (y + fz + f't + f''u)^2 + z^2 + t^2 - 4u^2$.

By a suitable unimodular transformation we can suppose that

(4.1) $$|f| \leqslant \frac{1}{2}, \qquad |f'| \leqslant \frac{1}{2}, \qquad |f''| \leqslant \frac{1}{2}.$$

Again for equality to occur in (3.15), the following inequality

$$-\frac{1}{4} < F(y, z, t, u) = \left(y + y_0 + f\left(z + \frac{1}{2}\right) + f'\left(t + \frac{1}{2}\right) + f''\left(u + \frac{1}{2}\right)\right)^2$$
$$+ \left(z + \frac{1}{2}\right)^2 + \left(t + \frac{1}{2}\right)^2 - 4\left(u + \frac{1}{2}\right)^2 < d - \frac{1}{4} = \frac{7}{4}$$

should not have any solution in integers y, z, t, and u. Now

$$-\frac{1}{4} < F(y, 0, 0, 0) = \left(y + y_0 + \frac{f}{2} + \frac{f'}{2} + \frac{f''}{2}\right)^2 + \frac{1}{4} + \frac{1}{4} - 1 < \frac{7}{4}$$

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is solvable for integer y unless

$$y_0 + f/2 + f'/2 + f''/2 \equiv \frac{1}{2} \pmod 1. \tag{4.2}$$

Similarly considering $F(y, -1, 0, 0)$, $F(y, 0, -1, 0)$ and $F(y, 0, 0, -1)$ for equality to occur we must have

$$y_0 - f/2 + f'/2 + f''/2 \equiv \frac{1}{2} \pmod 1, \tag{4.3}$$

$$y_0 + f/2 - f'/2 + f''/2 \equiv \frac{1}{2} \pmod 1, \tag{4.4}$$

$$y_0 + f/2 + f'/2 - f''/2 \equiv \frac{1}{2} \pmod 1. \tag{4.5}$$

Subtracting (4.3), (4.4) and (4.5) from (4.2) we get

$$f \equiv f' \equiv f'' \equiv 0 \pmod 1.$$

Then from (4.1) we have

$$f = f' = f'' = 0, \qquad y_0 \equiv \frac{1}{2} \pmod 1.$$

Therefore $\varphi(y, z, t, u) = y^2 + z^2 + t^2 - 4u^2$, and $(y_0, z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pmod 1$. Hence

$$Q(x, y, z, t, u) = (x + hy + gz + h't + g'u)^2 + y^2 + z^2 + t^2 - 4u^2.$$

Again if equality is necessary in (3.7), the following inequality

$$0 < Q\left(x + x_0, y + \frac{1}{2}, z + \frac{1}{2}, t + \frac{1}{2}, u + \frac{1}{2}\right) < d = 2$$

should not have any solution in integers x, y, z, t, u. Proceeding as above, one can see that this is solvable unless

$$h \equiv g \equiv h' \equiv 0 \pmod 1.$$

Since $|h| < \frac{1}{2}$, $|g| < \frac{1}{2}$, $|h'| < \frac{1}{2}$, $|g'| < \frac{1}{2}$ from (3.4), we must have

$$h = g = h' = g' = 0 \quad \text{and} \quad x_0 \equiv \frac{1}{2} \pmod 1.$$

Hence $Q(x, y, z, t, u) = x^2 + y^2 + z^2 + t^2 - 4u^2$ and $(x_0, y_0, z_0, t_0, u_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pmod 1$. Considering congruences modulo 8, one can see that the sign of equality is necessary in this case.

The proof of Theorem A follows from Lemmas 10–19, and thus our theorem is proved.

https://doi.org/10.1017/S1446788700033450 Published online by Cambridge University Press

Acknowledgements

The authors are very grateful to Professor R. P. Bambah and Professor V. C. Dumir for many useful discussions during the preparation of this paper.

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