ON THE UNIQUENESS OF THE GREEN'S FUNCTION ASSOCIATED WITH A SECOND-ORDER DIFFERENTIAL EQUATION

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SUMMARY

The Green's function $G(x, \xi, \lambda)$ associated with the differential equation

$$\frac{d^2\phi}{dx^2} + \{\lambda - q(x)\}\,\phi = 0 \qquad (x \geqq 0)$$

is of importance in the theory of the expansion of an arbitrary function in terms of the solutions of the differential equation. It is proved that this function is unique if $q(x) \geqq -Ax^2 - B$, where A and B are positive constants or zero. A similar theorem is proved for the Green's function $G(x, y, \xi, \eta, \lambda)$ associated with the partial differential equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \{\lambda - q(x, y)\}\,\phi = 0.$$

1. The problem of expanding an arbitrary function in terms of solutions of the differential equation

$$\frac{d^2\phi}{dx^2} + \{\lambda - q(x)\}\,\phi = 0 \tag{1.1}$$

depends on the properties of a function $G(x, \xi, \lambda)$ called the Green's function.[1] We shall consider here the case in which the interval of values of x is $(0, \infty)$, and where the solutions $\phi(x, \lambda)$ of (1.1) to be used in the expansion satisfy the boundary condition

$$\phi(0, \lambda)\cos\alpha + \phi_x(0, \lambda)\sin\alpha = 0, \tag{1.2}$$

suffixes denoting partial differentiations. In this case the Green's function has the following properties. Let ξ be a given positive number, λ a complex number whose imaginary part is not zero. Then if $x \neq \xi$, $G(x, \xi, \lambda)$ has continuous partial derivatives with respect to x up to the second order, and

$$\frac{\partial^2 G}{\partial x^2} + \{\lambda - q(x)\}\,G = 0. \tag{1.3}$$

Also

$$G(0, \xi, \lambda)\cos\alpha + G_x(0, \xi, \lambda)\sin\alpha = 0\,; \tag{1.4}$$

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[1]See D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Zweiter Abschnitt (Berlin, 1924); R. Courant and D. Hilbert, *Methoden der math. Physik* I, (Berlin, 1931), V, § 14; E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-order Differential Equations* (Oxford, 1946), p. 29; P. Hartman and A. Wintner, "A Criterion for the Non-degeneracy of the Wave Equation," *Amer. J. Math.*, vol. 71 (1949), 206-213.

$$\int_0^\infty |G(x, \xi, \lambda)|^2 \, dx \tag{1.5}$$

is convergent; and $G(x, \xi, \lambda)$ is continuous at $x = \xi$, while[2]

$$G_x(\xi + 0, \xi, \lambda) - G_x(\xi - 0, \xi, \lambda) = -1. \tag{1.6}$$

The Green's function is also symmetrical in x and ξ.

If $\phi(x, \lambda)$ is a solution of (1.1) satisfying (1.2), and $\psi(x, \lambda)$ is a solution of integrable square over $(0, \infty)$, and the Wronskian of ϕ and ψ is 1, then the above conditions are satisfied by the function

$$\begin{aligned} G(x, \xi, \lambda) &= -\phi(x, \lambda)\psi(\xi, \lambda) \qquad (x \leqq \xi), \\ &\quad -\psi(x, \lambda)\phi(\xi, \lambda) \qquad (x > \xi). \end{aligned}$$

It was shown by Weyl[3] that there are two different cases. In one case, called the limit-point case, there is only one $\psi(x, \lambda)$ with the required properties for each λ, and the Green's function is then unique. In the other case, called the limit-circle case, all solutions of (1.1) are of integrable square over $(0, \infty)$, and the Green's function depends on an arbitrary parameter.

The object of the present paper is to consider how the uniqueness of the Green's function depends on the nature of $q(x)$. I consider also the corresponding problem for functions of two variables, in which (1.1) is replaced by

$$\nabla^2\phi + \{\lambda - q(x, y)\} \phi = 0, \tag{1.7}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The extension to more than two variables seems to be immediate. No doubt more general differential equations could also be dealt with in a similar way.

2. For functions of one variable, we shall prove

THEOREM 1. *If $q(x) \geqq -Ax^2 - B$, where A and B are positive constants (or zero), then the Green's function is unique.*

Thus in particular the result holds if $q(x)$ is bounded below.

Suppose that two functions $G_1(x, \xi, \lambda)$ and $G_2(x, \xi, \lambda)$ satisfy the above conditions, and let

$$g(x, \lambda) = G_1(x, \xi, \lambda) - G_2(x, \xi, \lambda),$$

ξ now being fixed. Then $g(x, \lambda)$ also satisfies the same conditions, except that $g_x(x, \lambda)$ is continuous at $x = \xi$. Hence

$$\begin{aligned} &\int_0^X \{g(x, \lambda)g_{xx}(x, \mu) - g(x, \mu)g_{xx}(x, \lambda)\} \, dx \\ &= [g(x, \lambda)g_x(x, \mu) - g(x, \mu)g_x(x, \lambda)]_0^X \\ &= g(X, \lambda)g_x(X, \mu) - g(X, \mu)g_x(X, \lambda). \end{aligned}$$

[2]Different constant factors are attached to the Green's function by different writers. The definition adopted here is that of Courant and Hilbert. The function in my book had the opposite sign.

[3]See chap. II of my book.

The left-hand side is also equal to

$$\int_0^X [g(x,\lambda)\,\{q(x)-\mu\}\,g(x,\mu) - g(x,\mu)\,\{q(x)-\lambda\}\,g(x,\lambda)]\,dx$$
$$= (\lambda-\mu)\int_0^X g(x,\lambda)g(x,\mu)\,dx.$$

Taking $\mu = \bar{\lambda}$ (the conjugate of λ), and writing $g(x,\lambda) = g(x)$, $g_x(x,\lambda) = g'(x)$, $g(x,\bar{\lambda}) = \bar{g}(x)$, we obtain

$$\Im(\lambda)\int_0^X |g(x)|^2\,dx = \Im\,\{g(X)\bar{g}'(X)\}. \tag{2.1}$$

If the right-hand side tended to zero as $X \to \infty$, we should have

$$\int_0^\infty |g(x)|^2\,dx = 0.$$

Hence we should have $g(x) = 0$ for all values of x, and the uniqueness of the Green's function would follow.

To prove the theorem stated, integrate (2.1) with respect to X over $(0, T)$ and divide by $T\Im(\lambda)$. We obtain

$$\int_0^T \left(1-\frac{x}{T}\right)|g(x)|^2\,dx = \frac{1}{T\Im(\lambda)}\int_0^T \Im\,\{g(x)\bar{g}'(x)\}\,dx$$
$$\leqq \frac{1}{T|\Im(\lambda)|}\int_0^T |g(x)g'(x)|\,dx$$
$$\leqq \frac{1}{T|\Im(\lambda)|}\left\{\int_0^T |g(x)|^2\,dx\int_0^T |g'(x)|^2\,dx\right\}^{\frac{1}{2}}$$
$$= O\left[\frac{1}{T}\left\{\int_0^T |g'(x)|^2\,dx\right\}^{\frac{1}{2}}\right] \tag{2.2}$$

as $T \to \infty$, λ remaining fixed, since $g(x)$ is of integrable square over $(0, \infty)$. The result will now follow if

$$\int_0^T |g'(x)|^2\,dx = o(T^2) \tag{2.3}$$

as $T \to \infty$, at least through some sequence of values; for then

$$\int_0^{\frac{1}{2}T} |g(x)|^2\,dx \leqq 2\int_0^T \left(1-\frac{x}{T}\right)|g(x)|^2\,dx \to 0,$$

and hence $g(x) = 0$ as before.

Now

$$\int_0^T |g'(x)|^2\,dx \leqq 2\int_0^{2T}\left(1-\frac{x}{2T}\right)|g'(x)|^2\,dx,$$

and

$$\int_0^{2T}\left(1-\frac{x}{2T}\right)|g'(x)|^2\,dx = \int_0^{2T}\left(1-\frac{x}{2T}\right)g'(x)\bar{g}'(x)\,dx$$

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$$= \left[\left(1 - \frac{x}{2T}\right) g(x)\bar{g}'(x) \right]_0^{2T} - \int_0^{2T} \left(1 - \frac{x}{2T}\right) g(x)\bar{g}''(x)\, dx$$
$$+ \frac{1}{2T}\int_0^{2T} g(x)\bar{g}'(x)\, dx$$
$$= -g(0)\bar{g}'(0) - \int_0^{2T} \left(1 - \frac{x}{2T}\right) g(x) \left\{q(x) - \bar{\lambda}\right\} \bar{g}(x)\, dx$$
$$+ \frac{1}{2T}\int_0^{2T} g(x)\bar{g}'(x)\, dx.$$

Taking real parts, we obtain

$$\int_0^{2T} \left(1 - \frac{x}{2T}\right) |g'(x)|^2 dx = -\,\Re\left\{g(0)\bar{g}'(0)\right\}$$
$$- \int_0^{2T} \left(1 - \frac{x}{2T}\right) |g(x)|^2 \left\{q(x) - \Re\,(\lambda)\right\} dx + \frac{1}{4T}\int_0^{2T} \left\{g(x)\bar{g}'(x) + \bar{g}(x)g'(x)\right\} dx$$
$$= -\,\Re\left\{g(0)\bar{g}'(0)\right\} - \int_0^{2T} \left(1 - \frac{x}{2T}\right) |g(x)|^2 \left\{q(x) - \Re(\lambda)\right\} dx + \frac{|g(2T)|^2 - |g(0)|^2}{4T}.$$

Hence

$$\int_0^{T} |g'(x)|^2\, dx \leqq -\,2\Re\left\{g(0)\bar{g}'(0)\right\}$$
$$+\, 2\int_0^{2T} |g(x)|^2 \left\{Ax^2 + B + |\Re(\lambda)|\right\} dx + \frac{|g(2T)|^2}{2T}.$$

Writing the integral on the right-hand side as

$$\int_0^{\sqrt{T}} + \int_{\sqrt{T}}^{2T}$$

it is seen to be of the form

$$O\left\{T\int_0^{\sqrt{T}} |g(x)|^2\, dx\right\} + O\left\{T^2\int_{\sqrt{T}}^{2T} |g(x)|^2\, dx\right\} = o(T^2).$$

Finally

$$|g(2T)| < K$$

for some K and some arbitrarily large values of T, since otherwise $g(x)$ could not be of integrable square. Hence (2.3) holds for some arbitrarily large values of T, and the uniqueness result follows.

3. It is not clear whether the above result is the best possible of its kind. But it is not far from being so, since the result fails if we replace the x^2 of the theorem by $x^{2+\delta}$, where δ is any positive number. This follows from the analysis of § 5.8 of my book, *Eigenfunction Expansions*, where it is shown that the limit-circle case holds if $q(x) \leqq 0$, $q'(x) < 0$, $q(x) \to -\infty$, $q'(x) = O\left\{|q(x)|^c\right\}$, $0 < c < \frac{3}{2}$, $q''(x)$ is ultimately of one sign, and

$$\int^{\infty} |q(x)|^{-\frac{1}{2}}\, dx$$

is convergent. These conditions are satisfied if $q(x) = -\,x^{2+\delta} - x$.

4. We shall next consider the corresponding problem for functions of two variables, the region considered being the whole (x, y) plane. In this case the Green's function $G(x, y, \xi, \eta, \lambda)$ satisfies the following conditions. Let ξ and η be given real numbers, λ a complex number whose imaginary part does not vanish. Then G has continuous partial derivatives up to the second order with respect to x and y, and

$$\nabla^2 G + \{\lambda - q(x, y)\} G = 0, \tag{4.1}$$

except at $x = \xi$, $y = \eta$. The integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x, y, \xi, \eta, \lambda)|^2 \, dxdy \tag{4.2}$$

is convergent. In the neighbourhood of $x = \xi$, $y = \eta$,

$$G(x, y, \xi, \eta, \lambda) = \frac{1}{2\pi}\log\frac{1}{\rho} + O(1), \tag{4.3}$$

$$\frac{\partial G}{\partial \rho} = -\frac{1}{2\pi\rho} + O(1), \tag{4.4}$$

where

$$\rho = \sqrt{\{(x-\xi)^2 + (y-\eta)^2\}}.$$

It can be shown that such a function exists, at any rate if $q(x, y)$ is continuous and has continuous partial derivatives of the first order.[4]

THEOREM 2. *If $q(x, y) \geqq -Ar^2 - B$, where $r = \sqrt{(x^2 + y^2)}$, and A and B are positive constants (or zero), then the Green's function is unique.*

Suppose that two functions $G_1(x, y, \xi, \eta, \lambda)$ and $G_2(x, y, \xi, \eta, \lambda)$ satisfy the above conditions, and let

$$g(x, y, \lambda) = G_1(x, y, \xi, \eta, \lambda) - G_2(x, y, \xi, \eta, \lambda),$$

ξ and η now being fixed. Then $g(x, y, \lambda)$ satisfies the same conditions, except that g and $\partial g/\partial\rho$ are bounded as $\rho \to 0$.

If r, θ, are the polar coordinates corresponding to x, y, we write

$$g(x, y, \lambda) = g[r, \theta, \lambda].$$

Then[5]

$$\iint_{r \leqq R} \{g(x, y, \lambda)\nabla^2 g(x, y, \mu) - g(x, y, \mu)\nabla^2 g(x, y, \lambda)\}\, dxdy$$

$$= \int_0^{2\pi}\left\{g[R, \theta, \lambda]\frac{\partial}{\partial R}g[R, \theta, \mu] - g[R, \theta, \mu]\frac{\partial}{\partial R}g[R, \theta, \lambda]\right\} R\,d\theta.$$

The left-hand side is also equal to

$$\iint_{r \leqq R} [g(x, y, \lambda)\{q(x, y) - \mu\} g(x, y, \mu) - g(x, y, \mu)\{q(x, y) - \lambda\} g(x, y, \lambda)]\, dxdy$$

$$= (\lambda - \mu)\iint_{r \leqq R} g(x, y, \lambda)\, g(x, y, \mu)\, dxdy.$$

[4]See a forthcoming paper in the *Proc. London Math. Soc.*

[5]See e.g. E. Goursat, *Cours d'Analyse mathématique*, vol. 3 (Paris, 1927), § 506.

Taking $\mu = \bar{\lambda}$, we obtain

$$\Im(\lambda) \iint_{r \leqq R} |g(x, y, \lambda)|^2 \, dxdy = \int_0^{2\pi} \Im \left\{ g[R, \theta, \lambda] \frac{\partial}{\partial R} g[R, \theta, \bar{\lambda}] \right\} R d\theta .$$

Now integrate with respect to R over $(0, T)$ and divide by $T\Im(\lambda)$. Writing $g[r, \theta, \lambda] = g, \dfrac{\partial g}{\partial r} = g_r$, etc., we have

$$\iint_{r \leqq T} \left(1 - \frac{r}{T}\right) |g|^2 r dr d\theta = \frac{1}{T\Im(\lambda)} \int_0^T \int_0^{2\pi} \Im(g \bar{g}_r) \, r dr d\theta$$

$$\leqq \frac{1}{T|\Im(\lambda)|} \left\{ \int_0^T \int_0^{2\pi} |g|^2 r dr d\theta \int_0^T \int_0^{2\pi} |g_r|^2 \, r dr d\theta \right\}^{\frac{1}{2}}$$

$$= O\left[\frac{1}{T} \left\{ \int_0^T \int_0^{2\pi} |g_r|^2 \, r dr d\theta \right\}^{\frac{1}{2}} \right],$$

as $T \to \infty$. It is therefore sufficient to prove that

$$\int_0^T \int_0^{2\pi} |g_r|^2 \, r dr d\theta = o(T^2) \tag{4.5}$$

as $T \to \infty$ through some sequence of values; for then

$$\iint_{r \leqq \frac{1}{2} T} |g|^2 \, r dr d\theta \leqq 2 \iint_{r \leqq T} \left(1 - \frac{r}{T}\right) |g|^2 \, r dr d\theta \to 0,$$

and hence $g = 0$ for all values of r and θ.

Now

$$\int_0^T \int_0^{2\pi} |g_r|^2 \, r dr d\theta \leqq \int_0^T \int_0^{2\pi} \left(|g_r|^2 + \frac{1}{r^2} |g_\theta|^2 \right) r dr d\theta$$

$$\leqq 2 \int_0^{2T} \left(1 - \frac{r}{2T}\right) r dr \int_0^{2\pi} \left(g_r \bar{g}_r + \frac{1}{r^2} g_\theta \bar{g}_\theta \right) d\theta$$

$$= 2 \iint_{r \leqq 2T} \left(1 - \frac{r}{2T}\right) (g_x \bar{g}_x + g_y \bar{g}_y) \, dxdy .$$

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By another known formula[6]

$$\iint_{r \leqq 2T} \left(1 - \frac{r}{2T}\right) g \nabla^2 \bar{g}\, dxdy$$

$$= -\iint_{r \leqq 2T} \left[\frac{\partial}{\partial x}\left\{\left(1 - \frac{r}{2T}\right) g\right\} \bar{g}_x + \frac{\partial}{\partial y}\left\{\left(1 - \frac{r}{2T}\right) g\right\} \bar{g}_y\right] dxdy$$

$$= -\iint_{r \leqq 2T} \left(1 - \frac{r}{2T}\right) (g_x \bar{g}_x + g_y \bar{g}_y)\, dxdy$$

$$+ \frac{1}{2T} \iint_{r \leqq 2T} \left(\frac{\partial r}{\partial x} \bar{g}_x + \frac{\partial r}{\partial y} \bar{g}_y\right) g dxdy$$

$$= -\iint_{r \leqq 2T} \left(1 - \frac{r}{2T}\right) (g_x \bar{g}_x + g_y \bar{g}_y)\, dxdy + \frac{1}{2T} \iint_{r \leqq 2T} g \bar{g}_r\, r dr d\theta,$$

since

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{\partial x}{\partial r},$$

etc. Hence

$$\iint_{r \leqq 2T} \left(1 - \frac{r}{2T}\right) (g_x \bar{g}_x + g_y \bar{g}_y)\, dxdy$$

$$= -\iint_{r \leqq 2T} \left(1 - \frac{r}{2T}\right) g \nabla^2 \bar{g} . r dr d\theta + \frac{1}{2T} \iint_{r \leqq 2T} g \bar{g}_r r dr d\theta$$

$$= -\iint_{r \leqq 2T} \left(1 - \frac{r}{2T}\right) g\, (q - \bar{\lambda})\, \bar{g} . r dr d\theta + \frac{1}{2T} \iint_{r \leqq 2T} g \bar{g}_r r dr d\theta.$$

Taking real parts, we obtain

$$\iint_{r \leqq 2T} \left(1 - \frac{r}{2T}\right) (g_x \bar{g}_x + g_y \bar{g}_y)\, dxdy$$

$$= -\iint_{r \leqq 2T} \left(1 - \frac{r}{2T}\right) |g|^2 \{q - \Re(\lambda)\}\, r dr d\theta + \frac{1}{2T} \iint_{r \leqq 2T} (g \bar{g}_r + \bar{g} g_r)\, r dr d\theta.$$

Also

$$\int_{r \leqq 2T} (g \bar{g}_r + \bar{g} g_r)\, r dr = [|g|^2 r]_0^{2T} - \int_0^{2T} |g|^2\, dr \leqq 2T\, |g\, [2T, \theta]|^2.$$

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[6]It can be verified at once by integration by parts.

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Altogether, we obtain

$$\int_0^T\int_0^{2\pi} |g_r|^2\, rdrd\theta \leqq -2\iint\limits_{r \leqq 2T}\left(1-\frac{r}{2T}\right)|g|^2\{q - \Re(\lambda)\}\, rdrd\theta + 2\int_0^{2\pi}|g[2T,\theta]|^2\, d\theta.$$

Using the given condition, it follows that

$$\int_0^T\int_0^{2\pi} |g_r|^2\, rdrd\theta \leqq 2\iint\limits_{r \leqq 2T}|g|^2\{Ar^2 + B + |\Re(\lambda)|\}\, rdrd\theta + 2\int_0^{2\pi}|g[2T,\theta]|^2\, d\theta\,.$$

The first integral on the right is

$$O\Big\{T\iint\limits_{r \leqq \sqrt{T}}|g|^2\, rdrd\theta\Big\} + O\Big\{T^2\iint\limits_{\sqrt{T} < r \leqq 2T}|g|^2\, rdrd\theta\Big\} = o(T^2)\,.$$

Also

$$\int_0^{2\pi}|g[2T,\theta]|^2\, d\theta < K$$

for some K and some arbitrarily large values of T, since otherwise

$$\int_0^R rdr\int_0^{2\pi}|g[r,\theta]|^2\, d\theta > K'R^2\,.$$

The theorem therefore follows as before.

It seems likely that this result also is not far from the best possible, but in the case of functions of two or more variables the different possibilities have not been explored in sufficient detail to say anything definite on this point.

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