COMPLETELY RIGID GRAPHS

J. S. V. SYMONS

(Received 25 October 1972)

Communicated by G. B. Preston

Introduction

In general, a structure is called rigid if it admits only the trivial structure preserving transformations. Of course, what is trivial depends on the context. In [7] the authors understand by a rigid graph one which has the property that the only edge preserving transformation of the vertices is the identity map. In other contexts, however, it is convenient to regard the constant maps as trivial also. (See [2] where a topological space is given which admits as continuous transformations only the identity and the constants.) The purpose of this note is to construct graphs rigid in the later sense; we call them completely rigid.

Completely rigid graphs yield a supply of rigid graphs since any completely rigid graph can be modified to a rigid graph (by deleting a certain subgraph). However the concerse is not true (see example below). Another distinction between the two concepts is that completely rigid graphs on a finite number of vertices are by no means easy to construct, whereas, for rigid graphs, a well order suffices. They do however possess a stronger connectedness property than those constructed in [7], and this enables us to give, in a manner similar to [5], a new representation theorem for semigroups with identity.

1. Definitions, simple results

Let X be a non-empty set. A (directed) graph Γ on X is any subset of $X \times X$. The elements of Γ are called *edges* and the elements of X *vertices* of Γ. We set $\Gamma^{-1} = \{(y,x);(x,y) \in \Gamma\}$. Then Γ is *symmetric* if $\Gamma = \Gamma^{-1}$, and *antisymmetric* if $(x, y) \in \Gamma \cap \Gamma^{-1}$ implies $x = y$. Let $\mathscr{T}(X), \mathscr{K}(X)$, and $\mathscr{K}^1(X)$ be the full transformation semigroup on X, the semigroup of all constant transformations of X, and this latter with the identity transformation adjoined, respectively. By End$(\Gamma; X)$ (or End Γ for short) we shall understand the set of endomorphisms of Γ, namely

$$\{\alpha \in \mathscr{T}(X); (x, y) \in \Gamma \Rightarrow (x\alpha, y\alpha) \in \Gamma\}.$$

Clearly End Γ is a semigroup of $\mathscr{T}(X)$ under composition. We shall use the symbol ι to denote both the identity of $\mathscr{T}(X)$ and the graph $\{(x,x); x \in X\}$. It is customary to call the elements of $\Gamma \cap \iota$ *loops*.

In [7] a *rigid* graph Γ is defined by the property End $\Gamma = \{\iota\}$. We shall call Γ *completely rigid* if End $\Gamma = \mathscr{K}^1(X)$. This is a convenient terminology for if Γ is completely rigid then $\Gamma \supseteq \iota$ and $\Gamma \backslash \iota$ is rigid. Complete rigidity is, in fact, stronger than rigidity.

EXAMPLE. Set $X = \{1, 2, \cdots, n\}$ and put

$$\Gamma = \{(r,s) \in X \times X; \, r \geqq s\}.$$

If $n \geqq 3$, End Γ contains mappings other than the identity or the constants, while $\Gamma \backslash \iota$ is obviously rigid.

It is easy to see that if Γ is completely rigid then Γ is antisymmetric. For if $x \neq y$ and $(x,y) \in \Gamma \cap \Gamma^{-1}$ then we may map X onto $\{x,y\}$ in any way we please and still preserve the edges of Γ. We shall prove a deeper property of completely rigid graphs.

Firstly, let a and b be vertices of $\Gamma \subseteq X \times X$. If there exists a sequence of vertices of Γ, $a = a_0, a_1, \cdots, a_n = b$, such that each $(a_i, a_{i+1}) \in \Gamma$, we say that b is *accessible* from a. We write $D(a)$ for the set of vertices accessible from a, together with a itself. If a and b are each accessible from the other then we say that a and b are *mutually accessible.* This is equivalent to $D(a) = D(b)$. Finally a directed graph is *mutually connected* when all its vertices are mutually accessible, and *connected* when $\Gamma \cup \Gamma^{-1}$ is mutually connected. We prove

THEOREM 1. *If* $|X| > 2$ *and* $\Gamma \subseteq X \times X$ *is completely rigid then* Γ *is mutually connected.*

PROOF. Let Γ be completely rigid with $|X| > 2$. We define an equivalence, ρ, on X by writing $a\rho b$ whenever $D(a) = D(b)$. We construct a graph, Γ_1, with the elements of X/ρ as vertices, by admitting the edge (A,B) whenever there exist $a \in A$ and $b \in B$ with $(a,b) \in \Gamma$. Now Γ_1 contains no cycles (excepting loops), for this would imply that elements in distinct ρ-classes were mutually accessible. It follows that the transitive closure of Γ_1 is a partial order, $\leqq$, say. If $|X/\rho| = 1$ we have nothing to prove: we assume $|X/\rho| > 1$. If $A \in X/\rho$ is not $\leqq$ comparable to any other element of X/ρ, then, a fortiori, there are no edges in Γ joining A and $X \backslash A$. Consider a transformation of X which maps all A to some vertex in $X \backslash A$ and vice versa. This mapping clearly preserves the edges of Γ and is neither the identity nor a constant, a contradiction. We conclude that A is comparable to some other element of X/ρ, so that there exists $B, C \in X/\rho$ and an edge $(b,c) \in \Gamma$ with $b \in B$, $c \in C$ and $B > C$. We put

$$D = \bigcup_{E \geqq B} E$$

and define $\alpha \in \mathscr{T}(X)$ by

$$x\alpha = b \text{ when } x \in D, \\ = c \text{ otherwise.}$$

It is straightforward to show that $\alpha \in \operatorname{End}\Gamma$ so that $\alpha = \iota$. Hence $|X| = 2$, contrary to hypothesis.

REMARK 1. The edges of the rigid graph constructed in [7] are a subset of well order. It is clear that such a graph cannot be mutually connected: one can go forwards, but not back. Thus one does not obtain a completely rigid graph by adjoining all loops to this graph.

REMARK 2. It is also possible to prove that a completely rigid graph has, in the terminology of [4], no separating vertices and hence is strongly cyclic edge connected.

2. Completely rigid graphs

In this section we prove

THEOREM 2. There are no completely rigid graphs with 3 or 4 vertices; for sets of all other cardinalities, there exist completely rigid graphs.

PROOF. The case $|X| = 1$ is trivial. If $|X| = 2$ it is easy to see that $\Gamma = \iota \cup \{(x, y)\}$ gives the desired graph. In Figure 2.1 we have drawn all graphs Γ on 3 and 4 vertices that are mutually connected and antisymmetric. We have omitted loops, and the broken edges may be directed in either sense, or deleted.

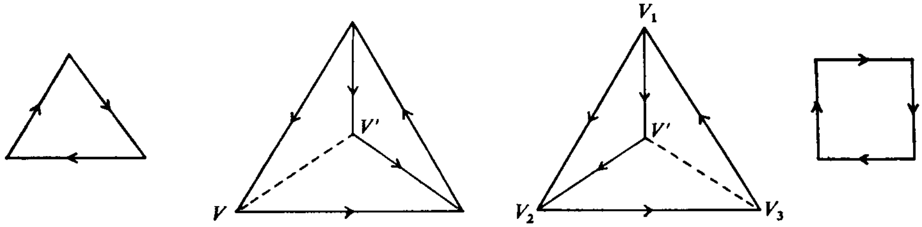

Figure 2.1.

The first and fourth graphs have obvious edge preserving transformations: the rotations. For the second graph, the trasformation which maps v' to v and fixes the other vertices preserves the edges, while in the third case we map v' to v_1 or v_2 according as edge (v', v_3) is directed towards v' or v_3. The result now follows from Theorem 1.

Now we assume X is of the form $\{a, b, c, d, e\} \cup A$ where A is an ordinal number considered as the set of its predecessors (we admit zero), Let $\mathscr{L}(A)$ and $\mathscr{E}(A)$ be the set of limit ordinals less than A, and the set of even ordinals less than A, respectively. We choose an injection

https://doi.org/10.1017/S1446788700022904 Published online by Cambridge University Press

$$f\colon \mathscr{L}(A) \to \mathscr{E}(A)\setminus\mathscr{L}(A)$$

with the property

$$f(Q) < Q \quad (Q \in \mathscr{L}(A)).$$

(The existence of such an f follows from a Zorn argument. One considers partial transformations of $\mathscr{L}(A)$ to $\mathscr{E}(A)\setminus\mathscr{L}(A)$ and orders them by extension. We omit the details. Also note than an ordinal is even if it can be represented as $2B$, odd if it can be represented as $2B+1$, and that all ordinals are even or odd but not both. In particular, limit ordinals are even. See [6]). We regard 0 as a limit ordinal and define $f(0) = d$. Our graph $\Gamma(A)$ is then the union of the following sets:

ι, J (the graph on the vertices a, b, c, d, e in Figure 2.2) together with

$$\begin{aligned} \sigma(A) &= \{(n, n+1);\, n < A\} \\ {}_{\circ}(A) &= \{(n, c);\, n < A,\, n \text{ odd}\} \\ \varepsilon(A) &= \{(n, a);\, n < A,\, n \text{ even}\} \\ \lambda(A) &= \{(L, f(L));\, L \in \mathscr{L}(A)\} \\ \tau(A) &= \{(c, L);\, L \in \mathscr{L}(A)\}. \end{aligned}$$

Two properties of $\Gamma(A)$ are fundamental:

(i) *$\Gamma(A)$ is antisymmetric.*

(ii) *The directed triangles (i.e. graphs on 3 vertices isomorphic to the first graph in Figure 2.1) either reside in J or have vertices $L, L+1, c$ where $L \in \mathscr{L}(A)$.*

Now let $\alpha \in \mathrm{End}(X; \Gamma(A))$ be neither the identity nor a constant map. We observe that J is not mapped to a single vertex. For then $c\alpha = d\alpha$ whereas $(0\alpha, d\alpha)$ and $(c\alpha, 0\alpha)$ are edges of $\Gamma(A)$. Antisymmetry implies $0\alpha = d\alpha$, and observing that each ordinal is the end-pont of an edge emanating from J or an edge emanating from a predecessor, as well as the starting-point of an edge to J, a transfinite induction shows α is constant—contrary to assumption.

Note that in general a directed triangle may not be mapped to an edge: its image is either a vertex or another directed triangle. Since the three directed triangles in J are, in pairs, adjacent or mutually adjacent to a third, it is clear that if any one is mapped to a vertex, the other two are mapped to the same vertex. Since this is prohibited, each directed triangle in J is mapped to a directed triangle. Further $(a\alpha, b\alpha)$ and $(b\alpha, c\alpha) \in \Gamma(A)$. Hence $c\alpha \neq a\alpha$ or $a\alpha = b\alpha = c\alpha$. We eliminate the latter possibility since abe is mapped to a triangle. Similarly $b\alpha \neq d\alpha$. Observe that any edge is the base of at most two directed triangles. Hence, in particular, $(e\alpha, a\alpha)$ has at most two directed triangles lying upon it, and since $b\alpha \neq d\alpha$, it has precisely two. In the same way $(d\alpha, e\alpha)$ is the base of two directed triangles. Noting that only the edges (e, a) and (d, e) have this prop-

https://doi.org/10.1017/S1446788700022904 Published online by Cambridge University Press

erty it is easy to see that $e\alpha = e$, $a\alpha = a$, and $d\alpha = d$. (Note that antisymmetry prohibits $a\alpha = d$, $d\alpha = a$). It is now immediate that $b\alpha = b$, $c\alpha = c$.

We have shown that α is the identity transformation on J. Moreover $0\alpha = 0$. For 0α begins an edge which terminates at $d\alpha = d$, so that $0\alpha = d, a, c$, or 0. If $0\alpha = c$, then $(0\alpha, a\alpha) = (c, \alpha) \in \Gamma(A)$ which is false. Similarly $0\alpha \neq a$, while if $0\alpha = d$, then $(0\alpha, a\alpha) = (d, a) \in \Gamma(A)$, again a contradiction.

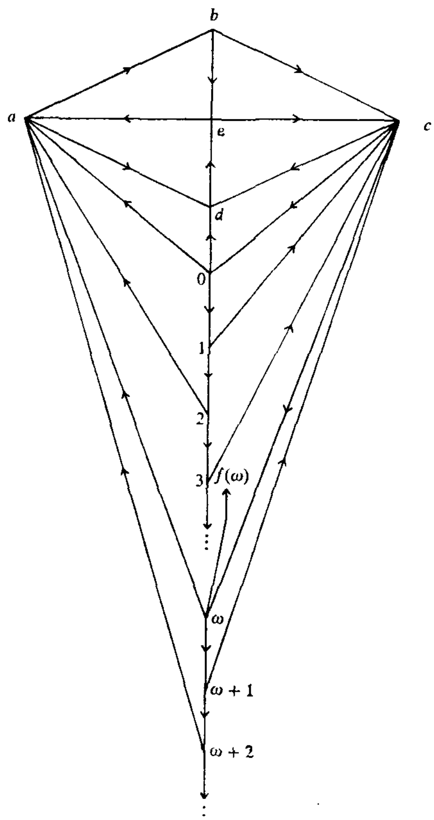

Figure 2.2. (Loops omitted)

Let B be the least ordinal $< A$ such that $B\alpha \neq B$. We distinguish two cases: *B is a limit ordinal.* Then $(B\alpha, f(B)\alpha) = (B\alpha, f(B)) \in \Gamma(A)$ and $f(B) \in \mathscr{E}(A)/\mathscr{L}(A)$. Inspection of $\Gamma(A)$ gives the possibilities $B\alpha = f(B)$, $f(B) - 1$, or L (when $f(B) = f(L)$). Moreover $(c\alpha, B\alpha) = (c, B\alpha) \in \Gamma(A)$ so that $B\alpha$ is a limit ordinal. But since neither $f(B)$ nor $f(B) - 1$ are limit ordinals, $B\alpha = L$. However, $f(B) = f(L)$ gives $B = L = B\alpha$, a contradiction.

B is not a limit ordinal. Then $(B-1, B) \in \Gamma(A)$ so that $(B-1, B\alpha) \in \Gamma(A)$. By assumption $(B-1, B\alpha) \notin \tau(A)$. If $B-1$ is even then $B\alpha = B-1, a$, or (when

https://doi.org/10.1017/S1446788700022904 Published online by Cambridge University Press

$B-1 \in \mathscr{L}(A))f(B-1)$. Since in this case B is odd, $(B\alpha, c\alpha) = (B\alpha, c) \in \Gamma(A)$. An inspection of $\Gamma(A)$ shows that $(a, c) \notin \Gamma(A)$ and, moreover, no even ordinal begins an edge which terminates at c. This observation eliminates all possibilities for $B\alpha$ when $B-1$ is even. The case $B-1$ odd is handled similarly and we conclude that there is no $B < A$ with $B\alpha \neq B$, a contradiction. (Since we have assumed $\alpha \neq \iota$.) The theorem is proved.

REMARK 1. Since $\Gamma(A)$ is completely rigid, it follows from Theorem 1 that $\Gamma(A)$ is mutually connected. To see this directly note that from any ordinal we may pass to c (perhaps via its successor). From c we may pass to any limit ordinal and it suffices to observe that any ordinal number is greater than a limit ordinal by a finite ordinal.

REMARK 2. We note that only a and c are ever incident to an infinite number of edges. Recall the following theorem from [4], page 28:

If N is an infinite cardinal and each vertex of the connected graph Γ is incident to at most N edges, then Γ has at most N vertices and N edges.

It follows that if Γ is completely rigid and $|X| > \aleph_0$ then Γ must contain at least one vertex incident to an infinite number of edges. Our graph $\Gamma(A)$ contains two. It is of interest to know whether there exists, in general, a completely rigid graph containing only one such vertex. This would be the "simplest" example possible.

3. Monoids

In this section we shall outline the result below. No proofs will be given since, in spirit at least, the constructions have been used by a number of authors. See [1], [2], [5].

THEOREM 3. *Let S be a monoid (semigroup with identity). Then there exists a mutually connected antisymmetric graph Γ with*

$$S \cong \operatorname{End}\Gamma.$$

Note. In the papers [3], [5], [7] the authors prove this result with "mutually connected antisymmetric" replaced by

(i) "antisymmetric connected", and

(ii) "symmetric connected".

CONSTRUCTIONS. We recall the notion of the Cayley graph S^* of S. We assume to each $s \in S$ three corresponds on object $c(s)$, distinct $s \in S$ giving rise to distinct $c(s)$. These objects are called colours, and we denote the set of colours by C. The vertices of S^* are the elements of S and an edge is drawn from $b \in S$ to $a \in S$ and assigned a colour $c(s)$ when $a = sb$. The resulting colured directed graph is S^*. Note that S^* may have multiple edges, but from each vertex there is precisely one edge of a given colour.

We define further Col. End. S^* to be those $a \in \mathscr{T}(S)$ which map each directed edge to a directed edge of the same colour. It is an easy matter to see $S \cong$ Col. End. S^*. We wish to have a set of $|C|$ distinct graphs $\{\gamma_i;\, i \in C\}$ with the following property

(P) For each $i \in C$ let X_i be the vertex set of γ_i. Then if $\alpha: X_i \to X_j$ maps edges to edges we have $i = j$ and $\alpha = \iota$. Let $\{A_i;\, i \in C\}$ be a collection of $|C|$ distinct ordinals, each at least three greater than a limit ordinal, and let f be the function of Theorem 2 constructed for the ordinal $\sup_i A_i$. If we now take $\gamma_i = \Gamma(A_i) \setminus \iota$ it is impossible to imbed γ_i in γ_j when $A_i < A_j$ in the obvious way. To exclude this possibility we attach another edge: we take

$$\gamma_i = \Gamma(A_i) \setminus \iota \cap \{(A_i - 3, A_i - 1)\}.$$

We claim the proof of Theorem 2 may now be adapted to prove (P). In fact, the task is easier, since we now cannot map an edge to a vertex. The vertices of the γ_i are not disjoint, but by taking $|C|$ copies of J and the class of ordinals it is clear we may ensure this. Accordingly we shall distinguish corresponding vertices of distinct γ_i by subscripts: $b_i, \omega_j, 1_k, \cdots$. We now construct our (uncoloured) graph Γ.

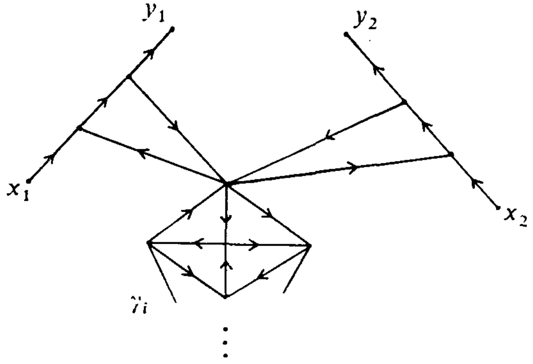

Figure 3

Along each edge (x, y) of S^* insert two vertices, orientating the new edges in the original direction of (x, y). We emphasise that edges joining the same vertices but of different colours (perforce) are to be regarded as distinct. The construction is completed by joining both of the inserted vertices to $b_i \in \gamma_i$ where i is the colour of (x, y), and orientating the two edges thus arising, so that, with their base, they form a directed triangle. In Figure 3 we illustrate the subgraph formed from edges (x_1, y_1) and (x_2, y_2), both of colour i. Then

$$\text{End}\, \Gamma \cong \text{Col. End. } S^* \cong S.$$

The proof of this is straightforward but not immediate. It is, of course, a matter

https://doi.org/10.1017/S1446788700022904 Published online by Cambridge University Press

of verifying that the inserted graphs, under endomorphisms, play the role of the colours of S^*.

By the nature of the construction, Γ is clearly antisymmetric. To see that Γ is mutually connected, note that we may pass, by a finite path, from any $a \in S$ to any $b \in S$. (We use the fact that each $a \in S^*$ has a loop of colour $c(1)$, and the directed triangles that we inserrt are all attached to $b_{c(1)}$.) It then suffices to note that all the γ_i are mutually connected.

References

[1] R. Frucht, 'Herstellung von Graphen mit vorgegebener abstrakter Gruppe', *Compos. Math.* 6 (1938), 239–250.

[2] J. De Groot, 'Groups represented by homeomorphism groups, I', *Math. Annalen* 138 (1959), 80–102.

[3] Z. Hedrlin and A. Pultr, 'Symmetric Relations (Undirected Graphs) with Given Semigroups', *Monatsh. Math.* 69, (1965), 318–322.

[4] O. Ore, *Theory of Graphs*, (Rhode Island, 1962.)

[5] A. Pultr and Z. Hedrlin, 'Relations (Graphs) with Given Infinite Semigroups,' *Monatsh. Math.* 68 (1964), 421–425.

[6] W. Sierpinski, *Cardinal and Ordinal Numbers*, (Warszawa, 1958.)

[7] P. Vopenka, A. Pultr, and Z. Hedrlin, 'A Rigid Relation Exists on Any Set', *Comment. Math. Univ. Carolinae* 6 (1965), 149–155.

Department of Mathematics
University of Western Australia
Nedlands, 6009