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THE RIEFFEL CORRESPONDENCE FOR EQUIVALENT FELL BUNDLES

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Abstract

We establish a generalized Rieffel correspondence for ideals in equivalent Fell bundles.

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1. Introduction

Morita equivalence is a fundamental tool in the study of C^*-algebras. For example, Morita equivalent C^*-algebras A and B share much of their fine structure and have equivalent representation theories. Many such properties are elucidated as the 'Rieffel correspondence' induced by an $A-B$-imprimitivity bimodule X. A summary of these properties is given in Theorem 2.1, but the key feature is that the Rieffel correspondence gives a natural lattice isomorphism between the ideal lattices of the two C^*-algebras. In the case of C^*-algebras associated to dynamical systems of various sorts, perhaps the fundamental tool used to generate useful Morita equivalences is the notion of Fell-bundle equivalence. In this article, we show that there is an analogous Rieffel correspondence induced by an equivalence $q: \mathscr{E} \to T$ between two Fell bundles $p_{\mathscr{B}}: \mathscr{B} \to H$ and $p_{\mathscr{C}}: \mathscr{C} \to K$ over locally compact groupoids H and K. Rather than work at the level of the Fell-bundle C^*-algebras $C^*(H;\mathscr{B})$ and $C^*(K;\mathscr{C})$, we work with the Fell bundles themselves. We introduce a natural notion of an ideal $\mathscr{J}$ of a Fell bundle $\mathscr{B}$. In the case where $\mathscr{B}$ is the Fell bundle corresponding to a group or groupoid G acting on a C^*-algebra A, these Fell-bundle ideals naturally correspond to G-invariant ideals of A in the standard sense. More generally, our ideals are the same as the Fell subbundles studied in [IW12]. We can form the quotient Fell bundles $\mathscr{B}/\mathscr{J}$, and if H has a Haar system and if our Fell bundles are separable, then

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the main result in [IW12] pushes the analogy of Fell bundle ideals with invariant ideals in crossed products; that is, we have a short exact sequence of C^*-algebras

$$0 \longrightarrow C^*(H;\mathscr{J}) \longrightarrow C^*(H;\mathscr{B}) \longrightarrow C^*(H;\mathscr{B}/\mathscr{J}) \longrightarrow 0.$$

However, as we do not work with C^*-algebras, we do not require our groupoids to have Haar systems.

In this article, our main result is that if $\mathscr{E}$ is an equivalence between $\mathscr{B}$ and $\mathscr{C}$ as above, then there is a lattice isomorphism between the ideals of $\mathscr{C}$ and those of $\mathscr{B}$. Furthermore, if $\mathscr{K}$ and $\mathscr{J}$ are corresponding ideals of $\mathscr{B}$ and $\mathscr{C}$, respectively, then $\mathscr{K}$ and $\mathscr{J}$ are equivalent Fell bundles as are the quotients $\mathscr{B}/\mathscr{K}$ and $\mathscr{C}/\mathscr{J}$. Naturally, these equivalences arise from submodules and quotients of the given equivalence $\mathscr{E}$.

We start in Section 2 with a detailed collection of preliminary material that summarizes and conveniently collects in one place the basics of Banach bundles, Fell bundles and Fell-bundle equivalence. We also introduce our notion of ideals of Fell bundles and develop some of their basic properties. In Section 3, we establish our basic Rieffel correspondence as Theorem 3.10. Then in Section 4, we establish the equivalence between corresponding ideals and their quotients.

We know that if two separable Fell bundles are equivalent, and if both of the underlying groupoids have Haar systems, then their corresponding Fell-bundle C^*-algebras are Morita equivalent and the classical Rieffel correspondence gives an isomorphism between the ideal lattices of the two Fell-bundle C^*-algebras. In Section 5, we confirm the natural conjecture that if two ideals correspond under our Rieffel correspondence for Fell-bundle ideals, then the corresponding ideals in the Fell-bundle C^*-algebras also correspond under the classical Rieffel correspondence.

Conventions. We use the standard conventions in the subject. In particular, homomorphisms between C^*-algebras are assumed to be $*$-preserving and ideals in C^*-algebras are two-sided and norm closed. Locally compact is meant to mean locally compact and Hausdorff, and our groupoids are always meant to be locally compact and Hausdorff. Suppose that A is an algebra and X is a (left) A-module. If $S \subset A$ and $\mathsf{Y} \subset \mathsf{X}$, then by convention, $S \cdot \mathsf{Y} = \operatorname{span}\{a \cdot x : a \in A \text{ and } x \in \mathsf{Y}\}$. Similarly, if $\langle \cdot, \cdot \rangle$ is an A-valued sesquilinear form on X, then $\langle \mathsf{Y}_1, \mathsf{Y}_2 \rangle = \operatorname{span}\{\langle x, y\rangle : x \in \mathsf{Y}_1 \text{ and } y \in \mathsf{Y}_2\}$. If A is a C^*-algebra and the A-module X is a Banach space, then we call X a Banach A-module if $\|a \cdot x\| \le \|a\|\|x\|$ for all $a \in A$ and $x \in \mathsf{X}$. Further, we say that X is nondegenerate if $A \cdot \mathsf{X}$ is dense in X.

2. Preliminaries

2.1. The Rieffel correspondence. If A is a C^*-algebra, then we let $\mathcal{I}(A)$ denote the lattice of ideals in A. Suppose that X is an A–B-imprimitivity bimodule, and let $C(\mathsf{X})$ be the lattice of closed A–B-submodules of X. Then the Rieffel correspondence asserts that there are natural lattice isomorphisms among $\mathcal{I}(A)$, $C(\mathsf{X})$ and $\mathcal{I}(B)$. Specifically, we have the following summary from [RW98, Section 3.3].

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THEOREM 2.1 (Rieffel correspondence). *Suppose that A and B are C^*-algebras and that X is an $A-B$-imprimitivity bimodule.*

(a) *Suppose that Y is a closed $A-B$-submodule of X. Then*

$$K = \overline{{}_A\langle \mathsf{Y}, \mathsf{X}\rangle} = \overline{{}_A\langle \mathsf{X}, \mathsf{Y}\rangle} = \overline{{}_A\langle \mathsf{Y}, \mathsf{Y}\rangle} \tag{2-1}$$

is an ideal in A, while

$$J = \overline{\langle \mathsf{Y}, \mathsf{X}\rangle_B} = \overline{\langle \mathsf{X}, \mathsf{Y}\rangle_B} = \overline{\langle \mathsf{Y}, \mathsf{Y}\rangle_B} \tag{2-2}$$

is an ideal in B. We have

$$K \cdot \mathsf{X} = \overline{K \cdot \mathsf{X}} = \mathsf{Y} = \overline{\mathsf{X} \cdot J} = \mathsf{X} \cdot J. \tag{2-3}$$

(b) *In particular, $J \mapsto \mathsf{X} \cdot J$ is a lattice isomorphism of $\mathcal{I}(B)$ onto $\mathcal{C}(\mathsf{X})$ with inverse $\mathsf{Y} \mapsto \overline{\langle \mathsf{Y}, \mathsf{Y}\rangle_B}$ and $K \mapsto K \cdot \mathsf{X}$ is a lattice isomorphism of $\mathcal{I}(A)$ onto $\mathcal{C}(\mathsf{X})$ with inverse $\mathsf{Y} \mapsto \overline{{}_A\langle \mathsf{Y}, \mathsf{Y}\rangle}$.*
(c) *If K, Y and J are as in part (a), then Y is a $K-J$-imprimitivity bimodule with respect to the restricted actions and inner products.*
(d) *If K, Y and J are as in part (a), then the quotient Banach space X/Y is an $A/K-B/J$-imprimitivity bimodule. In particular, the quotient norm on X/Y equals the imprimitivity-bimodule norm.*

REMARK 2.2. Suppose that J is an ideal in a C^*-algebra B, and that X is a right Hilbert B-module. Then $\mathsf{Y} = \overline{\mathsf{X} \cdot J}$ is a nondegenerate Banach J-module. Therefore, the Cohen factorization [RW98, Proposition 2.33] implies that every element of Y is of the form $x \cdot b$ with $x \in \mathsf{Y}$ and $b \in J$, so

$$\mathsf{Y} = \mathsf{X} \cdot J = \{x \cdot b : x \in \mathsf{X} \text{ and } b \in J\}.$$

As a result, we have $\overline{\mathsf{X} \cdot J} = \mathsf{X} \cdot J$ as in part (a), and similarly with $K \cdot \mathsf{X}$.

PROOF OF THEOREM 2.1. This is just a reworking of the basic results in [RW98, Section 3.3]. The equalities in Equations (2-1) and (2-2) follow from [RW98, Lemma 3.23]. The lattice isomorphisms follow from [RW98, Theorem 3.22], while Equation (2-3) follows from [RW98, Proposition 3.24] together with Remark 2.2. The statements about imprimitivity bimodules follow from [RW98, Proposition 3.25]. □

2.2. Banach bundles. Roughly speaking, a Banach bundle is a topological bundle in which each fibre is a Banach space. More precisely, we have the following definition.

DEFINITION 2.3. A *Banach bundle* over a topological space X is a topological space $\mathscr{B}$ together with a continuous, *open* surjection $p\colon \mathscr{B} \to X$ and complex Banach space structures on each fibre $B_x = p^{-1}(\{x\})$ satisfying the following axioms.

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(B1) The map $b \mapsto \|b\|$ is upper semicontinuous from $\mathscr{B}$ to $\mathbf{R}^+$ (this means that for all $\epsilon > 0$, $\{b \in \mathscr{B} : \|b\| < \epsilon\}$ is open).
(B2) The map $(a, b) \mapsto a + b$ from $\mathscr{B}^{(2)} = \{(a, b) \in \mathscr{B} \times \mathscr{B} : p(a) = p(b)\}$ to $\mathscr{B}$ is continuous.
(B3) The map $(\lambda, b) \mapsto \lambda b$ is continuous from $\mathbf{C} \times \mathscr{B}$ to $\mathscr{B}$.
(B4) If (b_i) is a net in $\mathscr{B}$ such that $p(b_i) \to x$ and $\|b_i\| \to 0$, then $b_i \to 0_x$ in $\mathscr{B}$ (where 0_x is the zero element in B_x).

We say that $p : \mathscr{B} \to X$ is *separable* if X is second countable and the Banach space $\Gamma_0(X; \mathscr{B})$ is separable. If the map in axiom (B1) is actually continuous, we call $\mathscr{B}$ a *continuous* Banach bundle.

REMARK 2.4. In some treatments, axiom (B3) in Definition 2.3 is replaced by the formally weaker axiom that $b \mapsto \lambda b$ is continuous for each $\lambda \in \mathbf{C}$. However, since $\{b \in \mathscr{B} : \|b\| < \epsilon\}$ is open, the proof of [FD88, Proposition II.13.10] shows the two definitions are equivalent.

REMARK 2.5 (The literature). Continuous Banach bundles are treated in detail in Sections 13–14 in [FD88, Ch. II] and many of the results there apply *mutatis mutandis* to Banach bundles. In the past, Banach bundles as defined above were called 'upper semicontinuous Banach bundles'. We have adopted the convention to drop the modifier in the general case. Banach bundles are discussed briefly in [MW08, Appendix A], which is where Definition 2.3 comes from, and the corresponding notion of a C^*-bundle is treated in detail in [Wil07, Appendix C].

The topology on the total space $\mathscr{B}$ of a Banach bundle might not be well behaved. For example, it need not be Hausdorff [Wil07, Example C.27]. However, we do have the following lemma.

LEMMA 2.6. *If $p: \mathscr{B} \to X$ is a Banach bundle, then the relative topology on B_x is the (Banach space) norm topology.*

PROOF. In the continuous case, this is [FD88, Proposition II.13.11], and proof carries over to the general case; see [DWZ22, Lemma 2.2]. □

If $p : \mathscr{B} \to X$ is a Banach bundle, we write $\Gamma(X; \mathscr{B})$ for the vector space of continuous sections. If X is locally compact, then we write $\Gamma_c(X; \mathscr{B})$ and $\Gamma_0(X; \mathscr{B})$ for the continuous sections that have compact support or that vanish at infinity, respectively. We say that $p : \mathscr{B} \to X$ *has enough sections* if given $b \in B_x$, there is an $f \in \Gamma(X; \mathscr{B})$ such that $f(x) = b$. Note that if X is locally compact, then since $\Gamma(X; \mathscr{B})$ is a $C(X)$-module by axiom (B3), we can take $f \in \Gamma_c(X; \mathscr{B})$.

THEOREM 2.7 [Laz18, Corollary 2.10]. *If $p : \mathscr{B} \to X$ is a Banach bundle over a locally compact space, then $\mathscr{B}$ has enough sections.*

While the notion of a Banach bundle is a natural mathematical object, generally Banach bundles arise in nature from their sections as described in the following result.

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THEOREM 2.8 (Hofmann–Fell). *Let X be a locally compact space and suppose that for each $x \in X$ we are given a Banach space B_x. Let $\mathscr{B}$ be the disjoint union $\coprod_{x\in X} B_x$ viewed as a bundle $p : \mathscr{B} \to X$. Suppose that Γ is a subspace of sections such that:*

(a) *for each $f \in \Gamma$, $x \mapsto \|f(x)\|$ is upper semicontinuous; and*
(b) *for each $x \in X$, $\{f(x) : f \in \Gamma\}$ is dense in B_x.*

Then there is a unique topology on $\mathscr{B}$ such that $p : \mathscr{B} \to X$ is a Banach bundle with $\Gamma \subset \Gamma(X; \mathscr{B})$. Furthermore, the sets of the form

$$W(f, U, \epsilon) = \{a \in \mathscr{B} : p(a) \in U \text{ and } \|a - f(p(a))\| < \epsilon\}$$

with $f \in \Gamma$, U open in X and $\epsilon > 0$ form a basis for this topology.

PROOF. In the continuous case, this is [FD88, Theorem II.13.18]. In general, it is stated in [DG83, Proposition 1.3] and also follows *mutatis mutandis* from [Wil07, Theorem C.25]. □

2.3. Banach subbundles. A subbundle of a Banach bundle is a Banach subbundle if it is a Banach bundle in the inherited structure.

DEFINITION 2.9. Let $p: \mathscr{B} \to X$ be a Banach bundle. We say that $\mathscr{C} \subset \mathscr{B}$ is a *Banach subbundle* if each $C_x = B_x \cap C$ is a closed vector subspace of B_x, and $p|_{\mathscr{C}} : \mathscr{C} \to X$ is a Banach bundle when we give C_x the Banach-space structure coming from B_x and we give $\mathscr{C}$ the relative topology.

REMARK 2.10. Since $0_x \in C_x$ for all x, we must have $p(\mathscr{C}) = X$. However, some fibres can be the zero Banach space.

REMARK 2.11. If $\{C_x\}$ is any collection of closed subspaces with $C_x \subset B_x$ and if we give $\mathscr{C} = \coprod C_x = \{b \in \mathscr{B} : b \in C_{p(b)}\}$ the relative topology, then $p: \mathscr{C} \to X$ is a continuous surjection satisfying axioms (B1), (B2), (B3), and (B4) of Definition 2.3. However, $p: \mathscr{C} \to X$ may fail to be a Banach subbundle unless we also have $p|_{\mathscr{C}}$ open.

Even if $p|_{\mathscr{C}}$ is not open, we write $\Gamma(X; \mathscr{C})$ for the continuous functions f from X to $\mathscr{C}$ such that $p(f(x)) = x$ for all $x \in X$. Of course, if $p|_{\mathscr{C}}$ is not open, there is no reason that $\Gamma(X; \mathscr{C})$ should contain anything other than the zero section, as shown in the next example.

EXAMPLE 2.12. Let B be a Banach space and $\mathscr{B} = X \times B$ the trivial bundle over X. Fix $x_0 \in X$ and let

$$C_x = \begin{cases} B & \text{if } x = x_0, \\ 0_x & \text{otherwise.} \end{cases}$$

Then in general, $p|_{\mathscr{C}} : \mathscr{C} \to X$ is not open and admits only the zero section.

PROPOSITION 2.13. *Let $p: \mathscr{B} \to X$ be a Banach bundle. Suppose that C_x is a closed subspace of B_x for all $x \in X$ and let $\mathscr{C} = \coprod C_x$ be as above. If $\{f(x) : f \in \Gamma(X; \mathscr{C})\}$ is dense in C_x for all $x \in X$, then the bundle $p|_{\mathscr{C}} : \mathscr{C} \to X$ is a Banach subbundle of $\mathscr{B}$.*

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PROOF. Suppose that $\{f(x) : f \in \Gamma(X; \mathscr{C})\}$ is dense in C_x for all x. Let U be a nonempty (relatively) open set in $\mathscr{C}$. In view of Remark 2.11, to show that $p|_{\mathscr{C}} : \mathscr{C} \to X$ is a Banach subbundle, it suffices to see that $p(U)$ is open in X. Let $x \in p(U)$ and suppose that (x_i) is a net in X converging to x in X. It suffices to see that (x_i) is eventually in $p(U)$. Let $b \in U$ be such that $p(b) = x$. Then for each n, the set $\{b' \in \mathscr{B} : \|b' - b\| < 1/n\}$ is an open neighbourhood of b in $\mathscr{B}$. Hence, there is $f_n \in \Gamma(X; \mathscr{C})$ such that $\|f_n(x) - b\| < 1/n$. Thus, $\|f_n(x) - b\| \to 0$. By axiom (B4), $f_n(x) - b \to 0_x$ in $\mathscr{B}$. However, by axiom (B2), $f_n(x) \to b$ in $\mathscr{B}$. Since everything in sight is in $\mathscr{C}$ and $\mathscr{C}$ has the relative topology, for some N, $f_N(x) \in U$. However, $f_N(x_i) \to f_N(x)$. So $f_N(x_i)$ is eventually in U. Therefore x_i is eventually in $p(U)$. □

REMARK 2.14. In [FD88, Problem 41 in Ch. II], Fell and Doran call a family $\{C_x\}$ of subspaces as in Proposition 2.13 in a continuous Banach bundle a *lower semicontinuous choice of subspaces*.

REMARK 2.15. If $p : \mathscr{B} \to X$ is a Banach bundle over a locally compact space and if $p|_{\mathscr{C}} : \mathscr{C} \to X$ is a Banach subbundle, then it has enough sections by Lazar's Theorem 2.7. Hence, $\{f(x) : f \in \Gamma(X; \mathscr{C})\}$ is not only dense, it is all of C_x.

2.4. Quotient Banach bundles. Let $p : \mathscr{B} \to X$ be a Banach bundle over a locally compact space X and let $\mathscr{C} \subset \mathscr{B}$ be a Banach subbundle as in Definition 2.3. Then we can formally form the quotient $\mathscr{B}/\mathscr{C} = \coprod_{x \in X} B_x/C_x$, where B_x/C_x is the usual Banach space quotient. We let $q : \mathscr{B} \to \mathscr{B}/\mathscr{C}$ be the quotient map so that if $b \in B_x$, then $q(b) = q_x(b)$, where $q_x : B_x \to B_x/C_x$ is the usual Banach space quotient map. In particular, q is norm reducing. If $f \in \Gamma_c(X; \mathscr{B})$, then we write $q(f)$ for the section of $\mathscr{B}/\mathscr{C}$ given by $q(f)(x) = q_x(f(x))$.

PROPOSITION 2.16. *Let $p : \mathscr{B} \to X$ be a Banach bundle and $\mathscr{C} \subset \mathscr{B}$ a Banach subbundle. Then $\bar{p} : \mathscr{B}/\mathscr{C} \to X$ is a Banach bundle in the quotient topology. Furthermore, the quotient map $q : \mathscr{B} \to \mathscr{B}/\mathscr{C}$ is continuous and open, and the quotient topology on $\mathscr{B}/\mathscr{C}$ is the unique topology such that $\Gamma = \{q(f) : f \in \Gamma_c(X; \mathscr{B})\} \subset \Gamma_c(X; \mathscr{B}/\mathscr{C})$.*

REMARK 2.17. As pointed out in [Laz18], Proposition 2.16 can be sorted out of [Gie82, Ch. 9]. We give the short proof for completeness.

PROOF. Let $f \in \Gamma_c(X; \mathscr{B})$. We claim $x \mapsto \|q_x(f(x))\|$ is upper semicontinuous. Fix $\epsilon > 0$. Suppose $\|q_x(f(x))\| < \epsilon$. Then by definition of the quotient norm, there is a $c \in C_x$ such that $\|f(x) + c\| < \epsilon$. Let $d \in \Gamma_c(X; \mathscr{C})$ be such that $d(x) = c$. Then there is a neighbourhood V of x such that $\|f(y) + d(y)\| < \epsilon$ if $y \in V$. Since $q(f) = q(f + d)$, it follows that $\|q(f)(y)\| < \epsilon$ for $y \in V$. This establishes the claim.

It follows from Theorem 2.8 that there is a unique topology on $\mathscr{B}/\mathscr{C}$ such that $\mathscr{B}/\mathscr{C}$ is a Banach bundle with $\Gamma := \{q(f) : f \in \Gamma_c(X; \mathscr{B})\} \subset \Gamma_c(X; \mathscr{B}/\mathscr{C})$.

Next we claim that the quotient map $q : \mathscr{B} \to \mathscr{B}/\mathscr{C}$ is continuous. Suppose that (a_i) is a net in $\mathscr{B}$ such that $a_i \in B_{x_i}$ and $a_i \to a_0$ in $\mathscr{B}$. Then $x_i \to x_0$ in X. Let $f \in \Gamma_c(X; \mathscr{B})$ be such that $f(x_0) = a_0$. Then

$$\|f(x_i) - a_i\| \to 0.$$

Since q is norm reducing,

$$\|q(f)(x_i) - q(a_i)\| \to 0.$$

Since $q(f) \in \Gamma_c(X; \mathscr{B}/\mathscr{C})$, [MW08, Lemma A.3] implies that $q(a_i) \to q(a_0)$. Thus, q is continuous as claimed.

To see that q is also open, let V be an open neighbourhood of $b \in \mathscr{B}$. Then in view of Theorem 2.8, there is a $f \in \Gamma_c(X; \mathscr{B})$, an open neighbourhood U of $p(b)$ and an $\epsilon > 0$ such that

$$b \in W(f, U, \epsilon) := \{a \in B : p(a) \in U \text{ and } \|a - f(p(a))\| < \epsilon\}.$$

We need to verify that $q(V)$ is a neighbourhood of $q(b)$. Since $q(f) \in \Gamma_c(X; \mathscr{B}/\mathscr{C})$, it suffices to see that

$$q(W(f, U, \epsilon)) = \{q(c) : p(c) \in U \text{ and } \|q(c) - q(f)(p(c))\| < \epsilon\}.$$

Since the left-hand side is clearly a subset of the right-hand side, it suffices to consider $q(c)$ in the right-hand side. If $x = p(c)$, then

$$\epsilon > \|q(c) - q(f)(x)\| = \|q_x(c - f(x))\| = \inf_{q_x(a)=c} \|a - f(x)\|.$$

Hence, there is an $a \in W(f, U, \epsilon)$ such that $q(a) = q(c)$. This suffices to show that q is open.

Since q is continuous and open, the topology on $\mathscr{B}/\mathscr{C}$ is the quotient topology. □

2.5. Fell bundles. Fell bundles are natural generalizations of Fell's Banach $*$-algebraic bundles from [FD88, Ch. VIII] and were introduced by Yamagami in [Yam87]. The following definition comes from [MW08, Definition 1.1].

DEFINITION 2.18. Suppose that $p : \mathscr{B} \to G$ is a Banach bundle over a second countable locally compact Hausdorff groupoid G. Let

$$\mathscr{B}^{(2)} = \{(a, b) \in \mathscr{B} \times \mathscr{B} : (p(a), p(b)) \in G^{(2)}\}.$$

We say that $p : \mathscr{B} \to G$ is a *Fell bundle* if there is a continuous, bilinear, associative multiplication map $(a, b) \mapsto ab$ from $\mathscr{B}^{(2)}$ to $\mathscr{B}$ and a continuous involution $b \mapsto b^*$ from $\mathscr{B}$ to $\mathscr{B}$ such that:

(FB1) $p(ab) = p(a)p(b)$;
(FB2) $p(a^*) = p(a)^{-1}$;
(FB3) $(ab)^* = b^* a^*$;
(FB4) for each $u \in G^{(0)}$, the fibre B_u is a C^*-algebra with respect to the inherited multiplication and involution on B_u; and

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(FB5) for each $g \in G$, B_g is an $B_{r(g)} - B_{s(g)}$-imprimitivity bimodule when equipped with the inherited actions and inner products given by

$$_{B_{r(g)}}\langle a, b\rangle = ab^* \quad \text{and} \quad \langle a, b\rangle_{B_{s(g)}} = a^*b. \tag{2-4}$$

We say that the Fell bundle $p : \mathscr{B} \to G$ is *separable* if it is separable as a Banach bundle.

REMARK 2.19 (Saturated). It should be noted that our Fell bundles are saturated in that whenever $(g, h) \in G^{(2)}$, then $B_g \cdot B_h := \operatorname{span}\{ab : a \in B_g \text{ and } b \in B_h\}$ is always dense in B_{gh} [MW08, Lemma 1.2]. This is a consequence of item (FB5). Some authors prefer to work with a weakened version of item (FB5) where the inner products in Equation (2-4) are not full.

REMARK 2.20. If $p: \mathscr{B} \to G$ is a Fell bundle, then the restriction $\mathscr{B}|_{G^{(0)}}$ is a C^*-bundle and $\Gamma_0(G^{(0)}; \mathscr{B})$ is a C^*-algebra called the *associated C^*-algebra* to $\mathscr{B}$. (The terminology is a bit challenging. If G has a Haar system, then one can also form the Fell-bundle C^*-algebra $C^*(G; \mathscr{B})$ by viewing $\Gamma_c(G; \mathscr{B})$ as a $*$-algebra and completing as in [MW08].)

2.6. Equivalence of Fell bundles. Suppose that T is a left G-space. Then we say that a Fell bundle $p: \mathscr{B} \to G$ acts on (the left of) a Banach bundle $q : \mathscr{E} \to T$ if there is a continuous map $(b, e) \mapsto b \cdot e$ from $\mathscr{B} * \mathscr{E} := \{(b, e) \in \mathscr{B} \times \mathscr{E} : s(b) = r(q(e))\}$ to $\mathscr{E}$ such that:

(a) $q(b \cdot e) = p(b) \cdot q(e)$;
(b) $a \cdot (b \cdot e) = (ab) \cdot e$ for appropriate $a, b \in B$ and $e \in \mathscr{E}$; and
(c) $\|b \cdot e\| \le \|b\|\|e\|$.

Right actions of a Fell bundle are defined similarly.

Let T be a (G, H) equivalence with open moment maps $\rho: T \to G^{(0)}$ and $\sigma: T \to H^{(0)}$ as in [Wil19, Definition 2.29]. It is shown in [Wil19, Lemma 2.42] that there are open continuous maps $\tau_G: T *_\sigma T \to G$ and $\tau_H: T *_\rho T \to H$ such that $\tau_G(e, f) \cdot f = e$ and $e \cdot \tau_H(e, f) = f$.

DEFINITION 2.21 [MW08, Definition 6.1]. Suppose that T is a (G, H)-equivalence, and that $p_{\mathscr{B}} \cdot \mathscr{B} \to G$ and $p_{\mathscr{C}}: \mathscr{C} \to H$ are Fell bundles. Then a Banach bundle $q: \mathscr{E} \to T$ is a *$\mathscr{B} - \mathscr{C}$-equivalence* if the following conditions hold.

(E1) There is a left $\mathscr{B}$-action and a right $\mathscr{C}$-action on $\mathscr{E}$ such that $b \cdot (e \cdot c) = (b \cdot e) \cdot c$ for composable $b \in \mathscr{B}$, $e \in E$ and $c \in \mathscr{C}$.
(E2) There are continuous sesquilinear maps $(e, f) \mapsto {}_{\mathscr{B}}\langle e, f\rangle$ from $\mathscr{E} *_\sigma \mathscr{E}$ to $\mathscr{B}$ and $(e, f) \mapsto \langle e, f\rangle_{\mathscr{C}}$ from $\mathscr{E} *_\rho \mathscr{E}$ to $\mathscr{C}$ such that:

   (i) $p_{\mathscr{B}}({}_{\mathscr{B}}\langle e, f\rangle) = \tau_G(q(e), q(f))$ and $p_{\mathscr{C}}(\langle e, f\rangle_{\mathscr{C}}) = \tau_H(q(e), q(f))$;
   (ii) ${}_{\mathscr{B}}\langle e, f\rangle^* = {}_{\mathscr{B}}\langle f, e\rangle$ and $\langle e, f\rangle^*_{\mathscr{B}} = \langle f, e\rangle_{\mathscr{B}}$;

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(iii) ${}_{\mathscr{B}}\langle b\cdot e, f\rangle = b\,{}_{\mathscr{B}}\langle e, f\rangle$ and $\langle e, f\cdot c\rangle_{\mathscr{C}} = \langle e, f > \rangle_{\mathscr{C}}$; and
(iv) ${}_{\mathscr{B}}\langle e, f\rangle\cdot g = e\cdot\langle f, g\rangle_{\mathscr{C}}$.

(E3) With the actions and inner products coming from conditions (E1) and (E2), each E_t is a $B_{\rho(t)} - C_{\sigma(t)}$-imprimitivity bimodule.

2.7. Fell subbundles and ideals. Naturally, a subbundle of a Fell bundle is called a Fell subbundle if it is a Fell bundle in the inherited structure.

DEFINITION 2.22. Let $p\colon \mathscr{B}\to G$ be a Fell bundle over a groupoid G. We call $\mathscr{C}\subset\mathscr{B}$ a *Fell subbundle* if $\mathscr{C}$ is a Banach subbundle such that $p|_{\mathscr{C}}\colon \mathscr{C}\to G$ is a Fell bundle with respect to the inherited operations. In particular, $\mathscr{C}$ must be closed under multiplication and involution.

We focus on Fell subbundles that are multiplicatively absorbing.

DEFINITION 2.23. A Fell subbundle $\mathscr{J}$ of a Fell bundle $\mathscr{B}$ is called an *ideal* if $ab\in\mathscr{J}$ whenever $(a,b)\in\mathscr{B}^{(2)}$ and either $a\in\mathscr{J}$ or $b\in\mathscr{J}$.

EXAMPLE 2.24. Suppose that $\alpha\colon \mathcal{G}\to \mathrm{Aut}(A)$ is a C^*-dynamical system for a group $\mathcal{G}$. Let $\mathscr{B} = A\times\mathcal{G}$ be the associated Fell bundle over $\mathcal{G}$: $(a,s)(b,r) = (a\alpha_s(b), sr)$. Let I be an *α-invariant ideal* of A. Then $\mathscr{J} = I\times\mathcal{G}$ is an ideal in $\mathscr{B}$.

EXAMPLE 2.25. Suppose that $\mathscr{A}$ is a C^*-bundle over X so that $A = \Gamma_0(X;\mathscr{A})$ is a C^*-algebra. Let J be an ideal in A and for each $x\in X$, let $J_x = \{a(x) : a\in J\}$ so that J_x is an ideal in A_x. Let

$$\mathscr{J} = \coprod_{x\in X} J_x.$$

It follows from Proposition 2.13 that $\mathscr{J}$ is a Banach subbundle and in fact is obviously an ideal of the Fell bundle $\mathscr{A}$. Clearly, $J\subset\Gamma_0(X;\mathscr{J})$. Since J is an ideal in the $C_0(X)$-algebra A, if $\phi\in C_0(X)$ and $b\in J$, then $\phi\cdot b\in J$. Now it follows from [Wil07, Proposition C.24] that J is dense in $\Gamma_0(X;\mathscr{J})$. Therefore, $J = \Gamma_0(X;\mathscr{J})$.

LEMMA 2.26. *Suppose that $\mathscr{J}$ is an ideal in $\mathscr{B}$. Then for each $g\in G$, J_g is a $J_{r(g)} - J_{s(g)}$-imprimitivity bimodule. Furthermore,*

$$J_g = B_g\cdot J_{s(g)} = J_{r(g)}\cdot B_g, \tag{2-5}$$

where we are taking advantage of Remark 2.2. Furthermore,

$$J_{r(g)} = \overline{J_gB_g^*} = \overline{B_gJ_g^*} = \overline{J_gJ_g^*} \quad \text{and} \quad J_{s(g)} = \overline{J_g^*B_g} = \overline{B_g^*J_g} = \overline{J_g^*J_g}.$$

PROOF. The first assertion is immediate since $\mathscr{J}$ is, by assumption, a Fell subbundle. The remaining statements follow from the Rieffel correspondence; see Theorem 2.1. □

DEFINITION 2.27. Let $p:\mathscr{B}\to G$ be a Fell bundle and $\mathscr{J}\subset\mathscr{B}$ a Banach subbundle. We call $\mathscr{J}$ a *weak ideal of $\mathscr{B}$* if whenever $(a,b)\in\mathscr{B}^{(2)}$, then $ab\in\mathscr{J}$ whenever either a or b is in $\mathscr{J}$.

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REMARK 2.28. If I is an ideal in a C^*-algebra, then the existence of approximate identities implies that I is $*$-closed and hence a C^*-subalgebra. A similar serendipity applies to weak ideals.

PROPOSITION 2.29. *If $p : \mathscr{B} \to G$ is a Fell bundle, then every weak ideal in $\mathscr{B}$ is an ideal.*

PROOF. Let $\mathscr{J}$ be a weak ideal in $\mathscr{B}$. Since J_g is closed with respect to the norm on B_g, it is a closed $B(r(g)) - B(s(g))$-submodule of the $B(r(g)) - B(s(g))$-imprimitivity bimodule B_g. In particular, J_u is an ideal in the C^*-algebra B_u for all $u \in G^{(0)}$. Then, applying the Rieffel correspondence, J_g is a $K_g - I_g$-imprimitivity bimodule where I_g is the closed linear span of elements of the form b^*a with $b \in B_g$ and $a \in J_g$. Similarly, K_g is the closed linear span of products ab^* with $a \in J_g$ and $b \in B_g$. Furthermore,

$$J_g = B_g \cdot I_g = K_g \cdot B_g. \tag{2-6}$$

Note that I_g is an ideal in $J_{s(g)}$. Fix $c \in J_{s(g)}$. If $b \in B_g$, then $bc \in J_g$ by the weak ideal property. Since $B_g^* \cdot B_g$ is a dense ideal in $B_{s(g)}$, we can find an approximate unit (e_i) in $B_{s(g)}$ where each $e_i = \sum_{k=1}^{n_i} b_k^* b_k$ with each $b_k \in B_g$. However, then $e_i c$ is in I_g and $e_i c \to c$. Hence, $c \in I_g$ and $I_g = J_{s(g)}$. A similar argument shows that $K_g = J_{r(g)}$.

Since $J_{s(g)}$ is an ideal in the C^*-algebra $B_{s(g)}$, we have $J_{s(g)}^* = J_{s(g)}$. Hence, using Equation (2-6),

$$J_g^* = (B_g \cdot J_{s(g)})^* = J_{s(g)}^* \cdot B_g^* = J_{r(g^{-1})} \cdot B_{g^{-1}} = J_{g^{-1}}.$$

In particular, $\mathscr{J}^* = \mathscr{J}$ and $\mathscr{J}$ is closed under taking adjoints. Since $\mathscr{J}$ is a weak ideal, it is closed under multiplication and we just showed it is also closed under the adjoint operation. Now we just have to observe that it is a Fell bundle. However, this follows from the above discussion and identification of I_g with $J_{s(g)}$ and K_g with $J_{r(g)}$. □

It is standard to think of a Fell bundle $p: \mathscr{B} \to G$ as a generalized groupoid crossed product of G acting on the associated C^*-algebra $A := \Gamma_0(G^{(0)}; \mathscr{B})$. As an example of this rubric, it is shown in [IW12, Proposition 2.2] that there is a natural action of G on $\operatorname{Prim} A$ given as follows. Note that $\operatorname{Prim} A$ is naturally fibred over $G^{(0)}$. Since B_g is a $B_{r(g)} - B_{s(g)}$-imprimitivity bimodule, the Rieffel correspondence induces a homeomorphism $\phi_g : \operatorname{Prim}(B_{s(g)}) \to \operatorname{Prim}(B_{r(g)})$ [RW98, Corollary 3.33]. Then the G-action is given by $g \cdot P_{s(g)} = \phi_g(P_{s(g)})$. Naturally, an ideal I in A is called G-invariant if $\operatorname{hull}(I) := \{P \in \operatorname{Prim} A : P \supset I\}$ is G-invariant. If I is an ideal in A, then we let $I_u = q_u(I)$, where $q_u : \Gamma_0(G^{(0)}; \mathscr{B}) \to B_u$ is the evaluation map.

PROPOSITION 2.30 [IW12]. *Suppose $p: \mathscr{B} \to G$ is a Fell bundle and that I is a G-invariant ideal in the associated C^*-algebra $\Gamma_0(G^{(0)}; \mathscr{B})$. Then*

$$\mathscr{B}_I := \{b \in \mathscr{B} : b^*b \in I_{s(b)}\}$$

is an ideal in $\mathscr{B}$. Conversely, if $\mathscr{J}$ is an ideal in $\mathscr{B}$, then $I = \Gamma_0(G^{(0)}; \mathscr{J})$ is a G-invariant ideal in $\Gamma_0(G^{(0)}; \mathscr{B})$ and $\mathscr{J} = \mathscr{B}_I$.

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PROOF. It follows from [IW12, proof of Lemma 3.1] that an ideal $I \subset \Gamma_0(G^{(0)}; \mathscr{B})$ is G-invariant if and only if for all g, we have $\phi_g(I_{s(g)}) = I_{r(g)}$. By the Rieffel correspondence, the latter is equivalent to

$$\mathscr{B}_g \cdot I_{s(b)} = I_{r(b)} \cdot B_g \quad \text{for all } g \in G. \tag{2-7}$$

Suppose that I is G-invariant. Then it follows from [IW12, Proposition 3.3] that $\mathscr{J} := \mathscr{B}_I$ is a Fell subbundle such that Equation (2-7) holds. Suppose that $a \in B_g$ and $b \in B_h$ are composable. If $a \in \mathscr{J}$, then

$$ab \in I_{r(g)} \cdot B_g B_h \subset I_{r(gh)} \cdot B_{gh} = J_{gh}.$$

Similarly, if $b \in \mathscr{J}$, then

$$ab \in B_g B_h \cdot I_{s(h)} \subset B_{gh} \cdot I_{s(gh)} = J_{gh}$$

and B_I is an ideal.

Now suppose that $\mathscr{J}$ is an ideal in $\mathscr{B}$. Let

$$I = \Gamma_0(X; \mathscr{J}).$$

Then as in Example 2.25, we have $I_u = J_u$. Now it follows from Lemma 2.26 and Equation (2-7) that I is G-invariant. Since $\mathscr{B}_I$ and $\mathscr{J}$ have the same fibres, clearly $\mathscr{B}_I = \mathscr{J}$. □

3. The Rieffel correspondence for Fell-bundle equivalences

In this section, we let $q_{\mathscr{E}} \colon \mathscr{E} \to T$ be an equivalence between $p_{\mathscr{B}} \colon \mathscr{B} \to H$ and $p_{\mathscr{C}} \colon \mathscr{C} \to K$. In particular, T is an (H, K)-equivalence and we let $\rho \colon T \to H^{(0)}$ and $\sigma \colon T \to K^{(0)}$ be the open moment maps.

We need the following observation from [MW08, Lemma 6.2].

LEMMA 3.1. *As above, let $q_{\mathscr{E}} \colon \mathscr{E} \to T$ be a Fell-bundle equivalence between $p_{\mathscr{B}} \colon \mathscr{B} \to H$ and $p_{\mathscr{C}} \colon \mathscr{C} \to K$. Then $(b, e) \mapsto b \cdot e$ induces an imprimitivity bimodule isomorphism of $B_h \otimes_{B_{\rho(t)}} E_t$ onto $E_{h \cdot t}$. Similarly, $(e, c) \mapsto e \cdot c$ induces an isomorphism between $E_t \otimes_{C_{\sigma(t)}} C_k$ and $E_{t \cdot k}$.*

COROLLARY 3.2. *Let $\mathscr{E}$, $\mathscr{B}$ and $\mathscr{C}$ be as above. Let $\mathscr{J}$ be an ideal in $\mathscr{C}$ and $\sigma(t) = r(k)$. Then $\overline{E_t \cdot J_k} = E_{t \cdot k} \cdot J_{s(k)}$.*

PROOF. Lemma 3.1 implies that $\overline{E_t \cdot C_k} = E_{t \cdot k}$. Therefore, by Lemma 2.26,

$$\overline{E_t \cdot J_k} = \overline{E_t \cdot C_k \cdot J_{s(k)}} = E_{t \cdot k} \cdot J_{s(k)}.$$ □

DEFINITION 3.3. Let $q_{\mathscr{E}} \colon \mathscr{E} \to T$ be an equivalence between $p_{\mathscr{B}} \colon \mathscr{B} \to H$ and $p_{\mathscr{C}} \colon \mathscr{C} \to K$. Then a Banach submodule $\mathscr{M}$ of $\mathscr{E}$ is called a *Banach $\mathscr{B}$ – $\mathscr{C}$-submodule* if $B_h \cdot M_t \subset M_{h \cdot t}$ whenever $s(h) = \rho(t)$ and $M_t \cdot C_k \subset M_{t \cdot k}$ whenever $\sigma(t) = r(k)$. We say that $\mathscr{M}$ is *full* if $\overline{B_h \cdot M_t} = M_{h \cdot t}$ and $\overline{M_t \cdot C_k} = M_{t \cdot k}$.

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PROPOSITION 3.4. *Let $\mathscr{E}$, $\mathscr{B}$ and $\mathscr{C}$ be as above. If $\mathscr{J}$ is an ideal in $\mathscr{C}$, then*

$$\mathscr{E} \cdot \mathscr{J} := \bigcup_{\{(t,k):\sigma(t)=r(k)\}} E_t \cdot J_k$$

is a full Banach $\mathscr{B}-\mathscr{C}$-submodule of $\mathscr{E}$ with $(\mathscr{E} \cdot \mathscr{J})_t = E_t \cdot J_{\sigma(t)}$. Similarly, if $\mathscr{K}$ is an ideal in $\mathscr{B}$, then $\mathscr{K} \cdot \mathscr{E}$ is a full Banach $\mathscr{B}-\mathscr{C}$-submodule with $(\mathscr{K} \cdot \mathscr{E})_t = K_{\rho(t)} \cdot E_t$.

PROOF. We have

$$\mathscr{E} \cdot \mathscr{J} = \bigcup_{\{(t,k):\sigma(t)=r(k)\}} E_t \cdot J_k \subset \bigcup_{\{(t,k):\sigma(t)=r(k)\}} \overline{E_t \cdot J_k}$$

which, by Corollary 3.2, is

$$= \bigcup_{\{(t,k):\sigma(t)=r(k)\}} E_{t\cdot k} \cdot J_{s(k)} = \bigcup_{t\in T} E_t \cdot J_{\sigma(t)} \subset \mathscr{E} \cdot \mathscr{J}.$$

Therefore, $\mathscr{E} \cdot \mathscr{J} = \bigcup_{t\in T} E_t \cdot J_{\sigma(t)}$ and $(\mathscr{E} \cdot \mathscr{J})_t = E_t \cdot J_{\sigma(t)}$ as claimed.

In particular, $\mathscr{E} \cdot \mathscr{J}$ is a bundle over T with closed fibres $E_t \cdot J_{\sigma(t)}$. However, if $f \in \Gamma_c(T; \mathscr{E})$ and $\phi \in \Gamma_c(K^{(0)}; \mathscr{J})$, then $\phi \cdot f$ given by $\phi \cdot f(t) = f(t) \cdot \phi(\sigma(t))$ is a section in $\Gamma_c(T; \mathscr{E} \cdot \mathscr{J})$. Now it follows from Proposition 2.13 that $\mathscr{E} \cdot \mathscr{J}$ is a Banach subbundle of $\mathscr{E}$.

We still need to see that $\mathscr{E} \cdot \mathscr{J}$ is a full $\mathscr{B}-\mathscr{J}$-submodule. However, if $s(h) = \rho(t)$, then

$$\begin{aligned}
\overline{B_h \cdot (\mathscr{E} \cdot \mathscr{J})_t} &= \overline{B_h \cdot (E_t \cdot J_{\sigma(t)})} \\
&= \overline{(B_h \cdot E_t) \cdot J_{\sigma(t)}} \\
&= E_{h\cdot t} \cdot J_{\sigma(t)} = (\mathscr{E} \cdot \mathscr{J})_{h\cdot t}.
\end{aligned}$$

However, if $\sigma(t) = r(k)$, then

$$\overline{(\mathscr{E} \cdot \mathscr{J})_t \cdot C_k} = \overline{E_t \cdot (J_{r(k)} \cdot C_k)}$$

which, by Lemma 2.26, is

$$\begin{aligned}
&= \overline{E_t \cdot (C_k \cdot J_{s(k)})} \\
&= E_{t\cdot k} \cdot J_{s(k)} = (\mathscr{E} \cdot \mathscr{J})_{t\cdot k}.
\end{aligned}$$

Thus, $\mathscr{E} \cdot \mathscr{J}$ is a full Banach $\mathscr{B}-\mathscr{C}$-submodule as claimed.

The corresponding statements for $\mathscr{K} \cdot \mathscr{E}$ are proved similarly. □

Now suppose that $\mathscr{M}$ is a full Banach $\mathscr{B}-\mathscr{C}$-submodule of $\mathscr{E}$. Then M_t is a closed $B_{\rho(t)}-C_{\sigma(t)}$-submodule of E_t. By the Rieffel correspondence, M_t is a L_t-R_t-imprimitivity bimodule for the ideals $L_t = \overline{{}_{\mathscr{B}}\langle M_t, E_t\rangle}$ in $B_{\rho(t)}$ and $R_t = \overline{\langle E_t, M_t\rangle_{\mathscr{C}}}$ in $C_{\sigma(t)}$. Furthermore, $L_t \cdot E_t = M_t = E_t \cdot R_t$.

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LEMMA 3.5. *In the current set-up, the ideal R_t depends only on $\sigma(t)$ and the ideal L_t depends only on $\rho(t)$. Hereafter, we denote them by $R_{\sigma(t)}$ and $L_{\rho(t)}$, respectively.*

PROOF. If $\sigma(t') = \sigma(t)$, then $t' = h \cdot t$. Since E_t is a $B_{\rho(t)} - C_{\sigma(t)}$-imprimitivity bimodule, and since R_t and $R_{h \cdot t}$ are both ideals in $C_{\sigma(t)}$, to see that $R_t = R_{h \cdot t}$, it suffices, by the Rieffel correspondence, to see that $E_t \cdot R_t = E_t \cdot R_{h \cdot t}$. Since $\mathscr{M}$ is full,

$$E_t \cdot R_{h \cdot t} = E_t \cdot \overline{\langle E_{h \cdot t}, M_{h \cdot t}\rangle_{\mathscr{C}}} = \overline{E_t \cdot \langle \overline{B_h \cdot E_t}, \overline{B_h \cdot M_t}\rangle_{\mathscr{C}}}.$$

Clearly,

$$\overline{E_t \cdot \langle B_h \cdot E_t, B_h \cdot M_t\rangle_{\mathscr{C}}} \subset \overline{E_t \cdot \langle \overline{B_h \cdot E_t}, \overline{B_h \cdot M_t}\rangle_{\mathscr{C}}}. \tag{3-1}$$

However, consider

$$e \cdot \langle f, g\rangle_{\mathscr{C}}$$

with $e \in E_t$, $f \in \overline{B_h \cdot E_t}$ and $g \in \overline{B_h \cdot M_t}$. Then there are sequences $(f_i) \subset B_h \cdot E_t$ and $(g_i) \subset B_h \cdot M_t$ such that $f_i \to f$ and $g_i \to g$ in norm in $E_{h \cdot t}$ and hence in $\mathscr{E}$. Therefore, $e \cdot \langle f_i, g_i\rangle_{\mathscr{C}} \to e \cdot \langle f, g\rangle_{\mathscr{C}}$ in $\mathscr{E}$. Since the convergence takes place in E_t, the convergence is in norm. It follows that

$$E_t \cdot \langle \overline{B_h \cdot E_t}, \overline{B_h \cdot M_t}\rangle_{\mathscr{C}} \subset \overline{E_t \cdot \langle B_h \cdot E_t, B_h \cdot M_t\rangle_{\mathscr{C}}}.$$

Therefore, we have equality in Equation (3-1), and

$$E_t \cdot R_{h \cdot t} = \overline{E_t \cdot \langle B_h \cdot E_t, B_h \cdot M_t\rangle_{\mathscr{C}}}$$

which, using condition (E2)(iv), is

$$= \overline{{}_{\mathscr{B}}\langle E_t, B_h \cdot E_t\rangle \cdot B_h \cdot M_t}$$

which, using conditions (E2)(ii) and (E2)(iii), is

$$= \overline{{}_{\mathscr{B}}\langle E_t, E_t\rangle \cdot B_h^* B_h \cdot M_t}$$

which, since $\overline{B_h^* B_h} = B_{s(h)} = B_{\rho(t)} = \overline{{}_{\mathscr{B}}\langle E_t, E_t\rangle}$ and since $\overline{B_{s(h)} \cdot M_t} = M_t$, is

$$= M_t = E_t \cdot R_t.$$

Thus, $R_t = R_{h \cdot t}$ as required.

The proof for L_t is similar. □

If $\sigma(t) = r(k)$, then $\langle M_t, E_{t \cdot k}\rangle_{\mathscr{C}}$ is the subspace of C_k spanned by inner products of elements in M_t with elements of $E_{t \cdot k}$. Then given $k \in K$, we let $\bigoplus_{\sigma(t)=r(k)} \langle M_t, E_{t \cdot k}\rangle_{\mathscr{C}}$ denote the subspace of C_k generated by the summands. Then

$$\langle \mathscr{M}, \mathscr{E}\rangle_{\mathscr{C}} := \coprod_{k \in K} \bigoplus_{\sigma(t)=r(k)} \langle M_t, E_{t \cdot k}\rangle_{\mathscr{C}} = \coprod_{k \in K} \bigoplus_{\sigma(t)=r(k)} \langle M_t, \overline{E_t \cdot C_k}\rangle_{\mathscr{C}} \tag{3-2}$$

where the closure takes place in the Banach space $E_{t \cdot k}$.

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LEMMA 3.6. *In the setting above, both* $\langle M_t, E_t \cdot C_k\rangle_{\mathscr{C}}$ *and* $\langle M_t, \overline{E_t \cdot C_k}\rangle_{\mathscr{C}}$ *are norm dense in* $R_{r(k)} \cdot C_k$*, where we have invoked Lemma 3.5 to realize that* R_t *depends only on* $r(k) = \rho(t)$.

PROOF. Clearly,

$$\langle M_t, E_t \cdot C_k\rangle_{\mathscr{C}} = \langle M_t, E_t\rangle_{\mathscr{C}} \cdot C_k \subset R_{r(k)} \cdot C_k.$$

Moreover,

$$\overline{\langle M_t, E_t\rangle_{\mathscr{C}} \cdot C_k} = \overline{R_{r(k)} \cdot C_k} = R_{r(k)} \cdot C_k.$$

This implies the first assertion.

For the second, we just need to see that $\langle M_t, \overline{E_t \cdot C_k}\rangle_{\mathscr{C}} \subset \overline{\langle M_t, E_t\rangle_{\mathscr{C}} \cdot C_k}$. To this end, suppose that (c_i) is a sequence in $E_t \cdot C_k$ converging to c in $E_{t \cdot k}$. Then for any $m \in M_t$, the sequence $(\langle m, c_i\rangle_{\mathscr{C}})$ converges to $\langle m, c\rangle_{\mathscr{C}}$ in $\mathscr{C}$ since $\langle \cdot, \cdot\rangle_{\mathscr{C}}$ is continuous on $\mathscr{E} *_\rho \mathscr{E}$. Since the convergence takes place in C_k, the convergence is in norm by Lemma 2.6. Since each $\langle m, c_i\rangle_{\mathscr{C}} \in \langle M_t, E_t\rangle_{\mathscr{C}} \cdot C_k$, the result follows. □

Using Lemma 3.6 and Equation (3-2),

$$\langle \mathscr{M}, \mathscr{E}\rangle_{\mathscr{C}} \subset \coprod_{k\in K} R_{r(k)} \cdot C_k. \tag{3-3}$$

Moreover, $\langle \mathscr{M}, \mathscr{E}\rangle_{\mathscr{C}} \cap C_k$ is norm dense in $R_{r(k)} \cdot C_k$.

LEMMA 3.7. *In the current set-up,* $\mathscr{J}_{\mathscr{M}} := \coprod_{k\in K} R_{r(k)} \cdot C_k$ *is a Banach subbundle of* $\mathscr{C}$.

PROOF. As in the proof of Proposition 2.13, the issue is to see that $p\colon \mathscr{J}_{\mathscr{M}} \to K$ is open. Let U be a nonempty (relatively) open set in $\mathscr{J}_{\mathscr{M}}$. Given $k \in p(U)$, it suffices to show that given a sequence (k_i) converging to k in K, (k_i) is eventually in U. If this fails, then after passing to a subsequence and relabelling, we can assume $k_i \notin U$ for all i.

Since $(\mathscr{J}_{\mathscr{M}})_k$ is $R_{r(k)} \cdot C_k$, we can find $c \in C_r$ and $c' \in R_{r(k)}$ such that $c'c \in U$ and $p(c'c) = k$. Let $t \in T$ be such that $\sigma(t) = r(k)$. Since $R_{r(k)}$ is the closure of $\langle E_t, M_t\rangle_{\mathscr{C}}$ in $C_{r(k)}$, there is a sequence $(c_i') \subset \langle E_t, M_t\rangle_{\mathscr{C}}$ converging to c' in norm. However, then $c_i'c \to c'c$ in norm. Then $(c_i'c)$ is eventually in U. Therefore, we may as well assume that $c' = \sum_{j=1}^n \langle e_j, m_j\rangle_{\mathscr{C}}$ with each $e_j \in E_t$ and each $m_j \in M_t$.

Since Banach bundles have enough sections, we can find $f \in \Gamma(K; \mathscr{C})$, $g_j \in \Gamma(T; \mathscr{E})$ and $h_j \in \Gamma(T; \mathscr{M})$ such that $f(k) = c$, $g_j(t) = e_j$ and $h_j(t) = m_j$.

Since $r(k_i) \to r(k) = \sigma(t)$ and since σ is open, we can pass to a subsequence, relabel and assume that there are $t_i \in T$ such that $t_i \to t$ and $\sigma(t_i) = r(k_i)$. Then

$$d(t_i) := \sum_{j=1}^n \langle g(t_i), h(t_i)\rangle_{\mathscr{C}} \in R_{\sigma(t_i)} \quad \text{and} \quad d(t_i)f(k_i) \in R_{\sigma(t_i)} \cdot C_{k_i} = (\mathscr{J}_{\mathscr{M}})_{k_i}.$$

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Furthermore, $d(t_i)f(k_i) \to c'c$ in $\mathscr{J}_{\mathscr{M}}$. Hence, $(d(t_i)f(k_i))$ is eventually in U. Since p is continuous, $p(d(t_i)f(k_i)) = k_i$ is eventually in $p(U)$, which contradicts our assumptions on (k_i) and completes the proof. □

PROPOSITION 3.8. *In the current set-up,* $\mathscr{J}_{\mathscr{M}} := \coprod_{k\in K} R_{r(k)} \cdot C_k$ *is an ideal in* $\mathscr{C}$.

PROOF. In view of Proposition 2.29, we just have to show that $\mathscr{J}_{\mathscr{M}}$ is a weak ideal. For convenience, let $J_k = R_{r(k)} \cdot C_k$.

Suppose that $(c, m) \in C_l \times J_k$ with $s(l) = r(k)$. By Lemma 3.6, there is a sequence (m_i) in $\langle M_t, E_t \cdot C_k\rangle_{\mathscr{C}}$ converging to m in norm (and hence in $\mathscr{C}$). However, then $cm_i \to cm$ in $\mathscr{C} \cap C_{lk}$ in $\mathscr{C}$ and hence in norm. Since $M_t c^* \subset M_{t\cdot l^{-1}}$, $cm_i \in \langle M_{t\cdot l^{-1}}, E_{t\cdot k}\rangle_{\mathscr{C}} \subset J_{lk}$ (using Equations (3-2) and (3-3)). Since J_{kl} is closed in norm in C_{kl}, it follows that $cm \in \mathscr{J}_{\mathscr{M}}$.

A similar argument shows that $mc \in \mathscr{J}_{\mathscr{M}}$ if $(m, c) \in J_k \times C_l$ with $s(k) = r(l)$. Hence, $\mathscr{J}_{\mathscr{M}}$ is a weak ideal as claimed. □

PROPOSITION 3.9. *We retain the current set-up. Let* $\mathscr{J}$ *be an ideal in* $\mathscr{C}$. *Then* $\mathscr{M} = \mathscr{E} \cdot \mathscr{J}$ *is a Banach* $\mathscr{B}$–$\mathscr{J}$*-submodule of* $\mathscr{E}$ *and* $\mathscr{J}_{\mathscr{M}} = \mathscr{J}$. *Hence,* $\mathscr{J} \mapsto \mathscr{E} \cdot \mathscr{J}$ *is a lattice isomorphism of the collection of ideals in* $\mathscr{C}$ *to the collection of closed* $\mathscr{B}$–$\mathscr{C}$*-submodules of* $\mathscr{E}$.

PROOF. By Proposition 3.4, $\mathscr{M} := \mathscr{E} \cdot \mathscr{J}$ is a full Banach $\mathscr{B}$–$\mathscr{J}$-submodule and $M_t = (\mathscr{E} \cdot \mathscr{J})_t = E_t \cdot J_{\sigma(t)}$. Thus, applying the Rieffel correspondence to E_t,

$$R_{\sigma(t)} = \overline{\operatorname{span}}\langle E_t, M_t\rangle_{\mathscr{C}} = \overline{\operatorname{span}}\langle E_t, E_t \cdot J_{\sigma(t)}\rangle_{\mathscr{C}} = J_{\sigma(t)}.$$

Thus, in Equation (3-3), $R_{r(k)} = J_{r(k)}$. Therefore,

$$\mathscr{J}_{\mathscr{M}} = \coprod_{k\in K} R_{r(k)} \cdot C_k = \coprod_{k\in K} J_{r(k)} \cdot C_k = \mathscr{J},$$

where the last equality comes from Equation (2-5) of Lemma 2.26.

Now suppose that $\mathscr{M}$ is a full closed $\mathscr{B}$–$\mathscr{C}$-submodule. Then Proposition 3.8 implies that $\mathscr{J}_{\mathscr{M}}$ is an ideal in $\mathscr{C}$ with $(\mathscr{J}_{\mathscr{M}})_k = R_{r(k)} \cdot C_k$. Let $\mathscr{M}' := \mathscr{E} \cdot \mathscr{J}_{\mathscr{M}}$. In particular, if $u \in K^{(0)}$, $(\mathscr{J}_{\mathscr{M}})_u = R_u \cdot C_u = R_u$. Thus, $(\mathscr{J}_{\mathscr{M}})_k = R_{r(k)} \cdot C_k = C_k \cdot R_{s(k)}$ by Lemma 2.26. Then $\mathscr{M}'_t = (\mathscr{E} \cdot \mathscr{J}_{\mathscr{M}})_t = E_t \cdot R_{\sigma(t)}$. This means

$$\begin{aligned} R'_{\sigma(t)} &= \overline{\langle E_t, \mathscr{M}'_t\rangle_{\mathscr{C}}} \\ &= \overline{\langle E_t, E_t\rangle_{\mathscr{C}} \cdot R_{\sigma(t)}} \\ &= \overline{\langle E_t, E_t\rangle_{\mathscr{C}} \cdot R_{\sigma(t)}} \\ &= \overline{C_{\sigma(t)} \cdot R_{\sigma(t)}} = R_{\sigma(t)}. \end{aligned}$$

Therefore, $M'_t = E_t \cdot R_{\sigma(t)} = E_t \cdot R'_{\sigma(t)} = M_t$. Hence, $\mathscr{E} \cdot \mathscr{J}_{\mathscr{M}} = \mathscr{M}$. □

By symmetry, we have a lattice isomorphism $\mathscr{K} \mapsto \mathscr{K} \cdot \mathscr{E}$ between the ideals in $\mathscr{B}$ and the full Banach $\mathscr{B}$–$\mathscr{C}$-submodules of $\mathscr{E}$. Then we have the following Rieffel correspondence for Fell bundle equivalence.

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THEOREM 3.10. *Suppose that $q_{\mathscr{E}}\colon \mathscr{E} \to T$ is a Fell-bundle equivalence between $p_{\mathscr{B}}\colon \mathscr{B} \to H$ and $p_{\mathscr{C}}\colon \mathscr{C} \to K$. Then there are lattice isomorphisms among the ideals of $\mathscr{B}$, the full Banach $\mathscr{B}$–$\mathscr{C}$-submodules of $\mathscr{E}$ and the ideals of $\mathscr{C}$. The correspondences are given as follows.*

(a) *If $\mathscr{J}$ is an ideal in $\mathscr{C}$, then the corresponding full Banach $\mathscr{B}$–$\mathscr{C}$-submodule is*

$$\mathscr{E}\cdot\mathscr{J} = \bigcup_{t\in T} E_t \cdot J_{\sigma(t)}.$$

(b) *If $\mathscr{M}$ is a full Banach $\mathscr{B}$–$\mathscr{C}$-submodule, then for each $t \in T$, M_t is a $L_{\rho(t)}$–$R_{\sigma(t)}$-imprimitivity bimodule for ideals $L_{\rho(t)} = \overline{{}_{\mathscr{B}}\langle M_t, E_t\rangle}$ in $B_{\rho(t)}$ and $R_{\sigma(t)} = \overline{\langle E_t, M_t\rangle_{\mathscr{C}}}$ in $\mathscr{C}_{\sigma(t)}$. Then the corresponding ideals $\mathscr{J}_{\mathscr{M}}$ in $\mathscr{C}$ and $\mathscr{K}^{\mathscr{M}}$ in $\mathscr{B}$ are given by*

$$\mathscr{J}_{\mathscr{M}} = \bigcup_{k\in K} R_{r(k)}\cdot C_k \quad \textit{and} \quad \mathscr{K}^{\mathscr{M}} = \bigcup_{h\in H} L_{r(h)}\cdot B_h.$$

(c) *If $\mathscr{K}$ is an ideal in $\mathscr{B}$, then the corresponding $\mathscr{B}$–$\mathscr{C}$-submodule is*

$$\mathscr{K}\cdot\mathscr{E} = \bigcup_{t\in T} K_{\rho(t)}\cdot E_t.$$

The following is a generalization of [RW98, Proposition 3.24].

COROLLARY 3.11. *Suppose that $q_{\mathscr{E}}\colon \mathscr{E} \to T$ is a Fell-bundle equivalence between $p_{\mathscr{B}}\colon \mathscr{B} \to H$ and $p_{\mathscr{C}}\colon \mathscr{C} \to K$. If $\mathscr{J}$ is an ideal in $\mathscr{C}$, then the corresponding ideal $\mathscr{K}$ in $\mathscr{B}$ is $\coprod_{h\in H} H_{r(h)}\cdot B_h$, where*

$$H_{\rho(t)} = \overline{{}_{\mathscr{B}}\langle E_t\cdot J_{\sigma(t)}, E_t\rangle}. \tag{3.4}$$

PROOF. If $\mathscr{J}$ is an ideal in $\mathscr{C}$, then according to Theorem 3.10(a), the corresponding full Banach $\mathscr{B}$–$\mathscr{C}$-submodule is $\mathscr{M} = \mathscr{E}\cdot\mathscr{J}$. Then using Theorem 3.10(b), the corresponding ideal $\mathscr{K}$ in $\mathscr{B}$ is $\bigcup_{h\in H} L_{r(h)}\cdot B_h$, where $L_{r(h)}$ is given by the right-hand side of Equation (3-4) for any $t \in T$ such that $\rho(t) = r(h)$. This gives the result. □

4. Extending the Rieffel correspondence

Now we want to state and prove the analogues for Fell bundles of parts (c) and (d) of Theorem 2.1.

PROPOSITION 4.1. *Suppose that $q_{\mathscr{E}}\colon \mathscr{E} \to T$ is a Fell-bundle equivalence between $p_{\mathscr{B}}\colon \mathscr{B} \to H$ and $p_{\mathscr{C}}\colon \mathscr{C} \to K$. Suppose that $\mathscr{J}$ is an ideal in $\mathscr{C}$, and that $\mathscr{M}$ and $\mathscr{K}$ are the corresponding full Banach $\mathscr{B}$–$\mathscr{C}$-submodule in $\mathscr{E}$ and ideal in $\mathscr{B}$. Then $\mathscr{M}$ is a Fell-bundle equivalence between $\mathscr{K}$ and $\mathscr{J}$.*

PROOF. Since $\mathscr{M}$ is a $\mathscr{B}$–$\mathscr{C}$-submodule of $\mathscr{E}$, we clearly have a left $\mathscr{K}$-action and a right $\mathscr{J}$-action satisfying condition (E1).

For condition (E2), we claim that it suffices to let ${}_{\mathscr{K}}\langle e, f\rangle = {}_{\mathscr{B}}\langle e, f\rangle$ and $\langle e, f\rangle_{\mathscr{J}} = \langle e, f\rangle_{\mathscr{C}}$. To see this, note that if $(e, f) \in \mathscr{M} *_\sigma \mathscr{M}$, then we can assume $(e, f) \in M_t \times M_{h^{-1}\cdot t}$ for some $h \in H$ and $t \in T$. However, $M_t = K_{\rho(t)} \cdot E_t$. Additionally, we have ${}_{\mathscr{B}}\langle K_{\rho(t)} \cdot E_t, M_{h^{-1}\cdot t}\rangle = K_{\rho(t)} \cdot {}_{\mathscr{B}}\langle E_t, M_{h^{-1}\cdot t}\rangle \subset K_{\rho(t)} \cdot B_h = K_h$. Therefore, ${}_{\mathscr{K}}\langle\cdot,\cdot\rangle$ is $\mathscr{K}$-valued. Similarly, $\langle\cdot,\cdot\rangle_{\mathscr{J}}$ is $\mathscr{J}$-valued. The rest of condition (E2) follows from the given properties of ${}_{\mathscr{B}}\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\mathscr{C}}$.

For condition (E3), the fact that M_t is a $K_{\rho(t)} - J_{\sigma(t)}$-imprimitivity bimodule follows from the Rieffel correspondence (part (c) of Theorem 2.1). □

Proposition 4.2. *Let $q_{\mathscr{E}} : \mathscr{E} \to T$ be an equivalence between $p_{\mathscr{B}} : \mathscr{B} \to H$ and $p_{\mathscr{C}} : \mathscr{C} \to K$. Suppose that $\mathscr{J}$ is an ideal in $\mathscr{C}$ and that $\mathscr{M}$ and $\mathscr{K}$ are the corresponding full Banach $\mathscr{B}$ – $\mathscr{C}$-submodule in $\mathscr{E}$ and ideal in $\mathscr{B}$, respectively. Then the quotient Banach bundle $\mathscr{E}/\mathscr{M}$ is an equivalence between $\mathscr{B}/\mathscr{K}$ and $\mathscr{C}/\mathscr{J}$.*

Proof. We let $q^{\mathscr{K}} : \mathscr{B} \to \mathscr{B}/\mathscr{K}$ and $q^{\mathscr{J}} : \mathscr{C} \to \mathscr{C}/\mathscr{J}$ be the quotient maps. Then the given left and right actions of $\mathscr{B}$ and $\mathscr{C}$ on $\mathscr{E}$ induce left and right actions of $\mathscr{B}/\mathscr{K}$ and $\mathscr{C}/\mathscr{J}$ on $\mathscr{E}/\mathscr{M}$ in the expected way:

$$q^{\mathscr{K}}(b) \cdot q(e) = q(b \cdot e) \quad \text{and} \quad q(e) \cdot q^{\mathscr{J}}(c) = q(e \cdot c)$$

assuming that $b \cdot e$ and $e \cdot c$ are defined.

To see that these actions are continuous, we use the fact that q, $q^{\mathscr{K}}$ and $q^{\mathscr{J}}$ are open as well as continuous (Proposition 2.16). Suppose that $q(e_i) \to q(e)$ while $q^{\mathscr{K}}(b_i) \to q^{\mathscr{K}}(b)$ with $b_i \cdot e_i$ defined for all i. We need to verify that $q(b_i \cdot e_i) \to q(b \cdot e)$. For this, it suffices to see that every subnet has a subnet converging to $q(b \cdot e)$. However, after passing to a subnet and relabelling, the openness of the quotient maps means we can pass to another subnet and assume that $e'_i \to e$ and $b'_i \to b$ with $q(e'_i) = q(e)$ and $q^{\mathscr{K}}(b'_i) = q^{\mathscr{K}}(b_i)$. Then the continuity of the quotient maps implies that $q(b_i \cdot e_i) = q(b'_i \cdot e'_i) \to q(b \cdot e)$ as required.

We also have

$$\begin{aligned}\|q(b \cdot e)\| &\le \inf\{\|b' \cdot e'\| : q^{\mathscr{K}}(b') = q^{\mathscr{K}}(b) \text{ and } q(e') = q(e)\}\\ &\le \inf\{\|b'\|\|e'\| : q^{\mathscr{K}}(b') = q^{\mathscr{K}}(b) \text{ and } q(e') = q(e)\}\\ &= \|q^{\mathscr{K}}(b)\|\|q(e)\|.\end{aligned}$$

Therefore, $\mathscr{B}/\mathscr{K}$ acts on the left of $\mathscr{E}/\mathscr{M}$. The argument for the right action is similar.

Now we need to verify the axioms in Definition 2.21. Axiom (E1) is immediate since $\mathscr{E}$ is an equivalence. For Axiom (E2), we define

$$\langle q(e), q(f)\rangle_{\mathscr{C}/\mathscr{J}} := q^{\mathscr{J}}(\langle e, f\rangle_{\mathscr{C}}) \quad \text{and} \quad {}_{\mathscr{B}/\mathscr{K}}\langle q(e), q(f)\rangle = q^{\mathscr{K}}({}_{\mathscr{B}}\langle e, f\rangle).$$

It is not hard to check that these pairings are well defined. Then properties (i), (ii), (iii) and (iv) follow from the corresponding properties for $\mathscr{E}$ and the observation that

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the quotient maps are multiplicative. The continuity follows using the continuity and openness of the quotient maps as we did above for the left and right actions.

Of course, Axiom (E3) is clear. □

5. At the C^*-level

Since the previous exposition did not require it, we have purposely avoided discussing the Fell-bundle C^*-algebras that are associated to a Fell bundle. However, there is an obvious question: how is our Rieffel correspondence for ideals in equivalent Fell bundles related to the standard Rieffel correspondence for ideals in Morita equivalent C^*-algebras? In order that there be C^*-algebras, we now have to assume our groupoids have Haar systems. To apply the equivalence theorem, that is, [MW08, Theorem 6.4], we also need our Fell bundles to be separable.

We return to the set-up in Section 3: we let $q_{\mathscr{E}}\colon \mathscr{E} \to T$ be an equivalence between the separable Fell bundles $p_{\mathscr{B}}\colon \mathscr{B} \to H$ and $p_{\mathscr{C}}\colon \mathscr{C} \to K$. In particular, T is a (H, K)-equivalence (although it is not required, we note that T must be second countable since H and K are [Wil19, Proposition 2.53]) and we let $\rho\colon T \to H^{(0)}$ and $\sigma\colon T \to K^{(0)}$ be the open moment maps.

Then the equivalence theorem implies that $C^*(H^{(0)}; \mathscr{B})$ and $C^*(K^{(0)}; \mathscr{C})$ are Morita equivalent via an imprimitivity bimodule X, which is the completion of $\mathsf{X}_0 := \Gamma_c(T, \mathscr{E})$ with the actions and inner products given in [MW08, Theorem 6.4]. Then we can let

$$\mathsf{X}\text{–Ind} : \mathcal{I}(C^*(K^{(0)}; \mathscr{C})) \to \mathcal{I}(C^*(H^{(0)}; \mathscr{B}))$$

be the classical Rieffel lattice isomorphism.

If $\mathscr{J}$ is an ideal in $\mathscr{C}$, then as shown in [IW12, Lemma 3.5], the identity map ι induces an isomorphism of $C^*(K^{(0)}; \mathscr{J})$ onto the ideal $\mathrm{Ex}(\mathscr{J})$, which is the closure of $\iota(\Gamma_c(K^{(0)}; \mathscr{C}))$ in $C^*(K^{(0)}; \mathscr{C})$.

Let $\mathscr{J}$ be an ideal in $\mathscr{C}$ and $\mathscr{K}$ the corresponding ideal in $\mathscr{B}$ as in Theorem 3.10. The goal here is to establish that the two Rieffel correspondences are compatible in that

$$\mathsf{X}\text{–Ind}(\mathrm{Ex}(\mathscr{J})) = \mathrm{Ex}(\mathscr{K}). \tag{5-1}$$

By [RW98, Proposition 3.24], the left-hand side of Equation (5-1) is

$$\overline{\mathrm{span}}\{{}_*\langle\langle x \cdot b, y\rangle\rangle : x, y \in \mathsf{X} \text{ and } b \in \mathrm{Ex}(\mathscr{J})\},$$

where $\langle\cdot,\cdot\rangle$ is the $\Gamma_c(H^{(0)}; \mathscr{B})$-valued inner product from [MW08, Equation (6.3) in Theorem 6.4]. In particular, if $x, y \in \mathsf{X}_0$ and $b \in \Gamma_c(K^{(0)}; \mathscr{J})$, then provided $\rho(t) = s(h)$,

$${}_*\langle\langle x \cdot b, y\rangle\rangle(h) = \int_K {}_{\mathscr{B}}\langle x \cdot b(h \cdot t \cdot k), y(t \cdot k)\rangle \, d\lambda_K^{\sigma(t)}(k), \tag{5-2}$$

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where according to [MW08, Equation (6.2) in Theorem 6.4],

$$x \cdot b(h \cdot t \cdot k) = \int_K x(h \cdot t \cdot kl) b(l^{-1}) \, d\lambda_K^{s(k)}(l). \tag{5-3}$$

Let $\mathscr{M} = \mathscr{E} \cdot \mathscr{J} = \mathscr{K} \cdot \mathscr{E}$. Note that $\mathscr{M}$ is a $\mathscr{K} - \mathscr{J}$-equivalence. Then the integrand in Equation (5-3) is in the Banach space $M_{h \cdot t \cdot k}$ for all l. Hence,

$$x \cdot b(h \cdot t \cdot k) \in M_{h \cdot t \cdot k} = K_{r(h)} \cdot E_{h \cdot t \cdot k}.$$

Plugging into Equation (5-2), and using condition (E2)(iii) of Definition 2.21, we clearly have

$${}_*\langle\langle x \cdot b, y \rangle\rangle(h) \in K_{r(h)} \cdot B_h = K_h.$$

It follows that

$$\mathsf{X}\text{–Ind}(\mathrm{Ex}(\mathscr{J})) \subset \mathrm{Ex}(\mathscr{K}).$$

However, we can also work with $\mathsf{X}\text{–Ind}^{-1}$. Then

$$\mathsf{X}\text{–Ind}^{-1}(\mathrm{Ex}(\mathscr{K})) = \overline{\mathrm{span}}\{\langle\langle x, c \cdot y \rangle\rangle_* : x, y \in \mathsf{X} \text{ and } c \in \mathrm{Ex}(\mathscr{K})\}.$$

A similar argument to the above shows that

$$\mathsf{X}\text{–Ind}^{-1}(\mathrm{Ex}(\mathscr{K})) \subset \mathrm{Ex}(\mathscr{J}). \tag{5-4}$$

Now Equation (5-1) follows by applying X–Ind to both sides of Display (5-4).

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