Can. J. Math., Vol. XXXII, No. 5, 1980, pp. 1261–1265

THE POINCARÉ SERIES OF STRETCHED COHEN-MACAULAY RINGS

JUDITH D. SALLY

There are relatively few classes of local rings (R, m) for which the question of the rationality of the Poincaré series

$$P_R(t) = \sum_{i=0}^{\infty} \dim_k \mathrm{Tor}_i^R(k, k)t^i,$$

where $k = R/m$, has been settled. (For an example of a local ring with non-rational Poincaré series see the recent paper by D. Anick, "Construction of loop spaces and local rings whose Poincaré—Betti series are non-rational", C. R. Acad. Sc. Paris *290* (1980), 729–732.) In this note, we compute the Poincaré series of a certain family of local Cohen-Macaulay rings and obtain, as a corollary, the rationality of the Poincaré series of d-dimensional local Gorenstein rings (R, m) of embedding dimension at least $e + d - 3$, where e is the multiplicity of R. It follows that local Gorenstein rings of multiplicity at most five have rational Poincaré series.

Recall [**1**] that the embedding dimension v of a d-dimensional local Cohen-Macaulay ring (R, m) of multiplicity e satisfies $d \leqq v \leqq e + d - 1$. If (R, m) is a local Cohen-Macaulay ring of embedding dimension d or $d + 1$ it is, of course, well known that $P_R(t)$ is rational. If (R, m) is a local Cohen-Macaulay ring of embedding dimension $e + d - 1$ or a local Gorenstein ring of embedding dimension $e + d - 2$, then $P_R(t)$ is also rational [**6**] but, as we shall see, such local rings are "stretched" so the rationality of $P_R(t)$ in these two cases will follow from the result in this paper.

Let (R, m) be a local Artin ring of length e and embedding dimension v $(= \dim_{R/m}(m/m^2))$. Then $m^{e-v+1} = 0$. R is said to be *stretched*, cf. [**7**], if $e - v$ is the least integer i such that $m^{i+1} = 0$. If R is not a field, R is stretched if and only if m^2 is principal. If (R, m) is a d-dimensional local Cohen-Macaulay ring of multiplicity e, R is said to be *stretched* if there is a minimal reduction $\mathbf{x} = x_1, \ldots, x_d$ of m (i.e., there exist d elements $x_1, \ldots, x_d$ of m such that $m^{r+1} = (x_1, \ldots, x_d)m^r$ for some non-negative integer r, cf. [**4**]) such that $R/\mathbf{x}R$ is stretched.

We will compute the Poincaré series of stretched local Cohen-Macaulay rings. We have not developed new methods for computing $\mathrm{Tor}_i^R(k, k)$ but rather we show that the structure of a stretched Cohen-Macaulay

Received September 11, 1978. The author is grateful to the Sloan Foundation and to the National Science Foundation for support during the preparation of this work.

ring is such that the computation of $\mathrm{Tor}_i^R(k, k)$ yields to "old" methods.

We use three changes of ring. The first is a result of Avramov and Levin [**3**] and of Rahbar-Rochandel [**5**].

(1) [**3**], [**5**]. *If (R, m) is a zero-dimensional local Gorenstein ring of embedding dimension greater than* 1, *then*

$$P_R(t) = P_{R/(0:m)}(t)/1 + t^2P_{R/(0:m)}(t).$$

(2) [**2**]. *Let (R, m) be a local ring and let $x \in m\backslash m^2$ such that $xm = 0$, then*

$$P_R(t) = P_{R/xR}(t)/1 - tP_{R/xR}(t).$$

(3) [**8**]. *Let x be a nonzero divisor in the local ring (R, m). If $x \in m\backslash m^2$, then*

$$P_R(t) = (1 + t)P_{R/xR}(t).$$

If $x \in m^2$, then

$$P_R(t) = (1 - t^2)P_{R/xR}(t).$$

We begin the computation by examining the structure of stretched Artin local rings. Let (R, m) be a stretched local Artin ring of length e and embedding dimension $e - h$. The structure of R is essentially determined by the dimension r of the socle $(0 : m)$ of R over R/m.

If $h = 1$, then $m = (0 : m)$ and $r = e - 1$. Assume that $h > 1$, i.e., that $m^2 \neq 0$. We have $m^h \subseteq (0 : m)$. If $m^h \neq (0 : m)$, i.e., if $r > 1$, there are $r - 1$ elements $w_1, \ldots, w_{r-1}$ in $m\backslash m^2$ such that $w_1, \ldots, w_{r-1}$ and some generator of m^h form a basis of $(0 : m)$ over R/m. It is clear that we may choose elements $z_1, \ldots, z_{e-h-r+1}$ in $m\backslash m^2$ so that $w_1, \ldots, w_{r-1}, z_1, \ldots, z_{e-h-r+1}$ is a minimal basis for m. Note that

$$(\tilde{R}, \tilde{m}) = (R/(w_1, \ldots, w_{r-1})R, m/(w_1, \ldots, w_{r-1})R)$$

is a stretched Gorenstein ring.

If $h = 2$, $m^3 = 0$ and for all $i, j \in \{1, \ldots, e - h - r + 1\}$ either $z_iz_j = 0$ or $z_iz_jR = m^2$. Moreover for each such i there is a j such that $z_iz_j \neq 0$.

If $h > 2$, there is an index i such that $m^2 = z_i^2R$. Otherwise, $z_i^2 \in m^3$ for all i and there indices p, q such that $m^2 = z_pz_qR$. This gives $m^3 = z_p^2z_qR \subseteq m^4$ and the contradiction $m^3 = 0$. Next, note that we may assume that $z_iz_j = 0$ for $j \neq i$ in $\{1, \ldots, e - h - r + 1\}$. For if $z_iz_j \neq 0$, $z_iz_j = uz_i^n$ with u a unit in R and $n > 1$. Then $z_i(z_j - uz_i^{n-1}) = 0$ so we may take $z_j - uz_i^{n-1}$ instead of z_j. In summary, we have the following theorem.

THEOREM 1. *Let (R, m) be a stretched local Artin ring of length e, embedding dimension $e - h$ and $\dim_{R/m}(0 : m) = r$. Assume that $h > 2$ and $e - h - r > 0$. Then there is a basis $w_1, \ldots, w_{r-1}, z_1, \ldots, z_{e-h-r+1}$ for m having the following properties*:

(i) $w_i m = 0$ *for all* $i \in \{1, \ldots, r-1\}$,

(ii) $m^s = z_1{}^s R$ *for* $s > 1$, *and* $z_1 z_j = 0$ *for* $j \in \{2, \ldots, e-h-r+1\}$,

(iii) *for* $i, j \in \{2, \ldots, e-h-r+1\}$, *either* $z_i z_j = 0$ *or there is a unit* u_{ij} *in* R *so that* $z_i z_j = u_{ij} z_1{}^h$. *Moreover, for each such* i, *there is a* j *in* $\{2, \ldots, e-h-r+1\}$ *so that* $z_i z_j \neq 0$. *If the characteristic of* R/m *is not* 2 *and if* R *is a homomorphic image of a regular local ring of dimension* $e-h$, *then there is such a regular local ring* S *with maximal ideal* n *generated by* $W_1, \ldots, W_{r-1}, Z_1, \ldots, Z_{e-h-r+1}$ *and units* u_p *in* S *such that*

$$R = S/(\{W_iW_j, W_iZ_1, W_iZ_p, Z_1Z_p, Z_pZ_q, Z_p{}^2 - u_pZ_1{}^h;\ i, j \in \{1, \ldots, r-1\}; p \neq q \in \{2, \ldots, e-h-r+1\}\}).$$

Proof. Only the final statement remains to be proved. This follows because $(\tilde{R}, \tilde{m})$ is a stretched Gorenstein ring. The images of $z_2, \ldots, z_{e-h-r+1}$ span the vector space $(0 : \tilde{m}^2)/\tilde{m}^{h-1}$ over $\tilde{R}/\tilde{m}$ and this vector space supports a nonsingular inner product induced by the products of the images of the z_p's in $(\tilde{R}, \tilde{m})$. If characteristic R/m is not 2, the inner product can be diagonalized.

Remark. If $h = 2$ and characteristic R/m is not 2, then a minimal basis for m can be diagonalized in an analogous fashion, cf. [**6**].

If (R, m) is a stretched local Artin ring of length e, embedding dimension $e - h, h > 2$, and $\dim_{R/m}(0 : m) = r$, a basis $w_1, \ldots, w_{r-1}, z_1, \ldots, z_{e-h-r+1}$ for m as in Theorem 1 will be called a *standard basis*. Such a basis for a stretched Gorenstein local ring was constructed in a slightly different way in [**7**].

Now we compute the Poincaré series of a d-dimensional stretched local Cohen-Macaulay ring. Recall that from (3) we get

(4) *If* (R, m) *is an Artin local ring with nonzero principal maximal ideal, then*

$$P_R(t) = 1/(1-t).$$

Also recall that the type of a d-dimensional local Cohen-Macaulay ring (R, m) is by definition $\dim_{R/m} \mathrm{Ext}_R{}^d(R/m, R)$.

THEOREM 2. *Let* (R, m) *be a* d*-dimensional stretched local Cohen-Macaulay ring of multiplicity* e, *embedding dimension* $e + d - h$, $1 \leqq h \leqq e$, *and type* r. *Then*

$$P_R(t) = \begin{cases} (1+t)^d/1-(e-h)t, & \text{if } r = e-h \\ (1+t)^d/1-(e-h)t+t^2, & \text{if } r \neq e-h. \end{cases}$$

Proof. Let $\mathbf{x} = x_1, \ldots, x_d$ be a minimal reduction of m such that $R/\mathbf{x}R$ is stretched. $P_R(t) = (1+t)^d P_{R/\mathbf{x}R}(t)$ by (3) so we may assume that $d = 0$, i.e., that (R, m) is a stretched local Artin ring. If $e - h = 1$, then $r = 1$, m is nonzero principal and (4) applies. Assume $e - h > 1$.

If $h = 1$, then $r = e - 1$, $m = (0 : m)$ is a vector space over R/m and

$$P_R(t) = \sum_{i=0}^{\infty} (e-1)^i t^i = 1/(1-(e-1)t).$$

Assume that $h > 1$. We take a basis $w_1, \ldots, w_{r-1}, z_1, \ldots, z_{e-h-r+1}$ for m with $w_1, \ldots, w_{r-1} \in (0 : m)$ and we set

$$(\tilde{R}, \tilde{m}) = (R/(w_1, \ldots, w_{r-1})R, m/(w_1, \ldots, w_{r-1})R).$$

By (2) and induction,

$$P_R(t) = P_{\tilde{R}}(t)/1 - (r-1)tP_{\tilde{R}}(t).$$

If $r = e - h$, we apply (4). We assume that $r \neq e - h$ so that $(\tilde{R}, \tilde{m})$ is a zero-dimensional Gorenstein ring with nonprincipal maximal ideal. By (1),

$$P_{\tilde{R}}(t) = P_S(t)/1 + t^2 P_S(t),$$

where

$$(S, n) = (\tilde{R}/(0 : \tilde{m}), \tilde{m}/(0 : \tilde{m})).$$

If $h = 2$, S has maximal ideal of square zero so

$$P_S(t) = 1/(1-(e-r-1)t)$$

and a simple computation gives the required result. Finally, take $h > 2$. We assume that the basis $w_1, \ldots, w_{r-1}, z_1, \ldots, z_{e-h-r+1}$ is standard and let $\bar{z}_1, \ldots, \bar{z}_{e-h-r+1}$ be the images of the z's in S. For $2 \leqq p \leqq e - h - r + 1$, $\bar{z}_p n = 0$. Consequently, again by (2) and induction,

$$P_S(t) = P_{\tilde{S}}(t)/1 - (e-h-r)tP_{\tilde{S}}(t),$$

where

$$(\tilde{S}, \tilde{n}) = (S/(\bar{z}_2, \ldots, \bar{z}_{e-h-r+1})S, n/(\bar{z}_2, \ldots, \bar{z}_{e-h-r+1})S).$$

But $(\tilde{S}, \tilde{n})$ has nonzero principal maximal ideal so $P_{\tilde{S}}(t) = 1/(1-t)$. Another simple computation then gives the desired result and concludes the proof.

COROLLARY 3. *Let (R, m) be a d-dimensional local Gorenstein ring of multiplicity e and embedding dimension $e + d - h$. If $h = 1$, 2 or 3, $P_R(t)$ is rational. More precisely, the following statements hold.*

(i) *If $h = 1$ and $e = 1$, $P_R(t) = (1+t)^d$. If $h = 1$ and $e > 1$, then $e = 2$ and $P_R(t) = (1+t)^d/(1-t)$.*

(ii) *If $h = 2$ or 3 and $e - h = 1$, then $P_R(t) = (1+t)^d/(1-t)$. If $h = 2$ or 3 and $e - h > 1$, then $P_R(t) = (1+t)^d/(1-(e-h)t+t^2)$.*

Proof. We may assume that R/m is infinite so that there exists $\mathbf{x} = x_1, \ldots, x_d$ a minimal reduction of m. We also assume $e > 1$. Then the fact that the socle of $R/\mathbf{x}R$ is one dimensional over R/m gives, in each of the cases $h = 1, 2, 3$ that $m/\mathbf{x}R \neq 0$ and that $(m/\mathbf{x}R)^2$ is principal so that R is stretched and Theorem 2 applies.

Many examples of stretched local rings can be found in [**6**] and [**7**]. We conclude this note with the following:

Example. If (R, m) is a d-dimensional local Gorenstein ring of multiplicity at most 5, then $P_R(t)$ is rational. For let the embedding dimension of R be $e + d - h$, where $1 \leqq h \leqq e$. Since $e \leqq 5$, the only case not covered by Corollary 3 is $h = 4$ and $e = 5$. But this case is no problem since the embedding dimension must then be $d + 1$.

References

1. S. S. Abhyankar, *Local rings of high embedding dimension*, Amer. J. Math. *89* (1967), 1073–1077.

2. T. H. Gulliksen and G. Levin, *Homology of local rings*, Queen's Papers in Pure and Applied Math *20* (Queen's University, Kingston, Ontario, 1969).

3. G. Levin, *Lectures on Golod homomorphisms*, preprint No. *15* (1976), Matematiska institutionem Stockholms universitet.

4. D. G. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Camb. Phil. Soc. *50* (1954), 145–158.

5. H. Rahbar-Rochandel, *Relation entre la serie de Betti d'un anneau local de Gorenstein R et celle de anneau R/socle* (R), C. R. Acad. Sc. Paris *284* (1977), 651–654.

6. J. D. Sally, *Cohen-Macaulay local rings of maximal embedding dimension*, J. of Alg. *56* (1979), 168–183.

7. ——— *Stretched Gorenstein rings*, J. London Math. Soc. *20* (1979), 19–26.

8. J. Tate, *Homology of Noetherian rings and local rings*, Ill. J. of Math. *1* (1957), 14–25.

Northwestern University,
Evanston, Illinois

https://doi.org/10.4153/CJM-1980-094-0 Published online by Cambridge University Press