A NOTE ON THE SEPARABILITY OF AN ORDERED SPACE

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An *open interval* of a simply ordered set S is a subset I of S such that either
(1) for some $a \in S$, $I = \{x \in S | x < a\}$,
(2) for some $a \in S$, $I = \{x \in S | a < x\}$, or
(3) for some $a \in S$ and $b \in S$, $I = \{x \in S | a < x < b\}$.
A simply ordered set with its interval topology (i.e., the topology in which "neighborhood of x" means "open interval containing x") will be called an *ordered space*.

It is shown that a connected ordered space S is separable provided it satisfies Souslin's condition **(2)** (i.e., there exists no uncountable collection of mutually exclusive open subsets of S) and there is a countable family F of continuous functions of S into itself such that each point p of S is a limit point of $\{f(p) | f \in F\}$. If S is not assumed to satisfy Souslin's condition, the existence of such a family F does not imply that S is separable; however, if no element of F has a fixed point or if the elements of F can be arranged in a sequence $\{f_n\}$ such that for each point p of S, $\{f_n(p)\} \to p$, then S must satisfy Souslin's condition and hence must be separable.

Notation. If S is an ordered space and a and b are elements of S such that $a < b$, then ab will denote the open interval of S with end points a and b; i.e., $ab = \{x \in S | a < x < b\}$. As usual, $S \times S$ will denote the topological product of S with itself and if f is a function of S into itself, then $G(f)$ will denote the "graph" of f in $S \times S$; that is, $G(f) = \{(x, f(x)) | x \in S\}$.

THEOREM 1. *Suppose S is a connected ordered space and F is a countable family of continuous functions of S into itself such that each point p of S is a limit point of $\{f(p) | f \in F\}$. If S satisfies Souslin's condition, then S is separable.*

LEMMA. *Under the above hypothesis, if a, b and c are elements of S such that $a < b < c$, then for some f in F, $G(f)$ intersects the subset $(ab \times bc) + (bc \times ab)$ of $S \times S$.*

Proof of Lemma. Since b is a limit of $\{f(b) | f \in F\}$, there exists an element f of F such that $f(b) \in ac$ and $f(b) \neq b$. Suppose $f(b) \in bc$. Since f is continuous, there exists a neighborhood V of b such that $f(V) \subset bc$. Since S is connected, there exists a point x of V such that $x \in ab$. Since $x \in ab$ and $f(x) \in bc$, $(x, f(x)) \in ab \times bc$. Similarly, if $f(b) \in ab$, there exists an x in bc such that $(x, f(x)) \in bc \times ab$. Hence $G(f)$ intersects $(ab \times bc) + (bc \times ab)$.

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Proof of Theorem 1. In **(1)** it is shown that if S is not separable there exists an uncountable collection $\mathfrak{U}$ of mutually exclusive open subsets of $S \times S$ such that if $U \in \mathfrak{U}$, there exist points a, b, c of S such that $a < b < c$ and $U = (ab \times bc) + (bc \times ab)$. Since F is countable and for each U in $\mathfrak{U}$ there is an f in F such that $G(f)$ intersects U, there exists an f_0 in F such that $G(f_0)$ intersects each of uncountably many elements of $\mathfrak{U}$. Since for each U in $\mathfrak{U}$, the intersection of $G(f_0)$ and U is an open subset of $G(f_0)$, $G(f_0)$ contains uncountably many mutually exclusive open sets. But since f_0 is continuous, $G(f_0)$ is homeomorphic to S and hence satisfies Souslin's condition.

That Theorem 1 does not remain true if the requirement that S satisfy Souslin's condition be dropped is shown by the following example.

Let the points of S be the ordered pairs (x, y) of real numbers such that $0 \leqslant y \leqslant 1$ and let (x_1, y_1) precede (x_2, y_2) in S if and only if either $x_1 < x_2$ or $x_1 = x_2$ and $y_1 < y_2$. Then S is a connected ordered space but is not separable since it does not satisfy Souslin's condition. For each positive integer n, let

$$g_n(x, y) = \left(x, y - \frac{y}{n} + \frac{y^2}{n}\right), \quad h_n(x, y) = \left(x + \frac{(-1)^n}{n}, y\right).$$

Then for each n, both g_n and h_n are continuous functions of S into itself. If $p = (x, y)$ then p is a limit point of $\{g_n(p)\}$ if $0 < y < 1$ and p is a limit point of $\{h_n(p)\}$ if $y = 0$ or $y = 1$. Hence if $F = \{\{g_n\} + \{h_n\}\}$, then for each p in S, p is a limit point of $\{f(p) | f \in F\}$.

THEOREM 2. *If S is a connected ordered space and F is a countable family of continuous functions of S into itself such that* (1) *no element of F has a fixed point and* (2) *each point p of S is a limit point of $\{f(p) | f \in F\}$, then S is separable.*

LEMMA 1. *Every uncountable subset of a connected ordered space has a limit point.*

Proof of Lemma 1. Suppose S is a connected ordered space and M is an uncountable subset of S which has no limit point. Let a be a point of S and suppose there are uncountably many points x of M such that $a < x$. Since S is connected, every infinite bounded subset of M has a limit point. Hence if b is a point of S such that $a < b$, then ab contains not more than a finite number of points of M. It follows that there exists a sequence $\{x_n\}$ of points of M such that for each n, $a < x_n < x_{n+1}$. Since for each n there are not more than a finite number of points of M in the interval ax_n, but there are uncountably many points x of M such that $a < x$, the sequence $\{x_n\}$ is bounded and hence has a limit point.

LEMMA 2. *If S is a connected ordered space and G is an uncountable collection of mutually exclusive intervals of S, then there exist a point p of S and an infinite countable subcollection G' of G such that every neighborhood of p contains all but a finite number of the elements of G'.*

Proof of Lemma 2. Let M denote a set consisting of one and only one point of each element of G. Since M is uncountable, it has a limit point. If q is a limit point of M, then either every open interval containing q contains a point x of M such that $x < q$ or every open interval containing q contains a point x of M such that $q < x$. Hence there exists a sequence $\{x_n\}$ of points of M such that either (1) for each n, $x_n < x_{n+1} < q$ or (2) for each n, $q < x_{n+1} < x_n$. Since S is connected and $\{x_n\}$ is bounded, $\{x_n\}$ has both a greatest lower bound in S and a least upper bound in S. In case (1), let p be the greatest lower bound of $\{x_n\}$ and in case (2), let p be the least upper bound of $\{x_n\}$. In either case it is clear that the sequence $\{x_n\}$ converges to p. For each n, let g_n denote the element of G which contains x_n. It is easily seen that since the elements of G are mutually exclusive, every neighborhood of p contains all but a finite number of the intervals $g_1, g_2, g_3, \ldots$.

Proof of Theorem 2. Suppose G is an uncountable collection of mutually exclusive open intervals of S. For each element g of G, there exists an element f_g of F such that $f_g(g)$ intersects g. Hence there exist an element f of F and an uncountable subcollection G' of G such that for each element g of G', $f(g)$ intersects g. From Lemma 2 it follows that there exist a point p of S and a sequence $g_1, g_2, g_3, \ldots$ of elements of G' such that every neighborhood of p contains all but a finite number of the intervals $g_1, g_2, g_3, \ldots$. For each n, let p_n be a point of g_n such that $f(p_n) \in g_n$. Then $\{p_n\} \to p$ and hence since f is continuous, $\{f(p_n)\} \to f(p)$. But since for each n, $f(p_n) \in g_n$, $\{f(p_n)\} \to p$. Hence $f(p) = p$ and f has a fixed point. Hence S satisfies Souslin's condition. It follows from Theorem 1 that S is separable.

NOTE. If in the hypothesis of Theorem 2 the elements of F are required to be homeomorphisms of S onto itself, it can be shown by a direct argument that for each point p of S the set

$$\{f^n(p) | f \in F, \ n = 0, \pm 1, \pm 2, \ldots\}$$

is a countable dense subset of S.

THEOREM 3. *If S is a connected ordered space and $\{f_n\}$ is a sequence of continuous functions of S into itself such that for each point p of S, $\{f_n(p)\} \to p$ and for infinitely many integers n, $f_n(p) \neq p$, then S is separable.*

Proof. Suppose G is an uncountable collection of mutually exclusive open intervals of S. For each element g of G there exist a point p_g of g and an integer n_g such that for $n \geqslant n_g$, $f_n(p_g) \in g$. Hence there exist an integer n and an uncountable subcollection G' of G such that for each element g of G', $n = n_g$. By Lemma 2 to Theorem 2, there exist a point p of S and a sequence $g_1, g_2, g_3, \ldots$ of elements of G' such that every neighborhood of p contains all but a finite number of the intervals $g_1, g_2, g_3, \ldots$. It follows as in the proof of Theorem 2 that for $k \geqslant n$, $f_k(p) = p$. But this is impossible by hypothesis. Hence S satisfies Souslin's condition. Hence, by Theorem 1, S is separable.

References

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