TOEPLITZ OPERATORS AND ALGEBRAS OF BOUNDED ANALYTIC FUNCTIONS ON THE DISK

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1. Introduction. Here and throughout, A is a closed subalgebra of H^∞ that contains the disk algebra and $M(A)$ denotes the maximal ideal space of A. Because A contains the function $f_0(z) = z$, we can define the fiber $M_\lambda(A)$ of $M(A)$ for $\lambda \in \partial D$ (the unit circle) in the usual way; i.e., $M_\lambda(A) = \{\phi \in M(A) : f_0(\phi) = \lambda\}$. The Bergman space $L^2_a(D)$ of the unit disk D is the $L^2(D,\ dx\,dy)$-closure of A. Let $P : L^2(D, dx\,dy) \to L^2_a(D)$ be the orthogonal projection. For $f \in L^\infty(D, dx\,dy)$, define the multiplication operator $M_f : L^2(D, dx\,dy) \to L^2(D, dx\,dy)$ by

$$M_f g = fg, \qquad g \in L^2(D, dx\,dy)$$

and define the Toeplitz operator $T_f : L^2_a(D) \to L^2_a(D)$ by

$$T_f g = PM_f g, \qquad g \in L^2_a(D).$$

Let $T(A)$ be the C^*-algebra of bounded operators on $L^2_a(D)$ generated by $\{T_f : f \in C(M(A))\}$ and let $C(A)$ be the commutator ideal of $T(A)$. Denote the maximal ideal space of $T(A)/C(A)$ by $E(A)$. The McDonald-Sundberg theorem (**[8]**) asserts that $E(H^\infty)$ consists of the one point Gleason parts of $M(H^\infty)$. At the other extreme, if A is the disk algebra, then $E(A)$ is the unit circle by a theorem of Coburn (**[2]**). The unit circle consists of the one point Gleason parts of $M(A)$ if A is the disk algebra, so a natural question arises: does $E(A)$ always consist of the one point Gleason parts of $M(A)$? As we see below, the answer to this question is no. However, we can see that $E(A)$ consists of the one point Gleason parts of $M(A)$ when $A = H^\infty \cap C(D \cup K)$, where K is a closed set in ∂D. Thus there is a class of algebras (albeit of somewhat limited interest) that includes the disk algebra and H^∞ allows us to "interpolate" between Coburn's theorem and the McDonald-Sundberg theorem.

For any unexplained notions from the theory of function algebras (e.g., maximal ideal space, Gleason parts, Shilov boundary) see Gamelin's book **[5]**.

2. Sundberg's criterion and applications. For $f \in A$ and $g \in L^\infty(D, dx\,dy)$ we can easily see that $T_{\bar{f}g} = T_{\bar{f}}T_g$ and $T_{fg} = T_g T_f$. By an argument in Chapter 7 of **[4]**, the commutator ideal $C(A)$ coincides with the semicommutator ideal of $T(A)$. This allows us to use the main result of **[11]** to assert that $E(A) = \{\phi \in M(A) : f \in C(M(A))$ with $f(\phi) = 0$ implies M_f is not bounded below on $L^2_a(D)\}$. As a consequence of his proof, Sundberg obtains the spectral inclusion $f(E(A)) \subset \sigma(T_f)$ for $f \in C(M(A))$ without any ancillary work and so each theorem identifying $E(A)$ has an immediate corollary giving a spectral inclusion result. This criterion of Sundberg for membership in $E(A)$ is the crucial ingredient in what follows. Indeed, part of the purpose of this note is to display the utility of Sundberg's criterion.

Let $R : M(H^\infty) \to M(A)$ be the restriction map $R\phi = \phi \mid A$, $\phi \in M(H^\infty)$.

THEOREM 2.1. *$R(E(H^\infty)) = E(A)$.*

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Proof. Let $\phi \in E(H^\infty)$ and suppose $f \in C(M(A))$ with $f(R\phi) = 0$. Then $f \circ R \in C(M(H^\infty))$ and $f \circ R$ vanishes at ϕ whereby $M_{f\circ R}$ is not bounded below on $L^2_a(D)$. Because $f = f \circ R$ on D, $M_f = M_{f\circ R}$ and so $R\phi \in E(A)$.

For the other inclusion, suppose $\phi \in M(A)$ does not belong to $R(E(H^\infty))$. Clearly $E(A)$ is contained in the $M(A)$-closure of D and so without loss of generality, assume ϕ lies in the $M(A)$-closure of D. There are disjoint open sets W, V in $M(A)$ such that $\phi \in W$ and $R(E(H^\infty)) \subset V$. Now $R^{-1}(W)$ and $R^{-1}(V)$ are disjoint open sets in $M(H^\infty)$ and $W \cap D = R^{-1}(W) \cap D$ so $U = W \cap D$ is an open set in D whose $M(H^\infty)$-closure does not meet $E(H^\infty)$. By exercise 2 from Chapter X of [6], there is a finite union S of interpolating sequences such that $U \subset \{z \in D : \rho(z, w) < \frac{1}{2}$ for some $w \in S\}$, where

$$\rho(z, w) = \left|\frac{z - w}{1 - \bar{w}z}\right|.$$

By [7], it follows that there is some $C > 0$ such that

$$\int_{D\setminus U} |g(z)|^2\,dx\,dy \geq C\,\|g\|^2, \qquad z = x + iy$$

for all $g \in L^2_a(D)$. Take $f \in C(M(A))$ with $f(\phi) = 0$ and $f \equiv 1$ off W. Then

$$\|M_f g\|^2 \geq \int_{D\setminus U} |g(z)|^2\,dx\,dy \geq C\,\|g\|^2$$

for $g \in L^2_a(D)$. By Sundberg's Criterion, $\phi \notin E(A)$.

This theorem characterizes the set $E(A)$, but is unsatisfying in comparison to Coburn's theorem and the McDonald-Sundberg theorem which characterize $E(A)$ (for A = disk algebra or H^∞) in terms of a natural notion from the theory of function algebras. Such a result does not seem possible in our context. Example 2.2 shows that $E(A)$ does not have to consist of one point Gleason parts of $M(A)$ and the other examples point out more subtle phenomena.

EXAMPLE 2.2. Let u be an inner function not in the disk algebra and let A be the algebra generated by the disk algebra and u. Suppose $\{\psi_0\}$ is a one point Gleason part of $M(H^\infty)$ and that $u(\psi_0) = 0$. Then $R\psi_0$ belongs to a nontrivial Gleason part of $M(A)$. Thus $E(A)$ does not consist of one point Gleason parts of $M(A)$.

Proof. Let $\rho(\phi, \psi) = \sup\{|f(\phi)| : f \in A,\ \|f\|_\infty \leq 1$ and $f(\psi) = 0\}$ denote the pseudo-hyperbolic distance on $M(A)$. We want to find $\phi \in M(A)$ such that $\phi \neq R\psi_0$ and $\rho(\phi, R\psi_0) < 1$. Let $\lambda \in \partial D$ such that $\psi_0 \in M_\lambda(H^\infty)$ and take $\phi \in M_\lambda(H^\infty)$ such that $0 < |u(\phi)| < 1$. This can be done because the cluster set of u at λ is the closed unit disk. Now A contains a dense set of elements of the form $F = g + fp(u)$ where f and g belong to the disk algebra and p is a polynomial with $p(0) = 0$ and $\|p\|_\infty = 1$. Suppose $F(R\psi_0) = 0$. Then $0 = F(R\psi_0) = g(\lambda) + f(\lambda)p(u(\psi_0)) = g(\lambda)$ so $F = fp(u)$ on $M_\lambda(A)$. Thus $|f(\lambda)| \leq \|F\|_\infty$ and taking $\|F\|_\infty \leq 1$, we get

$$|F(R\phi)| = |f(\lambda)|\,|p(u(\phi))| \leq |p(u(\phi))| \leq |u(\phi)|$$

by Schwarz's Lemma. It follows that $\rho(R\phi, R\psi_0) = |u(\phi)| < 1$; i.e., $R\phi \neq R\psi_0$ belongs to the same Gleason part as $R\psi_0$. The McDonald-Sundberg theorem and our Theorem 2.1 imply $R\psi_0 \in E(A)$.

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EXAMPLE 2.3. Let A be the algebra from example 2.2 with

$$u(z) = \exp\frac{z+1}{z-1}.$$

Then $E(A) = (\partial D\backslash\{1\}) \cup E_1(A)$ where $E_1(A) = M_1(A) \cap E(A)$ consists of those $\phi \in M_1(A)$ such that $|u(\phi)| = 1$ or 0.

Proof. If $|u(\phi)| = 1$ then $\phi \in \partial(A)$, the Shilov boundary of A, so $\phi \in E(A)$ by ([**10**]). By Theorem 2.1, if $\phi_0 \in M_1(A) \cap R(E(H^\infty))$, then $\phi_0 \in E(A)$. It is well known (see Chapter X of [**6**]) that if u is a singular inner function and $\psi \in M(H^\infty)$ satisfies $u(\psi) = 0$, then $\{\psi\}$ is a Gleason part of $M(H^\infty)$. Thus $\phi_0 \in M_1(A)$ with $u(\phi_0) = 0$ implies $\phi_0 \in E(A)$, as in Example 2.2. We can say more here; namely $E(A) = \partial(A) \cup \{\phi_0\}$. To show this, we need the observation by A. Matheson that

$$b(z) = \frac{u(z) - w}{1 - \bar{w}u(z)}$$

is an interpolating Blaschke product for all $w \in D\backslash\{0\}$. Because this result is unpublished, we sketch a proof. The function b extends analytically across $\partial D\backslash\{1\}$ and so the singular factor of b must be of the form

$$S(z) = \exp t\frac{z+1}{z-1}$$

for some $t \geq 0$. If $t > 0$, then $|S(z)| \to 0$ as z tends radially to 1. But $b(z) \to -w \neq 0$ as z tends radially to 1 and so $t = 0$; i.e., b is a Blaschke product. To see that the zero sequence of b is an interpolating sequence, consider the conformal map $f(\zeta) = (\zeta - i)/(\zeta + i)$ from the upper half plane to the unit disk. Then

$$(b \circ f)^{-1}(\{w\}) = \left\{\zeta \in \mathbb{C} : \operatorname{Im} \zeta = \log\frac{1}{|w|} \text{ and } \operatorname{Re} \zeta = \arg w + 2n\pi,\ n \text{ an integer}\right\}$$

which is easily seen to be an interpolating sequence for the upper half plane and it follows that $b^{-1}(\{w\})$ is an interpolating sequence in D. Now M_b is bounded below on $L^2_a(D)$ ([**8**]). Thus if $\phi \in M_1(A)$ and $u(\phi) = w \in D\backslash\{0\}$, then $\phi \notin E(A)$.

EXAMPLE 2.4. Let u be an inner function belonging to the little Bloch space. That is, $\lim_{|z|\to 1} (1 - |z|^2)\,|u'(z)| = 0$. Then

$$b(z) = \frac{u(z) - w}{1 - \bar{w}u(z)}$$

belongs to the little Bloch space for all $w \in D$. Such a function b cannot be a finite product of interpolating Blaschke products ([**6**, Chapter X, Exercise 11]) and, by [**8**], it follows that there is some $\phi \in E(H^\infty)$ such that $b(\phi) = 0$. Let A be the algebra generated by the disk algebra and u. Then $E(A) = M(A)\backslash D$ by Theorem 2.1.

Note that the following algebra was studied by Dawson [**3**], who showed that D is not dense in $M(A)$.

EXAMPLE 2.5. (The Gramophone algebra) Let A be the algebra generated by the disk algebra and the outer function $G(z) = (1 - z)^i$ where we take $-(\pi/2) < \arg(1 - z) < \pi/2$. Then $E(A) = \partial(A)$.

Proof. The cluster set of G at 1 is $\{z \in \mathbb{C}: e^{-\pi/2} \le |z| \le e^{\pi/2}\}$. If $\phi \in M_1(A)$ and $|G(\phi)| < e^{-\pi/2}$, then ϕ lies outside the $M(A)$-closure of D and so $\phi \notin E(A)$. Suppose $\phi \in M_1(A)$ with $e^{-\pi/2} < |G(\phi)| < e^{\pi/2}$. Now the level sets of $|G|$ in D are secants terminating at 1 and so ϕ lies in the $M(A)$-closure of the interior of a Stolz angle at 1. Let U be a truncation (so that $U \cap \partial D = \{1\}$) of this region. By **[7]**, there is some $C > 0$ such that

$$\int_{D \setminus U} |g(z)|^2 \, dx \, dy \ge C \, \|g\|^2 \quad \text{for} \quad g \in L_a^2(D).$$

As in the proof of Theorem 2.1 we see that $\phi \notin E(A)$. Now $\partial(A) = (\partial D \setminus \{1\}) \cup \{\phi \in M_1(A): |G(\phi)| = e^{\pm\pi/2}\}$ and so $E(A) = \partial(A)$ by **[10]**. It is worth noting that $\phi \in M_1(A)$ and $|G(\phi)| = e^{-\pi/2}$ imply that ϕ lies in a nontrivial Gleason part of $M(A)$.

Let K be a closed set in ∂D and let $A = H^\infty \cap C(D \cup K)$, the algebra of bounded analytic functions on D that extend continuously to K. For $\lambda \in K$ we clearly get $M_\lambda(A) = \{\lambda\}$. Suppose $\lambda \notin K$. Take a closed arc Γ in ∂D centered at λ such that $\Gamma \cap K = \varnothing$. There is a conformal map h of D onto a domain in D such that $h(\Gamma)$ is an arc in ∂D with $h(\lambda) = \lambda$, $h(K) \subset D$ and such that h extends continuusly to D ([**9**, Chapter V, Section 7]). Given $f \in H^\infty$, $g = f \circ h$ belongs to A and $\lim_{z \to \lambda} \sup_{|z|<1} |f(z) - g(z)| = 0$. Thus $A \mid M_\lambda(H^\infty) = H^\infty \mid M_\lambda(H^\infty)$ and $R \mid M_\lambda(H^\infty)$ is the identity map. Put $E_\lambda(H^\infty) = E(H^\infty) \cap M_\lambda(H^\infty)$. Applying Theorem 2.1 we obtain the following.

THEOREM 2.6. *Let K be a closed set in ∂D and let $A = H^\infty \cap C(D \cup K)$. Then $E(A) = \left(\bigcup_{\lambda \notin K} E_\lambda(H^\infty)\right) \cup K$. In particular, $E(A)$ consists of the one point Gleason parts of $M(A)$.*

REMARKS. Any of the examples 2.2–2.5 suffices to show that $E(A)$ does not necessarily consist of one point Gleason parts of $M(A)$. In fact, in examples 2.2–2.4, $E(A)$ is not even contained in the closure of the one point Gleason parts. Of course, the pseudohyperbolic distance from $R\phi$ to $R\psi$ in $M(A)$ is no larger than the pseudohyperbolic distance from ϕ to ψ in $M(H^\infty)$ whence $E(A)$ contains the one point Gleason parts of $M(A)$ that lie in the $M(A)$-closure of D.

In examples 2.2–2.4, we see the restriction map R sending one point Gleason parts of $M(H^\infty)$ to points belonging to nontrivial parts of $M(A)$, but in each of 2.2–2.4, we have $R(\partial(H^\infty)) = \partial(A)$ which consists of peak points for these algebras. However, in example 2.5 we see that $R(\partial(H^\infty))$ can meet a nontrivial Gleason part of $M(A)$. The first example (2.2) shows that Theorem 2.6 does not directly generalize to the algebras considered by Chang and Marshall in **[1]** and examples 2.3 and 2.4 serve to point out the difficulty of formulating a nice result in this context. To be more specific, we see that $E(A)$ can be as small (example 2.3) or as large (example 2.4) as is allowed by Theorem 2.1 and example 2.2.

Sundberg shows that $E(A)$ is the largest set in $M(A)$ such that $f(E(A)) \subset \sigma(T_f)$ for all $f \in C(M(A))$ ([**11**]). Because $T_z^* T_z - T_z T_z^*$ is a compact operator on $L_a^2(D)$ (see [**8**]) and the compact operators form a minimal closed two-sided ideal in the bounded operators ([**4**]), it follows that $C(A)$ contains the compact operators on $L_a^2(D)$. From standard facts about the spectrum ([**4**]), we obtain $f(E(A)) \subset \sigma_e(T_f)$ for all $f \in C(M(A))$.

If we define our operators on the Hardy space H^2 of the unit circle instead of the Bergman space, then ([**4**, Chapter 7]) the sets $E(A)$ in question turn out to be $\partial(H^\infty)$ when $A = H^\infty$ and $\partial D = \partial(A)$ when A is the disk algebra. This Hardy space case is better behaved than the situation for the Bergman space. In fact ([**10**]), if μ is a probability measure with $\operatorname{supp}\mu = \partial(A)$ then $\partial(A) = E(A) = \{\phi \in M(A) : f \in C(M(A))$ with $f(\phi) = 0$ implies M_f is not bounded below on the $L^2(\mu)$-closure of $A\}$.

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