UNIQUENESS OF COMPLEMENTING PAIRS IN PARTIALLY ORDERED ABELIAN GROUPS

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Let A and B denote subsets of an abelian group $(G, +)$. The pair (A, B) is called a *complementing pair* for $A+B$ provided each element of the sum is uniquely represented. A complementing pair (A, B) for C is denoted $(A, B) \sim C$.

A pair $(A, B) \sim C$ is said to be *unique* provided that for any pair $(A', B) \sim C$ we have $A = A'$. R. M. McLeod and R. Spira [1] showed that complementing pairs for certain subsets of $N = \{0, 1, 2, \cdots\}$, including N and $N_n = \{0, 1, 2, \cdots, n-1\}$, are unique. In this paper we shall first generalize this result to certain subsets of a partially ordered abelian group and then deduce several related results as special cases. The necessity of the conditions placed on the subsets is shown by constructing a non-unique complementing pair that violates only a single condition.

By a partially ordered abelian group (additively written), we mean an abelian group $(G, +)$ together with an order relation $\leqq$ (reflexive, antisymmetric, and transitive) such that $x \leqq y$ implies $x+z \leqq y+z$ for all $x, y, z \in G$. The basic result is now formulated.

THEOREM. *Let $(G, +, \leqq)$ be a partially ordered abelian group. Let A_1, A_2, and B be subsets of G satisfying.*

1. $(A_1, B) \sim A_1+B$
2. $(A_2, B) \sim A_2+B$
3. $A_1+B = A_2+B$
4. *B has a minimum element.*
5. *Each nonempty subset of $A_1 \cup A_2$ has a minimal element. Then $A_1 = A_2$.*

PROOF. Let $D = A_1 \triangle A_2$, where '$\triangle$' denotes the symmetric difference of the two sets. Assuming the proposition is false, we have $D \neq \emptyset$. Then by condition 5, D has a least element. We denote the least element of D by d. Without loss of generality assume $d \in A_1 \cap A_2'$, where $A_2' = \{x | x \in G \text{ and } x \notin A_2\}$. Further, let us assume that b_1 is the minimum element of set B. Then $d+b_1 \in A_1+B$, and since $A_1+B = A_2+B$ we have $d+b_1 \in A_2+B$.

Thus $d+b_1 = a_2+b_2$ for elements $a_2 \in A_2$ and $b_2 \in B$. If $a_2 \in A_1$, then $d+b$ has two representations in A_1+B - contrary to condition 2. Hence $a_2 \in A_1' \cap A_2 \subset D$

which implies $d < a_2$. Now

$$d < a_2 \Rightarrow d+b_2 < a_2+b_2.$$

Therefore

$$d+b_2 < a_2+b_2 = d+b_1$$

and so $b_2 < b_1$, contradicting the minimality of b_1. We conclude that our original assumption of $D \neq \emptyset$ is false, and hence $A_1 = A_2$ as desired.

Let us now see an application of this result by considering subsets of Z^n, where Z denotes the set of integers and n is a positive integer. Characterizing properties of complementing pairs in Z^n (principally in Z^2) may be found in [2]. To prove the uniqueness results in Z^n we first need to define a partially ordered abelian group on Z^n. The ordered triple $(Z^n, +, \leqq)$, where $+$ is n-dimensional vector addition and $\leqq$ denotes the lexicographic order, is such a group. The corollaries follow.

COROLLARY 1. *Let A_1, A_2, and B be subsets of any subset of Z^n which is bounded below. If $(A_1, B) \sim A_1+B$, $(A_2, B) \sim A_2+B$, and $A_1+B = A_2+B$, then $A_1 = A_2$.*

COROLLARY 2. *If $(A, B) \sim A+B \subset N^n$, for any positive integer n, then (A, B) is unique.*

COROLLARY 3. *If $(A, B) \sim A+B \subset Z^n$, for any positive integer n, and if $A+B$ is finite, then (A, B) is unique.*

The proof of each of the above results is strongly dependent upon the complementing pairs satisfying conditions 4 and 5 of the basic theorem. The necessity of these conditions is shown by the following subsets of Z. Let m be any positive integer greater than 1. Then defining

$$A = N_m \text{ and } B = \{km | k \in Z\},$$

we have $(A, B) \sim Z$. However, this pair lacks uniqueness as is easily seen by replacing A by any other complete residue system modulo m.

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References

[1] R. M. McLeod and R. Spira, 'Direct sum decomposition of the non-negative integers', *Unpublished*, (1966).

[2] R. T. Hansen, 'Complementing Pairs of Subsets of the Plane', *Duke Math. J.* 36 (1969), 441–449.

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