Compositio Mathematica **121**: 183-204, 2000.

Solvable Lie Algebras, Lie Groups and Polynomial Structures

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(Received: 17 March 1998; accepted in final form: 16 November 1998)

Abstract. In this paper, we study polynomial structures by starting on the Lie algebra level, then passing to Lie groups to finally arrive at the polycyclic-by-finite group level. To be more precise, we first show how a general solvable Lie algebra can be decomposed into a sum of two nilpotent subalgebras. Using this result, we construct, for any simply connected, connected solvable Lie group G of dim n, a simply transitive action on $\mathbb{R}^n$ which is polynomial and of degree $\leqslant n^3$. Finally, we show the existence of a polynomial structure on any polycyclic-by-finite group Γ, which is of degree $\leqslant h(\Gamma)^3$ on almost the entire group ($h(\Gamma)$ being the Hirsch length of Γ).

Mathematics Subject Classifications (1991): 17B30, 22E25.

Key words: solvable Lie group, solvable Lie algebra, polycyclic group, polynomial structure.

1. Introduction

In this paper, we continue our study of polynomial structures (see [7, 9–11]). Let us briefly recall the context of this topic.

In 1977 John Milnor [14] formulated the following question, which became widely known as Milnor's conjecture: Does any torsion-free polycyclic-by-finite group occur as the fundamental group of a complete, affinely flat manifold M?

An equivalent, but more algebraic formulation is the following: Is it true that any torsion-free, polycyclic-by-finite group Γ admits a morphism $\varphi\colon \Gamma \to \mathrm{Aff}(\mathbb{R}^n)$ ($\mathrm{Aff}(\mathbb{R}^n)$ is the group of invertible affine mappings of $\mathbb{R}^n$) letting Γ act properly discontinuously and with compact quotient on $\mathbb{R}^n$? Such a morphism φ has been called *an affine structure* on Γ.

For a long time, one expected that the answer to the above question was positive, until 1992 when Yves Benoist ([2–3]) produced a counter-example, which was later even generalized to a family of examples ([4–5]).

The study of polynomial structures arose in the search for the best possible alternative to Milnor's question. As the reader might suspect, a *polynomial structure* on a group Γ is a properly discontinuous action $\varphi\colon \Gamma \to \mathrm{P}(\mathbb{R}^n)$ with compact

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quotient. Here, $P(\mathbb{R}^n)$ stands for the group of polynomial diffeomorphisms of $\mathbb{R}^n$, i.e. the maps $\mu: \mathbb{R}^n \to \mathbb{R}^n$, which are bijective and for which both μ and μ^{-1} are expressed by means of polynomials (in the usual coordinates of $\mathbb{R}^n$).

It turned out that even in the cases where the affine structures fail to exist, there is always a polynomial structure. In fact, the two most important results obtained thus far are:

(1) Let Γ be any polycyclic-by-finite group, then Γ admits a polynomial structure (which is of canonical type, see Section 4 and [9]). It is known that this polynomial structure is of bounded degree, meaning that there is an upperbound on the degrees of the polynomials expressing this polynomial structure. However, there is no knowledge at all concerning the value of such an upperbound.

(2) Let Γ be any polycyclic-by-finite group, then Γ has a subgroup Γ', which is of finite index in Γ and which admits a polynomial structure of degree $\leqslant h(\Gamma)$ $(= h(\Gamma'))$. (In this paper, we will always use $h(\Gamma)$ to denote the Hirsch length of a polycyclic-by-finite group Γ (see [10]).) Unfortunately, it is not known if the polynomial structure on Γ' can be extended to the whole group Γ.

Knowing this result, one might formulate the following conjecture ([7]):

CONJECTURE 1.1. *There exists a function* ν: $\mathbb{N} \to \mathbb{N}$, (*most likely* $\nu(n) = n$ *will do*), *such that for any polycyclic-by-finite group* Γ, *there exists a polynomial structure of degree* $\leqslant \nu(h(\Gamma))$ *on* Γ.

This paper should be seen as a first step towards a possible solution of this problem. Indeed, we prove a result which can be seen as a combination of the two results mentioned above, namely we show that any polycyclic-by-finite group Γ admits a polynomial structure which restricts to a polynomial structure of degree $\leqslant h(\Gamma)^3$ on a subgroup of finite index. So, although the bound we get for the degree of the polynomial structure on the finite index subgroup is worse than the bound obtained in the second result mentioned above, we do know that the polynomial structure really exists on the whole group Γ and not only on the finite index subgroup.

Finally, we wish to remark that the results obtained in Sections 2 and 3 can also be useful, if you are not immediately interested in polynomial structures of polycyclic-by-finite subgroups. In fact, in Section 2 we describe how any solvable Lie algebra $\mathfrak{g}$ can be seen as a sum $\mathfrak{g} = \mathfrak{n} + \mathfrak{c}$ of two nilpotent subalgebras, where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$. In the following section, we use this result to decompose a simply connected, connected solvable Lie group G as a product $G = NC$, of two nilpotent Lie subgroups, where N is the nilradical of G. Moreover, we describe some interesting coordinate systems on such a group G, based on this decomposition and show how the product in G is expressed by means of a function which is polynomial in some of its variables. This leads to the construction of a simply transitive and polynomial action of G on some space $\mathbb{R}^n$.

2. A Decomposition of Solvable Lie Algebras

In this section we will proof a decomposition result for solvable Lie algebras. In fact, the result we present here is the analogue of the result for polycyclic groups as can be found in Segal's book ([16]) (from page 45 to page 51). The proofs we present here are translations from the proofs on the group level to the Lie algebra level. The basis of the decomposition result of solvable Lie algebras is the following cohomology vanishing result.

PROPOSITION 2.1. *Let* $\mathfrak{n}$ *be any nilpotent Lie algebra over any field. If* M *is a finite dimensional* $\mathfrak{n}$*-module with* $H^0(\mathfrak{n}, M) = 0$*, then* $H^i(\mathfrak{n}, M) = 0,\ \ \forall i \in \mathbb{N}$.

Proof. We proceed by induction on the dimension n of M. If $n = 0$, then $M = 0$ and the proposition is trivially true. So suppose that $n > 0$ and that the proposition holds for modules of lower dimension.

As $H^0(\mathfrak{n}, M) = \{m \in M \mid x \cdot m = 0,\ \ \forall x \in \mathfrak{n}\} = 0$, there exists a $x \in \mathfrak{n}$ and a $m \in M$ such that $x \cdot m \neq 0$. It follows that the ideal $\mathfrak{k} = \{x \in \mathfrak{n} \mid x \cdot m = 0,\ \ \forall m \in M\}$ of $\mathfrak{n}$ is not the whole of $\mathfrak{n}$.

Choose an element $x_0 \in \mathfrak{n}$ such that its image $\overline{x}_0$ in $\mathfrak{n}/\mathfrak{k}$ is non zero and belongs to the center $Z(\mathfrak{n}/\mathfrak{k})$ of the Lie algebra $\mathfrak{n}/\mathfrak{k}$. We claim that $x_0 \cdot M$ is a $\mathfrak{n}$-submodule of M. Indeed, for any $x \in \mathfrak{n}$ and any $m \in M$, we have that

$$x \cdot (x_0 \cdot m) = x_0 \cdot (x \cdot m) - \underbrace{[x_0, x]}_{\in \mathfrak{k}} \cdot m = x_0 \cdot (x \cdot m) \in x_0 \cdot M.$$

If we let K denote the kernel of the $\mathfrak{n}$-linear map $p: M \to x_0 \cdot M: m \mapsto x_0 \cdot m$, we obtain the following short exact sequence of $\mathfrak{n}$-modules $0 \to K \to M \to x_0 \cdot M \to 0$.

This short exact sequence gives rise to the long exact sequence of cohomology spaces

$$\begin{aligned} 0 \to \underbrace{H^0(\mathfrak{n}, K)}_{=0} \to \underbrace{H^0(\mathfrak{n}, M)}_{=0} \to \underbrace{H^0(\mathfrak{n}, x_0 \cdot M)}_{=0} \to \\ \to H^1(\mathfrak{n}, K) \to H^1(\mathfrak{n}, M) \to H^1(\mathfrak{n}, x_0 \cdot M) \to \\ \to H^2(\mathfrak{n}, K) \to H^2(\mathfrak{n}, M) \to H^2(\mathfrak{n}, x_0 \cdot M) \to \cdots . \end{aligned}$$

We now distinguish two cases.

Case 1: $K \neq 0$. As $x_0 \notin \mathfrak{k}$, $x_0 \cdot M \neq 0$ and therefore both $\dim(K) < \dim(M)$ and $\dim(x_0 \cdot M) < \dim(M)$. By the induction hypothesis, we know that $H^i(\mathfrak{n}, K) = 0 = H^i(\mathfrak{n}, x_0 \cdot M)$ for all $i \in \mathbb{N}$. The long exact cohomology sequence above now implies the exactness of the sequences $0 \to H^i(\mathfrak{n}, M) \to 0,\ \ \forall i \in \mathbb{N}$ proving that $H^i(\mathfrak{n}, M) = 0,\ \ \forall i \in \mathbb{N}$.

Case 2: $K = 0$, i.e. $M = x_0 \cdot M$. In this case, $x_0 \cdot m = 0$ (for some $m \in M$) implies that $m = 0$ (since the linear transformation $M \to M: m \mapsto x_0 \cdot m$ is

https://doi.org/10.1023/A:1001738932743 Published online by Cambridge University Press

bijective). It follows immediately that $H^0(\langle x_0\rangle, M) = 0$, where $\langle x_0\rangle$ denotes the vector space (in this case subalgebra) spanned by x_0. As $\langle x_0\rangle$ is a 1-dimensional Lie algebra $H^i(\langle x_0\rangle, M) = 0$, for all $i \geqslant 2$. It is also easy to show by a direct argument, using $x_0 \cdot M = M$, that $H^1(\langle x_0\rangle, M) = 0$.

Now, let $Z_i = Z_i(\mathfrak{n})$, with $i \in \mathbb{N}$, denote the ith term of the upper central series of $\mathfrak{n}$ (i.e $Z_0 = 0$ and $Z_{i+1}(\mathfrak{n})/Z_i(\mathfrak{n}) = Z(\mathfrak{n}/Z_i(\mathfrak{n}))$). In $\mathfrak{n}$ we can consider the subalgebra $Z_{i+1} + \langle x_0\rangle$, which contains $Z_i + \langle x_0\rangle$ as an ideal. We will now show by induction on i that $H^q(Z_i + \langle x_0\rangle, M) = 0$ for all $q \in \mathbb{N}$. For $i = 0$, $Z_i + \langle x_0\rangle = \langle x_0\rangle$ and the argument is given above.

Now fix an i and assume that $H^q(Z_i + \langle x_0\rangle, M) = 0$, for all $q \in \mathbb{N}$. The first terms of the Hochschild–Serre (5-term) exact sequence for $q \geqslant 1$ are of the form

$$0 \to H^q\left(\frac{Z_{i+1} + \langle x_0\rangle}{Z_i + \langle x_0\rangle}, \underbrace{H^0(Z_i + \langle x_0\rangle, M)}_{=0}\right) \to$$

$$\to H^q(Z_{i+1} + \langle x_0\rangle, M) \to \underbrace{H^q(Z_i + \langle x_0\rangle, M)}_{=0} \to \cdots$$

implying that $H^q(Z_{i+1} + \langle x_0\rangle, M) = 0$ for $q \geqslant 1$. The case $q = 0$ is trivially true as $x_0 \in Z_{i+1} + \langle x_0\rangle$.

The proposition is now proved since $Z_i + \langle x_0\rangle = Z_i = \mathfrak{n}$ for i sufficiently large. □

We are now ready to prove the decomposition result mentioned above.

THEOREM 2.2. *Suppose that* $\mathfrak{g}$ *is a finite dimensional Lie algebra* (*over any field*) *with a nilpotent ideal* $\mathfrak{n}$, *such that the quotient* $\mathfrak{g}/\mathfrak{n}$ *is also nilpotent. Then there exists a nilpotent subalgebra* $\mathfrak{c}$ *of* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{n} + \mathfrak{c}$ (*the above sum not necessarily being a direct sum!*)

Proof. We will prove this theorem by induction on the dimension of $\mathfrak{g}$. Let us denote the quotient $\mathfrak{g}/\mathfrak{n}$ by $\mathfrak{t}$. The theorem is trivially true for a Lie algebra of dimension 1. So, let us assume that the dimension of $\mathfrak{g}$ is greater than 1 and that the theorem is valid for Lie algebras of smaller dimension.

If $\mathfrak{n} = 0$, there is nothing to show, otherwise $\mathfrak{n}/Z(\mathfrak{n})$ is a nilpotent ideal of $\mathfrak{g}/Z(\mathfrak{n})$ and $(\mathfrak{g}/Z(\mathfrak{n}))/(\mathfrak{n}/Z(\mathfrak{n})) \cong \mathfrak{g}/\mathfrak{n} = \mathfrak{t}$ is nilpotent. We know by the induction hypothesis that there exists a nilpotent subalgebra $\mathfrak{h}/Z(\mathfrak{n})$ of $\mathfrak{g}/Z(\mathfrak{n})$, with $\mathfrak{g}/Z(\mathfrak{n}) = \mathfrak{n}/Z(\mathfrak{n}) + \mathfrak{h}/Z(\mathfrak{n}) \Rightarrow \mathfrak{g} = \mathfrak{n} + \mathfrak{h}$.

Again we consider two cases:

Case 1: $Z(\mathfrak{n}) \cap Z(\mathfrak{h}) \neq 0$. If $\mathfrak{k} = Z(\mathfrak{n}) \cap Z(\mathfrak{h})$, then $\mathfrak{h}/\mathfrak{k}$ is a Lie algebra with ideal $(\mathfrak{n} \cap \mathfrak{h})/\mathfrak{k}$ inducing a quotient

$$\frac{(\mathfrak{h}/\mathfrak{k})}{((\mathfrak{n} \cap \mathfrak{h})/\mathfrak{k})} \cong \frac{\mathfrak{h}}{(\mathfrak{n} \cap \mathfrak{h})} \cong \frac{(\mathfrak{h} + \mathfrak{n})}{\mathfrak{n}} \cong \frac{\mathfrak{g}}{\mathfrak{n}} \cong \mathfrak{t},$$

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which is nilpotent. By the induction hypothesis, there exists a nilpotent subalgebra $\mathfrak{c}/\mathfrak{k}$ of $\mathfrak{h}/\mathfrak{k}$ with

$$\frac{\mathfrak{h}}{\mathfrak{k}} = (\mathfrak{n} \cap \mathfrak{h})/\mathfrak{k} + \mathfrak{c}/\mathfrak{k} \Rightarrow \mathfrak{h} = \mathfrak{n} \cap \mathfrak{h} + \mathfrak{c}.$$

As $\mathfrak{k} \subseteq Z(\mathfrak{c})$ and $\mathfrak{c}/\mathfrak{k}$ is nilpotent, we can conclude that $\mathfrak{c}$ itself is nilpotent. In other words there exists a nilpotent subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ with

$$\mathfrak{g} = \mathfrak{n} + \mathfrak{h} = \mathfrak{n} + (\mathfrak{n} \cap \mathfrak{h}) + \mathfrak{c} = \mathfrak{n} + \mathfrak{c},$$

which finishes the proof in this case.

Case 2: $Z(\mathfrak{n}) \cap Z(\mathfrak{h}) = 0$. Consider the short exact sequence of Lie algebras

$$0 \to Z(\mathfrak{n}) \to \mathfrak{h} \to \mathfrak{h}/Z(\mathfrak{n}) \to 0. \tag{1}$$

This short exact sequence gives rise to a $\mathfrak{h}/Z(\mathfrak{n})$-module structure of $Z(\mathfrak{n})$ which, for any $x \in \mathfrak{h}$ is given by $\forall z \in Z(\mathfrak{n}) : \overline{x} \cdot z = [x, z]$, where $\overline{x} = x + Z(\mathfrak{n})$ and $[x, z]$ denotes the Lie bracket in $\mathfrak{h}$. From $Z(\mathfrak{n}) \cap Z(\mathfrak{h}) = 0$ it follows that $H^0(\mathfrak{h}/Z(\mathfrak{n}), Z(\mathfrak{n})) = 0$, and thus by Proposition 2.1 also $H^i(\mathfrak{h}/Z(\mathfrak{n}), Z(\mathfrak{n})) = 0$ for any $i \in \mathbb{N}$. In particular, from $H^2(\mathfrak{h}/Z(\mathfrak{n}), Z(\mathfrak{n})) = 0$, it follows that the extension (1) must split. This implies that there exists a subalgebra $\mathfrak{c} \cong \mathfrak{h}/Z(\mathfrak{n})$ (so $\mathfrak{c}$ is nilpotent) of $\mathfrak{h}$ with $\mathfrak{h} = Z(\mathfrak{n}) \oplus \mathfrak{c}$.

Note that the direct sum decomposition above is meant to be a direct sum of vector spaces not of Lie algebras ($\mathfrak{h}$ is only a semi-direct product). Moreover, $\mathfrak{g} = \mathfrak{n} + \mathfrak{h} = \mathfrak{n} + Z(\mathfrak{n}) + \mathfrak{c} = \mathfrak{n} + \mathfrak{c}$ which finishes the proof in the second case. □

DEFINITION 2.3. (1) Let $\mathfrak{g}$ be a Lie algebra with ideal $\mathfrak{n}$. A (nilpotent, ...) almost supplement for $\mathfrak{n}$ in $\mathfrak{g}$ is a (nilpotent, ...) subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ with $\mathfrak{g} = \mathfrak{n} + \mathfrak{c}$.

(2) Let G be a (Lie) group with normal (Lie) subgroup N. A (nilpotent, ...) almost supplement for N in G is a (Lie) subgroup C of G with $G = N.C$.

COROLLARY 2.4. *Let* $\mathfrak{g}$ *be a solvable Lie algebra over a field of characteristic* 0. *Then there exists a nilpotent almost supplement* $\mathfrak{c}$ *for the nilradical* $\mathfrak{n}$ *of* $\mathfrak{g}$.

Proof. For any solvable Lie algebra $\mathfrak{g}$ over a field of characteristic zero, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent ([17]). It follows that $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{n}$. The corollary is now a direct consequence of Theorem 2.2, since $\mathfrak{g}/\mathfrak{n}$ is Abelian. □

3. Simply Transitive Polynomial Actions

In this section we will apply the results obtained thusfar to construct a simply transitive polynomial action for any simply connected, connected solvable Lie group.

PROPOSITION 3.1. *Let* N *be any connected, simply connected nilpotent Lie group of nilpotency class* $\leqslant c$ *with Lie algebra* $\mathfrak{n}$. *Suppose that there is a vector space* *decomposition* $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b}$, *where* $\mathfrak{a}$ *is an ideal of* $\mathfrak{n}$ *and* $\mathfrak{b}$ *is a sub (vector) space of* $\mathfrak{n}$. *Then for any basis* $A_1, A_2, \ldots, A_r$ *of* $\mathfrak{a}$ *and any basis* $B_1, B_2, \ldots, B_s$ *of* $\mathfrak{b}$, *the maps* $\varphi_1: \mathbb{R}^{r+s} \to N$ *and* $\varphi_2: \mathbb{R}^{r+s} \to N$ *defined by* $\forall x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s \in \mathbb{R}$:

$$\varphi_1(x_1, \ldots, x_r, y_1, \ldots, y_s) = \exp(x_1 A_1 + \cdots + x_r A_r + y_1 B_1 + \cdots y_s B_s),$$
$$\varphi_2(x_1, \ldots, x_r, y_1, \ldots, y_s) = \exp(x_1 A_1 + \cdots + x_r A_r) \exp(y_1 B_1 + \cdots y_s B_s)$$

are analytical diffeomorphisms. Moreover,

$$\varphi_1^{-1} \circ \varphi_2: \mathbb{R}^{r+s} \to \mathbb{R}^{r+s} \quad \text{and} \quad \varphi_2^{-1} \circ \varphi_1: \mathbb{R}^{r+s} \to \mathbb{R}^{r+s}$$

are polynomial of total degree $\leqslant c$ *and of degree* $\leqslant \operatorname{Max}(c-1, 1)$ *in the variables* $x_1, x_2, \ldots x_r$ (resp. $y_1, y_2, \ldots, y_s$) *alone.*

Proof. The proof of this proposition can be obtained via a straightforward application of the Campbell–Baker–Haussdorf formula. □

Remark 3.2. Let

$$p: \mathbb{R}^{r+s} \to \mathbb{R}^s: (x_1, \ldots, x_r, y_1, \ldots, y_s) \mapsto (y_1, \ldots, y_s)$$

denote the projection on the last s components. There are commutative diagrams

$$\begin{array}{ccc} \mathbb{R}^{r+s} & \xrightarrow{\varphi_1^{-1} \circ \varphi_2} & \mathbb{R}^{r+s} \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle p} & \\ & \mathbb{R}^s & \end{array} \qquad \begin{array}{ccc} \mathbb{R}^{r+s} & \xrightarrow{\varphi_2^{-1} \circ \varphi_1} & \mathbb{R}^{r+s} \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle p} & \\ & \mathbb{R}^s & \end{array}$$

i.e. the maps $\varphi_1^{-1} \circ \varphi_2$ and $\varphi_2^{-1} \circ \varphi_1$ are constant on the last s components.

The following lemma seems to be well known (e.g., see [15]). However, because of its importance to the rest of the paper we like to present it here with a short proof in full detail.

LEMMA 3.3. *Let* G *be a simply connected, connected solvable Lie group with nilradical* N. *Let* $\mathfrak{g}$ *be the Lie algebra of* G *and denote by* $\mathfrak{n}$ *the subalgebra of* $\mathfrak{g}$ *corresponding to* N. *Let* $\mathfrak{t}$ *be a subspace of* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t}$. *Then* $G/N \cong \mathbb{R}^k$ *for some* k *and there is a commutative diagram (where the upper row is a short exact sequence of Lie groups and the bottom row is a short exact sequence of Lie algebras)*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & G & \xrightarrow{p} & \mathbb{R}^k & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \exp = 1_{\mathbb{R}^k}} & & \\ 1 & \longrightarrow & \mathfrak{n} & \longrightarrow & \mathfrak{g} & \xrightarrow{p'} & \mathbb{R}^k & \longrightarrow & 1, \end{array}$$

where $\varphi: \mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \to G: (x, y) \mapsto \exp(x)\exp(y)$, $\forall x \in \mathfrak{n}$, $\forall y \in \mathfrak{t}$.

Moreover, φ *is an analytic diffeomorphism.* (p *and* p' *are the natural projection homomorphisms.*)

Proof. The commutativity of the left-hand square is obvious. The commutativity of the right-hand square follows from the fact that $p \circ \exp = \exp \circ p'$. The map φ is an analytic map, since it is a product of analytic maps. The only thing left to show is the fact that φ is a bijection. Note that the two exponential maps on the outer sides of the diagram above are bijective. We will first show that φ is a surjective map. Let $g \in G$. There exists a unique element $t \in \mathfrak{t}$ such that $\exp(p'(t)) = p'(t) = p(g)$. It follows that $p(g\exp(t)^{-1}) = 1$, implying that $g\exp(t)^{-1} \in N$, from which it follows that there exists a $n \in \mathfrak{n}$ with $\exp(n) = g\exp(t)^{-1}$. This means that we have found $n \in \mathfrak{n}$ and $t \in \mathfrak{t}$ with $\varphi(n, t) = g$.

To show the injectivity of φ, we consider $n_1, n_2 \in \mathfrak{n}$ and $t_1, t_2 \in \mathfrak{t}$ with $\varphi(n_1, t_1) = \varphi(n_2, t_2)$. The commutativity of the diagram implies that $p(\varphi(n_1, t_1)) = \exp(t_1) = p(\varphi(n_2, t_2)) = \exp(t_2) \Rightarrow t_1 = t_2$. Using this, we find that $\exp(n_1) = \exp(n_2) \Rightarrow n_1 = n_2$, which was to be shown. □

We will now use the lemma above for a suitable $\mathfrak{t}$. Consider any solvable Lie algebra $\mathfrak{g}$ (over the field $\mathbb{R}$). Then, by Theorem 2.2, $\mathfrak{g} = \mathfrak{n} + \mathfrak{c}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$ and $\mathfrak{c}$ is a nilpotent almost supplement for $\mathfrak{n}$ in $\mathfrak{g}$. Take a subspace $\mathfrak{t}$ of $\mathfrak{g}$ such that $\mathfrak{c} = (\mathfrak{n} \cap \mathfrak{c}) \oplus \mathfrak{t}$ (direct sum of vector spaces).

We can choose a basis $B_1, B_2, \ldots, B_l$ of the vector space $\mathfrak{n} \cap \mathfrak{c}$, a basis $C_1, C_2, \ldots, C_m$ of the vector space $\mathfrak{t}$ and vectors $A_1, A_2, \ldots, A_k$ of $\mathfrak{n}$ such that the total set of vectors

$$A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_l, C_1, C_2, \ldots, C_m$$

forms a basis of $\mathfrak{g}$. It also follows then that $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_l$ is a basis of $\mathfrak{n}$. After having chosen this basis for $\mathfrak{g}$, we can identify $\mathfrak{g}$ with $\mathbb{R}^{k+l+m}$. By means of the map φ of Lemma 3.3 (with respect to the decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{t}$) we can therefore analytically identify the Lie group G with $\mathbb{R}^{k+l+m}$:

$$\begin{array}{ccc}
\mathbb{R}^{k+l+m} & & (x_1, \ldots, x_k, y_1, \ldots, y_l, z_1, \ldots, z_m) \\
\downarrow & & \downarrow \\
\mathfrak{g} = \mathfrak{n} + \mathfrak{t} & & \displaystyle\sum_{i=1}^{k} x_i A_i + \sum_{i=1}^{l} y_i B_i + \sum_{i=1}^{m} z_i C_i \\
\downarrow \varphi & & \downarrow \\
G & & \exp\left(\displaystyle\sum_{i=1}^{k} x_i A_i + \sum_{i=1}^{l} y_i B_i\right) \exp\left(\displaystyle\sum_{i=1}^{m} z_i C_i\right).
\end{array}$$

The *coordinate* map

$$\mathrm{co}\colon G \to \mathbb{R}^{k+l+m} \colon \exp\left(\sum_{i=1}^{k} x_i A_i + \sum_{i=1}^{l} y_i B_i\right) \exp\left(\sum_{i=1}^{m} z_i C_i\right) \mapsto (x_1, \ldots, x_k, y_1, \ldots, z_m)$$

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obtained as the inverse of the above identification is a global chart on G.

DEFINITION 3.4. A global chart $\mathrm{co}\colon G \to \mathbb{R}^{k+l+m}$ on a simply connected, connected solvable Lie group G, obtained in the way described above will be called a *Mal'cev-like coordinate map*. For an element $g \in G$, the $(k+l+m)$-tuple $\mathrm{co}(g)$ will be referred to as the Mal'cev-like coordinates of g.

Of course, when no confusion can arise, we will call $\mathrm{co}(g)$ simply the coordinates of g. The importance of these coordinates with respect to polynomial structures becomes clearer with the following theorem.

THEOREM 3.5. *Let G be a simply connected, connected solvable Lie group equipped with a Mal'cev-like coordinate map* co. *Then, the coordinate expression for the product in G is polynomial in the second variable, i.e. the map*

$$\mu\colon\ \mathbb{R}^{k+l+m} \times \mathbb{R}^{k+l+m} \to \mathbb{R}^{k+l+m}\colon\ (\boldsymbol{\alpha}, \boldsymbol{\beta}) \mapsto \mathrm{co}(\mathrm{co}^{-1}(\boldsymbol{\alpha})\, \mathrm{co}^{-1}(\boldsymbol{\beta}))$$

is polynomial in the $k+l+m$ *components of* $\boldsymbol{\beta}$.

Moreover, $\deg(\mu) \leqslant (k+l+m)^3$*, where* $\deg(\mu)$ *denotes the total degree of* μ *in the components of* $\boldsymbol{\beta}$.

Proof. Let

$$\boldsymbol{\alpha} = (a_1, \ldots, a_k, b_1, \ldots, b_l, c_1, \ldots, c_m) \quad \text{and}$$
$$\boldsymbol{\beta} = (x_1, \ldots, x_k, y_1, \ldots, y_l, z_1, \ldots, z_m).$$

We have to show that

$$\mathrm{co}(\exp(\mathbf{aA} + \mathbf{bB}) \exp(\mathbf{cC}) \exp(\mathbf{xA} + \mathbf{yB}) \exp(\mathbf{zC}))$$

is polynomial in the variables $x_1, \ldots, x_k, y_1, \ldots, y_l, z_1, \ldots, z_m$. Here, the vector expressions such as $\mathbf{aA}$ are used to denote sums like $a_1A_1 + a_2A_2 + \cdots + a_kA_k$ and so on.

First of all remember that the exponential map $\exp\colon \mathfrak{n} \to N$ from the nilradical $\mathfrak{n}$ of $\mathfrak{g}$ to the nilradical N of G is an analytical bijection. Moreover, for any continuous automorphism σ of N, there exists a linear map l of $\mathfrak{n}$ (namely the differential of σ), such that

$$\sigma(\exp(X)) = \exp(l(X)), \quad \forall X \in \mathfrak{n}. \tag{2}$$

(For more information, see, e.g., [8].)

Let $\mathfrak{h}$ be the Lie algebra of a connected and simply connected nilpotent Lie group H of nilpotency class c_h. Suppose that $T_1, T_2, \ldots, T_j$ is a basis of $\mathfrak{h}$. It is known that there exist polynomial functions $q_i\colon \mathbb{R}^{2j} \to \mathbb{R}$ of total degree $\leqslant c_h$ for which

$$\begin{aligned} &\exp(x_1T_1 + \cdots + x_jT_j) \exp(y_1T_1 + \cdots + y_jT_j) \\ &\quad = \exp(q_1(\mathbf{x}, \mathbf{y})T_1 + \cdots + q_j(\mathbf{x}, \mathbf{y})T_j) \end{aligned} \tag{3}$$

for all $x_1, x_2, \ldots, x_j, y_1, y_2, \ldots, y_j \in \mathbb{R}$. A proof of this fact is already given in [13]. (See also [8] and [10]). We will use this fact for two simply connected nilpotent Lie groups, namely $N = \exp(\mathfrak{n})$ and the Lie group $C = \exp(\mathfrak{c}) = \exp(\mathfrak{n} \cap \mathfrak{c}) \exp(\mathfrak{t})$. Let us denote the nilpotency class of N by c_1 and the nilpotency class of C by c_2. Let $n = k + l + m$, then $c_1 \leqslant n$ and $c_2 \leqslant n$.

In the course of this proof, we will use symbols $p_{i,j}$ to denote maps from some space $\mathbb{R}^p$ to a space $\mathbb{R}^q$ which are polynomial of total degree $\leqslant j$ in the variables mentioned. The i is merely meant as an index to distinguish the maps between one another. A symbol like $p_{i,j}^{\mathbf{c}}(\mathbf{x}, \mathbf{y})$ denotes a map which is polynomial in the variables $\mathbf{x}, \mathbf{y}$ (of degree $\leqslant j$), but which depends also on $\mathbf{c}$ in a way which is possibly not polynomial. The notation $l_{i,\mathbf{c}}$ is equivalent to $p_{i,1}^{\mathbf{c}}$ and is used for maps which are linear in the variables mentioned between brackets, but which depend also on $\mathbf{c}$.

We now compute the product above:

$$\mathrm{co}(\exp(\mathbf{aA} + \mathbf{bB}) \exp(\mathbf{cC}) \exp(\mathbf{xa} + \mathbf{yB}) \exp(\mathbf{zC}))$$

$$= \mathrm{co}\left(\exp(\mathbf{aA} + \mathbf{bB}) \underbrace{\exp(\mathbf{cC}) \exp(\mathbf{xa} + \mathbf{yB}) \exp(-\mathbf{cC})}_{\text{inner auto. of } G \text{ restricts to auto. on } N} \exp(\mathbf{cC}) \exp(\mathbf{zC}) \right)$$

$$= \mathrm{co}(\exp(\mathbf{aA} + \mathbf{bB}) \exp(l_{1,\mathbf{c}}(\mathbf{x}, \mathbf{y})\mathbf{A} + l_{2,\mathbf{c}}(\mathbf{x}, \mathbf{y})\mathbf{B}) \times$$
$$\times \exp[p_{1,n}(\mathbf{c}, \mathbf{z})\mathbf{B} + p_{2,n}(\mathbf{c}, \mathbf{z})\mathbf{C}])$$

(Use formula (2) and formula (3) with $H = C$)

$$= \mathrm{co}(\exp[p_{3,n}(\mathbf{a}, \mathbf{b}, l_{1,\mathbf{c}}(\mathbf{x}, \mathbf{y}), l_{2,\mathbf{c}}(\mathbf{x}, \mathbf{y}))\mathbf{A} +$$
$$+ p_{4,n}(\mathbf{a}, \mathbf{b}, l_{1,\mathbf{c}}(\mathbf{x}, \mathbf{y}), l_{2,\mathbf{c}}(\mathbf{x}, \mathbf{y}))\mathbf{B}] \times$$
$$\times \exp[p_{5,n}(p_{1,n}(\mathbf{c}, \mathbf{z}), p_{2,n}(\mathbf{c}, \mathbf{z}))\mathbf{B}] \exp[p_{6,n}(p_{1,n}(\mathbf{c}, \mathbf{z}), p_{2,n}(\mathbf{c}, \mathbf{z}))\mathbf{C}])$$

(Use formula (3) for $H = N$ and Proposition 3.1 for C)

$$= \mathrm{co}(\exp[p_{7,n}^{\mathbf{c}}(\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y})\mathbf{A} + p_{8,n}^{\mathbf{c}}(\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y})\mathbf{B}] \times$$
$$\times \exp[p_{9,n^2}(\mathbf{c}, \mathbf{z})\mathbf{B}] \exp[p_{10,n^2}(\mathbf{c}, \mathbf{z})\mathbf{C}])$$

$$= \mathrm{co}(\exp[p_{11,n}(p_{7,n}^{\mathbf{c}}(\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}), p_{8,n}^{\mathbf{c}}(\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}), p_{9,n^2}(\mathbf{c}, \mathbf{z}))\mathbf{A} +$$
$$+ p_{12,n}(p_{7,n}^{\mathbf{c}}(\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}), p_{8,n}^{\mathbf{c}}(\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}), p_{9,n^2}(\mathbf{c}, \mathbf{z}))\mathbf{B}] \times$$
$$\times \exp[p_{10,n^2}(\mathbf{c}, \mathbf{z})\mathbf{C}])$$

$$= (p_{13,n^3}^{\mathbf{c}}(\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}, \mathbf{z}), p_{9,n^2}(\mathbf{c}, \mathbf{z})).$$

This computation shows that the map μ depends in a polynomial way on the variables $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$. Moreover, the total degree of the polynomial in these three variables is $\leqslant n^3$, where $n = k + l + m$ is the dimension of G. □

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Remark 3.6. The careful reader will have noticed that via a detailed analysis of the above proof it is possible to find an even better upper-bound for the degree of the polynomial map μ. However, the quality of such a bound does not really improve, because it remains a cubic polynomial in n. For our purposes, the existence of such a bound is more important than its actual value!

Remark 3.7. At first sight one can be disappointed that the coordinate expression for the product is only polynomial in the second (set of) variable(s) $\boldsymbol{\beta}$. However, one can not hope to get anything better because it is known that any Lie group G admitting a global chart $c: G \to \mathbb{R}^n$ (for some n), for which the product is expressed by means of a polynomial function in both variables is a nilpotent Lie group (see [1]), i.e. such a global chart does not exist for general solvable Lie groups.

To construct a simply transitive action of a simply connected solvable Lie group on a space $\mathbb{R}^n$, we can consider the action obtained by left multiplication in G and identify G with $\mathbb{R}^n$ by means of a Mal'cev-like coordinate chart. We make this explicit in the following corollary to Theorem 3.5.

COROLLARY 3.8. *Let G be a connected and simply connected solvable Lie group of dimension n equipped with a Mal'cev-like coordinate map* $\mathrm{co}: G \to \mathbb{R}^n$. *The map*

$$\rho: G \times \mathbb{R}^n \to \mathbb{R}^n: (g, \mathbf{y}) \mapsto \mathrm{co}(g \cdot \mathrm{co}^{-1}(\mathbf{y}))$$

is a simply transitive action of G on $\mathbb{R}^n$. *Moreover, the action of any element* $g \in G$ *is expressed by a polynomial of degree* $\leqslant n^3$.

Proof. This is an immediate consequence of Theorem 3.5. □

We remark here that we will improve this result in a forthcoming paper [6] by showing that any connected and simply connected solvable Lie group G admits a simply transitive polynomial action of degree $\leqslant n$, where $n = \dim(G)$. The importance of the approach we developed here will become apparent in the following section. Therefore, we need a refinement of Theorem 3.5 dealing with more general coordinate systems on G.

Let $\mathfrak{g}$ be a solvable Lie algebra equipped with a fixed basis

$$A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_l, C_1, C_2, \ldots, C_m \tag{4}$$

as on page 189 i.e. $A_1, A_2, \ldots, B_l$ is a basis of the nilradical $\mathfrak{n}$ of $\mathfrak{g}$, $B_1, B_2, \ldots, C_m$ is a basis of a nilpotent almost-supplement $\mathfrak{c}$ for $\mathfrak{n}$ in $\mathfrak{g}, \ldots$).

Now, let $D_1, D_2, \ldots, D_{k+l+m}$ be any basis of $\mathfrak{g}$. There is an invertible linear map $L = (L_1, L_2, \ldots, L_{k+l+m}) \in \mathrm{Gl}(k + l + m, \mathbb{R})$ describing the change of coordinates for these two bases

$$\sum_{i=1}^{k+l+m} x_i D_i = \sum_{i=1}^{k} L_i(\mathbf{x}) A_i + \sum_{i=1}^{l} L_{k+i}(\mathbf{x}) B_i + \sum_{i=1}^{m} L_{k+l+i}(\mathbf{x}) C_i.$$

https://doi.org/10.1023/A:1001738932743 Published online by Cambridge University Press

Using this map L we can construct another coordinate map on the corresponding simply connected solvable Lie group G by taking the inverse map of the following identification of $\mathbb{R}^{k+l+m}$ with G

$$\begin{array}{ccc}
\mathbb{R}^{k+l+m} & & (x_1, x_2, x_3, \ldots, x_{k+l+m}) \\
\downarrow & & \downarrow \\
\mathfrak{g} & & \sum_{i=1}^{k+l+m} x_i D_i \\
\| & & \| \\
\mathfrak{g} = \mathfrak{n} + \mathfrak{t} & & \sum_{i=1}^{k} L_i(\mathbf{x})A_i + \sum_{i=1}^{l} L_{k+i}(\mathbf{x})B_i + \sum_{i=1}^{m} L_{k+l+i}(\mathbf{x})C_i \\
\downarrow \varphi & & \downarrow \\
G & & \exp\left(\sum_{i=1}^{k} L_i(\mathbf{x})A_i + \sum_{i=1}^{l} L_{k+i}(\mathbf{x})B_i\right) \exp\left(\sum_{i=1}^{m} L_{k+l+i}(\mathbf{x})C_i\right).
\end{array}$$

If we denote the inverse of the above map by Co, then we find that $\mathrm{Co} = L^{-1} \circ \mathrm{co}$.

DEFINITION 3.9. Let $\mathrm{Co}\colon G \to \mathbb{R}^{k+l+m}$ be a map obtained as described above, then Co is called a *generalized Mal'cev-like coordinate map* on G.

It is easy to see that for any linear map M in $\mathrm{Gl}(k+l+m, \mathbb{R})$ there exists a change of basis such that the map $M^{-1} \circ \mathrm{co}$ is a generalized Mal'cev-like coordinate map. So the generalized Mal'cev-like coordinate maps are nothing but the linear alterations of the usual Mal'cev-like coordinate maps.

For our purposes it is useful to choose the basis for constructing a generalized Mal'cev-like coordinate map in the following way. Let $\mathfrak{g}$ be the solvable Lie algebra under consideration. Consider any central series (e.g. the upper or lower central series)

$$0 \subset \mathfrak{z}_1 \subset \mathfrak{z}_2 \subset \cdots \subset \mathfrak{z}_c = \mathfrak{n} \tag{5}$$

of characteristic ideals of the nilradical $\mathfrak{n}$ of $\mathfrak{g}$. Note that the spaces $\mathfrak{z}_i$ are also characteristic ideals of $\mathfrak{g}$.

We now choose the basis to construct a generalized Mal'cev-like coordinate map to consist of vectors

$$Z_{1,1}, Z_{1,2}, \ldots Z_{1,k_1}, Z_{2,1}, \ldots, Z_{2,k_2}, Z_{3,1}, \ldots Z_{c,k_c}, C_1, \ldots, C_m, \tag{6}$$

in such a way that the set of vectors

$$Z_{1,1}, Z_{1,2}, \ldots Z_{1,k_1}, Z_{2,1}, \ldots, Z_{2,k_2}, Z_{3,1}, \ldots Z_{i,k_i}$$

is a basis of the ideal $\mathfrak{z}_i$ and the vectors C_i come from our original basis (4).

DEFINITION 3.10. The generalized Mal'cev-like coordinate map obtained by considering the basis (6) (related to (5)) is called a *structured coordinate map.*

Remark 3.11. From now onwards we will write the coordinates in a structured coordinate map as a column vector rather then a row vector. It turns out that this choice facilitates the notations used further on.

A typical coordinate of an element with respect to the basis (6) will be written as

$$\mathbf{x} = \begin{pmatrix} x_{1,1} \\ x_{1,2} \\ \vdots \\ x_{1,k_1} \\ x_{2,1} \\ \vdots \\ x_{c,k_c} \\ y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}.$$

We will also use $\mathbf{x}_i$ (resp. $\mathbf{y}$) to denote the ith (resp. last) block of this coordinate vector consisting of the elements

$$\mathbf{x}_i = \begin{pmatrix} x_{i,1} \\ x_{i,2} \\ \vdots \\ x_{i,k_i} \end{pmatrix}, \quad \left(\text{resp. } \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} \right).$$

The following refinement of Theorem 3.5 will be the basis of our next section.

THEOREM 3.12. *Let G be a simply connected, connected, n-dimensional solvable Lie group equipped with a generalized Mal'cev-like coordinate map* Co. *Then, the coordinate expression for the product in G is polynomial of degree $\leqslant n^3$ in the second variable, i.e. the map*

$$\mu\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\colon (\boldsymbol{\alpha}, \mathbf{x}) \mapsto \mathrm{Co}(\mathrm{Co}^{-1}(\boldsymbol{\alpha}) \cdot \mathrm{Co}^{-1}(\mathbf{x}))$$

is polynomial in the n components of $\mathbf{x}$.

Moreover, if Co *is a structured coordinate map, say with respect to a basis*

$$Z_{1,1}, Z_{1,2}, \dots Z_{1,k_1}, Z_{2,1}, \dots, Z_{2,k_2}, Z_{3,1}, \dots Z_{c,k_c}, C_1, \dots, C_m,$$

(corresponding to a series of ideals (5)) we have that for any $\boldsymbol{\alpha} = (\alpha_{1,1}, \dots, \alpha_{c,k_c}, \beta_1, \dots, \beta_m)^T$ *and for any* $\mathbf{x} = (x_{1,1}, x_{1,2}, \dots, x_{1,k_1}, x_{2,1}, \dots, x_{c,k_c}, y_1, \dots, y_m)^T$

$$\mu(\boldsymbol{\alpha}, \mathbf{x}) = \begin{pmatrix} p_1^{\boldsymbol{\alpha}}(\mathbf{x}) \\ p_2^{\boldsymbol{\alpha}}(\mathbf{x}) \\ \vdots \\ p_c^{\boldsymbol{\alpha}}(\mathbf{x}) \\ p_{c+1}^{\boldsymbol{\alpha}}(\mathbf{x}) \end{pmatrix},$$

where

(1) *For all* $1 \leqslant i \leqslant c$: $p_i^{\boldsymbol{\alpha}}(\mathbf{x}) = A_i^{\boldsymbol{\alpha}}\mathbf{x}_i + q^{\boldsymbol{\alpha}}(\mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \dots, \mathbf{x}_c, \mathbf{y})$ *for some* $A_i \in \mathrm{Gl}(k_i, \mathbb{R})$ *and some polynomial map* $q^{\boldsymbol{\alpha}}: \mathbb{R}^{k_{i+1}+\cdots+k_c+m} \to \mathbb{R}^{k_i}$.

(2) $$p_{c+1}^{\boldsymbol{\alpha}}(\mathbf{x}) = \begin{pmatrix} y_1 + \beta_1 \\ y_2 + \beta_2 \\ \vdots \\ y_m + \beta_m \end{pmatrix}.$$

(3) *If* $\alpha = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_i, 0, 0, \dots, 0)^T$ *for some* $i \in \{1, 2, \dots, c\}$ *then*

$$p_i^{\boldsymbol{\alpha}}(\mathbf{x}) = \begin{pmatrix} x_{i,1} + \alpha_{i,1} \\ x_{i,2} + \alpha_{i,2} \\ \vdots \\ x_{i,k_i} + \alpha_{i,k_i} \end{pmatrix},$$

$$p_j^{\boldsymbol{\alpha}}(\mathbf{x}) = \begin{pmatrix} x_{j,1} \\ x_{j,2} \\ \vdots \\ x_{j,k_i} \end{pmatrix} \quad (\text{for } i < j \leqslant c) \quad \text{and} \quad p_{c+1}^{\boldsymbol{\alpha}}(\mathbf{x}) = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}.$$

Proof. Let the generalized Mal'cev coordinate map be written as a composition $L^{-1} \circ \mathrm{co}$, where $L \in \mathrm{Gl}(n, \mathbb{R})$ and co is a genuine Mal'cev like coordinate map.

By Theorem 3.5 we know that $\forall \boldsymbol{\alpha} \in \mathbb{R}^n$ the map

$$\lambda_{\boldsymbol{\alpha}}: \mathbb{R}^n \to \mathbb{R}^n: \mathbf{x} \mapsto \mathrm{co}(\mathrm{co}^{-1}(\boldsymbol{\alpha}), \mathrm{co}^{-1}(\mathbf{x}))$$

is polynomial of degree $\leqslant n^3$. It follows that the map $L^{-1} \circ \lambda_{L(\boldsymbol{\alpha})} \circ L: \mathbb{R}^n \to \mathbb{R}^n$, as a composition of two linear maps and a polynomial map of degree $\leqslant n^3$ is also a polynomial map of degree $\leqslant n^3$, for any $\boldsymbol{\alpha} \in \mathbb{R}^n$.

This proves the first claim of the theorem since

$$\begin{aligned}(L^{-1}\circ\lambda_{L(\boldsymbol{\alpha})}\circ L)(\mathbf{x}) &= (L^{-1}\circ \mathrm{co})(\mathrm{co}^{-1}(L(\boldsymbol{\alpha})),\mathrm{co}^{-1}(L(\mathbf{x})))\\ &= \mathrm{Co}(\mathrm{Co}^{-1}(\boldsymbol{\alpha}),\mathrm{Co}^{-1}(\mathbf{x}))=\mu(\boldsymbol{\alpha},\mathbf{x}).\end{aligned}$$

From now onwards, we assume that Co is a structured coordinate map on G. To prove the last assertions of the theorem, we re-use the proof of Theorem 3.5. However, in stead of focussing on the polynomiality of the expression (a fact we already know now), we pay special attention to how the expression depends on the variables involved.

Consider the basis

$$Z_{1,1},Z_{1,2},\ldots,Z_{1,k_1},Z_{2,1},Z_{2,2},\ldots,Z_{c,k_c},$$

(relative to a central series $0\subset\mathfrak{z}_1\subset\mathfrak{z}_2\subset\cdots\subset\mathfrak{z}_c=\mathfrak{n}$) of $\mathfrak{n}$. It is known that there is a rather nice expression for the product in the corresponding simply connected, connected Lie group N. To be precise, there exist polynomial maps $p_{i,j}$ (depending on the variables indicated below) such that ([11, 13] and [7, p. 84])

$$\begin{aligned}&\exp\left(\sum_{i=1}^{c}\sum_{j=1}^{k_i}u_{i,j}Z_{i,j}\right)\exp\left(\sum_{i=1}^{c}\sum_{j=1}^{k_i}v_{i,j}Z_{i,j}\right)\\ &\quad=\exp\left(\sum_{i=1}^{c}\sum_{j=1}^{k_i}(u_{i,j}+v_{i,j}+p_{i,j}(u_{i+1,1},u_{i+1,2},\ldots,\right.\\ &\qquad\qquad\left. u_{c,k_c},v_{i+1,1},v_{i+1,2},\ldots,v_{c,k_c}))Z_{i,j}\right).\end{aligned}$$

Moreover, as the $\mathfrak{z}_i$ are characteristic ideals of $\mathfrak{n}$, any automorphism of $\mathfrak{n}$ is expressed by means of a blocked upper triangular matrix

$$\begin{pmatrix}A_1 & * & \cdots & *\\ 0 & A_2 & \cdots & *\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & A_c\end{pmatrix},$$

where $A_i\in \mathrm{Gl}(k_i,\mathbb{R})$ for all $i\in\{1,2,\ldots,c\}$.

In the computation below we will use symbols $f_{s,i}$ to denote functions (in fact they will all be polynomial) which have their image in $\mathbb{R}^{k_i}$. The counter s is used to distinguish the functions amongst each other. As usual a notation like $\mathbf{x}_i\mathbf{Z}_i$ is used

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as a shorthand to $\sum_{j=1}^{k_i} x_{i,j} Z_{i,j}$.

$$
\begin{aligned}
&\mathrm{Co}^{-1}(\boldsymbol{\alpha}) \cdot \mathrm{Co}^{-1}(\mathbf{x}) \\
&= \exp\left(\sum_{i=1}^{c} \boldsymbol{\alpha}_i \mathbf{Z}_i\right) \exp(\boldsymbol{\beta}\mathbf{C}) \exp\left(\sum_{i=1}^{c} \mathbf{x}_i \mathbf{Z}_i\right) \exp(\mathbf{y}\mathbf{C}) \\
&= \exp\left(\sum_{i=1}^{c} \boldsymbol{\alpha}_i \mathbf{Z}_i\right) \underbrace{\exp(\boldsymbol{\beta}\mathbf{C}) \exp\left(\sum_{i=1}^{c} \mathbf{x}_i \mathbf{Z}_i\right) \exp(-\boldsymbol{\beta}\mathbf{C})}_{\text{automorphism of } \mathfrak{n}} \exp(\boldsymbol{\beta}\mathbf{C}) \exp(\mathbf{y}\mathbf{C}) \\
&= \exp\left(\sum_{i=1}^{c} \boldsymbol{\alpha}_i \mathbf{Z}_i\right) \exp\left(\sum_{i=1}^{c} (A_i^{\boldsymbol{\alpha}} \mathbf{x}_i + f_{1,i}(\mathbf{x}_{i+1}, \ldots, \mathbf{x}_c, \boldsymbol{\beta}) \mathbf{Z}_i\right) \times \\
&\quad \times \exp\left(\sum_{i=1}^{c} f_{2,i}(\mathbf{y}, \boldsymbol{\beta}) \mathbf{Z}_i\right) \cdot \exp((\mathbf{y} + \boldsymbol{\beta})\mathbf{C}) \\
&= \exp\left(\sum_{i=1}^{c} \boldsymbol{\alpha}_i \mathbf{Z}_i\right) \cdot \exp\left(\sum_{i=1}^{c} (A_i^{\boldsymbol{\alpha}} \mathbf{x}_i + f_{1,i}(\mathbf{x}_{i+1}, \ldots, \mathbf{x}_c, \boldsymbol{\beta}) + f_{2,i}(\mathbf{y}, \boldsymbol{\beta}) + \right. \\
&\qquad\qquad \left. + f_{3,i}(\mathbf{x}_{i+1}, \ldots, \mathbf{x}_c, \mathbf{y}, \boldsymbol{\beta})) \mathbf{Z}_i \right) \exp((\mathbf{y} + \boldsymbol{\beta})\mathbf{C}) \\
&= \exp\left(\sum_{i=1}^{c} \boldsymbol{\alpha}_i \mathbf{Z}_i\right) \exp\left(\sum_{i=1}^{c} (A_i^{\boldsymbol{\alpha}} \mathbf{x}_i + f_{4,i}(\mathbf{x}_{i+1}, \ldots, \mathbf{x}_c, \mathbf{y}, \boldsymbol{\beta})) \mathbf{Z}_i\right) \times \\
&\quad \times \exp((\mathbf{y} + \boldsymbol{\beta})\mathbf{C}) \\
&= \exp\left(\sum_{i=1}^{c} (A_i^{\boldsymbol{\alpha}} \mathbf{x}_i + \boldsymbol{\alpha}_i + \right. \\
&\qquad \left. + f_{5,i}(\mathbf{x}_{i+1}, \ldots, \mathbf{x}_c, \mathbf{y}, \boldsymbol{\alpha}_{i+1}, \ldots, \boldsymbol{\alpha}_c, \boldsymbol{\beta})) \mathbf{Z}_i \right) \exp((\mathbf{y} + \boldsymbol{\beta})\mathbf{C}).
\end{aligned}
$$

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It follows that for a fixed $\boldsymbol{\alpha}$

$$\mathrm{Co}(\mathrm{Co}^{-1}(\boldsymbol{\alpha})\cdot\mathrm{Co}^{-1})(\mathbf{x}) = \begin{pmatrix} p_1^{\boldsymbol{\alpha}}(\mathbf{x}) \\ p_2^{\boldsymbol{\alpha}}(\mathbf{x}) \\ \vdots \\ p_c^{\boldsymbol{\alpha}}(\mathbf{x}) \\ p_{c+1}^{\boldsymbol{\alpha}}(\mathbf{x}) \end{pmatrix}$$

with

$$p_i^{\boldsymbol{\alpha}}(\mathbf{x}) = A_i^{\boldsymbol{\alpha}}\mathbf{x}_i + \underbrace{\boldsymbol{\alpha}_i + f_{5,i}(\mathbf{x}_{i+1},\dots,\mathbf{x}_c,\mathbf{y},\boldsymbol{\alpha}_{i+1},\dots,\boldsymbol{\alpha}_c,\boldsymbol{\beta})}_{=q_i^{\boldsymbol{\alpha}}(\mathbf{x}_{i+1},\dots,\mathbf{x}_c,\mathbf{y})},$$

$$\forall i \in \{1,2,\dots,c\}$$

and

$$p_{c+1}^{\boldsymbol{\alpha}}(\mathbf{x}) = \begin{pmatrix} y_1+\beta_1 \\ y_2+\beta_2 \\ \vdots \\ y_m+\beta_m \end{pmatrix}.$$

Finally, assume $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\dots,\boldsymbol{\alpha}_i,0,0,\dots,0)^T$ for some $i \in \{1,2,\dots,c\}$. As $\boldsymbol{\beta} = 0$ it follows that $A_s^{\boldsymbol{\alpha}}$ is the identity matrix and the maps $f_{1,s} = f_{2,s} = f_{3,s} = f_{4,s} = 0$ for all $s \in \{1,2,\dots,c\}$. The fact that $\boldsymbol{\alpha}_{i+1} = \cdots = \boldsymbol{\alpha}_c = 0$ implies that the maps $f_{5,i+1} = \cdots = f_{5,c} = 0$. This information proves the last claim of the theorem. □

Remark 3.13. The theorem above shows that for a structured coordinate system the maps $\mu(\alpha,\ .\ )$ belong to the blocked Jonquière group of type $(k_1,k_2,\dots,k_c,m)$ ([9]).

4. Polynomial Structures on Polycyclic-by-Finite Groups

Let G be any connected and simply connected solvable Lie group acting polynomially and simply transitively on some space $\mathbb{R}^n$. We denote this action by $\tilde{\rho}: G \to \mathrm{P}(\mathbb{R}^n)$. (Recall that $\mathrm{P}(\mathbb{R}^n)$ is the group of polynomial diffeomorphisms of $\mathbb{R}^n$.)

Assume Γ is a uniform lattice (i.e. a discrete and cocompact subgroup) of G. Then, the restriction of $\tilde{\rho}$ to Γ determines a polynomial structure on Γ. In other words $\rho = \tilde{\rho}_{|\Gamma}: \Gamma \to \mathrm{P}(\mathbb{R}^n): \gamma \mapsto \rho(\gamma) = \tilde{\rho}(\gamma)$ defines a properly discontinuous action with compact quotient.

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Therefore, if one aims to construct a polynomial structure on a polycyclic group, it is often useful to consider lattices in solvable Lie groups. Before we continue, let us recall a few facts about such lattices:

If Γ is a uniform lattice of a simply connected, connected solvable Lie group G with nilradical N, then $\Gamma \cap N$ is a uniform lattice of N and $\Gamma/(\Gamma \cap N)$ is a uniform lattice of G/N (see [12]). $\Gamma \cap N$ does not need to coincide with $\mathrm{Fitt}(\Gamma)$, where $\mathrm{Fitt}(\Gamma)$ is the unique maximal normal nilpotent subgroup of Γ. An example of this situation is obtained when $G = \mathbb{R}^2 \rtimes \mathbb{R}$, where the action of $t \in \mathbb{R}$ on $\mathbb{R}^2$ is given by the matrix $\begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix}$. In this case, the set Γ consisting of all integral triples (z_1, z_2, z_3) in G is a uniform lattice with $\Gamma = \mathbb{Z}^2 \rtimes \mathbb{Z} = \mathbb{Z}^3$.

However, as $N = \mathbb{R}^2$ for this Lie group G, we find that $N \cap \Gamma = \mathbb{Z}^2 \neq \mathrm{Fitt}(\Gamma) = \mathbb{Z}^3$.

This is a rather unfortunate situation for our purposes, but is not of much influence because we can regard G as being a wrong choice of Lie group to contain $\mathbb{Z}^3$ as a uniform lattice. A better choice is of course $G = \mathbb{R}^3$. We make this argument more precise in the following more or less well known theorem.

THEOREM 4.1. *Let Γ be any polycyclic-by-finite group, then Γ contains a normal subgroup of finite index Γ' such that there exists a connected and simply connected solvable Lie group G containing Γ' as a uniform lattice and such that $N \cap \Gamma' = \mathrm{Fitt}(\Gamma')$, where N denotes the nilradical of G.*

Proof. The proof of this theorem is exactly the same as the somewhat weaker formulated version of this theorem in [15, Thm. 4.2.8]. □

In view of the theorem above it makes sense to pay special attention to those lattices Γ of a connected and simply connected solvable Lie group such that $\Gamma \cap N = \mathrm{Fitt}(\Gamma)$ (N is the nilradical of G). The group $\Gamma_n = \Gamma \cap N$ is a lattice of N, as such it is a finitely generated torsion free nilpotent group, and $\overline{\Gamma} = \Gamma/\Gamma_n$ is free Abelian (being a lattice of G/N). Let

$$\zeta_0(\Gamma_n) = 0 \subseteq \zeta_1(\Gamma_n) \subseteq \zeta_2(\Gamma_n) \subseteq \cdots \subseteq \zeta_c(\Gamma_n) = \Gamma_n$$

be the upper central series of Γ_n (we suppose that the nilpotency class of Γ_n, which equals the nilpotency class of N, is c). Choose a set of generators of Γ_n

$$\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{1,k_1}, \gamma_{2,1}, \ldots, \gamma_{2,k_2}, \gamma_{3,1}, \ldots, \gamma_{c,k_c},$$

in such a way that the cosets

$$\gamma_{i,1}\zeta_{i-1}(\Gamma_n), \gamma_{i,2}\zeta_{i-1}(\Gamma_n), \ldots, \gamma_{i,k_i}\zeta_{i-1}(\Gamma_n)$$

freely generate the free Abelian group $\zeta_i(\Gamma_n)/\zeta_{i-1}(\Gamma_n)(1 \leqslant i \leqslant c)$.

It is well-known that the vectors

$$Z_{1,1}, Z_{1,2}, \ldots Z_{1,k_1}, Z_{2,1}, \ldots, Z_{2,k_2}, Z_{3,1}, \ldots Z_{c,k_c}, \quad \text{with } \exp(Z_{i,j}) = \gamma_{i,j}$$

form a basis of the Lie algebra $\mathfrak{n}$ of N. Moreover, the vectors

$$Z_{1,1}, Z_{1,2}, \dots Z_{1,k_1}, Z_{2,1}, \dots, Z_{2,k_2}, Z_{3,1}, \dots Z_{i,k_i}$$

form a basis of the ith term $\mathfrak{z}_i(\mathfrak{n})$ of the upper central series of $\mathfrak{n}$.

We now complete the above basis to a basis

$$Z_{1,1}, Z_{1,2}, \dots Z_{1,k_1}, Z_{2,1}, \dots, Z_{2,k_2}, Z_{3,1}, \dots Z_{c,k_c},$$
$$C_1, C_2, \dots, C_m$$

of $\mathfrak{g}$ by choosing vectors $C_1, C_2, \dots, C_m$ in a nilpotent almost supplement $\mathfrak{c}$ of $\mathfrak{n}$ in $\mathfrak{g}$. In this way we constructed a basis satisfying the necessary conditions to obtain a structured coordinate map which we again denote by Co. By construction, we have the following lemma.

LEMMA 4.2. *With the notations of above we have that* $\mathrm{Co}(\gamma_{i,j}) = (0, 0, 0, \dots, 0, 1, 0, \dots, 0)^T$ *where the* 1 *appears on the* (i, j)*th place. More generally,*

$$\mathrm{Co}(\gamma_{1,1}^{a_{1,1}} \gamma_{1,2}^{a_{1,2}} \dots \gamma_{i,k_i}^{a_{i,k_i}}) = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_i, 0, 0, \dots, 0)^T,$$
$$\textit{with } \boldsymbol{\alpha}_i = (a_{i,1}, a_{i,2}, \dots, a_{i,k_i})^T.$$

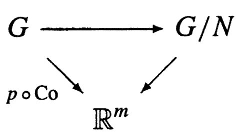
Let $\gamma_{c+1,1}, \gamma_{c+1,2}, \dots, \gamma_{c+1,m} \in \Gamma$ be elements of Γ such that $\gamma_{c+1,1}\Gamma_n$, $\gamma_{c+1,2}\Gamma_n, \dots, \gamma_{c+1,m}\Gamma_n$ freely generated the free Abelian group $\overline{\Gamma}$. Again by construction, we have the following lemma.

LEMMA 4.3. *Let* $p\colon \mathbb{R}^{k_1+k_2+\cdots+k_c+m} \to \mathbb{R}^m$ *be the projection on the last* m *components, then we have that* $p \circ \mathrm{Co}\colon G \to \mathbb{R}^m$ *is a surjective morphism of Lie groups with kernel* N. *The image of* Γ *under this projection is a uniform lattice of* $\mathbb{R}^m$ *(isomorphic to* $\overline{\Gamma}$*).*

Proof. Let $g_1, g_2 \in G$, then $g_1 = \exp(n_1)\exp(r_1C_1 + r_2C_2 \cdots + r_mC_m)$ and $g_2 = \exp(n_2)\exp(s_1C_1 + s_2C_2 + \cdots + s_mC_m)$ for some unique $n_1, n_2 \in \mathfrak{n}$, $r_1, \dots, r_m$, $s_1, \dots, s_m \in \mathbb{R}$. The following small computation

$$\begin{aligned} &p \circ \mathrm{Co}(g_1g_2) \\ &\quad = p \circ \mathrm{Co}(\exp(n_1)\exp(r_1C_1 + \cdots + r_mC_m)\exp(n_2) \times \\ &\qquad \times \exp(s_1C_1 + \cdots + s_mC_m)) \\ &\quad = (r_1 + s_1, r_2 + s_2, \dots, r_m + s_m) \\ &\quad = p \circ \mathrm{Co}(g_1) + p \circ \mathrm{Co}(g_2) \end{aligned}$$

shows that $p \circ \mathrm{Co}$ is a surjective homomorphism of Lie groups, with kernel N. The factorization of this morphism



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induces an isomorphism of G/N with $\mathbb{R}^m$. Via this isomorphism, the statement of $p \circ \mathrm{Co}(\Gamma)$ being a lattice in $\mathbb{R}^m$ is equivalent to $\overline{\Gamma}$ being a lattice in G/N, which is known to be true. □

The importance of this structured coordinate map Co lies in the fact that the polynomial structure $\rho: \Gamma \to \mathrm{P}(\mathbb{R}^n)$ $(n = k_1 + \cdots + k_c + m)$ obtained as a restriction of the simply transitive action

$$\tilde{\rho}: G \to \mathrm{P}(\mathbb{R}^n): g \mapsto \tilde{\rho}(g), \quad \text{with } \tilde{\rho}(g)(\mathbf{x}) = \mathrm{Co}(g\,\mathrm{Co}^{-1}(\mathbf{x})), \tag{7}$$

is of the so-called *canonical type*. To explain this notion, we recall the necessary concepts (see also [7, 9], and [11]).

For any polycyclic-by-finite group Γ there exists an ascending sequence (or filtration) of normal subgroups Γ_i $(0 \leqslant i \leqslant s+1)$ of Γ ([16, Lem. 6, pp. 16])

$$\Gamma_* : \Gamma_0 = 1 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Gamma_{s-1} \subseteq \Gamma_s \subseteq \Gamma_{s+1} = \Gamma, \tag{8}$$

for which

$$\Gamma_i / \Gamma_{i-1} \cong \mathbb{Z}^{l_i} \quad \text{for } 1 \leqslant i \leqslant s \text{ and some } l_i \in \mathbb{N}_0 \text{ and } \Gamma/\Gamma_s \text{ is finite.}$$

Such a filtration of Γ is called a *torsion-free filtration* (of length s). We will also use $L_i = l_i + l_{i+1} + \cdots + l_s$ and $L_{s+1} = 0$. It follows that the Hirsch length of Γ, $h(\Gamma) = L_1$.

We denote the real vector space of polynomial mappings from $\mathbb{R}^L$ to $\mathbb{R}^l$ by $\mathrm{P}(\mathbb{R}^L, \mathbb{R}^l)$. An element $p(x_1, \ldots, x_L)$ of $\mathrm{P}(\mathbb{R}^L, \mathbb{R}^l)$ consists of l polynomials in L variables

$$p(x_1, \ldots, x_L) = \begin{pmatrix} p_1(x_1, x_2, \ldots, x_L) \\ p_2(x_1, x_2, \ldots, x_L) \\ \vdots \\ p_l(x_1, x_2, \ldots, x_L) \end{pmatrix},$$

$$\text{with } p_i(x_1, \ldots, x_L) \in \mathrm{P}(\mathbb{R}^L, \mathbb{R}).$$

The vector space $\mathrm{P}(\mathbb{R}^L, \mathbb{R}^l)$ contains $\mathbb{R}^l$ as the subspace of constant mappings. $\mathrm{P}(\mathbb{R}^L, \mathbb{R}^l)$ is made into a $\mathrm{Gl}(\mathbb{R}^l) \times \mathrm{P}(\mathbb{R}^L)$-module, via

$$\forall g \in \mathrm{Gl}(\mathbb{R}^l), \ \ \forall h \in \mathrm{P}(\mathbb{R}^L), \ \ \forall p \in \mathrm{P}(\mathbb{R}^L, \mathbb{R}^l) : {}^{(g,h)}p = g \circ p \circ h^{-1}.$$

The resulting semi-direct product $\mathrm{P}(\mathbb{R}^L, \mathbb{R}^l) \rtimes (\mathrm{Gl}(\mathbb{R}^l) \times \mathrm{P}(\mathbb{R}^L))$ embeds into $\mathrm{P}(\mathbb{R}^{l+L})$ as follows: $\forall p \in \mathrm{P}(\mathbb{R}^L, \mathbb{R}^l)$, $\forall g \in \mathrm{Gl}(\mathbb{R}^l)$, $\forall h \in \mathrm{P}(\mathbb{R}^L)$

$$\forall x \in \mathbb{R}^l, \ \ \forall y \in \mathbb{R}^L : (p, g, h) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g(x) + p(h(y)) \\ h(y) \end{pmatrix}.$$

Using the notations above, we are ready to define the notion of a canonical type polynomial representation (or a canonical type polynomial structure).

DEFINITION 4.4. Assume Γ is a polycyclic-by-finite group with a torsion-free filtration Γ_*. For every i, write $\varphi_i\colon \Gamma/\Gamma_i \to \mathrm{Aut}(\mathbb{Z}^{l_i})$ for the morphism induced by the short exact sequence

$$1 \to \mathbb{Z}^{l_i}\ (\cong \Gamma_i/\Gamma_{i-1}) \to \Gamma/\Gamma_{i-1} \to \Gamma/\Gamma_i \to 1.$$

A polynomial representation $\rho = \rho_0\colon \Gamma \to \mathrm{P}(\mathbb{R}^{h(\Gamma)})$ will be called *of canonical type* with respect to Γ_* (or simply of canonical type) iff it induces a sequence of representations

$$\rho_i\colon \Gamma/\Gamma_i \to \mathrm{P}(\mathbb{R}^{L_{i+1}}), \quad (1 \leqslant i \leqslant s)$$

and a sequence of morphisms $j_i\colon \mathbb{Z}^{l_i} \to \mathrm{P}(\mathbb{R}^{L_{i+1}}, \mathbb{R}^{l_i})$, $(1 \leqslant i \leqslant s)$ mapping each $z \in \mathbb{Z}^{l_i}$ onto a constant mapping $j_i(z)\colon \mathbb{R}^{L_{i+1}} \to \mathbb{R}^{l_i}\colon x \mapsto r_i(z)$, with the property that the $r_i(z)$ span $\mathbb{R}^{l_i}$ as a vector space, such that for all i the following diagram commutes

$$\begin{array}{ccccccccc}
1 \longrightarrow & \mathbb{Z}^{l_i} \cong \Gamma_i/\Gamma_{i-1} & \longrightarrow & \Gamma/\Gamma_{i-1} & \longrightarrow & \Gamma/\Gamma_i & \longrightarrow 1 \\
& \downarrow j_i & & \downarrow \rho_{i-1} & & \downarrow \psi_i \times \rho_i & \\
1 \longrightarrow & \mathrm{P}(\mathbb{R}^{L_{i+1}}, \mathbb{R}^{l_i})) & \longrightarrow & \mathrm{P}(\mathbb{R}^{L_{i+1}}, \mathbb{R}^{l_i}) \rtimes (\mathrm{Gl}(\mathbb{R}^{l_i}) \times \mathrm{P}(\mathbb{R}^{L_{i+1}}) & \longrightarrow & \mathrm{Gl}(\mathbb{R}^{l_i}) \times \mathrm{P}(\mathbb{R}^{L_{i+1}}) & \longrightarrow 1,
\end{array} \qquad (9)$$

where ψ_i is the unique morphism $\psi_i\colon \Gamma/\Gamma_i \to \mathrm{Gl}(\mathbb{R}^{l_i})$ satisfying

$$\forall \overline{\gamma} \in \Gamma/\Gamma_i, \quad \forall z \in \mathbb{Z}^{l_i} : \psi_i(\overline{\gamma})(j_i(z)) = j_i(\varphi_i(\overline{\gamma})z).$$

Remark 4.5. To get a better understanding of this concept we refer the reader to [7, 9] and [11].

We are now ready to prove the following theorem.

THEOREM 4.6. *Let Γ be a lattice of a simply connected, connected solvable Lie group G, with $N \cap \Gamma = \mathrm{Fitt}(\Gamma)$, where N denotes the nilradical of G.*

Define a structured coordinate map $\mathrm{Co}\colon G \to \mathbb{R}^n$ on G by following the procedure starting above.

Then, the polynomial structure $\rho\colon \Gamma \to \mathrm{P}(\mathbb{R}^n)$ obtained as the restriction of the simply transitive polynomial action $\widetilde{\rho}\colon G \to \mathrm{P}(\mathbb{R}^n)$ defined in (7) *is of canonical type with respect to the torsion-free filtration* $(\Gamma_n = \mathrm{Fitt}(\Gamma))$

$$\begin{aligned}
\Gamma_* : \Gamma_0 &= 1 \subseteq \Gamma_1 = \zeta_1(\Gamma_n) \subseteq \Gamma_2 = \zeta_2(\Gamma_n) \subseteq \cdots \\
&\subseteq \Gamma_c = \zeta_c(\Gamma_n) = \Gamma_n \subseteq \Gamma_{c+1} = \Gamma \subseteq \Gamma_{c+2} = \Gamma.
\end{aligned}$$

Proof. Using the result of Theorem 3.12, the reader should be able to supply a detailed proof of this theorem. □

https://doi.org/10.1023/A:1001738932743 Published online by Cambridge University Press

We are now ready to prove the theorem mentioned in the Introduction and which states that any polycyclic-by-finite group Γ admits a polynomial structure, which is of degree $\leqslant h(\Gamma)^3$ on virtually the entire group. This theorem should be seen as a step towards Conjecture 1.1.

THEOREM 4.7. *Let Γ be any polycyclic-by-finite group of Hirsch length $h(\Gamma)$. Then there exists a polynomial structure $\rho\colon \Gamma \to \mathrm{P}(\mathbb{R}^{h(\Gamma)})$ for Γ and a finite index subgroup Γ' of Γ such that the restriction of the polynomial structure to Γ' is of degree $\leqslant h(\Gamma)^3$.*

Proof. The proof of this theorem is a combination of the results in [9] and Theorem 4.6. Let Γ be any polycyclic-by-finite group, then, by Theorem 4.1, there exists a normal subgroup Γ' of Γ which is of finite index in Γ and such that Γ' is a lattice of a simply connected, connected solvable Lie group G and such that $G \cap N = \mathrm{Fitt}(\Gamma')$ (N being the nilradical of G, say of nilpotency class c). By Theorem 4.6 there exists a polynomial structure $\overline{\rho}\colon \Gamma' \to \mathrm{P}(\mathbb{R}^{h(\Gamma')})$ of Γ' which is of canonical type with respect to the torsion-free filtration

$$\begin{aligned}\Gamma'_* : \Gamma'_0 &= 1 \subseteq \Gamma'_1 = \zeta_1(\mathrm{Fitt}(\Gamma')) \subseteq \cdots \\ &\subseteq \Gamma'_c = \zeta_c(\mathrm{Fitt}(\Gamma')) = \mathrm{Fitt}(\Gamma') \subseteq \Gamma'_{c+1} = \Gamma' \subseteq \Gamma'_{c+2} = \Gamma'.\end{aligned}$$

Moreover, this polynomial structure is of degree $\leqslant (\dim G)^3 = h(\Gamma')^3 = h(\Gamma)^3$. As all Γ'_i's in the torsion-free filtration Γ'_* are characteristic subgroups of Γ', they are normal in Γ. Therefore, the following sequence is a torsion-free filtration of Γ

$$\begin{aligned}\Gamma_* : \Gamma_0 &= 1 \subseteq \Gamma_1 = \zeta_1(\mathrm{Fitt}(\Gamma')) \subseteq \cdots \\ &\subseteq \Gamma_c = \zeta_c(\mathrm{Fitt}(\Gamma')) = \mathrm{Fitt}(\Gamma') \subseteq \Gamma_{c+1} = \Gamma' \subseteq \Gamma_{c+2} = \Gamma.\end{aligned}$$

By Theorem 4.1 of [9], there exists a polynomial structure $\breve{\rho}\colon \Gamma \to \mathrm{P}(\mathbb{R}^{h(\Gamma)})$, which is of canonical type with respect to Γ_*. By Theorem 2.2 of [9], we know that the restriction $\breve{\rho}_{|\Gamma'}$ of $\breve{\rho}$ to the subgroup Γ' is a polynomial representation of canonical type with respect to Γ'_*. Again by Theorem 4.1 of [9], we know that there exists a polynomial diffeomorphism $p \in \mathrm{P}(\mathbb{R}^{h(\Gamma')})$ such that $\overline{\rho} = p \circ \breve{\rho}_{|\Gamma'} \circ p^{-1}$. Now, let $\rho = p \circ \breve{\rho} \circ p^{-1}$. As ρ is conjugated inside $\mathrm{P}(\mathbb{R}^{h(\Gamma)})$ to a polynomial structure of Γ, ρ is itself a polynomial structure. Moreover, when restricted to Γ', this polynomial structure is of degree $\leqslant h(\Gamma)^3$, which was to be shown. □

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