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STRUCTURE OF CERTAIN PERIODIC RINGS

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ABSTRACT. Let R be a periodic ring, N the set of nilpotents, and D the set of right zero divisors of R. Suppose that (i) N is commutative, and (ii) every x in R can be *uniquely* written in the form $x = e + a$, where $e^2 = e$ and $a \in N$. Then N is an ideal in R and R/N is a Boolean ring. If (i) is satisfied but (ii) is now assumed to hold merely for those elements $x \in D$, and if $1 \in R$, then N is still an ideal in R and R/N is a subdirect sum of fields. It is further shown that if (i) is satisfied but (ii) is replaced by: "every right zero divisor is either nilpotent or idempotent," and if $1 \in R$, then N is still an ideal in R and R/N is either a Boolean ring or a field.

Throughout, N denotes the set of nilpotents and D denotes the set of right zero divisors of R. The ring R is called *periodic* if for every x in R, there exist distinct positive integers $m = m(x)$, $n = n(x)$ such that $x^m = x^n$. A Boolean ring is trivially a periodic ring with commuting nilpotents and, of course, every x in R can be *uniquely* written as a sum of an idempotent and a nilpotent. That these properties are *not* confined just to Boolean rings can be seen by considering the ring of integers, modulo 4. In Theorem 1 below, we show that a periodic ring R with the above properties, while not necessarily Boolean, is the next best thing to being Boolean in the sense that its factor ring R/N is indeed Boolean (and hence a subdirect sum of copies of GF(2)). Next, we consider a periodic ring R with identity 1 and with commuting nilpotents such that every right zero divisor x can be *uniquely* written in the form $x = e + a$, where $e^2 = e$ and $a \in N$. Here again N turns out to be an ideal in R but R/N is now a subdirect sum of (not necessarily identical) fields. On the other hand, if we replace the last hypothesis above by "every right zero divisor is either nilpotent or idempotent," then N is still an ideal in R and R/N is now necessarily a Boolean ring or a field.

We begin this note with the following

THEOREM 1. *Let R be a periodic ring (not necessarily with identity). Suppose that* (i) *N is commutative, and* (ii) *every x in R can be uniquely written in the form $x = e_0 + a_0$, where $e_0^2 = e_0$ and $a_0 \in N$. Then N is an ideal in R, and R/N is Boolean (and hence a subdirect sum of copies of* GF(2)). *In fact, R is commutative.*

PROOF. Let $e^2 = e \in R$, $x \in R$, and let $f = e + ex - exe$. Then $f^2 = f$. Moreover, since

$$f = e + (ex - exe); \quad ex - exe \in N;$$

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and

$$f = f + 0,$$

it follows from (ii) that $ex - exe = 0$, and hence $ex = exe$. Similarly, $xe = exe$, and thus

(1) All idempotents of R are central.

Combining (1) with hypotheses (ii) and (i), we see that R is commutative and hence N is an ideal in R. Let $x \in R$. By (ii),

$$x = e_0 + a_0; \qquad e_0^2 = e_0, \qquad a_0 \in N,$$

and hence $x + N$ is idempotent. Thus, R/N is Boolean.

THEOREM 2. *Let R be a periodic ring with identity* 1. *Suppose that (i) N is commutative, (ii) every $x \in D$ can be uniquely written in the form $x = e + a$, where $e^2 = e$ and $a \in N$. Then N is an ideal in R and R/N is isomorphic to a subdirect sum of fields.*

PROOF. Let $e^2 = e \in R$, $x \in R$, and let $f = e + ex - exe$. If $f = 1$, then $ex = exe$. Now, suppose $f \neq 1$. Then, $f^2 = f$, $f \neq 1$, and hence $f \in D$. Since

$$f = e + a, \quad \text{where } a = ex - exe \in N;$$

and

$$f = f + 0,$$

it follows from (ii) that $a = 0$ (since $f \in D$), and thus $ex = exe$. Similarly, $xe = exe$, and hence

(2) All idempotents are central.

Let $x \in R$. Since R is periodic, $x^m = x^n$ for some integers $m > n \geqq 1$, and hence $x^{(m-n)n}$ is idempotent. Therefore, by (2), for all y in R,

(3) $$[x^{(m-n)n}, y] = 0$$

where $[u, v] = uv - vu$. A well known Theorem of Herstein [2] asserts that (3) implies that the commutator ideal of R is nil and hence the nilpotents N of R form an ideal in R. Also, since $x^m = x^n$, for some polynomial $g(\lambda) \in \mathbb{Z}[\lambda]$,

$$(x^{m-n+1} - x)^n = (x^{m-n+1} - x)x^{n-1}g(x) = 0$$

and hence $x^{m-n+1} - x \in N$, $m > n \geqq 1$. Thus,

(4) $$(x + N)^{m-n+1} = x + N; \; m - n + 1 > 1, \qquad x \in R.$$

By a well known theorem of Jacobson [3], (4) implies that R/N is a subdirect sum of fields.

THEOREM 3. *Let R be a periodic ring with identity* 1. *Suppose that (i) N is commutative, and (ii) every x in D is either idempotent or nilpotent. Then N is an ideal of R, and R/N is either Boolean or a field.*

PROOF. Suppose $x \in R$, $x \notin D$. Since R is periodic, let $x^m = x^n$, $m > n \geqq 1$. Then $(x^{m-n} - 1)x^n = 0$. Since $x \notin D$, $x^{m-n} - 1 = 0$, and hence by (ii),

(5) For every x in R, x is nilpotent or idempotent or a unit.

CLAIM A. *If $a \in N$ and e is an idempotent, then $ae \in N$ and $ea \in N$.*

PROOF. Since N is commutative, we have

(6) N is a subring of R.

Let $a \in N$ and $e^2 = e$. Then $ae - eae \in N$, and hence by (i) we have $(ae - eae)a = a(ae - eae)$. So

(7) $$aea - eaea = a^2e - aeae.$$

Multiplying (7) by e from left and right we get $ea^2e = eaeae$. So $(eae)^2 = eaeeae = ea^2e$. Hence we have shown that

(8) $$(eae)^2 = ea^2e \text{ for every } a \in N,$$

and every idempotent e in R.

If $(eae)^{2^k} = ea^{2^k}e$, then $(eae)^{2^{k+1}} = (ea^{2^k}e)^2 = ea^{2^{k+1}}e$ (by (8)). The above induction shows that

(9) $$(eae)^{2^n} = ea^{2^n}e$$

for all positive integers n.

Since $a \in N$, (9) implies that $eae \in N$. But $ae - eae \in N$, and hence, by (6), we get $ae \in N$. Similarly, $ea \in N$. This proves Claim A.

CLAIM B. *Let $a \in N$ and x be a unit in R. Then $ax \in N$ and $xa \in N$.*

PROOF. Suppose $ax \notin N$. Then

(10) $$ax \neq xa.$$

Also, ax is not a unit in R (since a is nilpotent and x is invertible). So ax is idempotent, by (5), and hence $axax = ax$. So

(11) $$axa = a \text{ (since } x \text{ is invertible).}$$

Now, $1 + x \notin N$, since $a(1 + x) \neq (x + 1)a$ and N is commutative. If $(1 + x)^2 = 1 + x$, then $x^2 = -x$. So $x = -1$, which contradicts (10). Hence $(1 + x)$ is not idempotent, and since $1 + x \notin N$, we get from (5) that

(12) $1 + x$ is a unit in R.

Since $ax \notin N$, it follows that $a(1 + x) = a + ax \notin N$. Clearly, by (12), $a(1 + x)$ is not a unit in R, and hence $a(1 + x)$ is idempotent, by (5). Thus, $(a + ax)^2 = a + ax$. So $a^2 + a^2x + axa + (ax)^2 = a + ax$. Using (11) and $(ax)^2 = ax$ we get $a^2(1 + x) = 0$. Then (12) implies that

(13) $$a^2 = 0.$$

Since $a \in N$ and $x^{-1}ax \in N$, therefore, by (i) and (11), $a(x^{-1}ax) = (x^{-1}ax)a = x^{-1}(axa) = x^{-1}a$. Hence

(14) $$ax^{-1}ax = x^{-1}a.$$

Multiplying (14) by a from the left, and using (13) we get $ax^{-1}a = 0$. Then (14) implies that $x^{-1}a = 0$. Hence $a = 0$, which contradicts (10). Therefore, $ax \in N$. Similarly, $xa \in N$ and Claim B is proved.

Now we can complete the proof of Theorem 3. Clearly, since N is commutative, the product of two nilpotent elements is nilpotent. So it follows from (5) and Claims A and B that N is an ideal of R.

Let $x + N$ be any nonzero right zero divisor in R/N. Then $(y + N)\cdot(x + N) = N$, $x \notin N$, $y \notin N$. Thus $yx + N = N$, and hence

$$yx \in N, \qquad x \notin N, \qquad y \notin N. \tag{15}$$

Note that x is not a unit; otherwise, $y \in N$ (see (15)). Thus, by (5) x is idempotent and hence $(x + N)^2 = x^2 + N = x + N$. This shows that

(16) Every right zero divisor of R/N is idempotent.

Moreover, by (5), we see that

(17) Every $x + N$ in R/N is idempotent or a unit in R/N.

CLAIM C. *If R/N has an idempotent different from N and $1 + N$, then R/N is Boolean.*

PROOF. Let $(f + N)^2 = f + N; f \notin N; f - 1 \notin N$. Suppose $u + N$ is not idempotent. Then, by (17), $u + N$ is a unit in R/N and, of course, $u + N \neq 1 + N$. Note that $(f + N)(u + N)$ is not a unit in R/N; otherwise, $f + N$ would be a unit in R/N. Hence, by (17),

$$(f + N)(u + N) \text{ is idempotent.} \tag{18}$$

Now, since R/N is periodic and has no nonzero nilpotents, by a well known theorem of Herstein [1], R/N is commutative. Combining this with (18), we see that

$$(f + N)(u + N) = \{(f + N)(u + N)\}^2 = fu^2 + N$$

and hence $f(u - u^2) + N = N$. But $u + N$ is a unit and hence $f(1 - u) + N = N$. Thus, $(1 - u) + N$ is a right zero divisor (since $f \notin N$), and hence by (16), $(1 - u) + N$ is idempotent. Thus, $u^2 + N = u + N$ and hence $u + N = 1 + N$, a contradiction. This contradiction proves Claim C. Combining (17), Claim C, and the fact that R/N is commutative, the theorem follows.

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