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Classification of Regular Parametrized One-relation Operads

In memoriam Jean-Louis Loday (1946–2012)

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Abstract. Jean-Louis Loday introduced a class of symmetric operads generated by one bilinear operation subject to one relation making each left-normed product of three elements equal to a linear combination of right-normed products: $(a_1a_2)a_3 = \sum_{\sigma\in S_3} x_\sigma\, a_{\sigma(1)}(a_{\sigma(2)}a_{\sigma(3)})$. Such an operad is called a parametrized one-relation operad. For a particular choice of parameters $\{x_\sigma\}$, this operad is said to be regular if each of its components is the regular representation of the symmetric group; equivalently, the corresponding free algebra on a vector space V is, as a graded vector space, isomorphic to the tensor algebra of V. We classify, over an algebraically closed field of characteristic zero, all regular parametrized one-relation operads. In fact, we prove that each such operad is isomorphic to one of the following five operads: the left-nilpotent operad defined by the relation $((a_1a_2)a_3) = 0$, the associative operad, the Leibniz operad, the dual Leibniz (Zinbiel) operad, and the Poisson operad. Our computational methods combine linear algebra over polynomial rings, representation theory of the symmetric group, and Gröbner bases for determinantal ideals and their radicals.

1 Introduction

Jean-Louis Loday introduced the class of operads which he called parametrized one-relation operads. Each of these operads is generated by one binary operation satisfying one ternary relation which states that every monomial of the form $(a_1a_2)a_3$ can be rewritten as a linear combination of permutations of the monomial $a_1(a_2a_3)$. This can be regarded as a natural generalization of associativity, since it says that in each product of three arguments we can reassociate parentheses to the right; the new feature is that we permit permutations of the arguments.

Definition 1.1 An operad $\mathcal{O}$ generated by one bilinear operation $a_1, a_2 \mapsto (a_1a_2)$ is called a *parametrized one-relation operad* if its ideal of relations is generated by a single relation of the form

$$\text{(LR)}\quad (a_1a_2)a_3 = x_1a_1(a_2a_3) + x_2a_1(a_3a_2) + x_3a_2(a_1a_3) + x_4a_2(a_3a_1) + x_5a_3(a_1a_2) + x_6a_3(a_2a_1).$$

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It is called the *LR relation*, since it allows us to re-associate parentheses in products of three elements from the left to the right.

Example 1.2 The most familiar examples of parametrized one-relation operads are the following five special cases:

- $(a_1a_2)a_3 = 0$ [left-nilpotent],
- $(a_1a_2)a_3 = a_1(a_2a_3)$ [associative],
- $(a_1a_2)a_3 = a_1(a_2a_3) - a_2(a_1a_3)$ [left Leibniz],
- $(a_1a_2)a_3 = a_1(a_2a_3) + a_1(a_3a_2)$ [right Zinbiel],
- $(a_1a_2)a_3 = a_1(a_2a_3) + \frac{1}{3}[a_1(a_3a_2) - a_2(a_1a_3) + a_2(a_3a_1) - a_3(a_1a_2)]$ [Poisson].

The last identity defines the one-operation presentation of the Poisson operad discovered by Livernet and Loday [14]. The usual definition of Poisson algebras is obtained by polarization [17].

Notation 1.3 For given coefficients $\mathbf{x} = [x_i]$ in Relation (LR), we write:

- $\mathcal{O}_\mathbf{x}$ for the quadratic symmetric binary operad defined by that relation,
- $\mathcal{O}_\mathbf{x}(n)$ for the arity n component of that operad (viewed as a right S_n-module),
- $\mathcal{O}_\mathbf{x}(V)$ for the free $\mathcal{O}_\mathbf{x}$-algebra generated by the vector space V.

See Section 2 for a brief review of the theory of algebraic operads.

Not much is known about parametrized one-relation operads in general. One natural question asked by Loday [22] was to determine the values of parameters for which the operad $\mathcal{O}_\mathbf{x}$ is Koszul. The five examples above are all Koszul, and they have one more common feature: all components of each of these operads are regular representations of the corresponding symmetric groups (it is obvious for the first of them, and is well known for the others [15]). This observation naturally leads to an attempt to search for other examples of Koszul operads among the operads satisfying the same property.

Definition 1.4 We say that the vector of coefficients $\mathbf{x} = [x_i]$ in Relation (LR) is *regular* if the following equivalent conditions hold.

(1) For each finite-dimensional vector space V, the free algebra $\mathcal{O}_\mathbf{x}(V)$ is isomorphic as a graded vector space (*not* as a graded algebra) to the non-unital tensor algebra $\overline{T}(V)$.
(2) For all $n \geq 1$, the S_n-module $\mathcal{O}_\mathbf{x}(n)$ is isomorphic to the regular module $\mathbb{F}S_n$.

Remark 1.5 It is often the case that the term "regular" is used to describe symmetric operads obtained from nonsymmetric operads by symmetrization. We choose to break that tradition and use this more general notion that includes symmetrizations of nonsymmetric operads but is wider, *i.e.*, the operads Leib and Zinb are not symmetrizations of nonsymmetric operads. The class of operads whose free algebras have the tensor algebras as underlying vector spaces is very natural, and the term "regular" is most appropriate for that property.

In this paper we give a complete classification of regular parametrized one-relation operads over an algebraically closed field $\mathbb{F}$ of characteristic 0. The answer turns out to be wonderfully simple, however disappointing from the viewpoint of hunting for new Koszul operads: up to isomorphism, every such operad is one of those in Example 1.2. It is worth mentioning though, that for four of those operads, there is a one-parameter family of regular parametrized one-relation operads isomorphic to it.

Main Theorem (Theorem 7.1 (ii)) *Over an algebraically closed field of characteristic* 0*, every regular parametrized one-relation operad is isomorphic to one of the following five operads: the left-nilpotent operad defined by the identity* $((a_1a_2)a_3) = 0$*, the associative operad, the Leibniz operad* Leib*, the dual Leibniz (Zinbiel) operad* Zinb*, and the Poisson operad.*

It is an entertaining exercise to check that the five operads of Example 1.2 are pairwise nonisomorphic. One way to do that is as follows. The left-nilpotent operad, the associative operad, and the Poisson operad are easily seen to be isomorphic to their Koszul duals. The Koszul dual of the operad Leib is isomorphic to the operad Zinb; these two operads are not isomorphic because the suboperad generated by the S_2-invariants of $\mathrm{Zinb}(2)$ is the operad Com of commutative associative algebras, whereas in the case of Leib, we have $(a_1a_2 + a_2a_1)a_3 = 0$, which implies the identity $\{\{a_1, a_2\}, \{a_3, a_4\}\} = 0$ for the symmetrized product $\{a_1, a_2\} = a_1a_2 + a_2a_1$. (In fact, it is possible to show that each identity satisfied by the symmetrized product follows from that identity). The suboperads generated by the S_2-invariants and S_2-anti-invariants of $\mathrm{Poisson}(2)$ are the operad Com and the operad Lie of Lie algebras, respectively. Only the second of these claims holds for the associative operad, and neither is true for the left-nilpotent operad.

The proof of the main theorem uses algorithms for linear algebra over polynomial rings, the representation theory of the symmetric group, and commutative algebra, especially Gröbner bases for determinantal ideals and their radicals. It is worth mentioning that in fact, our proof of the main theorem shows that this classification result holds over a field $\mathbb{F}$ of characteristic zero where every quadratic equation has solutions (equivalently, $\mathbb{F}^\times = (\mathbb{F}^\times)^2$). The assumption on the characteristic is more fundamental: for example, the suboperad Com of Poisson naturally splits off as a direct summand, and this implies that the corresponding S_n-modules are, in general, not regular in positive characteristic.

Our main technical result classifies all parametrized one-relation operads that are regular in arity 4; it then turns out that such operads are necessarily regular in all arities. It is an open problem to provide a theoretical proof that explains conceptually why this should be true. In a way, this phenomenon makes one think of Bergman's Diamond Lemma [2] in the context of operads [11,15], although, there seems to be no obvious way to formalize that intuition. A related remark is that our results recover the family of operads from [17], which interpolates between the associative and the Poisson operad. This family provides some supporting evidence for the operadic analogue of the Koszul deformation principle for quadratic algebras [12, 20]; currently it is unknown if such an analogue exists.

At first glance, it is natural to expect that most relations (LR) define an operad whose components are regular modules: one can say that re-association would permit rewriting every product as a combination of right-normed products

$$a_1(a_2(\cdots(a_{n-1}a_n)\cdots))$$

that transform according to the regular representation. However, this strategy, when inspected more closely, exhibits many subtle phenomena: there are many ways to begin such rewriting, and at the same time, owing to the presence of all permutations on the right side of (LR), it is not at all clear that such a re-association process will terminate. In fact, it turns out that the generic operad defined by (LR) is as far from having regular modules as components as possible.

Nilpotency Theorem (Theorem 4.6) *Let $\mathcal{N}$ be the set of all points* $\mathbf{a}$ *in the parameter space $\mathbb{F}^6$ for which the operad $\mathcal{O}_{\mathbf{a}}$ is nilpotent of index* 3*. Then $\mathcal{N}$ is a Zariski open subset of the parameter space $\mathbb{F}^6$.*

In a nutshell, this follows from the fact that the Stasheff associahedron [15] of dimension 2, the pentagon, has the same number of vertices and edges; its vertices correspond to basis elements of the free operad in arity 4, and its edges are in one-to-one correspondence with the formal consequences of one ternary relation. Since the two numbers coincide, it is natural to expect that for a generic relation all operations of arity 4 will vanish.

1.1 Outline of This Paper

Section 2 recalls the necessary background on algebraic operads. We focus on binary quadratic operads, since they are the only type of operad that we consider.

Section 3 reviews basics of linear algebra over polynomial rings; we recall the notion of a determinantal ideal that is used to understand how the rank of a matrix with polynomial entries depends on the parameters.

Section 4 introduces the cubic relation matrix M, square of size 120, with entries in $\mathcal{C} = \mathbb{F}[x_1, \dots, x_6]$. This sparse matrix (over 94% zeros) is the main object of study throughout the paper. Its row module $\mathrm{Row}(M)$ over $\mathcal{C}$ is the S_4-module of relations satisfied by the general parametrized one-relation operad in arity 4. We use the algorithms from the previous section to obtain some basic information about the nullmodule of M: the $\mathcal{C}$-module $\{MH = O \mid H \in \mathcal{C}^{120}\}$. In particular, we prove the Nilpotency Theorem for parametrized one-relation operads.

Section 5 recalls basic concepts and methods from the representation theory of the symmetric group, emphasizing arity 4 and applications to polynomial identities. This allows us to replace the single large matrix M with five much smaller matrices that are much easier to study using computational commutative algebra.

In Section 6, we combine the approaches of the previous sections and prove the main technical result, a classification of all parametrized one-relation operads for which the arity 4 component is the regular module. This is done by a careful analysis of possible relations (LR) by increasing number of nonzero coefficients.

In Section 7, we establish that each of the operads in the previous section is regular, and isomorphic to one of the five operads from the Main Theorem, thus obtaining a full classification.

Section 8 outlines some further research directions and open problems.

2 Preliminaries on Algebraic Operads

In this section we recall basic background information from the theory of operads. All operads in this paper are generated by one binary operation, and we choose to keep this section within these limits. For general definitions and further details, we refer the reader to the recent comprehensive monograph by Loday and Vallette [15]. For the algorithmic aspects, see [5].

2.1 Nonsymmetric Operads

Operads encode multilinear operations with many arguments in the same way as associative algebras encode linear maps. The first level of abstraction is the notion of a nonsymmetric operad, where operations can be substituted into one another, but arguments of operations cannot be permuted. We may therefore choose a symbol such as $*$ to represent each of the n arguments of a given operation: $\omega(*,\dots,*)$. The different occurrences of $*$ represent different arguments, which are distinguished by their positions.

Throughout the paper, we only consider the case in which all operations are built out of one generating operation; therefore, we shall not give that operation a specific name, and write simply $(**)$, where it is understood that every pair of parentheses contains exactly two arguments, and each of these arguments is in turn either $*$ or another pair of parentheses containing exactly two …, etc. This notation remains unambiguous if we also omit the commas separating the arguments.

Definition 2.1 The *free nonsymmetric operad* Ω generated by one binary operation $(**)$ has *components* $\Omega(n)$, $n \geq 1$, where $\Omega(n)$ is spanned by the composite operations built out of $(**)$ that have exactly n arguments (in other words, it is of *arity* n). Such an operation must have exactly $n-1$ occurrences of $(**)$ (in other words, is of *weight* $n-1$).

Example 2.2 The following *balanced bracketings* form a basis of $\Omega(n)$ for $1 \leq n \leq 4$:

$$\left\{\begin{array}{ll} n & \text{monomials} \\ 1 & * \\ 2 & (**) \\ 3 & (*(**)),\quad ((**)*) \\ 4 & (*(*(**))),\quad (*((**)*)),\quad ((**)(**)),\quad ((*(**))*),\quad (((**)*)*) \end{array}\right.$$

Henceforth we will omit the outermost pair of parentheses.

Lemma 2.3 ([21]) *The dimension of $\Omega(n)$, or equivalently the number of distinct balanced bracketings using $n-1$ pairs of brackets, is equal to the Catalan number*

$$\dim \Omega(n) = \frac{1}{n}\binom{2n-2}{n-1} \quad (n \geq 1). \tag{2.1}$$

As a vector space, $\Omega(n)$ is the homogeneous subspace of degree n in the free nonassociative algebra with one binary operation ω and one generator $*$, but the collection

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of all components $\Omega(n)$ has a much richer structure to it that exemplifies the simplest case in the theory of algebraic operads.

Definition 2.4 The *composition maps* $\circ_i$ in the free nonsymmetric operad Ω are defined as follows. On basis monomials $\mu \in \Omega(n)$ and $\mu' \in \Omega(n')$, the i-th composition $\mu \circ_i \mu' \in \Omega(n + n' - 1)$ for $1 \le i \le n$, is the result of substituting μ' for the i-th argument $*$ in μ. This operation extends bilinearly to any elements $\alpha \in \Omega(n)$ and $\alpha' \in \Omega(n')$.

Definition 2.5 We inductively define a total order $\mu \prec \mu'$ on nonsymmetric basis monomials μ and μ'. The basis of the induction is the unique total order on the set $\{*\}$ which is a basis of $\Omega(1)$. Consider $\mu \in \Omega(n)$ and $\mu' \in \Omega(n')$, where n and n' are not both equal to 1. If $n < n'$, then we set $\mu \prec \mu'$. If $n' < n$, then we set $\mu' \prec \mu$. Otherwise, $n = n'$; write $\mu = \mu_1\mu_2$ and $\mu' = \mu_1'\mu_2'$. We have $\mu_i \in \Omega(p_i)$ for $p_i < n$ and $\mu_i' \in \Omega(p_i')$ for $p_i' < n'$,. Therefore, by induction we may assume that our total order is defined for μ_i and μ_i'. If $\mu_1 \neq \mu_1'$, we set $\mu \prec \mu'$ if and only if $\mu_1 \prec \mu_1'$, else we set $\mu \prec \mu'$ if and only if $\mu_2 \prec \mu_2'$. For example, the monomials in Example 2.2 follow this order.

2.2 Symmetric Operads

Of course, when one deals with actual multilinear operations, there is more structure to take into account, namely permutations of arguments. Formalizing that leads to the notion of a symmetric operad.

Definition 2.6 The *free symmetric operad* $\mathcal{T}$ generated by one binary operation has components

$$\mathcal{T}(n) = \Omega(n) \otimes \mathbb{F}S_n, \tag{2.2}$$

where S_n acts trivially on $\Omega(n)$ and $\mathbb{F}S_n$ is the right regular module. A basis for $\mathcal{T}(n)$ consists of all simple tensors $\psi \otimes \tau$, where $\psi \in \Omega(n)$ is a nonsymmetric basis monomial and $\tau \in S_n$ is a permutation of the arguments.

Remark 2.7 The natural interpretation of the simple tensor $\psi \otimes \tau$ is that ψ represents a certain bracketing (or placement of operation symbols) applied to the underlying multilinear monomial $a_{\tau(1)} \cdots a_{\tau(n)}$ that is the result of the action of τ on a decomposable tensor $a_1 \otimes \cdots \otimes a_n$. Since this action of S_n can lead quickly to a great deal of confusion, we include a few sentences to clarify it. Consider this left action of S_n on decomposable tensors $v_1 \otimes \cdots \otimes v_n$: $\tau(v_1 \otimes \cdots \otimes v_n) = v_{\tau^{-1}(1)} \otimes \cdots \otimes v_{\tau^{-1}(n)}$. This action moves the factor in position i to position $\tau(i)$, and induces a right action on $\mathcal{T}(n)$ that has the property that its extension to the tensor product $\mathcal{T}(n) \otimes_{\mathbb{F}S_n} V^{\otimes n}$ can be conveniently interpreted as applying operations to arguments. In other words, $(\psi \otimes \tau) \cdot \sigma = \psi \otimes \tau\sigma$. The total order of Definition 2.5 extends from the nonsymmetric case to the symmetric case: given basis monomials $\psi \otimes \tau$ and $\psi' \otimes \tau'$, we first compare the bracketings ψ, ψ', and if $\psi = \psi'$, then we compare the permutations τ, τ' in lexicographical order. It is straightforward to verify that the natural composition of operations in $\mathcal{T}$ is equivariant with respect to this action of the symmetric groups.

More concretely, one can view $\mathcal{T}(n)$ as the multilinear subspace of degree n in the free nonassociative algebra with one binary operation and n generators $a_1, \dots, a_n$.

Lemma 2.8 *The dimension of $\mathcal{T}(n)$, or equivalently the number of distinct multilinear n-ary nonassociative monomials, is given by the following formula:*

$$\dim \mathcal{T}(n) = \frac{1}{n}\binom{2n-2}{n-1} n!.$$

Proof This follows immediately from Definition 2.1 and equations (2.1) and (2.2). ∎

Definition 2.9 By a *quadratic relation* in the free symmetric operad $\mathcal{T}$ we mean an element of $\mathcal{T}(3)$, *i.e.*, a (nonzero) linear combination of simple tensors $\psi \otimes \tau$, where each bracketing $\psi \in \Omega(3)$ involves two occurrences of the generator $(*,*)$. Any S_3-submodule $\mathcal{R} \subseteq \mathcal{T}(3)$ is called a *module of quadratic relations*. To determine a module of relations $\mathcal{R}$, it suffices to give a set of module generators, not a linear basis (that is typically much larger).

Remark 2.10 When discussing relations in an operad, the word quadratic (and similarly cubic, quartic, etc.) refers to the weight $k-1$, not to the arity k. In particular, the quadratic relations are of arity three.

Definition 2.11 An *(operad) ideal* $\mathcal{J}$ in the free symmetric operad $\mathcal{T}$ is a family of S_n-submodules $\mathcal{J}(n) \subseteq \mathcal{T}(n)$, where $n \geq 1$, that is closed under composition with arbitrary elements of $\mathcal{T}$.

The *quotient operad* $\mathcal{T}/\mathcal{J}$ has components $(\mathcal{T}/\mathcal{J})(n) = \mathcal{T}(n)/\mathcal{J}(n)$ with the natural induced compositions.

The ideal $\mathcal{J} \subseteq \mathcal{T}$ *generated* by a subset $\Gamma \subseteq \mathcal{T}$ is the intersection of all the ideals containing Γ. Notation: $\mathcal{J} = (\Gamma)$.

Definition 2.12 Consider the operad ideal $\mathcal{J} = (\rho)$ generated by one quadratic relation $\rho \in \mathcal{T}(3)$. The S_3-module of quadratic relations is $\mathcal{J}(3) = \rho \cdot \mathbb{F}S_3$, the right S_3-module generated by ρ. We regard $\rho = \rho(a_1, a_2, a_3)$ as an operation with three arguments. The component $\mathcal{T}(4)$ contains three compositions $\rho \circ_i \omega$ and two compositions $\omega \circ_j \rho$ that vanish in $(\mathcal{T}/(\rho))(4)$:

$$\begin{gathered}\rho((a_1a_2), a_3, a_4), \quad \rho(a_1, (a_2a_3), a_4), \quad \rho(a_1, a_2, (a_3a_4)), \\ \rho(a_1, a_2, a_3)a_4, \quad a_1\rho(a_2, a_3, a_4).\end{gathered} \tag{2.3}$$

We call relations (2.3) the *cubic consequences* of the quadratic relation ρ. These five relations generate the S_4-module $\mathcal{J}(4) \subseteq \mathcal{T}(4)$. We can inductively repeat this generation of consequences into higher arities to compute every S_n-module in the ideal (ρ), but we will only require the cubic case.

Definition 2.13 We say that an operad $\mathcal{P} = \mathcal{T}/\mathcal{J}$ is *nilpotent* if there exists $k_0 \geq 0$ such that $\mathcal{P}(k+1) = \{0\}$ for all $k \geq k_0$. If k_0 is the least nonnegative integer satisfying this condition, then we say that $\mathcal{P}$ is nilpotent of *index* k_0. (This way, nilpotency of index

k_0 means that all operations made of k_0 or more copies of the generating operation vanish).

Clearly $\mathcal{P}(k) = \{0\}$ if and only if $\mathcal{J}(k) = \mathcal{T}(k)$. Hence $\mathcal{P}$ is nilpotent of index k_0 if and only if $\mathcal{J}(k_0) \neq \mathcal{T}(k_0)$ and $\mathcal{J}(k) = \mathcal{T}(k)$ for all $k > k_0$. Compositions of elements of $\mathcal{T}(j)$ with the generating operation produce all of $\mathcal{T}(j+1)$, so to check nilpotency it is enough to check that $\mathcal{P}(k+1) = 0$ just for $k = k_0$, and not for all $k \geq k_0$.

Example 2.14 ([18]) The simplest example of a nilpotent operad is the *anti-associative* operad $\mathcal{A}^+$ generated by one binary operation satisfying the relation

$$(a_1a_2)a_3 + a_1(a_2a_3) = 0;$$

this relation introduces a sign change every time we reassociate a product of three factors. This relation is the special case with parameters $[-1, 0, 0, 0, 0, 0]$ of Relation (LR); hence $\mathcal{A}^+$ is a parametrized one-relation operad. It is easy to show that $\mathcal{A}^+$ is nilpotent of index 3. Indeed, we note that the defining relation of our operad can be applied as a rewriting rule to the product $((a_1a_2)a_3)a_4$ in two different ways: by rewriting $(a_1a_2)a_3$ first, obtaining

$$((a_1a_2)a_3)a_4 = -(a_1(a_2a_3))a_4 = a_1((a_2a_3)a_4) = -a_1(a_2(a_3a_4)),$$

or by setting $b = (a_1a_2)$ and rewriting $(ba_3)a_4$ first, obtaining.

$$((a_1a_2)a_3)a_4 = -(a_1a_2)(a_3a_4) = a_1(a_2(a_3a_4)).$$

(This should remind the reader of computing an S-polynomial when calculating a Gröbner basis). We conclude that $a_1(a_2(a_3a_4)) = 0$. Since all five basis compositions (2.3) appear along the way, all of them are zero. Hence $\mathcal{A}^+(4) = \{0\}$, and the operad $\mathcal{A}^+$ is nilpotent.

2.3 Matrix Condition for Regularity

Relation (LR) is a special case of the following general binary quadratic relation [14]:

$$R(a_1, a_2, a_3) = \sum_{\tau \in S_3} w_\tau (a_{\tau(1)} a_{\tau(2)}) a_{\tau(3)} + \sum_{\tau \in S_3} y_\tau \, a_{\tau(1)} (a_{\tau(2)} a_{\tau(3)}), \tag{2.4}$$

where $w_\tau, y_\tau \in \mathbb{F}$. The S_3-submodule generated by R is the module $(R) \cap \mathcal{T}(3)$ of quadratic relations. If $H \subseteq S_3$ is the (normal) subgroup fixing R, then $(R) \cap \mathcal{T}(3) \cong \mathbb{F}(S_3/H)$ and so $\dim(R) \leq 6$, with equality if and only if only the identity permutation fixes R. The larger $\dim(R)$, the smaller H; dimension and symmetry are inversely related. For us the important case is $\dim(R) = 6$. Thus R generates an S_3-module isomorphic to $\mathbb{F}S_3$. Relation (LR) satisfies this condition. We shall return to this general relation (2.4) in Section 5 where it will serve as a toy example for the representation-theoretic method.

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We write out relation (2.4) term by term, replacing the permutation subscripts by integers, using the lex order in S_3. The relation R then has the form

$$
\begin{aligned}
(2.5)\qquad & w_1(a_1a_2)a_3 + w_2(a_1a_3)a_2 + w_3(a_2a_1)a_3 + w_4(a_2a_3)a_1 + w_5(a_3a_1)a_2 \\
& + w_6(a_3a_2)a_1 + y_1a_1(a_2a_3) + y_2a_1(a_3a_2) + y_3a_2(a_1a_3) \\
& + y_4a_2(a_3a_1) + y_5a_3(a_1a_2) + y_6a_3(a_2a_1).
\end{aligned}
$$

For a relation R of the form (LR) we have $w_1 = 1$ and $w_2 = \cdots = w_6 = 0$. Let $[W \mid Y]$ be the matrix whose rows are the coefficient vectors obtained by applying every $\sigma \in S_3$ to R:

$$
R \cdot \sigma = \sum_{\tau \in S_3} w_\tau (a_{\sigma\tau(1)} a_{\sigma\tau(2)}) a_{\sigma\tau(3)} + \sum_{\tau \in S_3} y_\tau (a_{\sigma\tau(1)} a_{\sigma\tau(2)}) a_{\sigma\tau(3)}, \quad \sigma \in S_3.
$$

Working this out explicitly, where the columns correspond to the basis monomials in the order of (2.5), we obtain a matrix where the pattern of subscripts matches that of the celebrated Dedekind–Frobenius determinant for S_3:

(2.6)

$$
[\ W \mid Y\ ] = \left[\begin{array}{cccccc|cccccc}
w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\
w_2 & w_1 & w_5 & w_6 & w_3 & w_4 & y_2 & y_1 & y_5 & y_6 & y_3 & y_4 \\
w_3 & w_4 & w_1 & w_2 & w_6 & w_5 & y_3 & y_4 & y_1 & y_2 & y_6 & y_5 \\
w_5 & w_6 & w_2 & w_1 & w_4 & w_3 & y_5 & y_6 & y_2 & y_1 & y_4 & y_3 \\
w_4 & w_3 & w_6 & w_5 & w_1 & w_2 & y_4 & y_3 & y_6 & y_5 & y_1 & y_2 \\
w_6 & w_5 & w_4 & w_3 & w_2 & w_1 & y_6 & y_5 & y_4 & y_3 & y_2 & y_1
\end{array}\right].
$$

Lemma 2.15 *Suppose that for the given 6×6 matrices W and Y with coefficients in $\mathbb{F}$, the rows of the matrix $[W \mid Y]$ form a single orbit for the right action of S_3, as in (2.6) above. The subspace they generate contains a relation of the type (LR) if and only if W is invertible.*

Proof Note that every matrix representing the orbit of a relation of the type (LR) is a matrix of the form

$$
(2.7)\qquad \left[\begin{array}{cccccc|cccccc}
1 & 0 & 0 & 0 & 0 & 0 & -x_1 & -x_2 & -x_3 & -x_4 & -x_5 & -x_6 \\
0 & 1 & 0 & 0 & 0 & 0 & -x_2 & -x_1 & -x_5 & -x_6 & -x_3 & -x_4 \\
0 & 0 & 1 & 0 & 0 & 0 & -x_3 & -x_4 & -x_1 & -x_2 & -x_6 & -x_5 \\
0 & 0 & 0 & 1 & 0 & 0 & -x_5 & -x_6 & -x_2 & -x_1 & -x_4 & -x_3 \\
0 & 0 & 0 & 0 & 1 & 0 & -x_4 & -x_3 & -x_6 & -x_5 & -x_1 & -x_2 \\
0 & 0 & 0 & 0 & 0 & 1 & -x_6 & -x_5 & -x_4 & -x_3 & -x_2 & -x_1
\end{array}\right],
$$

and this matrix is in row canonical form (RCF). For any matrix $[W \mid Y]$, its RCF is a matrix of the form $[I \mid Z]$ if and only if W is invertible, and in this case $Z = W^{-1}Y$. Finally, in (2.6) the matrices W and Y are, respectively, the matrices representing the action of $\sum_\sigma w_\sigma \sigma$ and $\sum_\sigma y_\sigma \sigma$ on the right regular module, and thus so is $W^{-1}Y$, justifying the same Dedekind–Frobenius determinant pattern of matrix elements in $W^{-1}Y$. ∎

2.4 Koszul Duality

The theory of Koszul duality for operads, due to Ginzburg and Kapranov [13], associates a quadratic operad $\mathcal{P}$ with another quadratic operad $\mathcal{P}^!$, its Koszul dual. In the case when $\mathcal{P}$ satisfies some good homological properties (such operads are called Koszul operads), the Koszul dual operad can be used to control deformation theory of $\mathcal{P}$-algebras. (Familiar examples are given by deformation complexes of associative algebras and Lie algebras). For an operad generated by a binary product, the operad $\mathcal{P}^!$ admits a very economic description that we recall here, referring the reader to [15] for general definitions and results on Koszul duality, as well as further motivation.

Proposition 2.16 ([15]) *Suppose that $\mathcal{P} \cong \mathcal{T}/(\mathcal{R})$ is a quotient operad of $\mathcal{T}$ by some module of quadratic relations $\mathcal{R}$. We define a scalar product on $\mathcal{T}(3)$ as follows:*

$$(\psi_1,\psi_1) = 1, (\psi_2,\psi_2) = -1, (\psi_1,\psi_2) = 0, \text{ where } \psi_1 = (**)*, \psi_2 = *(**). \tag{2.8}$$

This can be extended to an S_3-invariant scalar product on $\mathcal{T}(3)$ by the formula

$$(\psi_i \otimes \tau_j, \psi_k \otimes \tau_\ell) = (\psi_i,\psi_k)\delta_{j\ell}\varepsilon(\tau_j), \tag{2.9}$$

where $\varepsilon\colon S_3 \to \{\pm1\}$ is the sign, and $\delta_{j\ell}$ is the Kronecker symbol. We write $\mathcal{R}^\perp$ for its orthogonal complement with respect to (2.8). The Koszul dual operad *$\mathcal{P}^!$ is the quotient operad $\mathcal{T}/(\mathcal{R}^\perp)$.*

Lemma 2.17 ([14],[15, Proposition 7.6.8]) *The Koszul dual operad $\mathcal{P}^!$ of any parametrized one-relation operad $\mathcal{P}$ is isomorphic to a parameterized one-relation operad; if the operad $\mathcal{P}$ is defined by Relation (LR), the Koszul dual operad is isomorphic to the operad defined by the relation*

$$\begin{aligned}(a_1a_2)a_3 = x_1a_1(a_2a_3) - x_3a_1(a_3a_2) - x_2a_2(a_1a_3)\\ + x_4a_2(a_3a_1) + x_5a_3(a_1a_2) - x_6a_3(a_2a_1).\end{aligned}$$

In plain words, to obtain S, we switch and negate coefficients 2 *and* 3*, and negate coefficient* 6*.*

Proof We start from matrix (2.7) whose row space is the module $\mathcal{R}$ of quadratic relations. By Proposition 2.16, the computation of $\mathcal{R}^\perp$ is reduced to the computation of the nullspace of a modified matrix: we multiply columns 7–12 by −1 according to (2.8), and then multiply columns 2, 3, 6, 8, 9, 12 with odd permutations by −1 according to (2.9). We compute the RCF: for this we simply multiply the rows with odd permutations by −1:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -x_1 & x_2 & x_3 & -x_4 & -x_5 & x_6 \\ 0 & 1 & 0 & 0 & 0 & 0 & x_2 & -x_1 & -x_5 & x_6 & x_3 & -x_4 \\ 0 & 0 & 1 & 0 & 0 & 0 & x_3 & -x_4 & -x_1 & x_2 & x_6 & -x_5 \\ 0 & 0 & 0 & 1 & 0 & 0 & -x_5 & x_6 & x_2 & -x_1 & -x_4 & x_3 \\ 0 & 0 & 0 & 0 & 1 & 0 & -x_4 & x_3 & x_6 & -x_5 & -x_1 & x_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & x_6 & -x_5 & -x_4 & x_3 & x_2 & -x_1 \end{bmatrix}. \tag{2.10}$$

We compute the standard basis for the nullspace of (2.10) by setting free variables to unit vectors and solving for leading variables. We obtain another matrix whose

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row space is the nullspace of (2.10); this is the module $(\mathcal{R}^\perp)$. However, this matrix has the form $[X \mid I_6]$: it is not in row canonical form. Computing the RCF of this matrix requires dividing by polynomials in the x_i. However, this can be avoided by passing to the isomorphic operad for the opposite algebras, which interchanges ψ_1 and ψ_2, putting the columns back into the original order of the monomials, and then computing the RCF. We obtain the following result:

$$\left[\begin{array}{cccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & -x_1 & x_3 & x_2 & -x_4 & -x_5 & x_6 \\ 0 & 1 & 0 & 0 & 0 & 0 & x_3 & -x_1 & -x_5 & x_6 & x_2 & -x_4 \\ 0 & 0 & 1 & 0 & 0 & 0 & x_2 & -x_4 & -x_1 & x_3 & x_6 & -x_5 \\ 0 & 0 & 0 & 1 & 0 & 0 & -x_5 & x_6 & x_3 & -x_1 & -x_4 & x_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & -x_4 & x_2 & x_6 & -x_5 & -x_1 & x_3 \\ 0 & 0 & 0 & 0 & 0 & 1 & x_6 & -x_5 & -x_4 & x_2 & x_3 & -x_1 \end{array}\right]. \tag{2.11}$$

From the first row of (2.11), we easily read off the coefficients of S. ∎

3 Linear Algebra Over Polynomial Rings

Over a field $\mathbb{F}$, to determine whether two $m \times n$ matrices A and B belong to the same orbit under the left action of $\mathrm{GL}_m(\mathbb{F})$, we compute the row canonical forms $\mathrm{RCF}(A)$ and $\mathrm{RCF}(B)$ and check whether they are equal. Similarly, for the left-right action of $\mathrm{GL}_m(\mathbb{F}) \times \mathrm{GL}_n(\mathbb{F})$, we compute $\mathrm{Smith}(A)$ and $\mathrm{Smith}(B)$.

Over a Euclidean domain, in particular, the ring $\mathbb{F}[x]$ of polynomials in one variable x over a field $\mathbb{F}$, a modification of Gaussian elimination gives the desired result, since the coordinate ring is a PID and we can implement the Euclidean algorithm for GCDs using row (or column) operations. The analogue of the RCF in this case is called the Hermite normal form (HNF).

Once we go beyond Euclidean domains, these computations become much more difficult for two main reasons: we can no longer compute GCDs using row operations, and it would not help even if we could, since the coordinate ring is no longer a PID. In this setting, the existence of a normal form which determines when two matrices belong to the same orbit remains an open problem. We can nonetheless obtain some useful information about a multivariate polynomial matrix by elementary methods.

We consider the problem of computing the rank of an $m \times n$ matrix A with entries in the ring $\mathbb{F}[x_1, \dots, x_p]$ of polynomials in $p \geq 2$ variables (or parameters) over $\mathbb{F}$. In one sense, the rank of such a matrix is its rank when regarded as a matrix over the field $\mathbb{F}(x_1, \dots, x_p)$ of rational functions: since the coordinate ring is now a field again, we can use Gaussian elimination. However, crucial information is lost, since we are implicitly assuming that none of the denominators that arise in the matrix entries during this calculation ever become 0. Another definition of the rank of the matrix A is as follows.

Definition 3.1 Let A be an $m \times n$ matrix over $\mathbb{F}[x_1, \dots, x_p]$ regarded as a parametrized family of matrices over $\mathbb{F}$. We define the function $A| : \mathbb{F}^p \to \mathrm{Mat}_{mn}(\mathbb{F})$. For $a_1, \dots, a_p \in \mathbb{F}$ the matrix $A|(a_1, \dots, a_p)$ is obtained from A by setting $x_i = a_i$ for $i = 1, \dots, p$. Composing $A|$ with the rank on $\mathrm{Mat}_{mn}(\mathbb{F})$, gives the *substitution rank* function: $\mathrm{subrank}_A = \mathrm{rank} \circ A| : \mathbb{F}^p \to \{0, 1, 2, \dots, \min(m, n)\}$. The inverse images of the ranks $0 \le r \le \min(m,n)$ define the *inverse rank* function:

$$\mathrm{invrank}_A(r) = \{(a_1,\dots,a_p) \in \mathbb{F}^p \mid \mathrm{subrank}_A(a_1,\dots,a_p) = r\}.$$

We define the *minimal rank* $r_{\min}$ as follows:

$$r_{\min}(A) = \min\{r \mid 0 \le r \le \min(m,n), \mathrm{invrank}_A(r) \ne \varnothing\}.$$

The following very simple result will be useful to us later.

Proposition 3.2 *Let A be an $m \times n$ matrix over $\mathbb{F}[x_1,\dots,x_p]$. If there exist elements $a_1,\dots,a_p \in \mathbb{F}$ such that the matrix $A|(a_1,\dots,a_p) \in \mathrm{Mat}_{mn}(\mathbb{F})$ has full rank $r = \min(m,n)$, then A has full rank over the field $\mathbb{F}(x_1,\dots,x_p)$ of rational functions.*

Proof It is well known that the rank of an $m \times n$ matrix over $\mathbb{F}$ is r if and only if two conditions hold: at least one $r \times r$ minor is not 0, and every $(r+1) \times (r+1)$ minor is 0. Therefore, if A does not have full rank, then all minors of A of size r vanish, which of course would guarantee that all those minors vanish after specialisation to $(a_1,\dots,a_p)$, when they become the minors of $A|(a_1,\dots,a_p)$. ∎

Definition 3.3 Let A be an $m\times n$ matrix over $\mathbb{F}[x_1,\dots,x_p]$. The *determinantal ideals* $\mathrm{DI}_r(A)$ for $r = 0,\dots,\min(m,n)$ are defined as follows: $\mathrm{DI}_0(A) = \mathbb{F}[x_1,\dots,x_p]$, and if $r \ge 1$, then $\mathrm{DI}_r(A)$ is the ideal in $\mathbb{F}[x_1,\dots,x_p]$ generated by all $r \times r$ minors of A.

In terms of determinantal ideals, we can reformulate the classical formula for the rank of a matrix as follows.

Proposition 3.4 *Let A be an $m \times n$ matrix over $\mathbb{F}[x_1,\dots,x_p]$. For every value of r not exceeding* $\min(m,n)$, *we have* $\mathrm{invrank}_A(r) = V(\mathrm{DI}_{r+1}) \smallsetminus V(\mathrm{DI}_r)$.

The advantage of using determinantal ideals is that they allow us to study the rank of a matrix using only ring operations (without division). The classical theory of determinantal ideals is concerned almost exclusively with the homogeneous case, in which every minor is a homogeneous polynomial [19]. Since many entries of the cubic relation matrix M (to be defined in the next section) equal 1, the determinantal ideals we study in what follows will be inhomogeneous. We could reformulate our problem in homogeneous terms by introducing a new parameter x_0 to play the role of the coefficient of $(a_1a_2)a_3$ in Relation (LR). This leads into the theory of sparse determinantal ideals [3]. However, having many leading ones in the matrix will be very useful from a computational point of view.

Henceforth, most of our computations require a choice of monomial order.

Definition 3.5 For an element m of the monomial basis of $\mathbb{F}[x_1,\dots,x_p]$, we write

$$m = \prod_{k=1}^{p} x_k^{e_k}, \quad e(m) = [e_1,\dots,e_p], \quad \deg(m) = \sum_{k=1}^{p} e_k.$$

The *graded reverse lexicographic order* (see `grevlex` in `Maple`, `Magma` and `Macaulay`, `degrevlex` in `sage`, and `dp` in `Singular`) is defined by $m \prec m'$ if and only if either

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$\deg(m) < \deg(m')$, or $\deg(m) = \deg(m')$, and $e_k > e'_k$, where k is the smallest index such that $e_k \neq e'_k$. Note that $x_1 < \cdots < x_p$.

The leading monomial $\mathrm{LM}(f)$ of a nonzero polynomial $f \in \mathbb{F}[x_1, \dots, x_p]$ is the greatest with respect to $<$, and $\mathrm{LC}(f)$ is the coefficient of $\mathrm{LM}(f)$.

In what follows, we shall use this ordering of monomials for ordering lists of polynomials (term by term).

Definition 3.6 Given a monomial order $<$, every ideal $J \subseteq \mathbb{F}[x_1, \dots, x_p]$ contains a (finite) ordered set $G = \{f_1, \dots, f_t\}$ of (nonzero) polynomials, called a *Gröbner basis* with respect to $<$, satisfying the following conditions:

- $J = (G)$: the polynomials $f_1, \dots, f_t$ generate J.
- $(\{\mathrm{LM}(f) \mid f \in J\}) = (\{\mathrm{LM}(f) \mid f \in G\})$: the ideal generated by the leading monomials of the elements of J is generated by the leading monomials of the elements of G.

A *reduced* Gröbner basis satisfies the following additional conditions.

- The generators are monic: $\mathrm{LC}(f) = 1$ for every $f \in G$.
- For every $f \in G$, no monomial of any $f' \in G \setminus \{f\}$ is divisible by $\mathrm{LM}(f)$.

Every ideal has a unique reduced Gröbner basis with respect to a given monomial order. Of the many books on Gröbner bases, Cox et al.[9,10] are the most approachable.

Definition 3.7 For an ideal $J \subseteq \mathbb{F}[x_1, \dots, x_p]$, the *zero set* $V(J)$ is the set of points in $\mathbb{F}^p$ that are solutions to every polynomial in J:

$$V(J) = \{(a_1, \dots, a_p) \in \mathbb{F}^p \mid f(a_1, \dots, a_p) = 0 \text{ for all } f \in J.\}$$

The ideal $I(S)$ of the subset $S \subseteq \mathbb{F}^p$ consists of all polynomials that vanish on S:

$$I(S) = \{f \in \mathbb{F}[x_1, \dots, x_p] \mid f(a_1, \dots, a_p) = 0 \text{ for all } (a_1, \dots, a_p) \in S\}.$$

Clearly $J \subseteq I(V(J))$. The *radical* of J is the ideal $\sqrt{J} = I(V(J))$. We say that J is a *radical ideal* if $J = \sqrt{J}$. For our purposes, the value of these concepts is that often $\sqrt{J}$ is much larger than J and has a much smaller and simpler Gröbner basis.

For a matrix whose entries are multivariate polynomials, Algorithm 1 produces a *partial Smith form* based on elimination using nonzero scalar entries. The basic idea is rather naive, but this algorithm will be useful in reducing the size of matrices before computing determinantal ideals.

4 General Results on Parametrized One-relation Operads

4.1 The Cubic Relation Matrix M

Notation 4.1 The monomial basis of the quadratic space $\mathcal{T}(3)$ consists of the five elements from Example 2.2. We replace the argument symbols $*$ by the identity permutation of the variables a_1, a_2, a_3, a_4 obtaining a generating set for the S_4-module $\mathcal{T}(4)$:

$$\gamma_1 = ((a_1a_2)a_3)a_4, \quad \gamma_2 = (a_1(a_2a_3))a_4, \quad \gamma_3 = (a_1a_2)(a_3a_4),$$
$$\gamma_4 = a_1((a_2a_3)a_4), \quad \gamma_5 = a_1(a_2(a_3a_4)).$$

Algorithm 1 Partial Smith form

Input: an $m \times n$ matrix R with entries in $\mathbb{F}[x_1, \dots, x_p]$.

Output: an $m \times n$ matrix S equivalent to R over $\mathbb{F}[x_1, \dots, x_p]$ in the sense that $S = URV$, where U $(m \times m)$ and V $(n \times n)$ are invertible matrices over $\mathbb{F}[x_1, \dots, x_p]$, that is, $\det(U)$ and $\det(V)$ are nonzero scalars. Furthermore, S consists of two diagonal blocks: an identity matrix and a block B in which no entry is a nonzero scalar.

```
set S ← R   k ← 1.
while s_ij ∈ F ∖ {0} for some i, j ≥ k do
   Find the least i ≥ k for which s_ij ∈ F ∖ {0} for some j ≥ k.
   If i ≠ k, then interchange rows i and k of S.
   Find the least j ≥ k for which s_kj ∈ F ∖ {0}.
   If j ≠ k, then interchange columns j and k of S.
   If s_kk ≠ 1, then divide row k of S by s_kk.
   For i = k + 1, ..., m do: subtract s_ik times row k from row i.
   For j = k + 1, ..., n do: subtract s_kj times column k from column j.
   Set k ← k + 1.
end while
return S.
```

To each generator $\gamma_1, \dots, \gamma_5$ we apply all 24 permutations from S_4 to obtain a linear basis of $\mathcal{T}(4)$. We write these basis monomials using the notation $[\tau]_q = \tau \cdot \gamma_q$ for $\tau \in S_4$ and $q \in \{1, \dots, 5\}$. We impose a total order by defining monomial $j \in \{1, \dots, 120\}$ to be $[\tau]_q$, where $j - 1 = 24(q-1) + (r-1)$ and $r \in \{1, \dots, 24\}$ and τ is permutation r in lex order.

Let us consider the general relation of the type (LR):

$$\rho(a_1, a_2, a_3) = (a_1 a_2) a_3 - x_1 a_1 (a_2 a_3) - x_2 a_1 (a_3 a_2) - x_3 a_2 (a_1 a_3) \\ - x_4 a_2 (a_3 a_1) - x_5 a_3 (a_1 a_2) - x_6 a_3 (a_2 a_1).$$

In what follows, we denote by $\mathcal{J}$ the operad ideal in $\mathcal{T}$ generated by ρ. We regard the coefficients $x_1, \dots, x_6$ as indeterminates, and so $\mathcal{T}$ has become an operad not over $\mathbb{F}$, but instead over the polynomial ring $\mathcal{C} = \mathbb{F}[x_1, \dots, x_6]$. That is, we replace each S_n-module $\mathcal{T}(n)$ over $\mathbb{F}$ by the tensor product $\mathcal{C} \otimes \mathcal{T}(n)$ over $\mathcal{C}$, where every $\tau \in S_n$ acts as the identity map on $\mathcal{C}$. Thus $\mathcal{T}(n)$ has changed from a vector space of dimension $\frac{(2k-2)!}{(k-1)!}$ (Lemma 2.8) to a free $\mathcal{C}$-module of the same rank. In particular, $\mathcal{T}(4)$ is a free $\mathcal{C}$-module of rank 120.

According to Definition 2.12 the elements generate the S_4-module $\mathcal{J}(4) \subseteq \mathcal{T}(4)$

$$\rho(a_1 a_2, a_3, a_4) = ((a_1 a_2) a_3) a_4 - x_1 (a_1 a_2)(a_3 a_4) - x_2 (a_1 a_2)(a_4 a_3) \\ - x_3 a_3 ((a_1 a_2) a_4) - x_4 a_3 (a_4 (a_1 a_2)) - x_5 a_4 ((a_1 a_2) a_3) \\ - x_6 a_4 (a_3 (a_1 a_2)),$$

$$\rho(a_1, a_2a_3, a_4) = (a_1(a_2a_3))a_4 - x_1a_1((a_2a_3)a_4) - x_2a_1(a_4(a_2a_3)) - x_3(a_2a_3)(a_1a_4) - x_4(a_2a_3)(a_4a_1) - x_5a_4(a_1(a_2a_3)) - x_6a_4((a_2a_3)a_1),$$

$$\rho(a_1, a_2, a_3a_4) = (a_1a_2)(a_3a_4) - x_1a_1(a_2(a_3a_4)) - x_2a_1((a_3a_4)a_2) - x_3a_2(a_1(a_3a_4)) - x_4a_2((a_3a_4)a_1) - x_5(a_3a_4)(a_1a_2) - x_6(a_3a_4)(a_2a_1),$$

$$\rho(a_1, a_2, a_3)a_4 = ((a_1a_2)a_3)a_4 - x_1(a_1(a_2a_3))a_4 - x_2(a_1(a_3a_2))a_4 - x_3(a_2(a_1a_3))a_4 - x_4(a_2(a_3a_1))a_4 - x_5(a_3(a_1a_2))a_4 - x_6(a_3(a_2a_1))a_4,$$

$$a_1\rho(a_2, a_3, a_4) = a_1((a_2a_3)a_4) - x_1a_1(a_2(a_3a_4)) - x_2a_1(a_2(a_4a_3)) - x_3a_1(a_3(a_2a_4)) - x_4a_1(a_3(a_4a_2)) - x_5a_1(a_4(a_2a_3)) - x_6a_1(a_4(a_3a_2)).$$

Using the notation for basis elements described above, these expansions can be written as

(4.1)

$$\begin{gathered}
[1234]_1 - x_1[1234]_3 - x_2[1243]_3 - x_3[3124]_4 - x_4[3412]_5 - x_5[4123]_4 - x_6[4312]_5 \\
[1234]_2 - x_1[1234]_4 - x_2[1423]_5 - x_3[2314]_3 - x_4[2341]_3 - x_5[4123]_5 - x_6[4231]_4 \\
[1234]_3 - x_1[1234]_5 - x_2[1342]_4 - x_3[2134]_5 - x_4[2341]_4 - x_5[3412]_3 - x_6[3421]_3 \\
[1234]_1 - x_1[1234]_2 - x_2[1324]_2 - x_3[2134]_2 - x_4[2314]_2 - x_5[3124]_2 - x_6[3214]_2 \\
[1234]_4 - x_1[1234]_5 - x_2[1243]_5 - x_3[1324]_5 - x_4[1342]_5 - x_5[1423]_5 - x_6[1432]_5.
\end{gathered}$$

The following list of 120 relations generates $\mathcal{J}(4)$ as a $\mathcal{C}$-module:

$$\rho(a_1a_2, a_3, a_4).\tau, \quad \rho(a_1, a_2a_3, a_4).\tau, \quad \rho(a_1, a_2, a_3a_4).\tau,$$
$$\rho(a_1, a_2, a_3)a_4.\tau, \quad a_1\rho(a_2, a_3, a_4).\tau,$$

where $\tau \in S_4$ is an arbitrary permutation. These relations can be represented as row vectors of dimension 120 over $\mathcal{C}$ using the total order of Notation 4.1; each vector has the entries $\{1, -x_1, \dots, -x_6\}$ and 113 zeros. We sort these row vectors into semi-triangular form using the following *total order* $x < y$ on row vectors of the same but arbitrary length.

- Let $i, j \geq 1$ be the least integers for which $x_i \neq 0$ and $y_j \neq 0$.
- If $i \neq j$, then $x < y$ if and only if $i < j$.
- If $i = j$ but $x_i \neq y_j$, then $x < y$ if and only if $x_i < y_j$ according to

$$1 < -x_1 < -x_2 < -x_3 < -x_4 < -x_5 < -x_6.$$

- If $i = j$ and $x_i = y_j$, then $x < y$ if and only if $x' < y'$, where x' (resp. y') is obtained from x (resp. y) by deleting the first i entries.

Definition 4.2 The *cubic relation matrix* $M = (m_{ij})$ is the square matrix of size 120 in which entry m_{ij} is the coefficient of the j-th basis monomial (Notation 4.1) in the i-th row vector in the list of consequences of the relation ρ sorted as above.

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Lemma 4.3 *The cubic relation matrix M has minimal rank* 84*. The reduced Gröbner basis for the first nontrivial determinantal ideal* $\mathrm{DI}_{85}(M)$ *is as follows:*

$$x_2 + x_3, \quad x_1 + x_4, \quad x_6, \quad x_1^2, \quad x_2x_1, \quad x_1x_5 + x_2, \quad x_2^2, \quad x_5x_2 + x_1, \quad x_5^2 - 1.$$

The reduced Gröbner basis for the radical $\sqrt{\mathrm{DI}_{85}(M)}$ *is as follows:*

$$x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_6, \quad x_5^2 - 1.$$

The only parameter values which produce the minimal rank are $[0,0,0,0,\pm1,0]$*, and for these values we obtain the maximal dimension* $\mathrm{nullity}(M) = 36$ *for the* S_4*-module* $\mathcal{T}(4)/\mathcal{J}(4)$.

Proof The following properties of M were obtained using the computer algebra system `Maple`. Applying Algorithm 1 to the cubic relation matrix M, produces $\mathrm{diag}(I_{84}, B)$ where B is a 36×36 matrix in which no entry is a nonzero scalar. From this it follows that for all $a_1, \dots, a_6 \in \mathbb{F}$, the matrix $M|(a_1, \dots, a_6)$ has rank ≥ 84, and $\mathrm{DI}_k(M) = \mathrm{DI}_{k-84}(B)$ for $k > 84$. This formula easily allows computation of the Gröbner bases for the ideal $\mathrm{DI}_{85}(M) = \mathrm{DI}_1(B)$ and for its radical, which are as stated in the lemma. Examining these Gröbner bases, we see that $B = 0$ only for the values of parameters $[0,0,0,0,\pm1,0]$, which completes the proof. ∎

Remark 4.4 The computations for the proof of Lemma 4.3 also provided the following information about the block B. Every entry f of B has integer coefficients, and only 99 of the 1296 entries are zero. After normalizing these polynomials by making all leading coefficients equal to 1, there are 709 distinct polynomials with 665 distinct irreducible factors; in fact, 492 of these polynomials are irreducible. In a way, it is remarkable that the ideal generated by these polynomials has such a small and simple Gröbner basis.

Remark 4.5 The other determinantal ideals are much harder. A generating set for $\mathrm{DI}_r(B)$ contains $C(36,r)^2$ determinants of $r \times r$ submatrices. In particular, regularity requires $\mathrm{nullity}(M) = 24$ and hence $\mathrm{rank}(B) = 12$. To determine the parameter values satisfying this condition, Proposition 3.4 tells us to find the zero sets $V(\mathrm{DI}_r(B))$ for $r = 12, 13$. For $r = 12$ (and worse for $r = 13$), we must evaluate more than 10^{18} minors, and 12×12 determinants over $\mathbb{F}[x_1, \dots, x_6]$ are not easy to compute. Even supposing that this were possible, we would still have to compute Gröbner bases for the two ideals, and hope that these would make it possible to solve explicitly for the zero sets. We will be able to overcome these obstacles using the representation theory of the symmetric group, starting in Section 5.

4.2 Nilpotency Theorem

Theorem 4.6 *Let $\mathcal{N}$ be the set of all points* $\mathbf{a}$ *in the parameter space* $\mathbb{F}^6$ *for which the operad* $\mathcal{O}_{\mathbf{a}}$ *is nilpotent of index* 3*. Then $\mathcal{N}$ is a Zariski open subset of the parameter space* $\mathbb{F}^6$.

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Proof Example 2.14 showed that the anti-associative identity is a special case of Relation (LR) and that the anti-associative operad is nilpotent of index 3. Hence setting $x_1 = -1$ and $x_2 = \cdots = x_6 = 0$ in M produces an invertible matrix over $\mathbb{F}$. It follows from Proposition 3.2 that the cubic relation matrix M is invertible over the field of rational functions $\mathbb{F}(x_1, \dots, x_6)$. For $\mathbf{a} = (a_1, \dots, a_6) \in \mathbb{F}^6$, the parametrized one-relation operad $\mathcal{O}_{\mathbf{a}}$ is nilpotent of index 3 if and only if $\det(M|(a_1, \dots, a_6)) \neq 0$; this condition defines a Zariski open subset in the space of parameters. ∎

4.3 Towards Classifying Regular Parametrized One-Relation Operads

Lemma 4.7 *If the operad $\mathcal{O}_{\mathbf{x}}$ is regular for the values $x_k = a_k \in \mathbb{F}$ $(1 \le k \le 6)$, then*

- $\operatorname{rank}(M) = 96$ *and* $\operatorname{rowspace}(M) \cong (\mathbb{F}S_4)^4$ *as an S_4-module,*
- $\operatorname{nullity}(M) = 24$ *and* $\operatorname{nullspace}(M) \cong \mathbb{F}S_4$ *as an S_4-module.*

Proof Regularity means that $\mathcal{T}(n)/\mathcal{J}(n) \cong \mathbb{F}S_n$ for all $n \ge 1$, and so in particular we have $\mathcal{T}(4)/\mathcal{J}(4) \cong \mathbb{F}S_4$. Since $\Omega(4)$ is five-dimensional, we have $\mathcal{T}(4) \cong (\mathbb{F}S_4)^5$, which implies $\mathcal{J}(4) \cong (\mathbb{F}S_4)^4$. Since $\operatorname{rowspace}(M) \cong \mathcal{J}(4)$, we have $\operatorname{nullspace}(M) \cong \mathcal{T}(4)/\mathcal{J}(4)$. ∎

Consider any subset $A \subseteq \{1, \dots, 6\}$ and let $M(A)$ be the matrix obtained by setting $x_i = 0$ for all $i \in A$ in the cubic relation matrix M. If we apply Algorithm 1 to $M(A)$, then we obtain a block diagonal matrix $\operatorname{diag}(I_r, B_s)$ where $r = r(A)$, $s = s(A)$, and $r + s = 120$. As before, I_r is the identity matrix of size r, and B_s is a square matrix of size s in which no entry is a nonzero scalar. The following result can be obtained by a straightforward `Maple` computation.

Lemma 4.8 *The size s of B depends only on whether* 5 *or* 6 *is in A.*

- *If $x_5 = x_6 = 0$, then B has size* 24.
- *If $x_5 = 0$, but $x_6 \neq 0$, then B has size* 30.
- *If $x_5 \neq 0$, then B has size* 36.

We now consider the 16 cases in which $x_5 = x_6 = 0$; we can deal with them all at once by allowing $x_1, \dots, x_4$ to be free parameters. We shall be able to establish the following rather attractive result, which shows how the four most familiar cases of parametrized one-relation operads may be obtained directly from elementary observations using linear and commutative algebra. In fact, we shall provide two proofs of this result, since each of them is somewhat instructive.

Proposition 4.9 *The only cases of Relation* (LR) *with $x_5 = x_6 = 0$ that are regular are those defining the trivial, associative, Leibniz, and Zinbiel operads.*

First proof of Proposition 4.9 Algorithm 1 reduces M to an identity matrix of size 96 and a lower right block B of size 24. Thus, in order for the nullity of M to be 24, it is necessary and sufficient that $B = 0$, and this in turn is equivalent to $\mathrm{DI}_1(B) = \{0\}$. In B, $432/576 = 3/4$ of the entries are nonzero but there are only 18 distinct nonzero entries, with degrees $\{3, 4\}$, coefficients $\{\pm 1, 2\}$, and numbers of terms $\{2, 6, 7, 8\}$. Figure 1 lists these entries in `grevlex` order; those not factored are irreducible.

1	$x_4x_1(x_2+x_3)$
2	$x_4^2x_2 - x_3^2x_2 - x_3^2x_1 + x_4x_2x_1 - x_4x_1^2 - x_3x_1^2$
3	$(x_2+x_3)(x_1^2 - x_1x_2 + x_1x_3 - x_2x_4 + x_3x_4)$
4	$x_4^2x_3 - x_3x_2^2 + x_4x_3x_1 + x_2^2x_1 + x_4x_1^2 - x_2x_1^2$
5	$x_4x_3x_2^2 + x_4^2x_3x_1 + x_3x_2^2x_1 + x_4x_3x_1^2 + x_4x_3^2 + x_4^2x_1 - x_3x_2$
6	$x_4^2x_2^2 + x_3x_2^3 + x_4^3x_1 + x_4x_3x_2x_1 + x_3^3 + x_4x_3x_1 - x_3x_1$
7	$x_4x_3^2x_2 + x_4^2x_2x_1 + x_3^2x_2x_1 + x_4x_2x_1^2 - x_4x_2^2 - x_4^2x_1 + x_3x_2$
8	$x_4^2x_3x_2 + 2x_4x_3x_2x_1 + x_3x_2x_1^2 + x_4^2x_1 + x_3x_2x_1 - x_3x_2$
9	$x_4(x_1x_2^2 + x_1x_2x_3 + x_2^2x_4 + x_2x_3x_4 - x_1x_2 - x_1x_3 + x_2)$
10	$x_4^3x_2 + x_4x_3x_2^2 + x_4^2x_2x_1 + x_3x_2^2x_1 + x_4x_3x_1 + x_3x_1^2 - x_3x_1$
11	$x_4(x_2+x_3)(2x_1x_4 + x_2^2 + x_3^2)$
12	$x_4^2x_3^2 + x_3^3x_2 + x_4^3x_1 + x_4x_3x_2x_1 - x_2^3 - x_4x_2x_1 + x_2x_1$
13	$x_4(x_1x_2x_3 + x_1x_3^2 + x_2x_3x_4 + x_3^2x_4 + x_1x_2 + x_1x_3 - x_3)$
14	$x_4^2x_3^2 + 2x_4^2x_3x_2 + x_4^2x_2^2 + x_3^2x_2^2 + x_4x_3^2x_1 + x_4x_2^2x_1 + x_4^2x_1^2 - x_4$
15	$x_4^3x_3 + x_4x_3^2x_2 + x_4^2x_3x_1 + x_3^2x_2x_1 + x_4x_2x_1 + x_2x_1^2 - x_2x_1$
16	$x_4^3x_3 + x_4^3x_2 + x_4x_3^2x_2 + x_4x_3x_2^2 - x_3x_2x_1 - x_2^2x_1 + x_2^2$
17	$x_4^3x_3 + x_4^3x_2 + x_4x_3^2x_2 + x_4x_3x_2^2 + x_3^2x_1 + x_3x_2x_1 - x_3^2$
18	$x_4^4 + 2x_4^2x_3x_2 + x_3^2x_2^2 + x_3x_2x_1 + x_1^3 - x_1^2$

Figure 1: Nonzero entries of lower right block B when $x_5 = x_6 = 0$

The Gröbner basis for the ideal generated by these entries has seven elements:

$$x_4, \quad x_2(x_2-x_1), \quad x_3x_2, \quad x_3(x_3+x_1), \quad x_1^2(x_1-1), \quad x_2x_1(x_1-1), \quad x_3x_1(x_1-1).$$

The Gröbner basis for the radical also has seven elements:

$$x_4, \quad x_1(x_1-1), \quad x_2(x_1-1), \quad x_3(x_1-1), \quad x_2(x_2-1), \quad x_3x_2, \quad x_3(x_3+1).$$

From these results it is easy to verify that $\mathrm{DI}_1(B)$ is zero-dimensional and that its zero set $V(\mathrm{DI}_1(B))$ consists of exactly four points $(x_1, x_2, x_3, x_4) = (0,0,0,0)$, $(1,0,0,0)$, $(1,1,0,0)$, $(1,0,-1,0)$. We have seen these coefficients in Example 1.2: they correspond to the left nilpotent, associative, Zinbiel, and Leibniz operads. ∎

Second proof of Proposition 4.9 While Relation (LR) allows reassociation of parentheses to the right when we deal with products of three arguments, that does not, in general, help to reassociate parentheses in products of more than three arguments: since we allow all permutations of arguments on the right side, an infinite chain of reassociations might happen. However, if we assume that $x_5 = x_6 = 0$, that cannot happen, as the following lemma shows.

Lemma 4.10 *Suppose that $x_5 = x_6 = 0$. Then every operation in the corresponding operad is equal to a linear combination of right-normed products.*

Proof Let us consider some balanced bracketing of $k \geq 2$ arguments. It has the form (AB), where A and B are balanced bracketings of fewer arguments, with l arguments in A and $k-l$ arguments in B. We shall prove the statement by induction on k, and for a fixed k, by induction on l. In both cases, the basis of induction is trivial: for $k=2$, there is nothing to prove, and for each k and $l=1$, we may use the induction hypothesis and write B as a linear combination of right-normed products. The right-normed property does not change when we multiply by A.

Assume that $l \geq 2$, so that $A = (A_1 A_2)$; we are in a situation where we can apply the defining relation of our operad, obtaining

$$(A_1A_2)B = x_1A_1(A_2B) + x_2A_1(BA_2) + x_3A_2(A_1B) + x_4A_2(BA_1).$$

The first four permutations are exactly those which do not bring the third argument into the first position, so each of these terms has the parameter l smaller than the original one, and the induction hypothesis applies. ∎

This lemma shows that under the assumption $x_5 = x_6 = 0$ the spanning property of the right-normed products is trivially satisfied, so there is a surjective map from the regular representation of S_n onto the n-th component of our operad. It remains to check that this map has no kernel. Let us start with arity 4. Note that the defining relation of our operad can be applied as a rewriting rule to the product $((a_1a_2)a_3)a_4$ in two different ways: by rewriting $(a_1a_2)a_3$ first, or by rewriting $(ba_3)a_4$ and setting $b = (a_1a_2)$, as in Example 2.14. This leads to two *a priori* different expressions for $((a_1a_2)a_3)a_4$ as linear combinations of right-normed products.

We collect the nonzero coefficients of the difference of those in the the following table, where the polynomial in the row indexed $\tau \in S_4$ corresponds to the coefficient of $a_1(a_2(a_3a_4)).\tau$:

1234	$x_2^2x_3^2 + 2x_2x_3x_4^2 + x_4^4 + x_1^3 + x_1x_2x_3 - x_1^2$
1243	$x_1x_2x_3^2 + x_1x_3x_4^2 + x_2x_3^2x_4 + x_3x_4^3 + x_1^2x_2 + x_1x_2x_4 - x_1x_2$
1324	$x_1^2x_2 + x_1^2x_3 - x_1x_2^2 + x_1x_3^2 - x_2^2x_4 + x_3^2x_4$
1342	$-x_1^2x_2 + x_1^2x_4 + x_1x_2^2 + x_1x_3x_4 - x_2^2x_3 + x_3x_4^2$
1423	$x_1x_2x_3x_4 + x_1x_4^3 + x_2x_3^3 + x_3^2x_4^2 - x_1x_2x_4 - x_2^3 + x_1x_2$
1432	$x_2^2x_3x_4 + x_2x_3^2x_4 + x_2x_4^3 + x_3x_4^3 - x_1x_2^2 - x_1x_2x_3 + x_2^2$
2134	$x_1x_2^2x_3 + x_1x_2x_4^2 + x_2^2x_3x_4 + x_2x_4^3 + x_1^2x_3 + x_1x_3x_4 - x_1x_3$
2143	$x_1^2x_2x_3 + 2x_1x_2x_3x_4 + x_2x_3x_4^2 + x_1x_2x_3 + x_1x_4^2 - x_2x_3$
2314	$x_1^2x_3 + x_1^2x_4 - x_1x_2x_4 + x_1x_3^2 + x_2x_3^2 - x_2x_4^2$
2341	$x_1x_2x_4 + x_1x_3x_4$
2413	$x_1^2x_2x_4 + x_1x_2x_3^2 + x_1x_2x_4^2 + x_2x_3^2x_4 - x_1x_4^2 - x_2^2x_4 + x_2x_3$
2431	$x_1x_2^2x_4 + x_1x_2x_3x_4 + x_2^2x_4^2 + x_2x_3x_4^2 - x_1x_2x_4 - x_1x_3x_4 + x_2x_4$
3214	$x_2^2x_3x_4 + x_2x_3^2x_4 + x_2x_4^3 + x_3x_4^3 + x_1x_2x_3 + x_1x_3^2 - x_3^2$
3241	$x_1x_2x_3x_4 + x_1x_3^2x_4 + x_2x_3x_4^2 + x_3^2x_4^2 + x_1x_2x_4 + x_1x_3x_4 - x_3x_4$
3124	$x_1x_2x_3x_4 + x_1x_4^3 + x_2^3x_3 + x_2^2x_4^2 + x_1x_3x_4 + x_3^3 - x_1x_3$
3142	$x_1^2x_3x_4 + x_1x_2^2x_3 + x_1x_3x_4^2 + x_2^2x_3x_4 + x_1x_4^2 + x_3^2x_4 - x_2x_3$
3412	$2x_1x_2x_4^2 + 2x_1x_3x_4^2 + x_2^2x_3x_4 + x_2x_3^2x_4 + x_2^3x_4 + x_3^3x_4$
3421	$x_1^2x_4^2 + x_1x_2^2x_4 + x_1x_3^2x_4 + 2x_2x_3x_4^2 + x_2^2x_3^2 + x_2^2x_4^2 + x_3^2x_4^2 - x_1x_4$

If the right-normed products are linearly independent, all those coefficients must be equal to zero. The Gröbner basis for the corresponding system of polynomial equations is $x_1^4 - x_1^3$, $x_4^4 + x_1^3 - x_1^2$, $x_2^3 - x_2^2\ x_3^3 + x_3^2$, $x_1x_2 - x_2^2$, $x_1x_3 + x_3^2$, x_2x_3, x_1x_4, x_2x_4,

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x_3x_4. This implies that every solution to this system has

$$x_4 = 0, \quad x_1 \in \{0,1\}, \quad x_2 \in \{0,1\}, \quad x_3 \in \{-1,0\},$$

and $x_2x_3 = x_3(x_1+x_3) = x_2(x_1-x_2) = 0$, so the only solutions are $(0,0,0,0)$, the left-nilpotent operad, $(1,0,0,0)$, the associative operad, $(1,1,0,0)$, the Zinbiel operad, and $(1,0,-1,0)$, the Leibniz operad. ■

Proposition 4.9 shows that if we wish to find new regular solutions, we must consider the more difficult cases in which either x_5 or x_6 is nonzero. Examining the two proofs of that proposition, we see that since, according to Lemma 4.8, the matrix B has size either 30 or 36, in these cases we must deal with either impractically large numbers of minors (more than 10^{18} in the worst case of matrix of size 36), or a rewriting rule that has no termination property. We therefore need to introduce some more powerful techniques, and that is the topic of the next section.

5 Representation Theory of the Symmetric Groups

Because of the symmetric group actions on the components of any operad, it is to be expected that representation theory of symmetric groups can be utilized in operad theory. For an operad presented as a quotient of a free operad, the n-th component of the ideal of relations is an S_n-submodule of the direct sum of a finite number of copies of the regular S_n-module, $\mathbb{F}S_n$. In simplest terms, the motivation for using representation theory is to "divide and conquer": to split one large intractable problem into a number of smaller tractable pieces which are collectively equivalent to the original problem. We refer the reader to [7] for a systematic development of the necessary material using modern notation and terminology.

There are two significant advantages to using the representation theory of the symmetric group to study algebraic operads. We have already mentioned the first: this method allows us to study a set of multilinear relations "one representation at a time", which greatly reduces the sizes of the matrices involved. The second important reason is that using representation theory allows us to specify beforehand the S_n-module structure of the space of relations, not only its dimension, and this can save a great deal of further computation.

For example, the regular S_4-module $\mathbb{F}S_4$ has dimension 24, but there are other S_4-modules of dimension 24. Indeed, if $m_1,\dots,m_5 \geq 0$ are the multiplicities of the simple modules $[4], [31], [2^2], [21^2], [1^4]$ in the S_4-module T,

$$T \cong m_1[4] \oplus m_2[31] \oplus m_3[2^2] \oplus m_4[21^2] \oplus m_5[1^4],$$

then $\dim(T) = 24$ if and only if $m_1 + 3m_2 + 2m_3 + 3m_4 + m_5 = 24$. There are 1615 solutions to this equation, and no two of the corresponding modules are isomorphic, but only $\mathbb{F}S_4$ has multiplicities $[1,3,2,3,1]$. If we consider only submodules of $(\mathbb{F}S_4)^5$, then we still have 529 solutions. If we restrict further to modules T which are symmetric in the sense that $T \otimes [1^4] \cong T$, where $[1^4]$ is the sign module, or equivalently $m_1 = m_5$, $m_2 = m_4$, then the number of solutions decreases to a more manageable 21.

Without representation theory, if we encounter a module of dimension 24, we must determine its structure by computing the traces of the representation matrices for a set of conjugacy class representatives and then using the character table of S_4 to express the character as a linear combination (with non-negative integer coefficients) of the simple characters. With representation theory, this extra work is unnecessary.

5.1 Structure Theory

When the characteristic of $\mathbb{F}$ is 0 or $p > n$, the group algebra $\mathbb{F}S_n$ is semisimple, and classical structure theory applies. Let λ range over the partitions of n; we write $p(n)$ for the number of partitions. The regular module $\mathbb{F}S_n$ decomposes into the (orthogonal) direct sum of simple two-sided ideals $M(\lambda)$, each of which is isomorphic to a full matrix algebra $M_{d(\lambda)}(\mathbb{F})$, where d_λ is the dimension of the simple S_n-module $[\lambda]$:

$$\mathbb{F}S_n \cong \bigoplus_\lambda M(\lambda), \quad M(\lambda) \cong M_{d(\lambda)}(\mathbb{F}). \tag{5.1}$$

As a right (or left) ideal, $M(\lambda)$ decomposes as the direct sum of $d(\lambda)$ copies of $[\lambda]$ that correspond to the rows (or columns) of $M_{d(\lambda)}(\mathbb{F})$. Efficient algorithms are known for computing the isomorphism (5.1) in both directions [7]. We will only require the projections which take a partition λ and a permutation σ and produce the matrix $R_\lambda(\sigma)$ in $M_{d(\lambda)}(\mathbb{F})$ which represents the action of σ on $[\lambda]$. The simplest algorithm for computing the matrices $R_\lambda(\sigma)$ was discovered by Clifton [8].

The isomorphism (5.1) expresses $\mathbb{F}S_n$, a single vector space of dimension $n!$, as the direct sum of $p(n)$ subspaces of dimensions $d(\lambda)^2$, and these subspaces are orthogonal in the sense that $xy = 0$ if $x \in M(\lambda)$ and $y \in M(\lambda')$ with $\lambda \neq \lambda'$. Thus we have divided the original structure of size $n!$ into a list of $p(n)$ independent structures of average size $n!/p(n)$. But we have also converted the vector space $\mathbb{F}S_n$ (a tensor of rank 1) into a list of $p(n)$ full matrix algebras (tensors of rank 2). Thus the original problem has decomposed into $p(n)$ problems of size $\sqrt{n!/p(n)}$, which is the average dimension of a simple S_n-module.

5.2 Representation Matrices for Polynomial Identities of Arity 4

We now restrict to the case $n = 4$ which we need to continue our analysis of the cubic relation matrix M. For each partition λ of 4, the dimension d_λ of the simple module $[\lambda]$ is the number of standard tableaux; see Figure 2.

$\begin{smallmatrix}1&2&3&4\end{smallmatrix}$	$\begin{smallmatrix}1&2&3\\4\end{smallmatrix}$ $\begin{smallmatrix}1&2&4\\3\end{smallmatrix}$ $\begin{smallmatrix}1&3&4\\2\end{smallmatrix}$	$\begin{smallmatrix}1&2\\3&4\end{smallmatrix}$ $\begin{smallmatrix}1&3\\2&4\end{smallmatrix}$	$\begin{smallmatrix}1&2\\3\\4\end{smallmatrix}$ $\begin{smallmatrix}1&3\\2\\4\end{smallmatrix}$ $\begin{smallmatrix}1&4\\2\\3\end{smallmatrix}$	$\begin{smallmatrix}1\\2\\3\\4\end{smallmatrix}$
$d_\lambda = 1$	$d_\lambda = 3$	$d_\lambda = 2$	$d_\lambda = 3$	$d_\lambda = 1$
$\lambda = 4$	$\lambda = 31$	$\lambda = 2^2$	$\lambda = 21^2$	$\lambda = 1^4$

Figure 2: Partitions, dimensions, standard tableaux ($n = 4$)

The corresponding isomorphism (5.1) has the form

$$\mathbb{F}S_4 \cong \mathbb{F} \oplus M_3(\mathbb{F}) \oplus M_2(\mathbb{F}) \oplus M_3(\mathbb{F}) \oplus \mathbb{F}, \tag{5.2}$$

which can be viewed as a map from permutations σ to quintuples of matrices $R_\lambda(\sigma)$. The representation matrices for the generators $\sigma = (12), (23), (34) \in S_4$ are as follows:

$$(12) \longmapsto \left[\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 \end{bmatrix} \right],$$
$$(23) \longmapsto \left[\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 \end{bmatrix} \right],$$
$$(34) \longmapsto \left[\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 \end{bmatrix} \right].$$

Recall from Notation 4.1 that the S_4-module $\mathcal{T}(4)$ is isomorphic to the direct sum of five copies of $\mathbb{F}S_4$ generated by the five basis monomials $\gamma_1, \dots, \gamma_5$ of $\Omega(4)$. Thus every multilinear polynomial identity I of arity 4 can be decomposed into a sum of five components, $I = I_1 + \cdots + I_5$, where each I_i can be identified with an element of $\mathbb{F}S_4$ and each monomial in I_i has the same bracketing as γ_i. We combine this decomposition of $\mathcal{T}(4)$ with the decomposition (5.2) and rearrange the components to obtain the isotypic decomposition of $\mathcal{T}(4)$:

$$\begin{aligned} \mathcal{T}(4) &\cong \bigoplus_{j=1}^{5} \big(\mathbb{F} \oplus M_3(\mathbb{F}) \oplus M_2(\mathbb{F}) \oplus M_3(\mathbb{F}) \oplus \mathbb{F} \big) \\ &\cong \mathbb{F}^5 \oplus M_3(\mathbb{F})^5 \oplus M_2(\mathbb{F})^5 \oplus M_3(\mathbb{F})^5 \oplus \mathbb{F}^5. \end{aligned}$$

To obtain the analogous decomposition of the multilinear identity $I = I_1 + \cdots + I_5$, we compute the representation matrices $R_\lambda(I_j)$ for $\lambda = 4, \dots, 1^4$ and $j = 1, \dots, 5$. The isotypic decomposition of I is a sequence of five matrices indexed by λ of sizes $d_\lambda \times 5d_\lambda$:

$$I \longmapsto \begin{cases} [R_4(I_1)|R_4(I_2)|R_4(I_3)|R_4(I_4)|R_4(I_5)] & 1 \times 5, \\ [R_{31}(I_1)|R_{31}(I_2)|R_{31}(I_3)|R_{31}(I_4)|R_{31}(I_5)] & 3 \times 15, \\ [R_{2^2}(I_1)|R_{2^2}(I_2)|R_{2^2}(I_3)|R_{2^2}(I_4)|R_{2^2}(I_5)] & 2 \times 10, \\ [R_{21^2}(I_1)|R_{21^2}(I_2)|R_{21^2}(I_3)|R_{21^2}(I_4)|R_{21^2}(I_5)] & 3 \times 15, \\ [R_{1^4}(I_1)|R_{1^4}(I_2)|R_{1^4}(I_3)|R_{1^4}(I_4)|R_{1^4}(I_5)] & 1 \times 5. \end{cases}$$

If $\mathcal{G} = \{I^{(1)}, \dots, I^{(r)}\}$ is a set of multilinear identities of arity 4, then for each λ and each $i = 1, \dots, r$ we compute the $d_\lambda \times 5d_\lambda$ matrix as above and stack them together to obtain a matrix of size $rd(\lambda) \times 5d(\lambda)$:

$$R_\lambda(\mathcal{G}) = \left[\begin{array}{c|c|c|c|c} R_\lambda(I_1^{(1)}) & R_\lambda(I_2^{(1)}) & R_\lambda(I_3^{(1)}) & R_\lambda(I_4^{(1)}) & R_\lambda(I_5^{(1)}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ R_\lambda(I_1^{(i)}) & R_\lambda(I_2^{(i)}) & R_\lambda(I_3^{(i)}) & R_\lambda(I_4^{(i)}) & R_\lambda(I_5^{(i)}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ R_\lambda(I_1^{(r)}) & R_\lambda(I_2^{(r)}) & R_\lambda(I_3^{(r)}) & R_\lambda(I_4^{(r)}) & R_\lambda(I_5^{(r)}) \end{array} \right]. \tag{5.3}$$

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The row space of this matrix is the isotypic component for partition λ of the submodule of $\mathcal{T}(4)$ generated by $\mathcal{G}$, and the rank of this matrix is the multiplicity of the simple S_4-module $[\lambda]$ in that isotypic component.

5.3 Regularity in Terms of Representation Theory

Recall that Relation (LR) has five consequences (2.3) in arity 4 which generate the S_4-module $\mathcal{J}(4) \subseteq \mathcal{T}(4)$ of relations in arity 4 for parametrized one-relation algebras. We rewrite the expansions (4.1) of those five consequences by collecting terms corresponding to the same underlying bracketings:

$$\begin{aligned}
\rho(a_1a_2,a_3,a_4) &= [1234]_1 + [-x_1 1234 - x_2 1243]_3 + [-x_3 3124 - x_5 4123]_4 \\
&\qquad + [-x_4 3412 - x_6 4312]_5, \\
\rho(a_1,a_2a_3,a_4) &= [1234]_2 + [-x_3 2314 - x_4 2341]_3 + [-x_1 1234 - x_6 4231]_4 \\
&\qquad + [-x_2 1423 - x_5 4123]_5, \\
\rho(a_1,a_2,a_3a_4) &= [1234 - x_5 3412 - x_6 3421]_3 + [-x_2 1342 - x_4 2341]_4 \\
&\qquad + [-x_1 1234 - x_3 2134]_5, \\
\rho(a_1,a_2,a_3)a_4 &= [1234]_1 \\
&\qquad + [-x_1 1234 - x_2 1324 - x_3 2134 - x_4 2314 - x_5 3124 - x_6 3214]_2, \\
a_1\rho(a_2,a_3,a_4) &= [1234]_4 \\
&\qquad + [-x_1 1234 - x_2 1243 - x_3 1324 - x_4 1342 - x_5 1423 - x_6 1432]_5.
\end{aligned}$$

Each pair of square brackets in each of these expressions contains an element of the group algebra $\mathbb{F}S_4$ with coefficients extended to the polynomial ring $\mathbb{F}[x_1,\dots,x_6]$.

We now apply equation (5.3) to compute the representation matrices $R_\lambda(\mathcal{G})$ of the relations $\mathcal{G}$ in each partition λ. In this way we replace the original 120×120 cubic relation matrix M by five smaller matrices of sizes $5d_\lambda \times 5d_\lambda$ for $d_\lambda = 1,3,2,3,1$. Regularity holds if and only if the nullity of $R_\lambda(\mathcal{G})$ equals d_λ for all λ. This guarantees that $\mathcal{T}(4)$ contains exactly d_λ copies of the simple module $[\lambda]$, and is therefore isomorphic to the regular module $\mathbb{F}S_4$. Equivalently, the rank of $R_\lambda(\mathcal{G})$ must equal $4d_\lambda$ for all λ; in terms of determinantal ideals, this means that for $d = d_\lambda$ we have $\mathrm{DI}_{4d}(R_\lambda(\mathcal{G})) \neq \{0\}$ and $\mathrm{DI}_{4d+1}(R_\lambda(\mathcal{G})) = \{0\}$ for all λ. This proves the following result.

Lemma 5.1 *The parametrized one-relation operad is regular in arity 4 for particular values of the parameters $x_1,\dots,x_6$ if and only if all of the following conditions hold:*

$$\begin{array}{lll}
\lambda = 4: & \mathrm{DI}_4(R_\lambda(\mathcal{G})) \neq \{0\}, & \mathrm{DI}_5(R_\lambda(\mathcal{G})) = \{0\}, \\
\lambda = 31: & \mathrm{DI}_{12}(R_\lambda(\mathcal{G})) \neq \{0\}, & \mathrm{DI}_{13}(R_\lambda(\mathcal{G})) = \{0\}, \\
\lambda = 2^2: & \mathrm{DI}_8(R_\lambda(\mathcal{G})) \neq \{0\}, & \mathrm{DI}_9(R_\lambda(\mathcal{G})) = \{0\}, \\
\lambda = 21^2: & \mathrm{DI}_{12}(R_\lambda(\mathcal{G})) \neq \{0\}, & \mathrm{DI}_{13}(R_\lambda(\mathcal{G})) = \{0\}, \\
\lambda = 1^4: & \mathrm{DI}_4(R_\lambda(\mathcal{G})) \neq \{0\}, & \mathrm{DI}_5(R_\lambda(\mathcal{G})) = \{0\}.
\end{array}$$

Remark 5.2 The conditions in Lemma 5.1 may be combined and simplified. Let $G_1,\dots,G_k$ be Gröbner bases for ideals $I_1,\dots,I_k \subseteq \mathbb{F}[x_1,\dots,x_d]$. Consider these two equations:

$$V(I_1 + \cdots + I_k) = V(I_1) \cap \cdots \cap V(I_k), \quad V(\sqrt{I}) = V(I) = \bigcap_{i=1}^{k} V(I_i).$$

From the generating set $G = G_1 \cup \cdots \cup G_k$ for the ideal $I = I_1 + \cdots + I_k$ we compute a Gröbner basis H, and from this we compute a Gröbner basis K for the radical $\sqrt{I}$. We solve the system of equations $\{f = 0 \mid f \in K\}$ to find $V(\sqrt{I})$. To include the lower rank conditions $\mathrm{DI}_{4d}(R_\lambda(\mathcal{G})) \neq \{0\}$, we substitute each solution into the Gröbner bases for the lower ideals $\mathrm{DI}_{4d}(R_\lambda(\mathcal{G}))$, and retain a solution if and only if it is not in $Z(\mathrm{DI}_{4d}(R_\lambda(\mathcal{G})))$ for any λ.

We noted in Remark 4.5 that if the number of minors is too large, then it is not practical to compute a Gröbner basis for a determinantal ideal. Using representation theory allows us to go much further. To apply Lemma 5.1, we need to compute

- all minors of sizes 4 and 5 for the 5×5 matrix $R_\lambda(\mathcal{G})$ when $\lambda = 4$ and $\lambda = 1^4$,
- all minors of sizes 12 and 13 for the 15×15 matrix $R_\lambda(\mathcal{G})$ when $\lambda = 31$ and $\lambda = 21^2$,
- all minors of sizes 8 and 9 for the 10×10 matrix $R_\lambda(\mathcal{G})$ when $\lambda = 2^2$.

The total is extremely small compared to the numbers in Remark 4.5:

$$2\left[\binom{5}{4}^2 + \binom{5}{5}^2\right] + 2\left[\binom{15}{12}^2 + \binom{15}{13}^2\right] + \binom{10}{8}^2 + \binom{10}{9}^2 = 438277.$$

Notably, those matrices have many zero entries; furthermore, they have entries which are nonzero scalars (± 1), so we can apply Algorithm 1 to reduce their sizes even further, as we did in Section 4 when we extracted the 36×36 block B from the cubic relation matrix M.

5.4 Reduction of the Representation Matrices

The representation matrices $R_\lambda(\mathcal{G})$ are square matrices of sizes 5, 15, 10, 15, 5, respectively. Applying Algorithm 1 reduces each of these to a block diagonal matrix $[I, B]$, where I (identity matrix) and B (block with no nonzero scalars) have sizes r and s, respectively, where $[r, s]$ is one of the pairs $[3, 2]$, $[10, 5]$, $[6, 4]$, $[10, 5]$, and $[3, 2]$. We write $B(\lambda)$ for the block corresponding to partition λ. If $B(\lambda)$ is $s \times s$, then $\mathrm{DI}_s(B(\lambda))$ is the principal ideal generated by $\det(B(\lambda))$, and so $\mathrm{DI}_s(B(\lambda)) = \{0\}$ if and only if $\det(B(\lambda)) = 0$. The next result is Lemma 5.1 reformulated in terms of the reduced matrices $B(\lambda)$.

Lemma 5.3 *Regularity holds for particular values of the parameters $x_1, \ldots, x_6$ if and only if the following conditions on the determinantal ideals of $B(\lambda)$ hold for all λ:*

λ	$B(\lambda)$	$\mathrm{DI}_r(B(\lambda)) \neq \{0\}$	$\mathrm{DI}_{r+1}(B(\lambda)) = \{0\}$
4	2×2	$r = 1$	$r+1 = 2$, $\det(B(\lambda)) = 0$
31	5×5	$r = 2$	$r+1 = 3$
2^2	4×4	$r = 2$	$r+1 = 3$
21^2	5×5	$r = 2$	$r+1 = 3$
1^4	2×2	$r = 1$	$r+1 = 2$, $\det(B(\lambda)) = 0$

6 Main Technical Result

In this section we describe the computations which allow us to complete the classification of parametrized one-relation operads for which the arity 4 component is the regular S_4-module. These computations are based on the reduced representation matrices $B(\lambda)$ collated in the online addendum to this paper [6]. Essentially the same methods can be used to determine all instances of Relation (LR) that produce any desired S_4-module structure in arity 4, not necessarily the regular one.

We increase the complexity of the problem step by step, starting with the case of one nonzero parameter, and ending with the general case in which all six parameters are allowed to be nonzero. In order to avoid linguistic pedantry, when we say that the parameters in some subset $S \subseteq P = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ are nonzero, we mean that we are setting the parameters in $P \setminus S$ to zero and regarding those in S as free.

We call the ideals $\mathrm{DI}_{4d_\lambda+1}(R_\lambda(\mathcal{G}))$ *upper determinantal ideals*, and the ideals

$$\mathrm{DI}_{4d_\lambda}(R_\lambda(\mathcal{G}))$$

lower determinantal ideals; according to Lemma 5.1, for a parametrized one-relation operad to be regular in arity 4, the set of parameters must be a common zero of all upper determinantal ideals, and must be outside the zero set of each lower determinantal ideal. We denote by the symbols $\Sigma+$ and $\sqrt{\Sigma+}$ the sum of the upper determinantal ideals and its radical, respectively.

6.1 One Nonzero Parameter

When the only nonzero parameter is x_1, for every representation $[\lambda]$, the upper ideal is generated by $x_1^2(x_1-1)$ and the lower ideal is generated by 1. Then clearly the sum of the upper ideals is generated by $x_1^2(x_1-1)$ and its radical is generated by $x_1(x_1-1)$. For regularity, the sum of the upper ideals must be $\{0\}$, giving $x_1 = 0$ or $x_1 = 1$, and each lower ideal must be nonzero (which is clear). The solution $x_1 = 0$ corresponds to the left-nilpotent identity $(a_1a_2)a_3 = 0$, and $x_1 = 1$ corresponds to associativity $(a_1a_2)a_3 = a_1(a_2a_3)$.

When the only nonzero parameter is x_2, x_3, or x_4, the only regular solution is the zero solution (left-nilpotent identity).

When the only nonzero parameter is x_5, for every representation $[\lambda]$, the upper ideal is zero, and the lower ideals are generated by

$$x_5 - 1, \quad (x_5-1)(x_5+1)^2, \quad (x_5-1)^2, \quad (x_5-1)(x_5+1)^2, \quad x_5 - 1,$$

and we will have a regular solution if and only if every lower ideal is nonzero, and this happens if and only if $x_5 \neq \pm 1$.

When the only nonzero parameter is x_6, the upper ideals are generated by

$$x_6(x_6+1)(x_6-1)^2, \quad x_6(x_6-1)(x_6+1)^2, \quad x_6(x_6-1)^2(x_6+1)^2,$$
$$x_6(x_6+1)(x_6-1)^2, \quad x_6(x_6-1)(x_6+1)^2,$$

and the radical of their sum consists of all multiples of $x_6(x_6-1)(x_6+1)$ and hence will be zero if and only if $x_6 \in \{0, \pm 1\}$. The lower ideals are generated by

$$x_6 + 1, \quad x_6 - 1, \quad x_6^2 - 1, \quad x_6 - 1, x_6 + 1,$$

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and the only one of these values which does not make at least one lower ideal equal to zero is $x_6 = 0$, and so here again we recover only the zero solution.

Proposition 6.1 (Summary for (at most) one nonzero parameter) *When at most one of the parameters in Relation* (LR) *is nonzero, there are three solutions giving the regular module in arity* 4*, two isolated and one* 1*-dimensional (a one-parameter family):*

$$(a_1a_2)a_3 = 0, \quad (a_1a_2)a_3 = a_1(a_2a_3), \quad (a_1a_2)a_3 = x_5a_3(a_1a_2) \quad (x_5 \neq \pm 1).$$

6.2 Two Nonzero Parameters

Henceforth, ideals are not necessarily principal, so Gröbner bases typically contain two or more elements. There are 15 cases when we choose two parameters from six, but it will not be necessary to discuss all of them in detail. We begin with x_1, x_2 and continue in lex order.

x_1, x_2 **nonzero**:
The upper determinantal ideals have the following `grevlex` Gröbner bases:

$$\begin{aligned}
\mathrm{DI}_4^+ &= \big((x_2 - x_1)(x_2 + x_1)(x_2 + x_1 - 1) \big), \\
\mathrm{DI}_{31}^+ &= \big(x_1^3 + x_2^2 - x_2x_1 - x_1^2, x_2(x_1^2 - x_2), x_2x_1(x_2 - 1), x_2^2(x_2 - 1) \big), \\
\mathrm{DI}_{2^2}^+ &= \big(x_1^3 + x_2^2 - x_2x_1 - x_1^2, x_2(x_1^2 - x_2), x_2x_1(x_2 - 1), x_2^2(x_2 - 1) \big), \\
\mathrm{DI}_{21^2}^+ &= \big(x_2(x_2 - x_1), x_1^2(x_1 - 1), x_2x_1(x_1 - 1) \big), \\
\mathrm{DI}_{1^4}^+ &= \big((x_2 - x_1 + 1)(x_2 - x_1)^2 \big).
\end{aligned}$$

The sum of these ideals is the ideal $\Sigma+$ for $\lambda = 21^2$, and its radical has the following Gröbner basis and zero set:

$$\begin{aligned}
\sqrt{\Sigma+} &= \big(x_1(x_1 - 1), x_2(x_1 - 1), x_2(x_2 - 1) \big), \\
V\big(\sqrt{\Sigma+}\big) &= \big\{ (x_1, x_2) = (0,0), (1,0), (1,1) \big\}
\end{aligned}$$

Every lower ideal has Gröbner basis $\{1\}$, so all three of the solutions are regular. We have already seen the first and second, but the third is new: it defines the Zinbiel identity $(a_1a_2)a_3 = a_1(a_2a_3) + a_1(a_3a_2)$.

x_1, x_3 **nonzero**: This is the Koszul dual of the case x_1, x_2 nonzero. To derive the results in this case from those of the previous case, for each λ we replace x_2 by $-x_3$ and λ by its conjugate; this corresponds to tensoring with the sign module. We obtain again the trivial and associative identities, since they are self-dual, but the Zinbiel identity is transformed into the Leibniz identity: $(a_1a_2)a_3 = a_1(a_2a_3) - a_2(a_1a_3)$.

x_1, x_4 **nonzero**: The radical of the sum of the upper ideals is generated by the polynomials x_4 and $x_1(x_1 - 1)$, so $x_4 = 0$, and there are no new solutions.

x_1, x_5 **nonzero**: The radical of the sum of the upper ideals is $\sqrt{\Sigma+} = (x_1(x_1-1), x_5x_1)$, so $x_1x_5 = 0$, and there are no new solutions.

x_1, x_6 **nonzero**: The radical of the sum of the upper ideals is

$$\sqrt{\Sigma+} = \big(x_1(x_1 - 1), x_6x_1, x_6(x_6 - 1)(x_6 + 1) \big),$$

so $x_1x_6 = 0$, and there are no new solutions.

x_2, x_3 **nonzero**: We have $\sqrt{\Sigma+} = (x_2, x_3)$, so there are no new solutions.

x_2, x_4 **nonzero to** x_4, x_6 **nonzero**: No new features; we omit the details.

x_5, x_6 **nonzero**: The radical of the sum of the upper ideals is

$$\sqrt{\Sigma+} = \big(x_6 x_5, x_6(x_6 - 1)(x_6 + 1) \big),$$

so $x_5 x_6 = 0$, and there are no new solutions.

Proposition 6.2 (Summary for two nonzero parameters) *When exactly two parameters in Relation (LR) are different from zero, there are two regular solutions, both isolated, which are the Zinbiel and Leibniz identities:*

$$(a_1 a_2)a_3 = a_1(a_2 a_3) + a(a_3 a_2), \quad (a_1 a_2)a_3 = a_1(a_2 a_3) - a_2(a_1 a_3).$$

6.3 Three Nonzero Parameters

There are $\binom{6}{3} = 20$ cases, starting with x_1, x_2, x_3 in lex order and ending with x_4, x_5, x_6, but they produce no new regular solutions. We present details only for the first and last cases, since they illustrate the computations that are typical of all cases.

x_1, x_2, x_3 **nonzero**: Once we compute Gröbner bases for the radicals of the upper ideals, we note in particular that

$$\sqrt{\mathrm{DI}^+_{31}} = \big((x_1 - 1)(x_2 + x_1), x_3(x_1 - 1), x_2(x_2 - 1), x_3 x_2, x_3(x_3 + 1), x_1(x_1 - 1)(x_1 + 1) \big).$$

We see that $x_2 x_3 = 0$, so there are no new solutions.

x_1, x_2, x_4 **to** x_3, x_5, x_6 **nonzero**: No new features; we omit the details.

x_4, x_5, x_6 **nonzero**: Once we compute Gröbner bases for the radicals of the upper ideals, we note in particular that

$$\sqrt{\mathrm{DI}^+_{31}} = \big(x_4, x_6 x_5(x_6 + x_5 + 1), x_6(x_6 + x_5 + 1)(x_6 - x_5 - 1) \big).$$

We see that $x_4 = 0$, so there are no new solutions.

Proposition 6.3 (Summary for three nonzero parameters) *When exactly three parameters in Relation (LR) are nonzero, there are no solutions that are regular in arity* 4.

6.4 Four Nonzero Parameters

In this case, we obtain two new relations with irrational coefficients that are regular; but we will see shortly that these solutions belong to a one-parameter family, all of whose other solutions have five nonzero coefficients. We discuss these two cases, x_2, x_4, x_5, x_6 nonzero and x_3, x_4, x_5, x_6 nonzero, and one other case, x_2, x_3, x_4, x_6 nonzero that is remarkable for the complexity of the Gröbner bases that occur.

x_2, x_3, x_4, x_6 **nonzero**: The individual upper ideals have very complicated Gröbner bases with dozens of terms, some of which have coefficients of absolute value about 10^{23}. However, when we consider the sum of the upper ideals, the complexity vanishes: the Gröbner basis for the sum contains only seven polynomials of degrees 1,2,3 with one or two terms and all coefficients ±1. The radical is slightly simpler: only four polynomials, and none of degree 2: $\sqrt{\Sigma+} = \big(x_2, x_3, x_4, x_6(x_6+1)(x_6-1) \big)$. We see that $x_2 = x_3 = x_4 = 0$, so there are no new solutions.

x_2, x_4, x_5, x_6 **nonzero**: In this case the radical $\sqrt{\Sigma+}$ has the following Gröbner basis:

$$x_4 + x_2, \quad x_2(x_5 + x_2), \quad x_2(x_6 - 1), \quad x_5x_6 + x_2,$$
$$x_2(x_2^2 - x_2 - 1), \quad x_6(x_6 - 1)(x_6 + 1).$$

We assume $x_2 \neq 0$, so we may cancel the factor x_2 from three generators, obtaining

$$\big\{ x_4 + x_2, x_5 + x_2, x_6 - 1, x_5x_6 + x_2, x_2^2 - x_2 - 1, x_6(x_6 - 1)(x_6 + 1) \big\}.$$

If we set $x_4 = -x_2$, $x_5 = -x_2$, $x_6 = 1$, then this generating set reduces to $\{x_2^2 - x_2 - 1\}$. Therefore, we obtain solutions

$$[x_1, x_2, x_3, x_4, x_5, x_6] = [0, \phi, 0, -\phi, -\phi, 1], \tag{6.1}$$

where ϕ can be either of the roots of the polynomial $x^2 - x - 1$. Before we can verify that this is regular, we must consider the lower ideals, whose radicals are

$$\begin{array}{ll}
\lambda = 4 & (x_2, x_6 + x_5 - 1, x_4(x_4 + 1), x_5x_4), \\
\lambda = 31 & \big(x_2, x_4, x_5^2 - x_6 - 1, x_6x_5, x_6(x_6 + 1) \big), \\
\lambda = 2^2 & (x_2, x_4, x_5(x_5 - 1), x_6x_5, x_6^2 + x_5 - 1), \\
\lambda = 21^2 & \big(x_2, x_4, x_5^2 + x_6 - 1, x_6x_5, x_6(x_6 - 1) \big), \\
\lambda = 1^4 & \big(x_6 - x_5 + 1, x_2(x_4 - x_2), (x_4 - x_2)(x_4 + x_2 + 1), x_5(x_4 - x_2) \big).
\end{array}$$

For parameters equal to the values (6.1), some of these polynomials do not vanish: the first four ideals contain $x_2 = \phi \neq 0$, and the fifth contains $x_6 - x_5 + 1 = \phi + 2 \neq 0$.

x_3, x_4, x_5, x_6 **nonzero**: The calculations are similar to those of the previous case, and we obtain two new solutions: $[x_1, x_2, x_3, x_4, x_5, x_6] = [0, 0, -\phi, -\phi, -\phi, -1]$.

Proposition 6.4 (Summary for four nonzero parameters) *When exactly four parameters in Relation (LR) are different from zero, there are four solutions that are regular in arity* 4*, two for each root ϕ of the polynomial $x^2 - x - 1$:*

$$(a_1a_2)a_3 = \phi a_1(a_3a_2) - \phi a_2(a_3a_1) - \phi a_3(a_1a_2) + a_3(a_2a_1),$$
$$(a_1a_2)a_3 = -\phi a_2(a_1a_3) - \phi a_2(a_3a_1) - \phi a_3(a_1a_2) - a_3(a_2a_1).$$

6.5 Five Nonzero Parameters

We obtain a new one-parameter family involving the first five parameters. We present details of the computations in this case, and omit the others that do not produce any new solutions.

x_1, x_2, x_3, x_4, x_5 **nonzero**: Although the individual upper ideals have very complicated Gröbner bases with hundreds of terms, some of which have coefficients of absolute value about 10^{15}, the radical $\sqrt{\Sigma+}$ has the following simple Gröbner basis:

$$
\begin{array}{ll}
(x_3+x_2)(x_1-1), & (x_1-1)(x_4+x_1), \\
x_3x_2+x_2^2-x_5x_1-x_2, & x_4x_2-x_2^2+x_2x_1+x_1^2-x_4-x_1, \\
(x_2-x_1)(x_5+x_2+x_1-1), & (x_3+x_2)(x_3-x_2+1), \\
x_4x_3+x_2^2-x_2x_1-x_1^2+x_4+x_1, & x_5x_3-x_2^2+x_5x_1+x_1^2+x_2-x_1, \\
x_4^2-x_2^2+x_5x_1+x_2, & x_5x_4+x_2^2-x_1^2+x_4-x_2+x_1, \\
(x_1-1)(x_5x_1+x_2), & (x_1-1)(x_2-x_1)(x_2+x_1), \\
x_5^2x_1-x_2^2+x_5x_1+x_1^2+x_2-x_1. &
\end{array}
\tag{6.2}
$$

We note that several of these polynomials are divisible by x_1-1, so we can use a divide-and-conquer strategy to find the zero set of these polynomials.

Case 1: Setting $x_1 = 1$ in the polynomials (6.2) and recomputing the Gröbner basis produces the following nine polynomials:

$$
\begin{array}{lll}
x_2^2+x_2x_3-x_2-x_5, & (x_2-1)(x_4-x_2), & (x_2-1)(x_5+x_2), \\
(x_3+x_2)(x_3-x_2+1), & x_3x_4+x_2^2-x_2+x_4, & x_3x_5-x_2^2+x_2+x_5, \\
x_4^2-x_2^2+x_2+x_5, & x_4x_5+x_2^2-x_2+x_4, & (x_2+x_5)(x_5-x_2+1).
\end{array}
\tag{6.3}
$$

We note that two of the polynomials are divisible by x_2-1, so may use a divide-and-conquer strategy again.

Subcase 1a: Setting $x_2 = 1$ and recomputing the Gröbner basis produces

$$x_5-x_3, \quad x_3(x_3+1), \quad (x_3+1)x_4, \quad x_4^2+x_3.$$

Since $x_3 \neq 0$, we have $x_3 = -1$, so that $x_4 = \pm 1$ and $x_5 = -1$, giving the solutions

$$[x_1,x_2,x_3,x_4,x_5,x_6] = [1,1,-1,1,-1,0], \quad [1,1,-1,-1,-1,0].$$

Subcase 1b: If $x_2 \neq 1$, then we may divide by (= remove) the two factors x_2-1 in the polynomials (6.3) and recompute the Gröbner basis, obtaining

$$x_4-x_2, \quad x_2+x_5, \quad x_2(x_3+x_2), \quad (x_3+x_2)(x_3-x_2+1).$$

Since $x_2 \neq 0$, we have $x_3 = -x_2$, so that $x_4 = x_2$ and $x_5 = -x_2$, giving the solution

$$[x_1,x_2,x_3,x_4,x_5,x_6] = [1,x_2,-x_2,x_2,-x_2,0], \quad (x_2 \neq 1).$$

Case 2: If $x_1 \neq 1$, then we can remove the factors x_1-1 from the polynomials (6.2) and recompute the Gröbner basis, obtaining

$$x_3+x_2, \quad x_4+x_1, \quad x_1x_5+x_2, \quad (x_2-x_1)(x_2+x_1), \quad x_2x_5+x_1. \tag{6.4}$$

If $x_2 = x_1$, then (6.4) reduces to $\{x_3+x_1, x_4+x_1, x_1(x_5+1)\}$ and so we have one new solution $[x_1,x_2,x_3,x_4,x_5,x_6] = [x_1,x_1,-x_1,-x_1,-1,0]$, $(x_1 \neq 0,1)$. If $x_2 \neq x_1$, then (6.4) reduces to $\{x_3-x_1, x_4+x_1, x_1(x_5-1)\}$ and so we have one new solution

$$[x_1,x_2,x_3,x_4,x_5,x_6] = [x_1,-x_1,x_1,-x_1,1,0] \quad (x_1 \neq 0,1).$$

We sort the complete list of solutions by increasing number of nonzero parameters:

(6.5)

#	$[x_1, x_2, x_3, x_4, x_5, x_6]$	Comments
1	$[1, x_2, -x_2, x_2, -x_2, 0]$	including $x_2 = 1$
2	$[x_1, -x_1, x_1, -x_1, 1, 0]$	$x_1 \neq 0, 1$
3	$[x_1, x_1, -x_1, -x_1, -1, 0]$	$x_1 \neq 0$, including $x_1 = 1$

The solutions $[1, 1, -1, 1, -1, 0]$ and $[1, 1, -1, -1, -1, 0]$ now become special cases of #1 and #3, respectively. It is easy to verify by direct substitution that all these solutions belong to the zero set of every polynomial in the Gröbner basis (6.2).

To determine which of the solutions (6.5) are regular, we need to look at the lower ideals for the five partitions. Their radicals have the following Gröbner bases:

$$
\begin{array}{ll}
\mathrm{DI}_4^- & x_5 - 1,\ x_4x_1 + x_3x_1 + x_2x_1 + x_1^2 - x_4 - x_3 \\
& (x_2 + x_1)(x_3 - x_2 + 2x_1 - 2)\ x_4x_2 + x_2^2 - x_3x_1 - x_1^2 + x_4 + x_3 + x_2 + x_1, \\
& x_4x_3 + x_3^2 - x_2x_1 - x_1^2 + x_4 + x_3 - x_2 - x_1,\ x_4^2 - x_3^2 - x_4 - x_3 + 2x_2 + 2x_1, \\
& (x_2 + x_1)(x_2x_1 - x_1^2 + 3x_1 - 1), \\
\mathrm{DI}_{31}^- & x_4 + x_3 + x_2 + x_1,\ (x_1 + 1)(x_3 + x_2),\ x_5x_1 - x_3, \\
& (x_2 - x_1)(x_2 + x_1),\ x_3x_2 + x_1^2 - x_3 - x_2,\ x_5x_2 + x_3 + x_2 + x_1, \\
& (x_3 - x_1)(x_3 + x_1),\ x_5x_3 - x_1,\ (x_5 - 1)(x_5 + 1), \\
\mathrm{DI}_{2^2}^- & x_3 - x_2,\ x_4 - x_1,\ x_5 - 1,\ x_1(x_1 - 1),\ x_2(x_1 - 1),\ x_2^2 - x_1, \\
\mathrm{DI}_{21^2}^- & x_4 - x_3 - x_2 + x_1,\ (x_1 + 1)(x_3 + x_2),\ x_5x_1 + x_2, \\
& (x_2 - x_1)(x_2 + x_1),\ x_3x_2 + x_1^2 + x_3 + x_2,\ x_5x_2 + x_1, \\
& (x_3 - x_1)(x_3 + x_1),\ x_5x_3 + x_3 + x_2 - x_1,\ (x_5 - 1)(x_5 + 1), \\
\mathrm{DI}_{1^4}^- & x_5 - 1,\ x_4x_1 - x_3x_1 - x_2x_1 + x_1^2 - x_4 + x_2, \\
& x_4x_2 - x_2^2 - x_3x_1 + x_1^2 - x_4 - x_3 + x_2 + x_1,\ (x_3 - x_1)(x_3 - x_2 + 2x_1 - 2), \\
& x_4x_3 - x_3x_2 + x_3x_1 - x_1^2 - x_4 - x_3 + x_2 + x_1, \\
& x_4^2 - x_2^2 - x_4 - 2x_3 + x_2 + 2x_1,\ (x_3 - x_1)(x_2x_1 - x_1^2 - x_1 + 1).
\end{array}
$$

We substitute the three solutions (6.5) into these Gröbner bases, which makes all the polynomials univariate, and determine the ideals these univariate polynomials generate. For each of the solutions, we obtain a list of five ideals corresponding to the five partitions, and each ideal must be nonzero in order for regularity to hold.

$[x_1, x_2, x_3, x_4, x_5, x_6]$	$\lambda = 4$	$\lambda = 31$	$\lambda = 2^2$	$\lambda = 21^2$	$\lambda = 1^4$
$[1, x_2, -x_2, x_2, -x_2, 0]$	$(x_2 + 1)$	$(x_2 + 1)$	(1)	$(x_2 + 1)$	$(x_2 + 1)$
$[x_1, -x_1, x_1, -x_1, 1, 0]$	(0)	(0)	(x_1)	(0)	(0)
$[x_1, x_1, -x_1, -x_1, -1, 0]$	(1)	(0)	(1)	(0)	(1)

Thus the solution $[1, x_2, -x_2, x_2, -x_2, 0]$ with $x_2 \neq -1$ is the only regular one.

Proposition 6.5 (Summary for five nonzero parameters) *When exactly five parameters in Relation (LR) are nonzero, there is a one-dimensional family of solutions which are regular in arity* 4*:*

$$(a_1a_2)a_3 = a_1(a_2a_3) + x_2\big[\,a_1(a_3a_2) - a_2(a_1a_3) + a_2(a_3a_1) - a_3(a_1a_2)\big] \quad (x_2 \neq -1).$$

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Remark 6.6 The exceptional case $x_2 = -1$ gives a relation that is not regular:

$$(a_1a_2)a_3 = a_1(a_2a_3) - a_1(a_3a_2) + a_2(a_1a_3) - a_2(a_3a_1) + a_3(a_1a_2).$$

In this case, the cubic relation matrix M has nullity 32 and multiplicities [2,4,2,4,2], that is, the nullspace is isomorphic to the S_4-module

$$2[4] \oplus 4[31] \oplus 2[2^2] \oplus 4[21^2] \oplus 2[1^4].$$

6.6 Six Nonzero Parameters

There is only one case (all parameters are free), and we may assume that each parameter is nonzero, since, if any parameter is zero, then we return to one of the cases already considered.

Upper ideals: The sum $\Sigma+$ of the upper ideals has a `grevlex` Gröbner basis consisting of 83 elements, degrees 3 to 5, terms 2 to 117, and coefficients −1642727092 to 1636813156. There are 5 elements of degree 3, 62 of degree 4, and 16 of degree 5. Exactly two elements (numbers 2 and 5) have a parameter as an irreducible factor: in both cases x_6, $g_2 = -x_6(x_1x_4 - x_2x_3)$, $g_5 = x_6(x_1^2 - x_2^2 - x_3^2 + x_4^2)$. Since $x_6 \neq 0$ by assumption, we may divide both g_2 and g_5 by x_6 and replace them in the Gröbner basis by $g_2' = -x_1x_4 + x_2x_3$, $g_5' = x_1^2 - x_2^2 - x_3^2 + x_4^2$. We recompute the Gröbner basis and obtain 65 elements with degrees 2 to 5, terms 2 to 91, and coefficients −3024000276 to 2254275346. There are two elements of degree 2, one of degree 3, 49 of degree 4, and 13 of degree 5. The first two elements of this new basis are g_2' and g_5'. We call this the *simplified upper basis*.

Since $x_1 \neq 0$ by assumption, we solve for x_4 in $g_2' = 0$ and obtain $x_4 = x_2x_3/x_1$. We substitute this in g_5' and factor the result, obtaining

$$\frac{(x_1 - x_2)(x_1 + x_2)(x_1 - x_3)(x_1 + x_3)}{x_1^2}.$$

For every solution, this must vanish, so we may split the computation of the zero set of the simplified upper basis into four cases:

$$x_2 = x_1, \quad x_2 = -x_1, \quad x_3 = x_1, \quad x_3 = -x_1.$$

Making these substitutions into g_2' we obtain

$$x_1(x_3 - x_4), \quad -x_1(x_3 + x_4), \quad x_1(x_2 - x_4), \quad -x_1(x_2 + x_4).$$

Since $x_1 \neq 0$, in each case the other factor is 0, and so the four cases are defined as follows:

case	substitutions		relation coefficients
1	$x_2 = x_1,$	$x_4 = x_3$	$[\,x_1, x_1, x_3, x_3, x_5, x_6\,]$
2	$x_2 = -x_1,$	$x_4 = -x_3$	$[\,x_1, -x_1, x_3, -x_3, x_5, x_6\,]$
3	$x_3 = x_1,$	$x_4 = x_2$	$[\,x_1, x_2, x_1, x_2, x_5, x_6\,]$
4	$x_3 = -x_1,$	$x_4 = -x_2$	$[\,x_1, x_2, -x_1, -x_2, x_5, x_6\,]$

In this way we reduce the original problem with six free parameters to four much smaller problems each with four free parameters.

For each of these cases, we make the corresponding substitutions into the simplified upper basis, and recompute the Gröbner basis. We then repeatedly cancel irreducible factors in basis elements that are parameters, and recompute the Gröbner basis.

All these tricks seem necessary to be able to compute a Gröbner basis for the radical of the sum of the upper ideals in a reasonable time. We obtain the following results.

Case 1: The original basis of 65 elements reduces to 26, 21, 12 elements after cancelling x_1 five times; the resulting basis has 2, 4, 4, 2 elements of degrees 2, 3, 4, 5 respectively, terms from 9 to 34, and coefficients from -249 to 211. The radical of this ideal has the following Gröbner basis:

$$\begin{gathered} x_6+x_5-x_3-x_1+1, \quad (x_5-x_3)(2x_5-x_3-x_1+2), \\ 2x_3^2x_1-2x_1^3+x_5x_3-x_3^2-x_5x_1+3x_3x_1+4x_1^2-2x_1, \\ 6x_5x_3x_1+6x_5x_1^2-6x_3x_1^2-6x_1^3+x_5x_3-x_3^2-5x_5x_1+11x_3x_1+12x_1^2-6x_1, \\ 2x_3^3-2x_3x_1^2-x_5x_3+3x_3^2+x_5x_1+3x_3x_1-2x_3, \\ 2x_5x_3^2-2x_5x_1^2-x_5x_3+3x_3^2+x_5x_1+3x_3x_1-2x_3. \end{gathered}$$

The zero set of this radical ideal, excluding solutions in which any parameter is zero, and using the equations $x_2=x_1$ and $x_4=x_3$, consists of the point $\left[\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},-\frac{2}{3},\frac{1}{3}\right]$ and the family

$$\left[x_1,x_1,x_3,x_3,x_3,x_1-1\right], \text{ where } x_3^2+x_3-(x_1-1)^2=0 \text{ and } x_1\neq 0,1. \tag{6.6}$$

Case 2: The original basis of 65 elements reduces to 29 elements; the resulting basis has 25, four elements of degrees 4, 5, respectively, terms from 20 to 32, and coefficients from -2509 to 5018. The radical has the following Gröbner basis:

$$\begin{gathered} (3x_1-1)(x_3-x_1), \quad 3x_6x_1-3x_5x_1+2x_3+x_1, \quad (x_3-x_1)(x_3+x_1), \\ (x_5-1)(x_3+x_1), \quad 3x_6x_3+3x_5x_1-2x_3-x_1, \quad 3x_5x_6-x_1+x_3, \\ x_1(x_5-1)(3x_1-1), \quad 9x_5^2x_1+3x_5x_1-5x_3-7x_1, \quad 9x_6^3+18x_5x_1-9x_6-11x_3-7x_1. \end{gathered}$$

The zero set of this radical ideal, excluding solutions in which any parameter is zero, and using the equations $x_2=-x_1$ and $x_4=-x_3$, is as follows:

$$[x_1,-x_1,x_3,-x_3,x_5,x_6]=\left[\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{3},-\tfrac{2}{3},-\tfrac{1}{3}\right],\left[\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},\tfrac{2}{3}\right].$$

Case 3: The results are very similar to those of Case 2. The radical of the ideal has the following Gröbner basis:

$$\begin{gathered} (3x_1-1)(x_2+x_1), \quad 3x_6x_1+3x_5x_1+2x_2-x_1, \quad (x_2-x_1)(x_2+x_1), \\ (x_5-1)(x_2-x_1), \quad 3x_6x_2+3x_5x_1+2x_2-x_1, \quad 3x_5x_6+x_1+x_2, \\ x_1(x_5-1)(3x_1-1), \quad 9x_5^2x_1+3x_5x_1+5x_2-7x_1, \quad 9x_6^3-18x_5x_1-9x_6-11x_2+7x_1. \end{gathered}$$

The zero set of this radical ideal, excluding solutions in which any parameter is zero, and using the equations $x_3=x_1$ and $x_4=x_2$, is as follows:

$$[x_1,x_2,x_1,x_2,x_5,x_6]=\left[\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},-\tfrac{2}{3},\tfrac{1}{3}\right],\left[\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},-\tfrac{2}{3}\right].$$

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Case 4: The results are very similar to those of Case 1. The radical of the ideal has the following Gröbner basis:

$$
\begin{gathered}
x_6 - x_5 - x_2 + x_1 - 1, \quad (x_5 + x_2)(2x_5 + x_2 - x_1 + 2),\\
2x_2^2 x_1 - 2x_1^3 - x_5 x_2 - x_2^2 - x_5 x_1 - 3x_2 x_1 + 4x_1^2 - 2x_1,\\
6x_5 x_2 x_1 - 6x_5 x_1^2 - 6x_2 x_1^2 + 6x_1^3 + x_5 x_2 + x_2^2 + 5x_5 x_1 + 11x_2 x_1 - 12x_1^2 + 6x_1,\\
2x_2^3 - 2x_2 x_1^2 - x_5 x_2 - 3x_2^2 - x_5 x_1 + 3x_2 x_1 - 2x_2,\\
2x_5 x_2^2 - 2x_5 x_1^2 + x_5 x_2 + 3x_2^2 + x_5 x_1 - 3x_2 x_1 + 2x_2.
\end{gathered}
$$

The zero set of this radical ideal, excluding solutions in which any parameter is zero, and using the equations $x_3 = -x_1$ and $x_4 = -x_2$, consists of the point

$$\left[\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{1}{3}, -\tfrac{2}{3}, -\tfrac{1}{3}\right]$$

and the family

$$\left[x_1, x_2, -x_1, -x_2, -x_2, 1 - x_1\right], \quad x_2^2 - x_2 - (x_1 - 1)^2 = 0, \quad x_1 \neq 0, 1 \tag{6.7}$$

To decide which (if any) of the solutions we found are regular, we must compute Gröbner bases for the radicals of the lower determinantal ideals of the matrices $B(\lambda)$, and then substitute the solutions into the Gröbner bases.

Lower ideals: ideals:

$$
\begin{aligned}
\lambda = 4\colon\ & x_6 + x_5 - 1, \quad (x_4 + x_2 + 1)(x_2 + x_1), \quad (x_2 + x_1)(2x_5 - x_3 + x_2 - 2x_1),\\
& x_4 x_3 + x_3^2 + x_4 x_1 + x_3 x_1 - x_2 - x_1,\\
& x_4^2 - x_3^2 + x_3 x_2 - x_2^2 - 2x_4 x_1 - x_3 x_1 - x_2 x_1 + x_4 + x_3,\\
& 2x_5 x_4 + 2x_5 x_3 + x_3 x_2 - x_2^2 - 2x_4 x_1 - x_3 x_1 - x_2 x_1 - 2x_2 - 2x_1,\\
& (x_2 + x_1)(x_3^2 - x_3 x_2 + x_3 x_1 - x_2 x_1 - x_3 - x_2 - 2x_1)\\
\lambda = 31\colon\ & x_4 + x_3 + x_2 + x_1, \quad x_6 x_1 + x_5 x_1 - x_3, \quad (x_2 - x_1)(x_2 + x_1),\\
& (x_3 + x_1)(x_2 + x_1), \quad (x_5 + 1)(x_2 + x_1), \quad x_6 x_2 - x_5 x_1 + x_3,\\
& (x_3 - x_1)(x_3 + x_1), \quad (x_5 - 1)(x_3 + x_1), \quad x_6 x_3 - x_5 x_1 + x_3,\\
& 2x_5^2 - 4x_5 x_1 - x_3 x_1 - x_2 x_1 - 2x_6 + 3x_3 - x_2 - 2,\\
& 2x_6 x_5 + 4x_5 x_1 + x_3 x_1 + x_2 x_1 - 3x_3 + x_2,\\
& 2x_6^2 - 4x_5 x_1 - x_3 x_1 - x_2 x_1 + 2x_6 + 3x_3 - x_2\\
\lambda = 2^2\colon\ & x_3 - x_2, \qquad x_4 - x_1, \qquad (x_2 - x_1)(x_2 + x_1), \quad x_5 x_2 - x_6 x_1 - x_2,\\
& x_6 x_2 - x_5 x_1 + x_1, \quad (x_6 + x_5 - 1)(x_6 - x_5 + 1),\\
& (x_5 - x_1 - 1)(x_5 - x_1)(x_5 + 2x_1 - 1),\\
& x_6 x_5^2 - 3x_6 x_1^2 + 2x_2 x_1^2 - x_6 x_5 + x_6 x_1 - 2x_2 x_1
\end{aligned}
$$

$$
\begin{aligned}
\lambda = 21^2\colon\ & x_4 - x_3 - x_2 + x_1, \quad x_6x_1 - x_5x_1 - x_2, \quad (x_2 - x_1)(x_2 + x_1),\\
& (x_3 - x_1)(x_2 - x_1), \quad (x_5 - 1)(x_2 - x_1), \quad x_6x_2 - x_5x_1 - x_2,\\
& (x_3 - x_1)(x_3 + x_1), \quad (x_5 + 1)(x_3 - x_1), \quad x_6x_3 - x_5x_1 - x_2,\\
& 2x_5^2 - 4x_5x_1 + x_3x_1 + x_2x_1 + 2x_6 + x_3 - 3x_2 - 2,\\
& 2x_6x_5 - 4x_5x_1 + x_3x_1 + x_2x_1 + x_3 - 3x_2,\\
& 2x_6^2 - 4x_5x_1 + x_3x_1 + x_2x_1 - 2x_6 + x_3 - 3x_2\\
\lambda = 1^4\colon\ & x_6 - x_5 + 1, \qquad x_4x_2 - x_2^2 - x_4x_1 + x_2x_1 - x_3 + x_1,\\
& (x_4 - x_3 + 1)(x_3 - x_1), \quad (x_3 - x_1)(2x_5 - x_3 + x_2 - 2x_1),\\
& x_4^2 - x_3^2 + x_3x_2 - x_2^2 - 2x_4x_1 + x_3x_1 + x_2x_1 + x_4 - x_2,\\
& 2x_5x_4 - x_3^2 - 2x_5x_2 + x_3x_2 - 2x_4x_1 + x_3x_1 + x_2x_1 + 2x_3 - 2x_1,\\
& (x_3 - x_1)(x_3x_2 - x_2^2 - x_3x_1 + x_2x_1 - x_3 - x_2 + 2x_1)
\end{aligned}
$$

Comparison of Upper and Lower Ideals

Finally, we need to check if some of the parameter values (6.6)–(6.7) are common zeros for at least one of the Gröbner bases for the lower ideals. For example, substituting the solution $\left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, \frac{1}{3}\right]$ into the elements of the five Gröbner bases produces the following lists of scalars:

$$
\begin{array}{ll}
\lambda = 4 & -\frac{4}{3},\ \frac{10}{9},\ -\frac{4}{3},\ -\frac{2}{9},\ \frac{2}{9},\ -\frac{8}{3},\ -\frac{8}{9},\\
\lambda = 31 & \frac{4}{3},\ -\frac{4}{9},\ 0,\ \frac{4}{9},\ \frac{2}{9},\ \frac{2}{3},\ 0,\ -\frac{10}{9},\ \frac{2}{3},\ -\frac{4}{9},\ -\frac{16}{9},\ \frac{20}{9},\\
\lambda = 2^2 & 0,\ 0,\ 0,\ -\frac{2}{3},\ \frac{2}{3},\ -\frac{8}{3},\ -2,\ \frac{2}{9},\\
\lambda = 21^2 & 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\\
\lambda = 1^4 & 2,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0.
\end{array}
$$

The five ideals are therefore (1), (1), (1), (0), (1) and so regularity fails because the fourth ideal is zero. Similar calculations eliminate the other isolated points, and so it remains to check only the one-parameter solutions (6.6) and (6.7):

$$
\begin{array}{lll}
\text{(6.8)} & \left[x_1,\ x_1,\ x_3,\ x_3,\ x_3,\ x_1 - 1\right] & x_3^2 + x_3 - (x_1 - 1)^2 = 0, \quad x_1 \neq 0, 1,\\
& \left[x_1,\ x_2,\ -x_1,\ -x_2,\ -x_2,\ 1 - x_1\right] & x_2^2 - x_2 - (x_1 - 1)^2 = 0, \quad x_1 \neq 0, 1.
\end{array}
$$

Each of these solutions is a sextuple depending on two parameters subject to one equation. We substitute these solutions into the elements of the Gröbner bases of the radicals of the lower ideals, adjoin the equation relating the parameters to each of the five Gröbner bases, and solve the corresponding systems of equations. The union of all those solutions is precisely the set of values of parameters we must exclude.

An example will make this clear. Consider the first solution from (6.8). We substitute these values into the Gröbner basis for the radical of the lower determinantal ideal for $\lambda = 2^2$. The eight generators of that ideal become

$$0, \quad x_3 - x_1, \quad -x_1(x_1 - x_3), \quad x_1(x_1 - x_3), \quad (x_1 - 2 + x_3)(x_1 - x_3),$$
$$-(x_1 - 1)(x_1 - 1 + x_3)(x_1 - x_3), \quad (x_3 + 2x_1 - 1)(-x_3 + 1 + x_1)(x_1 - x_3),$$

and we see that all these are equal to zero only when $x_1 = x_3$. Taking into account the equation $x_3^2 + x_3 - (x_1 - 1)^2 = 0$, we see that $x_1 = x_3 = \frac{1}{3}$. Doing these for all determinantal ideals, we find that the points that have to be removed from the first family are $(1, -1)$ and $(\frac{1}{3}, \frac{1}{3})$, and from the second family, $(1, 1)$ and $(\frac{1}{3}, -\frac{1}{3})$. (In fact, the first point in each pair has already been removed, since we assume $x_1 \neq 1$.)

The formulation of the result becomes a little more elegant if we replace x_3 by $-x_3$ in the first family.

Proposition 6.7 (Summary for six nonzero parameters) *When all parameters in Relation* (LR) *are nonzero, there are two one-dimensional families of solutions that are regular in arity* 4*:*

$$\begin{aligned}(a_1a_2)a_3 = x_1\big[\,a_1(a_2a_3) + a_1(a_3a_2) + a_3(a_2a_1)\big]\\ - x_3\big[\,a_2(a_1a_3) + a_2(a_3a_1) + a_3(a_1a_2)\big] - a_3(a_2a_1),\\ (a_1a_2)a_3 = x_1\big[\,a_1(a_2a_3) - a_2(a_1a_3) - a_3(a_2a_1)\big]\\ + x_2\big[\,a_1(a_3a_2) - a_2(a_3a_1) - a_3(a_1a_2)\big] + a_3(a_2a_1),\end{aligned}$$

where both (x_1, x_2) *and* (x_1, x_3) *belong to the hyperbola* $y^2 - y - (x-1)^2 = 0$ *with five excluded points:* $(1,0)$, $(1,1)$, $(\frac{1}{3}, -\frac{1}{3})$, *and* $(0, \phi)$ *for both roots* ϕ *of the polynomial* $x^2 - x - 1$.

6.7 Statement of the Main Technical Result

After noticing that the excluded points $(0, \phi)$ in the last statement are precisely the points with four nonzero parameters that we found previously, and the excluded point $(1, 0)$ corresponds to the Zinbiel operad in the first case and to the Leibniz operad in the second case, we see that Propositions 6.1–6.7 lead to the following conclusion.

Theorem 6.8 *The parametrized one-relation operads with the regular module in arity* 4 *are precisely the operads from the following list:*

(i) $(a_1a_2)a_3 = s\, a_3(a_1a_2)$,

(ii) $(a_1a_2)a_3 = a_1(a_2a_3) + s[a_1(a_3a_2) - a_2(a_1a_3) + a_2(a_3a_1) - a_3(a_1a_2)]$,

(iii) $(a_1a_2)a_3 = u[a_1(a_2a_3) + a_1(a_3a_2) + a_3(a_2a_1)]$
$\qquad - v[a_2(a_1a_3) + a_2(a_3a_1) + a_3(a_1a_2)] - a_3(a_2a_1)$,

(iv) $(a_1a_2)a_3 = u[a_1(a_2a_3) - a_2(a_1a_3) - a_3(a_2a_1)]$
$\qquad + v[a_1(a_3a_2) - a_2(a_3a_1) - a_3(a_1a_2)] + a_3(a_2a_1)$,

where in (i) *we require* $s \neq \pm 1$, *in* (ii) *we require* $s \neq -1$, *and in both* (iii) *and* (iv) *the point* (u, v) *belongs to the hyperbola* $y^2 - y - (x-1)^2 = 0$ *with the points* $(1,1)$ *and* $(\frac{1}{3}, -\frac{1}{3})$ *excluded.*

7 Classification Theorem

In this section, we prove the following classification result which is the main result of this paper.

Theorem 7.1

(i) *Over any field* $\mathbb{F}$ *of characteristic* 0*, each regular parametrized one-relation operad is one of the operads of Theorem* 6.8.

(ii) *Over an algebraically closed field* $\mathbb{F}$ *of characteristic* 0*, every regular parametrized one-relation operad is isomorphic to one of the following five operads: the left-nilpotent operad defined by the identity* $((a_1a_2)a_3) = 0$*, the associative operad, the Leibniz operad* Leib*, the dual Leibniz (Zinbiel) operad* Zinb*, and the Poisson operad.*

Proof We shall go through the list of Theorem 6.8 and establish that each of the operads which have the regular module in arity 4 is in fact regular and isomorphic to one of the five operads listed above; we will use the following observation. For each $t \in \mathbb{F}$, one has the following endomorphism ϕ_t of the space of generators $\mathcal{T}(2)$ of the free operad: $\phi_t(a_1a_2) = a_1a_2 + ta_2a_1$, $\phi_t(a_2a_1) = a_2a_1 + ta_1a_2$. This endomorphism commutes with the symmetric group action, and is invertible if and only if $t \neq \pm 1$. (This change of basis was studied by Livernet and Loday in the context of relating the Poisson operad to the associative operad [17]. See also a similar change of basis in the space of operations in the work of Albert [1, §V] in the context of power-associative and quasiassociative rings.) It extends to a well-defined endomorphism of the free operad $\mathcal{T}$. We can replace Relation (LR) by its image under this endomorphism, which is one of relations of the general type (2.5). Recall from Formula (2.6) that the general space of relations is spanned by the rows of the 6×12 matrix $N = [W \mid X]$. If $\det W \neq 0$, then, according to Lemma 2.15, there exists an equivalent relation of the type (LR). Overall, this allows us to find, for each regular parametrized one-relation operad, a one-parameter family of regular parametrized one-relation operads which are isomorphic to it; the set of parameters is precisely the set of all t for which $\det W \neq 0$. We now make this outlined strategy more precise.

Note that the endomorphism of $\mathcal{T}(3)$ induced by ϕ_t is given by the matrix

$$A(t) = \begin{bmatrix}
1 & . & t & . & . & . & . & . & . & . & t & t^2 \\
. & 1 & . & . & t & . & . & . & t & t^2 & . & . \\
t & . & 1 & . & . & . & . & . & . & . & t^2 & t \\
. & . & . & 1 & . & t & t & t^2 & . & . & . & . \\
. & t & . & . & 1 & . & . & . & t^2 & t & . & . \\
. & . & . & t & . & 1 & t^2 & t & . & . & . & . \\
. & . & . & t & . & t^2 & 1 & t & . & . & . & . \\
. & . & . & t^2 & . & t & t & 1 & . & . & . & . \\
. & t & . & . & t^2 & . & . & . & 1 & t & . & . \\
. & t^2 & . & . & t & . & . & . & t & 1 & . & . \\
t & . & t^2 & . & . & . & . & . & . & . & 1 & t \\
t^2 & . & t & . & . & . & . & . & . & . & t & 1
\end{bmatrix}$$

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(zeros are replaced by dots for readability). This can be established by a direct calculation. For instance,

$$\begin{aligned}\phi_t((a_1a_2)a_3) &= (a_1a_2 + ta_2a_1)a_3 + t(a_3(a_1a_2 + t\,a_2a_1))\\ &= (a_1a_2)a_3 + t(a_2a_1)a_3 + ta_3(a_1a_2) + t^2a_3(a_2a_1),\end{aligned}$$

which precisely corresponds to the first column of the matrix $A(t)$.

Suppose that N_0 is the 6×12 matrix whose rows form the S_3-orbit of some relation of the type (LR). The change of basis we introduced amounts to multiplying N_0 by $A(t)$ on the right. We let $N(t) = N_0A(t) = [W(t) \mid Y(t)]$, where $W(t)$ and $Y(t)$ are 6×6 matrices with entries in $\mathbb{F}[t, x_1, \dots, x_6]$. The module of quadratic relations generated by the rows of this matrix contains a relation of type (LR) if and only if $\det W(t) \neq 0$, and that in this case the matrix $\widetilde{N}(t) = W(t)^{-1}N(t)$ encodes that relation.

We are now ready to investigate the isomorphism classes. We start with the parametric family $(a_1a_2)a_3 = sa_3(a_1a_2)$, $s \neq \pm 1$. We have

$$\det W(t) = (1-t)^3(t+1)^3(1-st)^6.$$

The change of basis given by $A(t)$ results in the following change of parametrization: $\widetilde{s} = \frac{t-s}{st-1}$. Clearly, if we put $t = s$, then $\det W(t) \neq 0$, and $\widetilde{s} = 0$. Therefore, each operad of this family is isomorphic to the left-nilpotent operad.

Next, we consider the parametric family

$$(a_1a_2)a_3 = a_1(a_2a_3) + s[a_1(a_3a_2) - a_2(a_1a_3) + a_2(a_3a_1) - a_3(a_1a_2)],$$

where $s \neq -1$. We have $\det W(t) = (1-t)^5(t+1)^3(3st + t^2 + t + 1)^2$. The change of basis given by $A(t)$ results in the following change of parametrization:

$$\widetilde{s} = \frac{st^2 - st + s + t}{3st + t^2 + t + 1}.$$

The resultant with respect to t of the product of irreducible factors of $\det W(t)$ and the numerator of $\widetilde{s}$ is equal to $(s+1)^3(3s-1)^3$, as one can check by an immediate computation. Thus, for each point $s \neq -1, \frac{1}{3}$, it is possible to find a value of t for which $\det W(t) \neq 0$, and $\widetilde{s} = 0$. For such t, we see that there is a change of basis that makes $\widetilde{s} = 0$, so each operad of this family, except for the operad for $s = \frac{1}{3}$, is isomorphic to the associative operad. The operad for $s = \frac{1}{3}$ is a fixed point for all changes of basis; it is the one-operation presentation of the operad of Poisson algebras [17].

Finally, we consider the parametric families

$$\begin{aligned}(a_1a_2)a_3 &= u[a_1(a_2a_3) + a_1(a_3a_2) + a_3(a_2a_1)]\\ &\qquad - v[a_2(a_1a_3) + a_2(a_3a_1) + a_3(a_1a_2)] - a_3(a_2a_1),\\ (a_1a_2)a_3 &= u[a_1(a_2a_3) - a_2(a_1a_3) - a_3(a_2a_1)]\\ &\qquad + v[a_1(a_3a_2) - a_2(a_3a_1) - a_3(a_1a_2)] + a_3(a_2a_1),\end{aligned}$$

where the parameters u and v are related by the equation $v^2 - v - (u-1)^2 = 0$, and $(u, v) \neq (1,1), (\frac{1}{3}, -\frac{1}{3})$. We have

$$\det W(t) = (1-t)^3(t+1)^5(ut + vt - t + 1)^3(1 + t - 3ut + 3vt).$$

The change of basis given by $A(t)$ is the following change of parametrization:

$$\widetilde{u} = \frac{2u^2t^2 + u^2t - ut^2 - u - 2v^2t^2 - v^2t - 2vt}{3u^2t^2 - 4ut^2 + 2ut - 3v^2t^2 + 2vt^2 - 4vt + t^2 - 1},$$
$$\widetilde{v} = \frac{u^2t^2 + 2u^2t - 2ut - v^2t^2 - 2v^2t - vt^2 - v}{3u^2t^2 - 4ut^2 + 2ut - 3v^2t^2 + 2vt^2 - 4vt + t^2 - 1}.$$

The resultant with respect to t of the product of irreducible factors of $\det W(t)$ and the numerator of $\widetilde{v}$ is $(u-v)^2(u+v)^2(u+v-2)^2(2u-v-1)^2(3u-3v-2)^2$. This polynomial has common roots with $v^2 - v - (u-1)^2 = 0$ if and only if $(u,v) = (1,1)$ or $(u,v) = (\frac{1}{3}, -\frac{1}{3})$, which are precisely the points we excluded. Therefore, for each operad in each of the two families, it is possible to find a value of t for which $\det W(t) \neq 0$, and $\widetilde{v} = 0$. For such t, we see that there is a change of basis that makes $\widetilde{v} = 0$, which in turn forces $\widetilde{u} = 1$. This proves that each operad of the first family is isomorphic to the Zinbiel operad, and each operad of the second family is isomorphic to the Leibniz operad. ∎

8 Further Directions

8.1 Further Questions About the Cubic Relation Matrix

It would be interesting to extend the nilpotency result of Section 4 and classify all parametrized one-relation operads that are nilpotent. There are two somewhat natural questions one may ask here.

Problem 8.1 Determine explicitly the factorization of the determinant of the cubic relation matrix M into the product of irreducible polynomials in $\mathbb{F}[x_1, \dots, x_6]$. Use this to determine explicitly all parameter values $a_1, \dots, a_6 \in \mathbb{F}$ for which the operad $\mathcal{O}_{\mathbf{a}}$ is nilpotent of index 3: these values form the complement $\mathbb{F}^6 \setminus V(\det(M))$.

Problem 8.2 For every $d \geq 3$, determine explicitly the set $\mathcal{N}_d \subseteq \mathbb{F}^6$ of all parameter values $a_1, \dots, a_6$ for which the operad $\mathcal{O}_{\mathbf{a}}$ is nilpotent of index d. For these values, we have $\mathcal{J}(d) \neq \mathcal{T}(d)$ and $\mathcal{J}(d+1) = \mathcal{T}(d+1)$. We have already seen in Theorem 4.6 that the set $\mathcal{N}_3$ is a Zariski open subset of $\mathbb{F}^6$.

We have been able to use representation theory in order to avoid dealing with the determinantal ideals of the cubic relation matrix M, or, equivalently, of the block B of its partial Smith normal form. Understanding the structure of those ideals remains an open problem.

Problem 8.3 For $r = 1$, the reduced Gröbner bases for the first determinantal ideal $\mathrm{DI}_1(B)$ and its radical were presented in Lemma 4.3. For $2 \leq r \leq 36$, an open problem (probably rather difficult, at least computationally) is to determine the reduced Gröbner bases for the r-th determinantal ideal $\mathrm{DI}_r(B)$ and its radical. For $r = 36$, the determinantal ideal $\mathrm{DI}_{36}(B)$ is the principal ideal generated by $\det(B)$, and by Algorithm 1 we know that $\det(B) = \pm\det(M)$, so this case overlaps with Problem 8.1.

8.1.1 Rank Distribution for Relations With Small Coefficients

Let us conclude this subsection with some experimental data that sheds some light on the rank distribution for the cubic relation matrix as a function of the parameter values. We consider the 729 relations (LR) with coefficients in $\{0, \pm 1\}$, and we partition this set by the number q of nonzero coefficients. In each case, we substitute the parameter values into B and compute $r = \operatorname{rank}(B)$, recalling that $\operatorname{rank}(M) = 84 + \operatorname{rank}(B)$. In the following table, the rows are indexed by q and the columns by r. The (q, r) entry is the number of relations for which $x_1, \ldots, x_6 \in \{0, \pm 1\}$ and $\left|\{ i \mid x_i \in \{\pm 1\}\}\right| = q$ and $\operatorname{rank}(B) = r$, where $0 \le q \le 6$ and $0 \le r \le 36$ (as above, zeros are replaced by dots for readability):

$q \backslash r$	0						6						12						18	19
0	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
1	2	.	.	.	.	.	2	.	.	.	.	.	1	.	.	.	.	.	.	.
2	.	.	.	.	.	.	.	.	.	.	.	.	2	.	.	.	2	.	.	.
3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	2	.	8	3	.
4	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
5	.	.	.	.	2	.	2	.	.	.	2	.	3	4	.	.	.	.	10	.
6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\sum$	2	.	.	.	2	.	4	.	.	.	2	.	7	4	2	2	2	8	13	.

$q \backslash r$					24						30						36
0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1	.	.	.	.	6	.	.	.	.	.	.	.	.	.	.	.	1
2	3	.	.	2	5	.	.	.	6	.	8	2	4	2	2	2	20
3	5	18	.	.	12	.	2	8	1	16	10	4	12	28	7	8	14
4	2	.	8	4	4	2	.	.	26	12	12	8	20	14	12	18	98
5	.	8	2	.	20	.	1	6	2	8	14	4	9	38	4	12	41
6	.	.	.	.	8	.	4	.	.	.	20	.	8	.	.	.	24
$\sum$	10	26	10	6	55	2	7	14	35	36	64	18	53	82	25	40	198

From column 36 we see that $198/729 \cong 27.16\%$ of these operads are nilpotent of index 3. Regularity implies $\operatorname{rank}(B) = 12$, but not conversely; column 12 indicates that there are respectively $1, 1, 2, 3$ relations for $q = 0, 1, 2, 5$ with $\operatorname{rank}(M) = 96$. In these seven cases, the parameter values are the rows of the following matrix, and the last column gives the multiplicities for the S_4-action on the nullspace of M:

$$
\left|\begin{array}{rrrrrr}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
1 & 1 & -1 & 1 & -1 & 0 \\
-1 & 1 & 1 & -1 & 0 & 1 \\
-1 & -1 & -1 & -1 & 0 & -1
\end{array}\right|
\qquad
\begin{array}{c}
[1,3,2,3,1] \\
[1,3,2,3,1] \\
[1,3,2,3,1] \\
[1,3,2,3,1] \\
[1,3,2,3,1] \\
[2,3,3,2,1] \\
[1,2,3,3,2]
\end{array}
$$

8.2 Koszul Operads With One Relation

A question of Loday that we mentioned in the introduction still remains open.

Problem 8.4 Which of the parametrized one-relation operads $\mathcal{O}_{\mathbf{x}}$ are Koszul?

Theorem 7.1, of course, implies that all regular parametrized one-relation operads are Koszul, while Theorem 4.6 easily implies that generic parametrized one-relation operads are not Koszul. The Hilbert series of an index 3 nilpotent parametrized one relation operad is $f(t) = t + t^2 + t^3$; the modified inverse series has negative coefficients:

$$-f^{\langle -1\rangle}(-t) = t + t^2 + t^3 - 4t^5 - 14t^6 - 30t^7 - 33t^8 + 55t^9 + O(x^{10}).$$

The Koszul-ness criterion of Ginzburg and Kapranov [13, 15] instantly implies that such an operad cannot be Koszul. Moreover, inspecting the list of 729 parametrized one-relation operads with coefficients in $\{0, \pm 1\}$ from §8.1.1, we discover that most of those operads are not Koszul either because the modified inverse of the Hilbert series has negative coefficients, or because the Hilbert series of the operad is not equal to the inverse of the modified Hilbert series of the Koszul dual operad (which is, as we know, isomorphic to a parametrized one-relation operad). Among those 729 operads, there are just six irregular cases where the Koszul-ness cannot be disproved using the Ginzburg–Kapranov criterion. Four of those, $(a_1a_2)a_3 = \pm a_1(a_2a_3)$ and $(a_1a_2)a_3 = \pm a_1(a_3a_2)$, are Koszul and in fact have quadratic Gröbner bases for the (weighted) `pathdeglex` ordering [11]. (We encountered two of those operads in Lemma 4.3; notably, the corresponding S_4-modules both have dimension 36 but are not isomorphic: the multiplicities are $[2, 4, 4, 4, 2]$ for the relation $(a_1a_2)a_3 = a_3(a_1a_2)$ and $[1, 5, 2, 5, 1]$ for the relation $(a_1a_2)a_3 = -a_3(a_1a_2)$). Two remaining operads for which the Koszul-ness remains an open question are

$$(a_1a_2)a_3 = \pm[a_1(a_2a_3) - a_1(a_3a_2) + a_2(a_1a_3) - a_2(a_3a_1)] + a_3(a_1a_2),$$

one of which we saw as an excluded point of an otherwise regular family of parametrized one-relation operads in Remark 6.6.

A Verification of Results in `Magma`

Our computer algebra system of choice for this project was `Maple`. Computations above use various tricks and divide-and-conquer methods designed to avoid asking `Maple` to compute the radical of an ideal: at least in `Maple 18`, which we were using at the crucial stage of this project, the implementation of radical computation seemed to have some bugs (which seem to have been fixed in `Maple 2016`). As an independent verification, we used the `RadicalDecomposition` function of `Magma` [4], which appears to be extremely efficient even in the free online calculator [16], which limits the input to 50Kb and the calculation time to 120 seconds. We fed in the respective blocks $B(\lambda)$ [6] obtained by partial reduction of representation matrices (which were obtained through simple linear algebra over the rational field by a direct computation not involving any complicated `Maple` functions, and thus represented the "foolproof" part of the computation), and requested the calculator to compute the following:

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- all the upper determinantal ideals $\mathrm{DI}_{r+1}(B(\lambda))$;
- all the lower determinantal ideals $\mathrm{DI}_{r}(B(\lambda))$;
- the prime decomposition of the radical of the sum of the upper determinantal ideals;
- the prime decompositions of the radicals of the five ideals obtained as sums of upper ideals for four out of five λ and the lower ideal for the remaining choice of λ.

(The simple `Magma` script that we used is given in the online addendum [6].) This computation took less than five seconds, and the result obtained was as follows.

Theorem A.1 *The zero set of the sum of the upper ideal has ten irreducible components:*

$$\left\{ [1-x_6, -x_5, x_6-1, x_5, x_5, x_6] \colon x_6^2 = x_5^2 + x_5 \right\}, \tag{A.1}$$

$$\left\{ [1+x_6, 1+x_6, x_5, x_5, x_5, x_6] \colon x_6^2 = x_5^2 + x_5 \right\}, \tag{A.2}$$

$$\left\{ [-x_4, -x_4, x_4, x_4, -1, 0] \right\}, \tag{A.3}$$

$$\left\{ [-x_4, x_4, -x_4, x_4, 1, 0 \right\}, \tag{A.4}$$

$$\left\{ [1, -x_5, x_5, -x_5, x_5, 0] \right\}, \tag{A.5}$$

$$\left\{ [0, 0, 0, 0, x_5, 0] \right\}, \tag{A.6}$$

$$\left\{ \left[\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{1}{3}, -\tfrac{2}{3}, -\tfrac{1}{3} \right] \right\}, \tag{A.7}$$

$$\left\{ \left[\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, -\tfrac{2}{3}, \tfrac{1}{3} \right] \right\}, \tag{A.8}$$

$$\left\{ [0, 0, 0, 0, 0, -1] \right\}, \tag{A.9}$$

$$\left\{ [0, 0, 0, 0, 0, 1] \right\}. \tag{A.10}$$

The zero sets of the five ideals obtained as sums of upper ideals for four out of five λ and the lower ideal for the remaining choice of λ are as follows.

- *for* $\lambda = 4$

$$\left\{ [-x_4, x_4, -x_4, x_4, 1, 0] \right\}, \tag{A.11}$$

$$\left\{ \left[\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{2}{3} \right] \right\}, \tag{A.12}$$

$$\{ [0, 0, 0, 0, 0, 1] \}. \tag{A.13}$$

- *for* $\lambda = 31$

$$\{ [-x_4, -x_4, x_4, x_4, -1, 0] \}, \tag{A.14}$$

$$\left\{ [-x_4, x_4, -x_4, x_4, 1, 0 \right\}, \tag{A.15}$$

$$\left\{ \left[\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{1}{3}, -\tfrac{2}{3}, -\tfrac{1}{3} \right] \right\}, \tag{A.16}$$

$$\left\{ [0, 0, 0, 0, 0, -1] \right\}. \tag{A.17}$$

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- *for* $\lambda = 2^2$

$$\left\{ \left[\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{2}{3}\right] \right\}, \tag{A.18}$$

$$\left\{ \left[\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, -\tfrac{2}{3}\right] \right\}, \tag{A.19}$$

$$\left\{ [0, 0, 0, 0, 1, 0] \right\}, \tag{A.20}$$

$$\left\{ [0, 0, 0, 0, -1, 0] \right\}, \tag{A.21}$$

$$\left\{ [0, 0, 0, 0, 0, 1] \right\}. \tag{A.22}$$

- *for* $\lambda = 21^2$

$$\left\{ [-x_4, -x_4, x_4, x_4, -1, 0] \right\}, \tag{A.23}$$

$$\left\{ [-x_4, x_4, -x_4, x_4, 1, 0 \right\}, \tag{A.24}$$

$$\left\{ \left[\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, -\tfrac{2}{3}, \tfrac{1}{3}\right] \right\}, \tag{A.25}$$

$$\left\{ [0, 0, 0, 0, 0, 1] \right\}. \tag{A.26}$$

- *for* $\lambda = 1^4$

$$\left\{ [-x_4, x_4, -x_4, x_4, 1, 0 \right\}, \tag{A.27}$$

$$\left\{ \left[\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, -\tfrac{2}{3}\right] \right\}, \tag{A.28}$$

$$\left\{ [0, 0, 0, 0, 0, -1] \right\}. \tag{A.29}$$

The answer to our problem is obtained by removing from the first zero set the union of the remaining ones. First, we note the following:

- the component (A.3) appears among the excluded ones as (A.14) and (A.23);
- the component (A.4) appears among the excluded ones as (A.11), (A.15), (A.24), and (A.27);
- the component (A.7) appears among the excluded ones as (A.16);
- the component (A.8) appears among the excluded ones as (A.25);
- the component (A.9) appears among the excluded ones as (A.17) and (A.29);
- the component (A.10) appears among the excluded ones as (A.13), (A.22), and (A.26).

This means that we just need to examine which points are to be removed from components (A.1), (A.2), (A.5), and (A.6). By a more careful inspection, we determine the following:

- for component (A.1), there are two points to be removed: the point corresponding to $x_5 = \frac{1}{3}$, $x_6 = \frac{2}{3}$ (it is the excluded component (A.12), same as (A.18)) and the point corresponding to $x_5 = -1$, $x_6 = 0$ (it corresponds to $x_4 = -1$ in the excluded component (A.3)),
- for component (A.2), there are two points to be removed: the point corresponding to $x_5 = \frac{1}{3}$, $x_6 = -\frac{2}{3}$ (it is the excluded component (A.19), same as (A.28)) and the point corresponding to $x_5 = -1$, $x_6 = 0$ (it corresponds to $x_4 = -1$ in the excluded component (A.3)),

- for component (A.5), there is one point to be removed: the point corresponding to $x_5 = 1$ (it corresponds to $x_4 = -1$ in the excluded component (A.4))
- for component (A.6), there are two points to be removed: the point corresponding to $x_5 = 1$ (it is the excluded component (A.20)) and the point corresponding to $x_5 = -1$ (it is the excluded component (A.21)).

By a direct inspection, this coincides with the set obtained in Theorem 6.8, which completes the verification.

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