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AN INTEGRAL CHARACTERIZATION OF EUCLIDEAN SPACE

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We show that recent integral versions of the classic Jordan-Von Neumann characterization of Euclidean space may be viewed as special cases of a general averaging principle over sets of isometries.

1.

Recently Stanojević and Suchanek [6] showed that a complex normed space X is an inner product space if and only if, given a compact group G with normalized Haar measure m and some non-trivial group character γ (that is a continuous homomorphism into the circle group),

$$\int_G \|x+\gamma(g)y\|^2 dm = \|x\|^2 + \|y\|^2 \tag{1}$$

for all x and y in X. Day [3] observed that it suffices for (1) to hold with X replaced by its unit sphere $S(X)$ and with "=" replaced by "$\sim$" where $\sim$ is one of $\leq, \geq$ or $=$. This then gives a broad generalization of the classic Schoenberg-Day characterizations of inner product spaces [2]. In this paper we show that (1) can be viewed as a special instance of an averaging condition involving sets of isometries.

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2.

Let X be a (real or complex) normed space and let H be a non-empty set of linear isometries. This is to say that $\|Tx\| = \|x\|$ for all T in H and all x in X. Let m be a Borel probability measure on H (in the induced strong operator topology). We say that H is *m-balanced* if the barycentre of H with respect to m, denoted m_H, exists and satisfies

$$(2) \qquad m_H := \int_{T \in H} T dm = 0 .$$

The integral in (2) is interpreted as a weak integral [5] and will exist whenever H is relatively compact and so whenever H is finite dimensional or compact. The next proposition motivates the definition.

PROPOSITION 1. *Let H be a non-empty subset of the isometries of an inner product space X. Let m be a Borel probability measure on H with respect to which H has a barycentre. Then for x and y in X,*

$$(3) \qquad \int_{T \in H} \|x+Ty\|^2 dm = \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\langle m_H y , x\rangle .$$

In particular,

$$(4) \qquad \int_{T \in H} \|x+Ty\|^2 dm \sim 2 \quad \text{for } x, y \text{ in } S(X) ,$$

if and only if H is m-balanced.

Proof. First observe that $\|x+Ty\|^2$ is continuous and is bounded by 4. The integral in (3) is thus well defined. Equation (3) is now a consequence of the fact that X is an inner product space, that H contains only isometries, and that $m_H := \int_{T \in H} T dm$. If $m_H = 0$, (4) follows easily. Conversely, if m_H is non-zero one can find a unit vector y with $m_H y \neq 0$. If we let $x := \pm m_H y / \|m_H y\|$, (3) shows that (4) is violated. □

It is standard and easily observed that $m_H \in \overline{\mathrm{conv}}\, H$. Thus when H is finite dimensional and closed, H is m-balanced with respect to some m if and only if $0 \in \mathrm{conv}\, H$. It follows that certain groups of isometries can never be used to obtain expressions like (4). For example, the isometry T of $l_2^n(\mathbb{R})$ which is given by

$$T(x_1, x_2, \ldots, x_n) := (x_2, x_3, \ldots, x_n, x_1)$$

generates a cyclic group of order n whose convex hull does not contain zero. Recall that a Borel measure on H is *strictly positive* if its support is H. Our main result is:

THEOREM 2. *Let H be a subset of the isometries of a finite dimensional normed space X. Suppose that H is m-balanced with respect to a strictly positive measure m. If*

$$(5) \qquad \int_{T\in H} \|x+Ty\|^2 dm \sim 2 \quad \text{for } x, y \text{ in } S(X) ,$$

then X is a Euclidean space.

Proof. A complex normed space of dimension n may be viewed (real isometrically) as a real normed space of dimension $2n$. The complex isometries remain real isometries, and since (5) is a real isometric invariant, it suffices to establish the real case of the theorem. We consider the "$\geq$" case and let E be the unique (Loewner) ellipsoid of maximal volume inside $C := \{x \mid \|x\| = 1\}$. Let $\|\cdot\|_E$ denote the associated Euclidean norm. The argument in [1, p. 90] shows that E inherits the isometries of X; and [2, p. 80] shows that $M := S(E) \cap S(X)$ spans X. Let x and y lie in M and choose T_0 in H. Then $z := T_0x$ also lies in M and (5) shows that

$$(6) \qquad \int_{T\in H} \|z+Ty\|^2 dm \geq 2 .$$

Since H is m-balanced and lies inside of the isometries of E, (3) shows that

$$(7) \qquad \int_{T\in H} \|z+Ty\|_E^2 dm = 2 .$$

Let $f(T) := \|z+Ty\|^2 - \|z+Ty\|_E^2$. Then f is a non-positive continuous function on H such that $\int_{T\in H} f(T) dm \geq 0$. Since m is strictly positive we must have $f(T_0) = 0$. But this says that

$$(8) \qquad \|x+y\| = \|T_0x+T_0y\| = \|T_0x+T_0y\|_E = \|x+y\|_E ,$$

because T_0 is an isometry of C and of E . It follows that the set of directions D in which the two norms coincide is midpoint-convex. Being closed and homogeneous, D must actually be a subspace. Since D contains M , D is the entire space and $\|\cdot\|$ coincides with $\|\cdot\|_E$. The "$\le$" case follows similarly from a minimality argument. □

We can replace the finite dimensionality hypothesis by the condition that, for some fixed n , H leaves n-dimensional subspaces of X invariant. This still allows us to show that every n-dimensional subspace of X is Euclidean; and so is X .

We also observe that the previous argument fails for any skew-norm.

The classical criterion uses $H := \{I, -I\}$ and the uniform two-point measure. More generally we have:

COROLLARY 3. *Let X be a finite dimensional normed space and let H be a closed subgroup of isometries which contains a non-trivial multiplication. Let m be normalized (left) Haar measure on H . Then H is m-balanced and* (5) *characterizes Euclidean space.*

Proof. Let S be a multiplication by α $(\alpha \ne 1)$ which lies in H . Then H is compact, whence m_H exists and

$$(9) \qquad m_H = \int_{T\in H} T dm = \int_{T\in H} ST dm = \alpha m_H .$$

Since $\alpha \ne 1$, $m_H = 0$ and H is m-balanced. Also, since H is compact and m is translation invariant, m is strictly positive. The result now follows from Theorem 2. □

A simple way of guaranteeing that a group H is balanced is to require that $H = -H$. Note that the full group of isometries is balanced. Our next corollary recaptures Day's version [3] of Stanojević and Suchanek's result [6] given in the introduction. Applications can be found in [6]. Observe that only abelian compact groups really appear in the corollary.

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COROLLARY 4. *Let* X *be a complex normed space and let* G *be a compact group endowed with normalized Haar measure. Let* γ *be a non-trivial group character on* G . *Then*

$$\int_{g\in G} \|x+\gamma(g)y\|^2 dm \sim 2 \quad \text{for all} \quad x, y \in S(X) \tag{10}$$

if and only if X *is an inner product space.*

Proof. For each g in G multiplication by $\gamma(g)$ is an isometry of X . Since G is compact the character γ induces a compact subgroup H of isometries of X . Since γ is non-trivial, H contains a non-trivial multiplication and, as in the previous corollary, is m-balanced. Proposition 1 now shows (10) to be necessary; and Theorem 2, which applies since H has one dimensional orbits, shows (10) to be sufficient. □

If one defines characters with respect to the underlying scalar field, Corollary 4 remains valid - if uninteresting - over the real field. Similarly, we have:

COROLLARY 5. *Let* X *be a normed space. Suppose that unit length scalars* $w_1, w_2, \ldots, w_m$ *and strictly positive real numbers* $\lambda_1, \lambda_2, \ldots, \lambda_m$ *are given such that*

$$\sum_{i=0}^{m} \lambda_i w_i = 0 , \quad \sum_{i=1}^{m} \lambda_i = 1 . \tag{11}$$

Then X *is an inner product space if and only if*

$$\sum_{i=1}^{m} \lambda_i \|x+w_i y\|^2 \sim 2 \quad \text{for} \quad x, y \quad \text{in} \quad S(X) . \tag{12}$$

Proof. Let H be the finite set of isometries T_i with $T_i x := w_i x$. Condition (11) shows that the discrete measure m with mass λ_i at T_i balances H , and is strictly positive. The result now follows as in Corollary 4. □

The proof is unchanged if $m = \infty$.

A special case of Corollary 5 (and of Corollary 4) is worth singling

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out. If $\omega \neq 1$ is any mth root of unity then $\sum_{i=1}^{m} \frac{1}{m} \omega^i = 0$, and

$$(13) \qquad \frac{1}{m} \sum_{i=1}^{m} \|x+\omega^i y\|^2 \sim 2 \quad \text{for} \quad x, y \in S(X)$$

characterizes inner-product spaces as in [3].

Similar extensions can be made to the integral inequalities given in [4] and extended to group characters in [3]. They do not, however, have the same completeness or simplicity as Theorem 2. Also, it seems worth observing that integration over a Haar measure gives a concise proof of the following classical result.

THEOREM 6 ([1]). *Let C be a convex body in finite dimensional normed space X . If any two points on the boundary of C are connected by a linear isometry then C is an ellipsoid.*

Proof. Let $\|\cdot\|_E$ be any Euclidean norm on X . Let H be the compact group of isometries on C and let $\|\cdot\|_F$ be defined by

$$(14) \qquad \|x\|_F^2 = \int_{T \in H} \|Tx\|_E^2 dm ,$$

where m is Haar measure on H . Then $\|\cdot\|_F$ is Euclidean and $\|Tx\|_F = \|x\|_F$ for each x in X and each T in H . Since H is transitive, all points on the boundary of C have the same value under $\|\cdot\|_F$. Thus C is an ellipsoid.

In the symmetric case the corollary follows easily from our results. Let H be the full group of isometries of X and let m be Haar measure on H . Define Φ by

$$(15) \qquad \Phi(x, y) := \int_{T \in H} \|x+Ty\|^2 dm \quad \text{for} \quad x, y \in S(X) .$$

If S_1 and S_2 are isometries in H then

$$(16) \qquad \Phi(S_1x, S_2y) = \Phi(x, S_2y) = \Phi(x, y) .$$

Since H is presumed transitive it follows that Φ is constant. Corollary 3 now applies since the constant, *mutatis mutandi*, is either no larger or no smaller than two. □

The mapping implicit in (14) has many pleasant properties when viewed as a mapping from the space of all n-dimensional norms into itself.

To apply Theorem 2 in other situations it is necessary to possess appropriate sets of isometries. We now give one such example. Recall that a norm on $\mathbb{R}^n$ is *absolute* if $\|x\| = \||x|\|$ for each x in $\mathbb{R}^n$. Here $|\cdot|$ is computed component-wise. It is a simple consequence of Caratheodory's theorem that such a norm is actually a lattice norm. Moreover, in this case the mappings π_k $(k = 1, \ldots, 2^n)$,

(17) $$\pi_k y := (\pm y_1, \ldots, \pm y_n) ,$$

where the signs range over all permutations of ± 1, are linear isometries. This leads to

COROLLARY 8. *An absolute norm on* $\mathbb{R}^n$ *is Euclidean if and only if*

(18) $$\frac{1}{2^n} \sum_{k=1}^{2^n} \|x+\pi_k y\|^2 \sim 2 \quad \text{for} \quad x, y \in S(X) .$$

Proof. Since $\frac{1}{2^n} \sum_{k=1}^{2^n} \pi_k = 0$, the set P of such isometries is balanced with respect to the uniform measure and so Theorem 2 applies. □

Obviously the corollary remains true for all balanced subsets of P.

Finally we observe, that as in [3], certain extensions may be made to replace measures by invariant means.

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