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LOCAL REGULARITY FOR NONLINEAR ELLIPTIC AND PARABOLIC EQUATIONS WITH ANISOTROPIC WEIGHTS

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Abstract The main purpose of this paper is to capture the asymptotic behaviour for solutions to a class of nonlinear elliptic and parabolic equations with the anisotropic weights consisting of two power-type weights of different dimensions near the degenerate or singular point, especially covering the weighted p-Laplace equations and weighted fast diffusion equations. As a consequence, we also establish the local Hölder estimates for their solutions in the presence of single power-type weights.

Keywords: local regularity; anisotropic weights; weighted Poincaré inequality; weighted p-Laplace equations; weighted fast diffusion equations

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1. Introduction and main results

In this paper, we will use the De Giorgi truncation method [16] to study the local behaviour of solutions for a class of nonlinear elliptic and parabolic equations with the weights comprising two power-type weights of different dimensions. For that purpose, we should first establish the corresponding anisotropic weighted Sobolev embedding theorems and Poincaré's inequality, which are fundamental tools to investigate relevant Sobolev spaces and partial differential equations. The former has recently been established by Li and Yan [30], whose results improve and extend the classical Caffarelli–Kohn–Nirenberg type inequalities in [10]. With regard to the latter, we prepare to prove that this type of anisotropic weights belongs to the *Muckenhoupt class* A_q under certain conditions, and then the anisotropic weighted Poincaré inequality is obtained by utilizing the theories of A_q-weights, $1 < q < \infty$, see Section 2 below for the finer details. This is another major novelty of this paper besides the regularity results with anisotropic weights. For more relevant investigations on weighted Sobolev and Poincaré inequalities, see [2, 3, 8, 9, 11, 14, 31, 33, 34] and the references therein.

The anisotropy of the weights considered in this paper comes from two power-type weights of different dimensions. This complex weighted form will bring great difficulties of analyses, computations and discussions in the following proofs, especially the findings for regular indices which make this type of anisotropic weights become A_q-weights. The mathematical formulations and main results for the considered nonlinear elliptic and parabolic problems with anisotropic weights are, respectively, presented as follows.

1.1. The nonlinear elliptic equations with anisotropic weights

Consider a bounded smooth domain $\Omega \subset \mathbb{R}^n$ with $0 \in \Omega$ and $n \geq 2$. With regard to the weighted elliptic equations, we mainly study the local regularity of solution to the following problem

$$\begin{cases} \operatorname{div}(Aw|\nabla u|^{p-2}\nabla u) = 0, & \text{in } \Omega, \\ 0 \leq u \leq \overline{M} < \infty, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $w = |x'|^{\theta_1}|x|^{\theta_2}$, the values of θ_1 and θ_2 are assumed in the following theorems, $1 < p < n + \theta_1 + \theta_2$, $\overline{M}$ is a given positive constant, $A(x) = (a_{ij}(x))_{n\times n}$ is symmetric and satisfies

$$\lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2, \quad \lambda \geq 1 \text{ for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n. \tag{1.2}$$

Here and throughout this paper, we use superscript prime to denote $(n-1)$-dimensional variables and domains, such as x' and B'. Moreover, in the following, we simplify the notations $B_R(0)$ and $B'_R(0')$ as B_R and B'_R, respectively, where $R>0$. The prototype equation is the anisotropic weighted p-Laplacian, that is, the equation in the case when $A=I$ in Equation (1.1). Remark that the origin can be called the degenerate or singular point of the weight. For example, if $\theta_1 > 0$, $\theta_2 > 0$, then the weight $w = |x'|^{\theta_1}|x|^{\theta_2} \to 0$ as $|x| \to 0$, while for $\theta_1 < 0$, $\theta_2 < 0$, it blows up as $|x|$ tends to zero. For the former, the origin is called the degenerate point of the weight, while it is called the singular point for the latter.

For the weighted elliptic problem (1.1), Fabes et al. [22] established the local Hölder regularity of weak solutions under the case of $\theta_1 = 0$, $\theta_2 > -n$ and $p=2$. However, the value of Hölder index α obtained in [22] is not explicit. Recently, Dong et al. [21] utilized spherical harmonic expansion to find the exact value of index α for the solution near the degenerate point of the weight. To be precise, for problem (1.1) with Ω replaced by B_R, $R>0$, let $n \geq 2$, $\theta_1 = 0$, $\theta_2 = p = 2$ and $A = \kappa(x)I$, where κ satisfies that $\lambda^{-1} \leq \kappa \leq \lambda$ in B_R and $\int_{\mathbb{S}^{n-1}} \kappa x_i = 0$, $i = 1, 2, \ldots, n$. Based on these assumed conditions, they derived

$$u(x) = u(0) + O(1)|x|^{\alpha}, \quad \alpha = \frac{-n + \sqrt{n^2 + 4\tilde{\lambda}_1}}{2} \quad \text{in } B_{R/2},$$

where $O(1)$ represents some quantity such that $|O(1)| \leq C = C(n, \lambda, \overline{M})$, $\tilde{\lambda}_1 \leq n-1$ is the first non-zero eigenvalue of the following eigenvalue problem:

$$-\operatorname{div}_{\mathbb{S}^{n-1}}(\kappa(\xi)\nabla_{\mathbb{S}^{n-1}}u(\xi)) = \tilde{\lambda}\kappa(\xi)u(\xi), \qquad \xi \in \mathbb{S}^{n-1}.$$

In particular, $\tilde{\lambda}_1 = n-1$ if $A = I$. See Lemmas 2.2 and 5.1 in [21] for more details. By finding the explicit exponent α, they succeeded in solving the optimal gradient blow-up rate for solution to the insulated conductivity problem in dimensions greater than two, which has been previously regarded as a challenging problem. By their investigations in [20, 21], we realize that the Hölder regularity for solutions to the weighted elliptic problem (1.1) is in close touch with the insulated conductivity problem arising from composite materials. Then the study on the regularity for weighted elliptic problem (1.1) is a topic of theoretical interest and also of great relevance to applications for the insulated composites. It is worth emphasizing that when $p > 2$, the exact value of index α still remains open. In addition, with regard to the Hölder regularity for nonlinear degenerate elliptic equations without weights, we refer to [32, 36, 38] and the references therein.

Before stating the definition of weak solution to problem (1.1), we first introduce some notations. Throughout this paper, we will use $L^p(\Omega, w)$ and $W^{1,p}(\Omega, w)$ to represent weighted L^p space and weighted Sobolev space with their norms, respectively, written as

$$\begin{cases} \|u\|_{L^p(\Omega,w)} = \left(\int_\Omega |u|^p w \,\mathrm{d}x\right)^{\frac{1}{p}}, \\ \|u\|_{W^{1,p}(\Omega,w)} = \left(\int_\Omega |u|^p w \,\mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_\Omega |\nabla u|^p w dx\right)^{\frac{1}{p}}. \end{cases}$$

We say that $u \in W^{1,p}(\Omega, w)$ is a weak solution of problem (1.1) if

$$\int_\Omega Aw|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \,\mathrm{d}x = 0, \quad \forall \varphi \in W_0^{1,p}(\Omega, w).$$

For later use, we introduce the following indexing sets:

$$\begin{cases} \mathcal{A} = \{(a,b) : a > -(n-1),\, b \geq 0\}, \\ \mathcal{B} = \{(a,b) : a > -(n-1),\, b < 0,\, a+b > -n\}, \\ \mathcal{C}_q = \{(a,b) : a < (n-1)(q-1),\, b \leq 0\}, \quad q > 1, \\ \mathcal{D}_q = \{(a,b) : a < (n-1)(q-1),\, b > 0,\, a+b < n(q-1)\}, \quad q > 1, \\ \mathcal{F} = \{(a,b) : a+b > -(n-1)\}. \end{cases}$$

The local behaviour of solution to problem (1.1) near the degenerate or singular point of the anisotropic weight is captured as follows.

Theorem 1.1. *For $n \geq 2$, $(\theta_1, \theta_2) \in (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{C}_q \cup \mathcal{D}_q) \cap \mathcal{F}$, $1 < q < p < n + \theta_1 + \theta_2$, let u be a weak solution of problem (1.1) with $\Omega = B_1$. Then there exists a constant $0 < \alpha < 1$ depending only on $n, p, q, \theta_1, \theta_2, \lambda$, such that*

$$u(x) = u(0) + O(1)|x|^\alpha \quad \text{for all } x \in B_{1/2}, \tag{1.3}$$

where O(1) denotes some quantity satisfying that $|O(1)| \leq C = C(n, p, q, \theta_1, \theta_2, \lambda, \overline{M})$.

Remark 1.2. If the considered domain B_1 is replaced with B_{R_0} for any given $R_0 > 0$ in Theorems 1.1 and 1.4, then by applying their proofs with minor modification, we

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obtain that Equations (1.3) and (1.4) also hold with $B_{1/2}$ replaced by $B_{R_0/2}$. In this case, the constant C will depend on R_0, but the index α depends not on it.

Remark 1.3. The result in Theorem 1.1 can be extended to general degenerate elliptic equations as follows:

$$\begin{cases} \operatorname{div}(\mathcal{G}(x,\nabla u)) = 0, & \text{in } B_{R_0}, \\ 0 \le u \le \overline{M} < \infty, & \text{in } B_{R_0}, \end{cases}$$

where $R_0 > 0$, $\mathcal{G} : B_{R_0} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function such that for a.e. $x \in B_{R_0}$ and any $\xi \in \mathbb{R}^n$, there holds

$$\lambda^{-1} w(x)|\xi|^p \le \mathcal{G}(x,\xi)\cdot\xi, \quad |\mathcal{G}(x,\xi)| \le \lambda w(x)|\xi|^{p-1}, \quad w(x) = |x'|^{\theta_1}|x|^{\theta_2}.$$

Here $\lambda \ge 1$, $(\theta_1,\theta_2) \in (\mathcal{A}\cup\mathcal{B})\cap(\mathcal{C}_q\cup\mathcal{D}_q)\cap\mathcal{F}$, $1<q<p<n+\theta_1+\theta_2$. In fact, it only needs to slightly modify the proof of Lemma 3.1 below for the purpose of achieving this generalization.

When $\theta_1 = 0$, the above weight becomes a single power-type weight. In this case, we establish the Hölder estimates as follows.

Theorem 1.4. *For $n \ge 2$, $\theta_1 = 0$, $\theta_2 > -(n-1)$, $1 < p < n+\theta_2$, let u be a bounded weak solution of problem (1.1) with $\Omega = B_1$. Then there exist a small constant $0 < \alpha = \alpha(n,p,\theta_2,\lambda) < 1$ and a large constant $0 < C = C(n,p,\theta_2,\lambda,\overline{M})$ such that*

$$|u(x) - u(y)| \le C|x-y|^{\alpha} \quad \textit{for all } x, y \in B_{1/2}. \tag{1.4}$$

Observe that when $\theta_1 = 0$, Equation (1.1) will become degenerate elliptic equation in any domain away from the origin, then we can directly establish its Hölder regularity in these regions by using the interior Hölder estimates for degenerate elliptic equation. This, in combination with Remark 1.2 and Theorem 1.4, gives the following corollary.

Corollary 1.5. *For $n \ge 2$, $\theta_1 = 0$, $\theta_2 > -(n-1)$, $1 < p < n+\theta_2$, let u be a weak solution of problem (1.1). Then u is locally Hölder continuous in Ω, that is, for any compact subset $K \subset \Omega$, there exists two constants $0 < \alpha = \alpha(n,p,\theta_2,\lambda) < 1$ and $C = C(\operatorname{dist}(K,\partial\Omega),\operatorname{dist}(0,\partial\Omega),n,p,\theta_2,\lambda,\overline{M}) > 0$ such that Equation (1.4) holds with $B_{1/2}$ replaced by K.*

1.2. The nonlinear parabolic equations with anisotropic weights

Let $0 \in \Omega \subset \mathbb{R}^n$, $n \ge 2$, be defined as above. The second problem of interest is concerned with studying the local regularity of solution to the weighted nonlinear parabolic equation as follows:

$$\begin{cases} w_1 \partial_t u^p - \operatorname{div}(A w_2 \nabla u) = 0, & \text{in } \Omega_T, \\ 0 < \overline{m} \le u \le \overline{M} < \infty, & \text{in } \Omega_T, \end{cases} \tag{1.5}$$

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where $\Omega_T = \Omega \times (-T, 0]$, $T > 0$, $w_1 = |x'|^{\theta_1}|x|^{\theta_2}$, $w_2 = |x'|^{\theta_3}|x|^{\theta_4}$, $p \geq 1$, the ranges of θ_i, $i = 1, 2, 3, 4$ are prescribed in the following theorems, $\overline{m}$ and $\overline{M}$ are two given positive constants, the symmetric matrix $A = (a_{ij}(x))_{n\times n}$ satisfies the uniformly elliptic condition in Equation (1.2). When $\theta_i = 0$, $i = 1, 2, 3, 4$, and $A = I$, Equation (1.5) becomes fast diffusion equation, whose relevant mathematical problem is modeled by

$$\begin{cases} \partial_t u^p - \Delta u = 0, & \text{in } \Omega \times (0, \infty), \\ u = 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) \geq 0. \end{cases} \tag{1.6}$$

In physics, Equation (1.6) can be used to describe fast diffusion phenomena occurring in gas kinetics, plasmas and thin liquid film dynamics. For more related applications and physical explanations, see [15, 39] and the references therein.

For problem (1.6), it is well known that when $u_0(x) \not\equiv 0$, there exists a finite extinction time $T^* > 0$ such that $u(\cdot, t) \equiv 0$ in Ω if $t \in [T^*, \infty)$ and $u(\cdot, t) > 0$ in Ω if $t \in (0, T^*)$. This, together with the continuity of u (see Chen-DiBenedetto [12]), indicates that for any $U \subset\subset \Omega \times (0, T^*)$, there exist two positive constants $\overline{m}$ and $\overline{M}$ such that $0 < \overline{m} \leq u \leq \overline{M} < \infty$ for $(x, t) \in U$. This fact motivates our investigation on the local regularity of weak solution for the corresponding weighted problem (1.5). In particular, it can be called the weighted fast diffusion equation when $A = I$ in Equation (1.5). For the fast diffusion problem (1.6), the regularity of solution and its asymptotic behaviour near extinction time have been extensively studied, for example, see [7, 12, 15, 17–19, 27–29, 35] for the regularity and [1, 4–6, 23] for the extinction behaviour, respectively. In particular, Jin and Xiong recently established a priori Hölder estimates for the solution to a weighted nonlinear parabolic equation in Theorem 3.1 of [28], which is critical to the establishment of optimal global regularity for fast diffusion equation with any $1 < p < \infty$. Their results especially answer the regularity problem proposed by Berryman and Holland [4]. It is worth pointing out that the degeneracy of weight in [28] is located at the boundary. By contrast, the degeneracy or singularity of the weights considered in this paper lies in the interior. This will lead to some distinct differences in terms of the establishments of Hölder estimates under these two cases. Moreover, since the weights considered in this paper take more sophisticated forms comprising two power-type weights of different dimensions, it greatly increases the difficulties of analyses and calculations. With regard to the regularity for weighted parabolic problem in the case when $p = 1$ in Equation (1.5), we refer to [13, 25, 37] and the references therein.

The weighted L^p space and weighted Sobolev spaces with respect to space variable have been defined above. Similarly, for a weight w, let $W^{1,p}(\Omega_T, w)$ represent the corresponding weighted Sobolev spaces in (x, t) with its norm given by

$$\|u\|_{W^{1,p}(\Omega_T,w)} = \left(\int_{\Omega_T} |u|^p w \,\mathrm{d}x \,\mathrm{d}t\right)^{\frac{1}{p}} + \left(\int_{\Omega_T} \left(|\partial_t u|^p + |\nabla u|^p\right) w \,\mathrm{d}x \,\mathrm{d}t\right)^{\frac{1}{p}}.$$

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We say that $u \in W^{1,2}(\Omega_T, w_2)$ is a weak solution of problem (1.5) if

$$\int_{t_1}^{t_2} \int_{\Omega} (w_1 \partial_t u^p \varphi + A w_2 \nabla u \nabla \varphi) \, dx \, dt = 0$$

for any $-T \leq t_1 < t_2 \leq 0$ and $\varphi \in C^1(\Omega_T)$, which vanishes on $\partial\Omega \times (-T, 0)$.

Introduce the following index conditions:

(S1) let $n \geq 4$ and $1 + 2/(n-1) < q < 2$, if $(\theta_1, \theta_2) \in (\mathcal{A} \cup \mathcal{B}) \cap \mathcal{C}_q$;
(S2) let $n \geq 3$ and $1 + 2/n < q < 2$, if $(\theta_1, \theta_2) \in \mathcal{A} \cap \mathcal{D}_q$.

For the local behaviour of solution to problem (1.5), we have

Theorem 1.6. *Suppose that $p \geq 1$, (θ_1, θ_2) satisfies condition* (**S**1) *or* (**S**2)*, $(\theta_3, \theta_4) \in \mathcal{A} \cup \mathcal{B}$, $\theta_1 + \theta_2 \geq \theta_3 + \theta_4 = 2$, $\theta_1/\theta_3 = \theta_2/\theta_4$, $\theta_3, \theta_4 \neq 0$. Let u be a weak solution of problem (1.5) with $\Omega \times (-T, 0] = B_1 \times (-1, 0]$. Then there exists a small constant $0 < \alpha = \alpha(n, p, q, \theta_1, \theta_2, \theta_3, \lambda, \overline{m}, \overline{M}) < 1$ such that for any $t_0 \in (-1/4, 0)$,*

$$u(x,t) = u(0,t_0) + O(1)\big(|x| + \sqrt[\theta_1+\theta_2]{|t - t_0|}\big)^{\alpha}, \quad \forall (x,t) \in B_{1/2} \times (-1/4, t_0], \tag{1.7}$$

where O(1) satisfies that $|O(1)| \leq C = C(n, p, q, \theta_1, \theta_2, \theta_3, \lambda, \overline{m}, \overline{M})$.

Remark 1.7. We provide here explanations for the index conditions (**S**1) and (**S**2). Observe that if $(\theta_1, \theta_2) \in C_q$, $1 < q < 2$ and $\theta_1 + \theta_2 \geq 2$, then we have $(n-1)(q-1) > 2$, which requires that $n \geq 4$ and $q > 2/(n-1) + 1$. Similarly, if $(\theta_1, \theta_2) \in D_q$, $1 < q < 2$ and $\theta_1 + \theta_2 \geq 2$, it requires that $n \geq 3$ and $q > 2/n + 1$.

Remark 1.8. For any fixed $R_0 > 0$, let $B_{R_0} \times (-R_0^{\theta_1+\theta_2}, 0]$ substitute for $B_1 \times (-1, 0]$ in Theorems 1.6 and 1.9. Then applying their proofs with a slight modification, we derive that Equations (1.7)–(1.8) still hold with $t_0 \in (-1/4, 0)$, $B_{1/2} \times (-1/4, t_0]$ and $B_{1/2} \times (-1/4, 0)$ replaced by $t_0 \in (-R_0^{\theta_1+\theta_2}/4, 0)$, $B_{R_0/2} \times (-R_0^{\theta_1+\theta_2}/4, t_0]$ and $B_{R_0/2} \times (-R_0^{\theta_2}/4, 0)$, respectively. A difference lies in that the constant C will depend on R_0, but not on α.

In the case of $\theta_1 = \theta_3 = 0$, w_1 and w_2 become single power-type weight. Then we have

Theorem 1.9. *For $p \geq 1$, $n \geq 2$, $\theta_1 = \theta_3 = 0$, $\theta_2 \geq \theta_4 = 2$, let u be a weak solution of problem (1.5) with $\Omega \times (-T, 0] = B_1 \times (-1, 0]$. Then there exist two constants $0 < \alpha < 1$ and $C > 0$, both depending only on $n, p, \theta_2, \lambda, \overline{m}, \overline{M}$, such that*

$$|u(x,t) - u(y,s)| \leq C\big(|x - y| + \sqrt[\theta_2]{|t - s|}\big)^{\alpha} \tag{1.8}$$

for any $(x,t), (y,s) \in B_{1/2} \times (-1/4, 0)$.

When $\theta_1 = \theta_3 = 0$ and $\theta_2 \geq \theta_4 = 2$, Equation (1.5) will be uniformly parabolic in any domain away from the origin. Then we can directly use the interior Hölder estimates

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for uniformly parabolic equation to obtain its Hölder regularity in these regions. This, together with Remark 1.8 and Theorem 1.9, leads to the following corollary.

Corollary 1.10. *For $p \geq 1$, $n \geq 2$, $\theta_1 = \theta_3 = 0$, $\theta_2 \geq \theta_4 = 2$, let u be a weak solution of problem (1.5). Then u is locally Hölder continuous in $\Omega \times (-T, 0)$, that is, for any compact subset $K \subset \Omega \times (-T, 0)$, there exist a small constant $0 < \alpha = \alpha(n, p, \theta_2, \lambda, \overline{m}, \overline{M}) < 1$ and a large constant $C = C(\mathrm{dist}(K, \partial(\Omega \times (-T, 0))), \mathrm{dist}(0, \partial\Omega), n, p, \theta_2, \lambda, \overline{m}, \overline{M}) > 0$ such that Equation (1.8) holds with K substituting for $B_{1/2} \times (-1/4, 0)$.*

The rest of this paper is organized as follows. In § 2, we establish the anisotropic weighted Poincaré type inequality and its corresponding isoperimetric inequality. Then we make use of the De Giorgi truncation method [16] to study the local regularity for solutions to the nonlinear elliptic and parabolic equations with anisotropic weights in § 3 and § 4, respectively.

2. Anisotropic weighted Poincaré inequality and its application to the isoperimetric inequality

As pointed out in the introduction, this section is devoted to establishing the anisotropic weighted Poincaré-type inequality. It will be achieved by proving that $w = |x'|^{\theta_1}|x|^{\theta_2}$ is an A_q-weight under the condition $(\theta_1, \theta_2) \in (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{C}_q \cup \mathcal{D}_q)$, see Theorem 2.6 and Corollary 2.8 below. As a consequence, we derive the isoperimetric inequality of De Giorgi type, which is critical to application for the De Giorgi truncation method in [16].

Denote by ω_n the volume of unit ball in $\mathbb{R}^n$. In this section, we employ the notation $a \sim b$ to represent that there exists a constant $C = C(n, \theta_1, \theta_2) > 0$ such that $\frac{1}{C}b \leq a \leq Cb$. To begin with, we have

Lemma 2.1. *$d\mu := wdx = |x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x$ is a Radon measure if $(\theta_1, \theta_2) \in \mathcal{A} \cup \mathcal{B}$. Moreover, $\mu(B_R) \sim R^{n+\theta_1+\theta_2}$ for any $R > 0$.*

Proof. It suffices to verify that the weight w is a locally integrable function in $\mathbb{R}^n$ under these above cases. Specifically, it only needs to prove that for any $r > 0$, there holds $\mu(B_r) < \infty$. Recall the following elemental inequalities: for $a, b \geq 0$,

$$\begin{cases} \frac{a^q+b^q}{2} \leq (a+b)^q \leq a^q + b^q, & 0 \leq q \leq 1, \\ a^q + b^q \leq (a+b)^q \leq 2^{q-1}(a^q + b^q), & q > 1. \end{cases} \tag{2.1}$$

Step 1. Consider the case when $\theta_2 \geq 0$. Then we have from Equation (2.1) that

$$\begin{aligned} \mu(B_r) =& 2\int_{B_r \cap \{x_n > 0\}} |x'|^{\theta_1} |x|^{\theta_2}\,\mathrm{d}x \sim \int_{B'_r} |x'|^{\theta_1}\,\mathrm{d}x' \int_0^{\sqrt{r^2-|x'|^2}} \left(|x'|^{\theta_2} + x_n^{\theta_2}\right) \mathrm{d}x_n \\ &\sim \int_0^r \left(s^{n+\theta_1+\theta_2-2}\left(r^2-s^2\right)^{\frac{1}{2}} + s^{n+\theta_1-2}\left(r^2-s^2\right)^{\frac{\theta_2+1}{2}}\right) \mathrm{d}s \\ &\sim r^{n+\theta_1+\theta_2} \int_0^1 \left(s^{n+\theta_1+\theta_2-2}\left(1-s\right)^{\frac{1}{2}} + s^{n+\theta_1-2}\left(1-s\right)^{\frac{\theta_2+1}{2}}\right) \mathrm{d}s. \end{aligned}$$

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Observe that this type of integration is called Beta function. It makes sense if and only if $n+\theta_1+\theta_2-1>0$ and $n+\theta_1-1>0$. Then the conclusion is proved in the case when $(\theta_1,\theta_2)\in\mathcal{A}$.

Step 2. Consider $\theta_2<0$. Then it follows from Equation (2.1) that

$$\begin{aligned}\mu(B_r) =&2\int_{B_r\cap\{x_n>0\}}|x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x\\ &\sim\int_{B'_r}|x'|^{\theta_1}\,\mathrm{d}x'\int_0^{\sqrt{r^2-|x'|^2}}\frac{1}{|x'|^{-\theta_2}+x_n^{-\theta_2}}\,\mathrm{d}x_n.\end{aligned}\tag{2.2}$$

For the last integration term in Equation (2.2), we further split it as follows:

$$\begin{aligned}I_1&=\int_{B'_{\frac{1}{\sqrt{2}}r}}|x'|^{\theta_1}\,\mathrm{d}x'\int_0^{|x'|}\frac{1}{|x'|^{-\theta_2}+x_n^{-\theta_2}}\,\mathrm{d}x_n,\\ I_2&=\int_{B'_{\frac{1}{\sqrt{2}}r}}|x'|^{\theta_1}\,\mathrm{d}x'\int_{|x'|}^{\sqrt{r^2-|x'|^2}}\frac{1}{|x'|^{-\theta_2}+x_n^{-\theta_2}}\,\mathrm{d}x_n,\\ I_3&=\int_{B'_r\setminus B'_{\frac{1}{\sqrt{2}}r}}|x'|^{\theta_1}\,\mathrm{d}x'\int_0^{\sqrt{r^2-|x'|^2}}\frac{1}{|x'|^{-\theta_2}+x_n^{-\theta_2}}\,\mathrm{d}x_n.\end{aligned}$$

Observe that $|x'|^{-\theta_2}\le|x'|^{-\theta_2}+x_n^{-\theta_2}\le2|x'|^{-\theta_2}$ if $0\le x_n\le|x'|$. Then for the first term I_1, we have

$$\begin{aligned}I_1\sim&\int_{B'_{\frac{1}{\sqrt{2}}r}}|x'|^{\theta_1}\,\mathrm{d}x'\int_0^{|x'|}|x'|^{\theta_2}\,\mathrm{d}x_n\sim\int_0^{\frac{r}{\sqrt{2}}}s^{n+\theta_1+\theta_2-1}\,\mathrm{d}s\\ &\sim r^{n+\theta_1+\theta_2}\int_0^1 s^{n+\theta_1+\theta_2-1}\,\mathrm{d}s.\end{aligned}$$

This integration makes sense iff $n+\theta_1+\theta_2>0$.

With regard to the second term I_2, we divide it into two cases to discuss as follow.

Case 1. If $\theta_2=-1$, then

$$\begin{aligned}I_2&\sim\int_{B'_{\frac{1}{\sqrt{2}}r}}|x'|^{\theta_1}\,\mathrm{d}x'\int_{|x'|}^{\sqrt{r^2-|x'|^2}}x_n^{-1}\,\mathrm{d}x_n\sim\int_0^{\frac{r}{\sqrt{2}}}s^{n+\theta_1-2}\ln\frac{\sqrt{r^2-s^2}}{s}\,\mathrm{d}s\\ &\sim r^{n+\theta_1-1}\int_0^{\frac{1}{\sqrt{2}}}s^{n+\theta_1-2}\ln\frac{\sqrt{1-s^2}}{s}\,\mathrm{d}s\sim-r^{n+\theta_1-1}\int_0^{\frac{1}{\sqrt{2}}}s^{n+\theta_1-2}\ln s\,\mathrm{d}s\end{aligned}$$

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$$\sim -r^{n+\theta_1+\theta_2}\left(s^{n+\theta_1+\theta_2}\ln s\Big|_0^{\frac{1}{\sqrt{2}}} - \int_0^{\frac{1}{\sqrt{2}}} s^{n+\theta_1+\theta_2-1}\,\mathrm{d}s\right).$$

It makes sense iff $n+\theta_1+\theta_2>0$.

Case 2. If $\theta_2\neq -1$, then

$$\begin{aligned} I_2 &\sim \int_{B'_{\frac{1}{\sqrt{2}}r}} |x'|^{\theta_1}\,\mathrm{d}x' \int_{|x'|}^{\sqrt{r^2-|x'|^2}} x_n^{\theta_2}\,\mathrm{d}x_n \\ &\sim \int_0^{\frac{1}{\sqrt{2}}r} s^{n+\theta_1-2}\left(\left(r^2-s^2\right)^{\frac{\theta_2+1}{2}} - s^{\theta_2+1}\right)\mathrm{d}s \\ &\sim r^{n+\theta_1+\theta_2}\int_0^{\frac{1}{\sqrt{2}}} s^{n+\theta_1-2}\left(\left(1-s^2\right)^{\frac{\theta_2+1}{2}} - s^{\theta_2+1}\right)\mathrm{d}s. \end{aligned}$$

Note that $\min\{1,2^{-\frac{\theta_2+1}{2}}\}\le |(1-s^2)^{\frac{\theta_2+1}{2}}|\le \max\{1,2^{-\frac{\theta_2+1}{2}}\}$ in $[0,\frac{1}{\sqrt{2}}]$. Then it makes sense iff $\theta_1>-(n-1)$ and $n+\theta_1+\theta_2>0$.

The last term I_3 remains to be analyzed. Note that $|x'|\ge\sqrt{r^2-|x'|^2}\ge x_n$ if $\frac{1}{\sqrt{2}}r\le |x'|\le r$ and $0\le x_n\le\sqrt{r^2-|x'|^2}$. Then we deduce

$$\begin{aligned} I_3 &\sim \int_{B'_r\setminus B'_{\frac{1}{\sqrt{2}}r}} |x'|^{\theta_1}\,\mathrm{d}x' \int_0^{\sqrt{r^2-|x'|^2}} |x'|^{\theta_2}\,\mathrm{d}x_n \\ &\sim \int_{\frac{1}{\sqrt{2}}r}^{r} s^{n+\theta_1+\theta_2-2}\sqrt{r^2-s^2}\,\mathrm{d}s \sim r^{n+\theta_1+\theta_2}\int_{\frac{1}{\sqrt{2}}}^{1} s^{n+\theta_1+\theta_2-2}\sqrt{1-s^2}\,\mathrm{d}s \\ &\sim r^{n+\theta_1+\theta_2}, \end{aligned}$$

where we used the fact that the integrand $s^{n+\theta_1+\theta_2}\sqrt{1-s^2}$ has no singular point in $[\frac{1}{\sqrt{2}},1]$. Consequently, combining these above facts, we obtain that if $(\theta_1,\theta_2)\in\mathcal{B}$, then $\mathrm{d}\mu$ is a Radon measure. The proof is complete. □

Definition 2.2. *A Radon measure* $\mathrm{d}\mu$ *is called doubling if there exists some constant* $0<C<\infty$ *such that* $\mu(B_{2R}(\bar{x}))\le C\mu(B_R(\bar{x}))$ *for any* $\bar{x}\in\mathbb{R}^n$ *and* $R>0$.

Theorem 2.3. *The Radon measure* $\mathrm{d}\mu=w\mathrm{d}x$ *is doubling if* $(\theta_1,\theta_2)\in\mathcal{A}\cup\mathcal{B}$.

Remark 2.4. $w=|x'|^{\theta_1}|x|^{\theta_2}$ degenerates to be an isotropic weight if $\theta_1=0$. In this case, it is doubling if $\theta_2>-n$. Its proof is simple and direct, see pages 505–506 in [24] for more details. By contrast, it will involve complex analyses, computations and discussions if $\theta_1\neq 0$.

Proof. For any $\bar{x}\in\mathbb{R}^n$ and $R>0$, we divide all balls $B_R(\bar{x})$ into two types as follows: the first type satisfies $|\bar{x}|\ge 3R$ and the second type satisfies $|\bar{x}|<3R$.

Step 1. Consider the case when $|\bar{x}| \geq 3R$. Observe that

$$\int_{B_R(\bar{x})} |x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x \geq \begin{cases} (|\bar{x}|-R)^{\theta_2}\int_{B_R(\bar{x})}|x'|^{\theta_1}\,\mathrm{d}x, & \text{if } \theta_2 \geq 0,\\ (|\bar{x}|+R)^{\theta_2}\int_{B_R(\bar{x})}|x'|^{\theta_1}\,\mathrm{d}x, & \text{if } \theta_2 < 0,\end{cases} \tag{2.3}$$

and

$$\int_{B_{2R}(\bar{x})} |x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x \leq \begin{cases} (|\bar{x}|+2R)^{\theta_2}\int_{B_{2R}(\bar{x})}|x'|^{\theta_1}\,\mathrm{d}x, & \text{if } \theta_2 \geq 0,\\ (|\bar{x}|-2R)^{\theta_2}\int_{B_{2R}(\bar{x})}|x'|^{\theta_1}\,\mathrm{d}x, & \text{if } \theta_2 < 0.\end{cases} \tag{2.4}$$

On one hand, we have

$$\int_{B_R(\bar{x})}|x'|^{\theta_1}\,\mathrm{d}x = 2\int_{B'_R(\bar{x}')}|x'|^{\theta_1}\sqrt{R^2-|x'-\bar{x}'|^2}\,\mathrm{d}x' \geq \sqrt{3}R\int_{B'_{R/2}(\bar{x}')}|x'|^{\theta_1}\,\mathrm{d}x'.$$

Observe that $|x'|^{\theta_1}$ increases radially if $\theta_1 \geq 0$, while it decreases radially for $\theta_1 < 0$. Then we obtain that

(i) for $|\bar{x}'| \geq \frac{3}{2}R$, then

$$\int_{B'_{R/2}(\bar{x}')}|x'|^{\theta_1}\,\mathrm{d}x' \geq \omega_{n-1}\left(\frac{R}{2}\right)^{n-1}\begin{cases}(|\bar{x}'|-R/2)^{\theta_1}, & \text{if } \theta_1 \geq 0,\\ (|\bar{x}'|+R/2)^{\theta_1}, & \text{if } \theta_1 < 0;\end{cases}$$

(ii) for $|\bar{x}'| < \frac{3}{2}R$, then

$$\begin{aligned}\int_{B'_{R/2}(\bar{x}')}|x'|^{\theta_1}\,\mathrm{d}x' \geq& \begin{cases}\int_{B'_{R/2}(0')}|x'|^{\theta_1}\,\mathrm{d}x', & \text{if } \theta_1 \geq 0,\\ \int_{B'_{R/2}\left(\frac{3}{2}R\frac{\bar{x}'}{|\bar{x}'|}\right)}|x'|^{\theta_1}\,\mathrm{d}x', & \text{if } \theta_1 < 0\end{cases}\\ \geq& \omega_{n-1}R^{n-1+\theta_1}\begin{cases}\frac{n-1}{2^{n-1+\theta_1}(n-1+\theta_1)}, & \text{if } \theta_1 \geq 0,\\ \frac{1}{2^{n-1}}, & \text{if } \theta_1 < 0.\end{cases}\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_{B_{2R}(\bar{x})}|x'|^{\theta_1}\,\mathrm{d}x =& 2\int_{B'_{2R}(\bar{x}')}|x'|^{\theta_1}\sqrt{4R^2-|x'-\bar{x}'|^2}\,\mathrm{d}x'\\ \leq& 4R\int_{B'_{2R}(\bar{x}')}|x'|^{\theta_1}\,\mathrm{d}x'.\end{aligned} \tag{2.5}$$

By the same argument as above, we deduce that

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(1) for $|\bar{x}'| \geq 3R$, then

$$\int_{B'_{2R}(\bar{x}')} |x'|^{\theta_1}\,\mathrm{d}x' \leq \omega_{n-1}(2R)^{n-1} \begin{cases} (|\bar{x}'|+2R)^{\theta_1}, & \text{if } \theta_1 \geq 0, \\ (|\bar{x}'|-2R)^{\theta_1}, & \text{if } \theta_1 < 0; \end{cases} \tag{2.6}$$

(2) for $|\bar{x}'| < 3R$, then

$$\int_{B'_{2R}(\bar{x}')} |x'|^{\theta_1}\,\mathrm{d}x' \leq \int_{B'_{5R}(0')} |x'|^{\theta_1}\,\mathrm{d}x' = \frac{(n-1)\omega_{n-1}}{n-1+\theta_1}(5R)^{n-1+\theta_1}. \tag{2.7}$$

Note that if $\theta_1 \geq 0$, then

$$\begin{cases} |\bar{x}'|+2R \leq 2(|\bar{x}'|-R/2), & \text{for } |\bar{x}'| \geq 3R, \\ R < |\bar{x}'|-R/2 < 5R/2, & \text{for } 3R/2 < |\bar{x}'| < 3R, \end{cases}$$

while, if $\theta_1 < 0$,

$$\begin{cases} |\bar{x}'|-2R \geq \frac{2}{7}(|\bar{x}'|+R/2), & \text{for } |\bar{x}'| \geq 3R, \\ 2R < |\bar{x}'|+R/2 < 7R/2, & \text{for } 3R/2 < |\bar{x}'| < 3R. \end{cases}$$

Then combining these above facts, we obtain

$$\int_{B_{2R}(\bar{x})} |x'|^{\theta_1}\,\mathrm{d}x \leq C(n,\theta_1,\theta_2) \int_{B_R(\bar{x})} |x'|^{\theta_1}\,\mathrm{d}x. \tag{2.8}$$

Since $|\bar{x}| \geq 3R$, then $|\bar{x}|+2R \leq 4(|\bar{x}-R|)$ and $|\bar{x}|+R \leq 4(|\bar{x}|-2R)$. This, in combination with Equations (2.3)–(2.8), reads that for $|\bar{x}| \geq 3R$,

$$\int_{B_{2R}(\bar{x})} |x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x \leq C(n,\theta_1,\theta_2) \int_{B_R(\bar{x})} |x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x. \tag{2.9}$$

Step 2. Let $|\bar{x}| < 3R$. Then we have

$$\int_{B_{2R}(\bar{x})} |x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x \leq \int_{B_{5R}(0)} |x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x \leq C(n,\theta_1,\theta_2)R^{n+\theta_1+\theta_2}.$$

First, if $\theta_1 < 0$, then

$$\begin{aligned} \int_{B_R(\bar{x})} |x'|^{\theta_1}|x|^{\theta_2} &\geq \int_{B_R(\bar{x})} |x|^{\theta_1+\theta_2}\,\mathrm{d}x \\ &\geq \begin{cases} \int_{B_R(0)} |x|^{\theta_1+\theta_2}\,\mathrm{d}x, & \text{if } \theta_2 \geq -\theta_1, \\ \int_{B_R\left(3R\frac{\bar{x}}{|\bar{x}|}\right)} |x|^{\theta_1+\theta_2}\,\mathrm{d}x, & \text{if } \theta_2 < -\theta_1 \end{cases} \end{aligned}$$

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$$\geq R^{n+\theta_1+\theta_2}\begin{cases}\frac{n\omega_n}{n+\theta_1+\theta_2}, & \text{if } \theta_2 \geq -\theta_1,\\ 2^{\theta_1+\theta_2}\omega_n, & \text{if } \theta_2 < -\theta_1,\end{cases}$$

where we utilized the fact that $|x|^{\theta_1+\theta_2}$ is radially increasing if $\theta_1+\theta_2 \geq 0$ and radially decreasing if $\theta_1+\theta_2 < 0$.

Second, if $\theta_1 \geq 0$, we discuss as follows:

(i) for $\theta_2 \geq 0$, similarly as before, we have

$$\begin{aligned}\int_{B_R(\bar{x})} |x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x &\geq \int_{B_R(\bar{x})} |x'|^{\theta_1+\theta_2}\,\mathrm{d}x \geq \sqrt{3}R\int_{B'_R(\bar{x}')} |x'|^{\theta_1+\theta_2}\,\mathrm{d}x'\\ &\geq \int_{B'_R(0')} |x'|^{\theta_1+\theta_2}\,\mathrm{d}x' = \frac{(n-1)\omega_{n-1}}{n+\theta_1+\theta_2-1}R^{n+\theta_1+\theta_2};\end{aligned}$$

(ii) for $\theta_2 < 0$, then

$$\begin{aligned}\int_{B_R(\bar{x})} |x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x &= \int_{B_R(\bar{x})} |x'|^{\theta_1+\theta_2}\left(\frac{|x'|}{|x|}\right)^{-\theta_2}\mathrm{d}x\\ &\geq \frac{8^{\theta_2}}{2^{\theta_1+\theta_2}}R^{\theta_1+\theta_2}\int_{B_R(\bar{x})\cap\{|x'|>R/2\}}\mathrm{d}x\\ &\geq \frac{8^{\theta_2}}{2^{\theta_1+\theta_2}}R^{\theta_1+\theta_2}\int_{B_R(0)\cap\{|x'|>R/2\}}\mathrm{d}x\\ &\geq \frac{8^{\theta_2}(\omega_n - 2^{2-n}\omega_{n-1})}{2^{\theta_1+\theta_2}}R^{n+\theta_1+\theta_2}.\end{aligned}$$

Then combining these aforementioned facts, we obtain that Equation (2.9) also holds if $|\bar{x}| < 3R$. The proof is complete.

□

Definition 2.5. *Let $1 < q < \infty$. We say that w is an A_q-weight, if there is a positive constant $C = C(n,q,w)$ such that*

$$⨍_B w\,\mathrm{d}x\left(⨍_B w^{-\frac{1}{q-1}}\,\mathrm{d}x\right)^{q-1} \leq C(n,q,w), \quad \text{with } ⨍_B = \frac{1}{|B|}\int_B$$

for any ball B in $\mathbb{R}^n$.

Theorem 2.6. *Let $1 < q < \infty$. If $(\theta_1,\theta_2) \in (\mathcal{A}\cup\mathcal{B})\cap(\mathcal{C}_q\cup\mathcal{D}_q)$, then $w = |x'|^{\theta_1}|x|^{\theta_2}$ is an A_q-weight.*

Remark 2.7. From Theorems 2.3 and 2.6, we see that the Radon measure $\mathrm{d}\mu = w\,\mathrm{d}x = |x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x$ is doubling on a larger range $(\theta_1,\theta_2) \in \mathcal{A}\cup\mathcal{B}$. This implies that

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when $(\theta_1,\theta_2)\in(\mathcal{A}\cup\mathcal{B})\setminus(\mathcal{C}_q\cup\mathcal{D}_q)$, the weight $w=|x'|^{\theta_1}|x|^{\theta_2}$ provides an example of a doubling measure but is not in A_q.

Proof. For $1<q<\infty$, according to the definition of A_q-weight, it needs to verify the following inequality:

$$\fint_B w\,\mathrm{d}x\left(\fint_B w^{-\frac{1}{q-1}}\,\mathrm{d}x\right)^{q-1}\leq C(n,q,\theta_1,\theta_2) \tag{2.10}$$

for any ball $B\subset\mathbb{R}^n$. For any $R>0$ and $\bar{x}\in\mathbb{R}^n$, the ball $B_R(\bar{x})$ must belong to one of the following two types: $|\bar{x}|\geq 3R$ and $|\bar{x}|<3R$. On the one hand, if $|\bar{x}|\geq 3R$, then we have

$$\frac{2}{3}|\bar{x}|\leq|\bar{x}|-R\leq|x|\leq|\bar{x}|+R\leq\frac{4}{3}|\bar{x}|,\quad\text{for } x\in B_R(\bar{x}). \tag{2.11}$$

Applying Equations (2.4)–(2.7) with $B_{2R}(\bar{x})$ and $B'_{2R}(\bar{x}')$ replaced by $B_R(\bar{x})$ and $B'_R(\bar{x}')$, it follows from Equation (2.11) that

$$\begin{aligned}\int_{B_R(\bar{x})}|x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x\leq& C(\theta_2)|\bar{x}|^{\theta_2}\int_{B_R(\bar{x})}|x'|^{\theta_1}\,\mathrm{d}x\leq C(\theta_2)|\bar{x}|^{\theta_2+1}\int_{B'_R(\bar{x}')}|x'|^{\theta_1}\,\mathrm{d}x'\\ \leq& C(n,\theta_1,\theta_2)|\bar{x}|^{n+\theta_1+\theta_2}\end{aligned}$$

and

$$\begin{aligned}\int_{B_R(\bar{x})}|x'|^{-\frac{\theta_1}{q-1}}|x|^{-\frac{\theta_2}{q-1}}\,\mathrm{d}x\leq& C(q,\theta_2)|\bar{x}|^{-\frac{\theta_2}{q-1}}\int_{B_R(\bar{x})}|x'|^{-\frac{\theta_1}{q-1}}\,\mathrm{d}x\\ \leq& C(q,\theta_2)|\bar{x}|^{-\frac{\theta_2}{q-1}+1}\int_{B'_R(\bar{x}')}|x'|^{-\frac{\theta_1}{q-1}}\,\mathrm{d}x'\\ \leq& C(n,q,\theta_1,\theta_2)|\bar{x}|^{n-\frac{\theta_1+\theta_2}{p-1}},\end{aligned}$$

where we require that $-(n-1)<\theta_1<(n-1)(q-1)$ and $\theta_2\in\mathbb{R}$. Combining these two relations, we obtain that Equation (2.10) holds in the case of $|\bar{x}|\geq 3R$.

On the other hand, if $|\bar{x}|<3R$, we have $|x|\leq 4R$ for $x\in B_R(\bar{x})$. Therefore, it follows from Lemma 2.1 that

$$\fint_{B_R(\bar{x})}|x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x\leq 4^n\fint_{B_{4R}(0)}|x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x\leq C(n,\theta_1,\theta_2)R^{\theta_1+\theta_2}$$

and

$$\begin{aligned}\left(\fint_{B_R(\bar{x})}|x'|^{-\frac{\theta_1}{q-1}}|x|^{-\frac{\theta_2}{q-1}}\,\mathrm{d}x\right)^{q-1}\leq& 4^{n(q-1)}\left(\fint_{B_{4R}(0)}|x'|^{-\frac{\theta_1}{q-1}}|x|^{-\frac{\theta_2}{q-1}}\,\mathrm{d}x\right)^{q-1}\\ \leq& C(n,q,\theta_1,\theta_2)R^{-\theta_1-\theta_2},\end{aligned}$$

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where these two relations hold if $(\theta_1, \theta_2), (-\frac{\theta_1}{q-1}, -\frac{\theta_2}{q-1}) \in \mathcal{A} \cup \mathcal{B}$, that is, $(\theta_1, \theta_2) \in (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{C}_q \cup \mathcal{D}_q)$. Therefore, Equation (2.11) holds for any $B \subset \mathbb{R}^n$. The proof is complete. □

Denote $\mathrm{d}\mu := w\,\mathrm{d}x = |x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x$. Combining Theorem 15.21 and Corollary 15.35 in [26] and Theorem 2.6 above, we obtain the following anisotropic weighted Poincaré inequality.

Corollary 2.8. *For $n \geq 2$ and $1 < q < \infty$, let $(\theta_1, \theta_2) \in [(\mathcal{A} \cup \mathcal{B}) \cap (C_q \cup \mathcal{D}_q)] \cup \{\theta_1 = 0,\, \theta_2 > -n\}$. Then for any $B := B_R(\bar{x}) \subset \mathbb{R}^n$, $R > 0$ and $\varphi \in W^{1,q}(B, w)$,*

$$\int_B |\varphi - \varphi_B|^q \,\mathrm{d}\mu \leq C(n, q, \theta_1, \theta_2) R^q \int_B |\nabla \varphi|^q \,\mathrm{d}\mu, \tag{2.12}$$

where $\varphi_B = \frac{1}{\mu(B)} \int_B \varphi \,\mathrm{d}\mu$.

Remark 2.9. It is worth emphasizing that according to Corollary 15.35 in [26], Equation (2.12) holds for any $(\theta_1, \theta_2) \in \{\theta_1 = 0,\, \theta_2 > -n\}$ and $1 < q < \infty$. This conclusion is very strong, which is achieved by combining the theories of A_q-weights and quasiconformal mappings; see Chapter 15 of [26] for further details.

Making use of the anisotropic weighted Poincaré inequality in Corollary 2.8, we can establish the corresponding weighted isoperimetric inequality of De Giorgi type as follows.

Proposition 2.10. *For $n \geq 2$ and $1 < q < \infty$, let $(\theta_1, \theta_2) \in [(\mathcal{A} \cup \mathcal{B}) \cap (C_q \cup \mathcal{D}_q)] \cup \{\theta_1 = 0,\, \theta_2 > -n\}$. Then for any $R > 0$, $l > k$ and $u \in W^{1,q}(B_R, w)$,*

$$\begin{aligned} &(l-k)^q \left(\int_{\{u \geq l\} \cap B_R} \mathrm{d}\mu \right)^q \int_{\{u \leq k\} \cap B_R} \mathrm{d}\mu \\ &\leq C(n, q, \theta_1, \theta_2) R^{q(n+\theta_1+\theta_2+1)} \int_{\{k < u < l\} \cap B_R} |\nabla u|^q \,\mathrm{d}\mu \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} &(l-k)^q \left(\int_{\{u \leq k\} \cap B_R} \mathrm{d}\mu \right)^q \int_{\{u \geq l\} \cap B_R} \mathrm{d}\mu \\ &\leq C(n, q, \theta_1, \theta_2) R^{q(n+\theta_1+\theta_2+1)} \int_{\{k < u < l\} \cap B_R} |\nabla u|^q \,\mathrm{d}\mu, \end{aligned} \tag{2.14}$$

where $\mathrm{d}\mu = w\,\mathrm{d}x = |x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x$.

Remark 2.11. Since the index $q > 1$, we have to establish two isoperimetric inequalities in Proposition 2.10, which are used to capture the decaying rates of the distribution function in Lemmas 3.4 and 4.5 below. Meanwhile, it also causes more complex calculations in the proofs of Lemmas 3.4 and 4.5.

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Proof. Step 1. Set

$$u_1 = \inf\{u, l\} - \inf\{u, k\}, \quad \bar{u}_1 = \frac{1}{\mu(B_R)} \int_{B_R} u_1 \,\mathrm{d}\mu.$$

First, we have

$$\begin{aligned}\int_{\{u_1=0\}\cap B_R} \bar{u}_1^q \,\mathrm{d}\mu =& \frac{1}{(\mu(B_R))^q} \left(\int_{B_R} u_1 \,\mathrm{d}\mu\right)^q \int_{\{u\le k\}\cap B_R} \mathrm{d}\mu \\ \ge& \frac{C(l-k)^q}{R^{q(n+\theta_1+\theta_2)}} \left(\int_{\{u\ge l\}\cap B_R} \mathrm{d}\mu\right)^q \int_{\{u\le k\}\cap B_R} \mathrm{d}\mu.\end{aligned}$$

Second, it follows from Corollary 2.8 that

$$\begin{aligned}\int_{\{u_1=0\}\cap B_R} |\bar{u}_1|^q \,\mathrm{d}\mu \le \int_{B_R} |u_1 - \bar{u}_1|^q \,\mathrm{d}\mu \le CR^q \int_{B_R} |\nabla u_1|^q \,\mathrm{d}\mu \\ =& CR^q \int_{\{k<u<l\}\cap B_R} |\nabla u|^q \,\mathrm{d}\mu.\end{aligned}$$

The proof of Equation (2.13) is finished.
Step 2. Denote

$$u_2 = \sup\{u, l\} - \sup\{u, k\}, \quad \bar{u}_2 = \frac{\int_{B_R} u_2 \,\mathrm{d}\mu}{|B_R|_\mu}.$$

By the same argument as before, we have

$$\begin{aligned}\int_{\{u_2=0\}\cap B_R} \bar{u}_2^q \,\mathrm{d}\mu =& \frac{1}{(\mu(B_R))^q} \left(\int_{B_R} u_2 \,\mathrm{d}\mu\right)^q \int_{\{u\ge l\}\cap B_R} \mathrm{d}\mu \\ \ge& \frac{C(l-k)^q}{R^{q(n+\theta_1+\theta_2)}} \left(\int_{\{u\le k\}\cap B_R} \mathrm{d}\mu\right)^q \int_{\{u\ge l\}\cap B_R} \mathrm{d}\mu\end{aligned}$$

and

$$\begin{aligned}\int_{\{u_2=0\}\cap B_R} |\bar{u}_2|^q \,\mathrm{d}\mu \le \int_{B_R} |u_2 - \bar{u}_2|^q \,\mathrm{d}\mu \le CR^q \int_{B_R} |\nabla u_2|^q \,\mathrm{d}\mu \\ =& CR^q \int_{\{k<u<l\}\cap B_R} |\nabla u|^q \,\mathrm{d}\mu.\end{aligned}$$

The proof is complete. □

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3. Regularity for solutions to degenerate elliptic equations with anisotropic weights

Throughout this section, denote $\mathrm{d}\mu := w\,\mathrm{d}x = |x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x$. The first step is to establish a Caccioppoli inequality for the truncated solution.

Lemma 3.1. *Let u be the solution of problem (1.1). Then for any non-negative $\eta \in C_0^\infty(B_R(x_0))$ with any $B_R(x_0) \subset B_1$,*

$$\int_{B_R(x_0)} |\nabla(v\eta)|^p w\,\mathrm{d}x \le C(n,p,\lambda)\int_{B_R(x_0)} |\nabla\eta|^p|v|^p w\,\mathrm{d}x,$$

where $v = (u-k)^+$ or $(u-k)^-$ with $k \ge 0$.

Proof. First, pick test function $\varphi = v\eta^p$ if $v = (u-k)^+$. Since

$$0 = \int_{B_R(x_0)} Aw|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi = \int_{B_R(x_0)} Aw|\nabla v|^{p-2}\nabla v\cdot\nabla\varphi,$$

then it follows from Young's inequality that

$$\begin{aligned}
&\lambda\int_{B_R(x_0)} |\nabla v|^p\eta^p w\,\mathrm{d}x \\
&\le \int_{B_R(x_0)} Aw|\nabla v|^p\eta^p = -p\int_{B_R(x_0)} Aw|\nabla v|^{p-2}\nabla v\cdot\nabla\eta v\eta^{p-1} \\
&\le p\lambda\int_{B_R(x_0)} |\eta\nabla v|^{p-1}|v\nabla\eta|w\,\mathrm{d}x \\
&\le \frac{\lambda}{2}\int_{B_R(x_0)} |\nabla v|^p\eta^p w\,\mathrm{d}x + C\int_{B_R(x_0)} |v|^p|\nabla\eta|^p w\,\mathrm{d}x,
\end{aligned} \tag{3.1}$$

which yields that

$$\begin{aligned}
\int_{B_R(x_0)} |\nabla(v\eta)|^p w\,\mathrm{d}x \le& 2^{p-1}\int_{B_R(x_0)} \left(|\eta\nabla v|^p + |v\nabla\eta|^p\right) w\,\mathrm{d}x \\
\le& C\int_{B_R(x_0)} |\nabla\eta|^p|v|^p w\,\mathrm{d}x.
\end{aligned}$$

Second, choose test function $\varphi = -v\eta^2$ if $v = (u-k)^-$. Then we have

$$0 = \int_{B_R(x_0)} Aw|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi = -\int_{B_R(x_0)} Aw|\nabla v|^{p-2}\nabla v\cdot\nabla\varphi.$$

Therefore, in exactly the same way to Equation (3.1), we obtain that Lemma 3.1 holds. □

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We now improve the oscillation of the solution u in a small domain provided that u is small on a large portion of a larger domain.

Lemma 3.2. *Assume that $n \geq 2$, $(\theta_1, \theta_2) \in (\mathcal{A} \cup \mathcal{B}) \cap \mathcal{F}$, $1 < p < n + \theta_1 + \theta_2$. For $R \in (0, 1)$, let $0 \leq m \leq \inf_{B_R} u \leq \sup_{B_R} u \leq M \leq \overline{M}$. Then there exists a small constant $0 < \tau_0 = \tau_0(n, p, \theta_1, \theta_2, \lambda) < 1$ such that for any $\varepsilon > 0$ and $0 < \tau < \tau_0$,*

(a) *if*

$$|\{x \in B_R : u > M - \varepsilon\}|_\mu \leq \tau |B_R|_\mu, \tag{3.2}$$

then

$$u \leq M - \frac{\varepsilon}{2}, \quad \text{for} \quad x \in B_{R/2}; \tag{3.3}$$

(b) *if*

$$|\{x \in B_R : u < m + \varepsilon\}|_\mu \leq \tau |B_R|_\mu,$$

then

$$u \geq m - \frac{\varepsilon}{2}, \quad \text{for} \quad x \in B_{R/2}. \tag{3.4}$$

Remark 3.3. The assumed condition in Equation (3.2) is natural, since

$$|\{x \in B_R : u > M - \varepsilon\}|_\mu \to 0, \quad \text{as } \varepsilon \to 0.$$

This fact also implies that the value of τ can be chosen to satisfy that $\tau \to 0$, as $\varepsilon \to 0$. Then the key to applying Lemma 3.2 lies in making clear the dependency between τ and ε in condition (3.2). The purpose will be achieved by establishing the explicit decaying estimates in terms of the distribution function of u in Lemma 3.4 below.

Proof. Step 1. For $\varepsilon > 0$ and $i = 0, 1, 2, \ldots$, let

$$r_i = \frac{R}{2} + \frac{R}{2^{i+1}}, \quad k_i = M - \varepsilon + \frac{\varepsilon}{2}(1 - 2^{-i}).$$

Take a cutoff function $\eta_i \in C_0^\infty(B_{r_i})$, satisfying that $\eta_i = 1$ in $B_{r_{i+1}}$, $0 \leq \eta_i \leq 1$, $|\nabla \eta_i| \leq C(r_i - r_{i+1})^{-1}$ in B_{r_i}. For $k \in [m, M]$ and $\rho \in (0, R]$, write $v_i = (u - k_i)^+$ and $A(k, \rho) = \{x \in B_\rho : u > k\}$. By Theorem 1.1 in [30], we have the following anisotropic Caffarelli–Kohn–Nirenberg type inequality:

$$\|u\|_{L^{\frac{(n+\theta_1+\theta_2)p}{n+\theta_1+\theta_2-p}}(B_R, w)} \leq C(n, p, \theta_1, \theta_2) \|\nabla u\|_{L^p(B_R, w)}, \quad \forall u \in W_0^{1,p}(B_R, w),$$

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which, together with Lemma 3.1, reads that

$$\left(\int_{B_R}|\eta_i v_i|^{p\chi}w\,\mathrm{d}x\right)^{\frac{1}{\chi}}\leq C\int_{B_R}|v_i|^p|\nabla\eta_i|^p w\,\mathrm{d}x,\quad \chi=\frac{n+\theta_1+\theta_2}{n+\theta_1+\theta_2-p}.$$

Since

$$\int_{B_R}|v_i|^p|\nabla\eta_i|^p w\,\mathrm{d}x\leq\frac{C(M-k_i)^p}{(r_i-r_{i+1})^p}|A(k_i,r_i)|_\mu$$

and

$$\left(\int_{B_R}|\eta_i v_i|^{p\chi}w\,\mathrm{d}x\right)^{\frac{1}{\chi}}\geq(k_{i+1}-k_i)^p|A(k_{i+1},r_{i+1})|_\mu^{\frac{1}{\chi}},$$

it then follows that there exists a positive constant $i_0=i_0(n,p,\theta_1,\theta_2)>0$ such that for $i\geq i_0$,

$$\begin{aligned}\frac{|A(k_{i+1},r_{i+1})|_\mu}{|B_R|_\mu}\leq&(C2^{2p(i+2)})^\chi R^{(\theta_1+\theta_2+n)(\chi-1)-p\chi}\left(\frac{|A(k_i,r_i)|_\mu}{|B_R|_\mu}\right)^\chi\\ =&(C2^{2p(i+2)})^\chi\left(\frac{|A(k_i,r_i)|_\mu}{|B_R|_\mu}\right)^\chi\\ \leq&\prod_{s=0}^{i}(C2^{2p(i+2-s)})^{\chi^{s+1}}\left(\frac{|A(k_0,r_0)|_\mu}{|B_R|_\mu}\right)^{\chi^{i+1}}\\ \leq&(C^*)^{i+1}\left(\frac{|A(k_0,r_0)|_\mu}{|Q_R|_\mu}\right)^{\chi(i+1)}.\end{aligned}$$

Fix $\tau_0=(C^*)^{-\chi}$. Then we deduce that for any $\varepsilon>0$ and $0<\tau<\tau_0$, if Equation (3.2) holds, then

$$\frac{|A(k_{i+1},r_{i+1})|_\mu}{|B_R|_\mu}\leq(C^*\tau^\chi)^{i+1}=\left(\frac{\tau}{\tau_0}\right)^{\chi(i+1)}\to 0,\quad\text{as } i\to\infty.$$

Hence, Equation (3.3) is proved.

Step 2. Similarly as above, set

$$r_i=\frac{R}{2}+\frac{R}{2^{i+1}},\quad \tilde{k}_i=m+\varepsilon-\frac{\varepsilon}{2}(1-2^{-i}),\quad i\geq 0.$$

For $k\in[m,M]$ and $\rho\in(0,R]$, let $\tilde{v}_i=(u-\tilde{k}_i)^-$ and $\tilde{A}(k,\rho)=\{x\in B_\rho:u<k\}$. Then applying the proof of Equation (3.3) with minor modification, we obtain that Equation (3.4) also holds. The proof is finished. □

The decaying estimates for the distribution function of the solution u are established as follows.

Lemma 3.4. *Suppose that* $n \geq 2$, $(\theta_1, \theta_2) \in [(\mathcal{A} \cup \mathcal{B}) \cap (C_q \cup \mathcal{D}_q) \cap \mathcal{F}] \cup \{\theta_1 = 0, \theta_2 > -(n-1)\}$, $1 < q < p < n + \theta_1 + \theta_2$, $0 < \gamma < 1$, $0 < R < \frac{1}{2}$ *and* $0 \leq m \leq \inf_{B_{2R}} u \leq \sup_{B_{2R}} u \leq M$. *Then for any* $\varepsilon > 0$,

(a) if

$$\frac{|\{x \in B_R : u > M - \varepsilon\}|_\mu}{|B_R|_\mu} \leq 1 - \gamma, \tag{3.5}$$

then for any $j \geq 1$,

$$\frac{|\{x \in B_R : u > M - \frac{\varepsilon}{2^j}\}|_\mu}{|B_R|_\mu} \leq \frac{C}{\sqrt[q]{\gamma} j^{\frac{p-q}{pq}}}; \tag{3.6}$$

(b) if

$$\frac{|\{x \in B_R : u < m + \varepsilon\}|_\mu}{|B_R|_\mu} \leq 1 - \gamma, \tag{3.7}$$

then for any $j \geq 1$,

$$\frac{|\{x \in B_R : u < m + \frac{\varepsilon}{2^j}\}|_\mu}{|B_R|_\mu} \leq \frac{C}{\sqrt[q]{\gamma} j^{\frac{p-q}{pq}}},$$

where $C = C(n, p, q, \theta_1, \theta_2, \lambda)$.

Proof. Step 1. For $i \geq 0$, let $k_i = M - \frac{\varepsilon}{2^i}$ and $A(k_i, R) = B_R \cap \{u > k_i\}$. From Equation (2.13), we know that for $q > 1$,

$$\begin{aligned} &(k_{i+1} - k_i)^q |A(k_{i+1}, R)|_\mu^q |B_R \setminus A(k_i, R)|_\mu \\ &\leq C R^{q(n+\theta_1+\theta_2+1)} \int_{A(k_i,R) \setminus A(k_{i+1},R)} |\nabla u|^q w \, \mathrm{d}x. \end{aligned} \tag{3.8}$$

Using Equation (3.5), we have

$$|B_R \setminus A(k_i, R)|_\mu \geq \gamma |B_R|_\mu = C(n, \theta_1, \theta_2) \gamma R^{n+\theta_1+\theta_2}.$$

This, together with Equation (3.8), shows that

$$|A(k_{i+1}, R)|_\mu \leq \frac{C 2^{i+1}}{\varepsilon \sqrt[q]{\gamma}} R^{\frac{(n+\theta_1+\theta_2)(q-1)}{q}+1} \left(\int_{A(k_i,R) \setminus A(k_{i+1},R)} |\nabla u|^q w \, \mathrm{d}x \right)^{\frac{1}{q}}.$$

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Since $1 < q < p < n+\theta_1+\theta_2$, we then have from Hölder's inequality that

$$\begin{aligned}&\int_{A(k_i,R)\setminus A(k_{i+1},R)} |\nabla u|^q w\,\mathrm{d}x\\ &\leq \left(\int_{A(k_i,R)\setminus A(k_{i+1},R)} |\nabla u|^p w\,\mathrm{d}x\right)^{\frac{q}{p}} \left(\int_{A(k_i,R)\setminus A(k_{i+1},R)} w\,\mathrm{d}x\right)^{\frac{p-q}{p}}\\ &\leq \left(\int_{B_R} |\nabla (u-k_i)^+|^p w\,\mathrm{d}x\right)^{\frac{q}{p}} \left(\int_{A(k_i,R)\setminus A(k_{i+1},R)} w\,\mathrm{d}x\right)^{\frac{p-q}{p}}.\end{aligned}$$

Choose a cutoff function $\eta \in C_0^\infty(B_{2R})$ satisfying that

$$\eta = 1 \text{ in } B_R, \quad 0 \leq \eta \leq 1,\ |\nabla\eta| \leq \frac{C(n)}{R} \text{ in } B_{2R}. \tag{3.9}$$

It then follows from Lemma 3.1 that

$$\left(\int_{B_R} |\nabla (u-k_i)^+|^p w\,\mathrm{d}x\right)^{\frac{1}{p}} \leq C\left(\int_{B_{2R}} |(u-k_i)^+|^p |\nabla\eta|^p w\,\mathrm{d}x\right)^{\frac{1}{p}} \leq \frac{C\varepsilon}{2^i} R^{\frac{n+\theta_1+\theta_2-p}{p}}.$$

A combination of these above facts shows that

$$|A(k_{i+1},R)|_\mu \leq \frac{C}{\sqrt[q]{\gamma}} R^{(n+\theta_1+\theta_2)(1-\frac{p-q}{pq})} |A(k_i,R)\setminus A(k_{i+1},R)|_\mu^{\frac{p-q}{pq}}.$$

This leads to that for $j \geq 1$,

$$\begin{aligned} j|A(k_j,R)|_\nu^{\frac{pq}{p-q}} &\leq \sum_{i=0}^{j-1} |A(k_{i+1},R)|_\mu^{\frac{pq}{p-q}} \leq \frac{C}{\gamma^{\frac{p}{p-q}}} R^{(n+\theta_1+\theta_2)\left(\frac{pq}{p-q}-1\right)} |B_R|_\mu\\ &\leq \frac{C}{\gamma^{\frac{p}{p-q}}} |B_R|_\mu^{\frac{pq}{p-q}}.\end{aligned}$$

Then Equation (3.6) is proved.

Step 2. For $i \geq 0$, denote $\tilde{k}_i = m + \frac{\varepsilon}{2^i}$ and $\tilde{A}(k_i,R) = B_R \cap \{u < k_i\}$. In light of Equation (2.14), we see that for $q > 1$,

$$\begin{aligned}&(\tilde{k}_i - \tilde{k}_{i+1})^q |\tilde{A}(\tilde{k}_{i+1},R)|_\mu^{\frac{q}{\mu}} |B_R \setminus \tilde{A}(\tilde{k}_i,R)|_\mu\\ &\leq CR^{q(n+\theta_1+\theta_2+1)} \int_{\tilde{A}(\tilde{k}_i,R)\setminus \tilde{A}(\tilde{k}_{i+1},R)} |\nabla u|^q w\,\mathrm{d}x.\end{aligned}$$

From Equation (3.7), we have

$$|B_R \setminus \tilde{A}(\tilde{k}_i,R)|_\mu \geq \gamma |B_R|_\mu = C(n,\theta_1,\theta_2)\gamma R^{n+\theta_1+\theta_2}.$$

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Hence, we obtain

$$|\tilde{A}(\tilde{k}_{i+1}, R)|_\mu \le \frac{C2^{i+1}}{\varepsilon \sqrt[q]{\gamma}} R^{\frac{(n+\theta_1+\theta_2)(q-1)}{q}+1} \left(\int_{\tilde{A}(\tilde{k}_i, R)\setminus \tilde{A}(\tilde{k}_{i+1}, R)} |\nabla u|^q w \, \mathrm{d}x \right)^{\frac{1}{q}}.$$

Analogously as before, it follows from Hölder's inequality and Lemma 3.1 that for $1 < q < p < n + \theta_1 + \theta_2$

$$\begin{aligned}
&\left(\int_{\tilde{A}(\tilde{k}_i, R)\setminus \tilde{A}(\tilde{k}_{i+1}, R)} |\nabla u|^q w \, \mathrm{d}x \right)^{\frac{1}{q}} \\
&\le \left(\int_{\tilde{A}(\tilde{k}_i, R)\setminus \tilde{A}(\tilde{k}_{i+1}, R)} |\nabla u|^p w \, \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_{\tilde{A}(\tilde{k}_i, R)\setminus \tilde{A}(\tilde{k}_{i+1}, R)} w \, \mathrm{d}x \right)^{\frac{p-q}{pq}} \\
&\le \left(\int_{B_R} |\nabla (u - \tilde{k}_i)^-|^p w \, \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_{\tilde{A}(\tilde{k}_i, R)\setminus \tilde{A}(\tilde{k}_{i+1}, R)} w \, \mathrm{d}x \right)^{\frac{p-q}{pq}} \\
&\le C \left(\int_{B_{2R}} |(u - \tilde{k}_i)^-|^p |\nabla \eta|^p w \, \mathrm{d}x \right)^{\frac{1}{p}} |\tilde{A}(\tilde{k}_i, R) \setminus \tilde{A}(\tilde{k}_{i+1}, R)|_\mu^{\frac{p-q}{pq}} \\
&\le \frac{C\varepsilon}{2^i} R^{\frac{n+\theta_1+\theta_2-p}{p}} |\tilde{A}(\tilde{k}_i, R) \setminus \tilde{A}(\tilde{k}_{i+1}, R)|_\mu^{\frac{p-q}{pq}},
\end{aligned}$$

where η is given by Equation (3.9). Then we obtain

$$|\tilde{A}(\tilde{k}_{i+1}, R)|_\mu \le \frac{C}{\sqrt[q]{\gamma}} R^{(n+\theta_1+\theta_2)(1-\frac{p-q}{pq})} |\tilde{A}(\tilde{k}_i, R) \setminus \tilde{A}(\tilde{k}_{i+1}, R)|_\mu^{\frac{p-q}{pq}},$$

and thus,

$$\begin{aligned}
j|\tilde{A}(\tilde{k}_j, R)|_\nu^{\frac{pq}{p-q}} &\le \sum_{i=0}^{j-1} |\tilde{A}(\tilde{k}_{i+1}, R)|_\mu^{\frac{pq}{p-q}} \le \frac{C}{\gamma^{\frac{p}{p-q}}} R^{(n+\theta_1+\theta_2)\left(\frac{pq}{p-q}-1\right)} |B_R|_\mu \\
&\le \frac{C}{\gamma^{\frac{p}{p-q}}} |B_R|_\mu^{\frac{pq}{p-q}} \quad \text{for } j \ge 1.
\end{aligned}$$

The proof is complete. □

A combination of Lemmas 3.2 and 3.4 yields the following improvement on oscillation of u in a small domain.

Corollary 3.5. *Assume that* $n \ge 2$, $(\theta_1, \theta_2) \in [(\mathcal{A} \cup \mathcal{B}) \cap (C_q \cup \mathcal{D}_q) \cap \mathcal{F}] \cup \{\theta_1 = 0,\ \theta_2 > -(n-1)\}$, $1 < q < p < n + \theta_1 + \theta_2$, $0 < \gamma < 1$, $0 < R < \frac{1}{2}$ *and* $0 \le m \le \inf_{B_{2R}} u \le \sup_{B_{2R}} u \le M$. *Then there exists a large constant* $k_0 > 1$ *depending only on* $n, p, q, \theta_1, \theta_2, \lambda, \gamma$ *such that for any* $\varepsilon > 0$,

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(i) *if*

$$\frac{|\{x \in B_R : u > M - \varepsilon\}|_\mu}{|B_R|_\mu} \le 1 - \gamma,$$

then

$$\sup_{B_{R/2}} u \le M - \frac{\varepsilon}{2^{k_0}};$$

(ii) *if*

$$\frac{|\{x \in B_R : u < m + \varepsilon\}|_\mu}{|B_R|_\mu} \le 1 - \gamma,$$

then

$$\inf_{B_{R/2}} u \ge m + \frac{\varepsilon}{2^{k_0}}.$$

Proof. Applying Lemmas 3.2 and 3.4, we obtain that Corollary 3.5 holds. In particular, in the case of $\theta_1 = 0$, $\theta_2 > -(n-1)$, $1 < p < n + \theta_2$, we fix $q = \frac{p+1}{2}$ in Lemma 3.4. □

We are now ready to prove Theorems 1.1 and 1.4, respectively.

Proof of Theorem 1.1. For $0 < R \le \frac{1}{2}$, denote

$$\overline{\mu}(R) = \sup_{B_R} u, \quad \underline{\mu}(R) = \inf_{B_R} u, \quad \omega(R) = \overline{\mu}(R) - \underline{\mu}(R).$$

Note that one of the following two statements must hold: either

$$|\{x \in B_R : u > \overline{\mu}(R) - 2^{-1}\omega(R)\}|_\mu \le \frac{1}{2}|B_R|_\mu \tag{3.10}$$

or

$$|\{x \in B_R : u < \underline{\mu}(R) + 2^{-1}\omega(R)\}|_\mu \le \frac{1}{2}|B_R|_\mu. \tag{3.11}$$

Using Corollary 3.5 with $\gamma = \frac{1}{2}$, we derive that there is a large constant $k_0 > 1$ such that

$$\overline{\mu}(R/2) \le \overline{\mu}(R) - \frac{\omega(R)}{2^{k_0+1}}, \quad \text{when Equation (3.10) holds}$$

and

$$\underline{\mu}(R/2) \ge \underline{\mu}(R) + \frac{\omega(R)}{2^{k_0+1}}, \quad \text{when Equation (3.11) holds.}$$

In either case, we have

$$\omega(R/2) \le \left(1 - \frac{1}{2^{k_0+1}}\right)\omega(R) = \frac{1}{2^\alpha}\omega(R), \quad \text{with } \alpha = -\frac{\ln\left(1 - \frac{1}{2^{k_0+1}}\right)}{\ln 2}.$$

Observe that for each $0 < R \le \frac{1}{2}$, there exists an integer $l \ge 1$ such that $2^{-(l+1)} < R \le 2^{-l}$. Since $\omega(R)$ is non-decreasing with respect to R, we then have

$$\omega(R) \le \omega(2^{-l}) \le 2^{-(l-1)\alpha}\omega(2^{-1}) = 4^\alpha 2^{-(l+1)\alpha}\omega(2^{-1}) \le CR^\alpha,$$

where $C = C(n, p, q, \theta_1, \theta_2, \lambda, \overline{M})$. The proof is complete. □

Proof of Theorem 1.4. First, by applying the proof of Theorem 1.1 with a slight modification, we also obtain that there exist a small constant $0 < \alpha = \alpha(n, p, \theta_2, \lambda) < 1$ and a large constant $C = C(n, p, \theta_2, \lambda, \overline{M}) > 0$ such that

$$|u(x) - u(0)| \le C|x|^\alpha \quad \text{for all } x \in B_{1/2}. \tag{3.12}$$

For $R \in (0, 1/2)$, $y \in Q_{1/R}$, denote

$$u_R(y) = u(Ry), \qquad A_R(y) = A(Ry).$$

Hence, u_R is the solution of

$$\operatorname{div}(A_R|y|^{\theta_2}\nabla u_R) = 0 \quad \text{in } Q_{1/R}.$$

After the change of variables, we see that this equation becomes degenerate elliptic equation in $B_{1/2}(\bar{y})$ for any $\bar{y} \in \partial B_1$. For any two given points $x, \tilde{x} \in B_{1/2}$, let $|\tilde{x}| \le |x|$ without loss of generality. Denote $R = |x|$. By the interior Hölder estimate for degenerate elliptic equation, we derive that there exist two constants $0 < \beta = \beta(n, p, \theta_2, \lambda) < 1$ and $0 < C = C(n, p, \theta_2, \lambda, \overline{M})$ such that for any $\bar{y} \in \partial B_1$,

$$|u_R(y) - u_R(\bar{y})| \le C|y - \bar{y}|^\beta, \quad \forall y \in B_{1/2}(\bar{y}). \tag{3.13}$$

Consequently, for $|x - \tilde{x}| \le R^2$, we have from Equation (3.13) that

$$|u(x) - u(\tilde{x})| = |u_R(x/R) - u_R(\tilde{x}/R)| \le C|(x - \tilde{x})/R|^\beta \le C|x - \tilde{x}|^{\beta/2},$$

while, for $|x - \tilde{x}| > R^2$, we deduce from Equation (3.12) that

$$|u(x) - u(\tilde{x})| \le |u(x) - u(0)| + |u(0) - u(\tilde{x})| \le C\big(R^\alpha + |\tilde{x}|^\alpha\big) \le CR^\alpha \le C|x - \tilde{x}|^{\frac{\alpha}{2}}.$$

Therefore, the proof of Theorem 1.4 is complete. □

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4. Regularity for solutions to nonlinear parabolic equations with anisotropic weights

Let $n \geq 2$, $R>0$ and $-T \leq t_1 < t_2 \leq 0$. For $u \in C((t_1,t_2);L^2(B_R,w_1)) \cap L^2((t_1,t_2);W_0^{1,2}(B_R,w_2))$, denote

$$\|u\|_{V_0^1(B_R\times(t_1,t_2))} = \sqrt{\sup_{t\in(t_1,t_2)}\int_{B_R}|u|^2 w_1\,\mathrm{d}x + \int_{t_1}^{t_2}\int_{B_R}|\nabla u|^2 w_2\,\mathrm{d}x\,\mathrm{d}t},$$

where the anisotropic weights w_1 and w_2 are defined in Equation (1.5). The parabolic Sobolev inequality with anisotropic weights is now given as follows.

Proposition 4.1. *For $n \geq 2$, $R>0$, $\theta_1+\theta_2 > -(n-2)$ and $-T \leq t_1 < t_2 \leq 0$, let $u \in C((t_1,t_2);L^2(B_R,w_1)) \cap L^2((t_1,t_2);W_0^{1,2}(B_R,w_2))$. Then*

$$\|u\|_{L^{2\chi}(B_R\times(t_1,t_2),w_2)} \leq C\|u\|_{V_0^1(B_R\times(t_1,t_2))}, \quad \text{with } \chi = \frac{n+\theta_1+\theta_2+2}{n+\theta_1+\theta_2},$$

where $C = C(n,\theta_1,\theta_2)$.

Proof. Applying the anisotropic version of the Caffarelli–Kohn–Nirenberg inequality in [30], we obtain that for any $u \in W_0^{1,2}(B_R,w_2)$,

$$\left(\int_{B_R}|u|^{\frac{2(n+\theta_1+\theta_2)}{n+\theta_1+\theta_2-2}}|x'|^{\frac{\theta_3(n+\theta_1+\theta_2)-2\theta_1}{n+\theta_1+\theta_2-2}}|x|^{\frac{\theta_4(n+\theta_1+\theta_2)-2\theta_2}{n+\theta_1+\theta_2-2}}\,\mathrm{d}x\right)^{\frac{n+\theta_1+\theta_2-2}{n+\theta_1+\theta_2}}$$
$$\leq C\int_{B_R}|\nabla u|^2|x'|^{\theta_3}|x|^{\theta_4}\,\mathrm{d}x.$$

This, in combination with the Hölder's inequality, leads to

$$\begin{aligned}
&\int_{B_R}|u|^{2\chi}|x'|^{\theta_3}|x|^{\theta_4}\,\mathrm{d}x\\
&= \int_{B_R}|u|^2|x'|^{\theta_3-\theta_1(\chi-1)}|x'|^{\theta_4-\theta_2(\chi-1)}|u|^{2(\chi-1)}|x'|^{\theta_1(\chi-1)}|x|^{\theta_2(\chi-1)}\,\mathrm{d}x\\
&\leq \left(\int_{B_R}|u|^{\frac{2}{2-\chi}}|x'|^{\frac{\theta_3-\theta_1(\chi-1)}{2-\chi}}|x|^{\frac{\theta_4-\theta_2(\chi-1)}{2-\chi}}\,\mathrm{d}x\right)^{2-\chi}\left(\int_{B_R}|u|^2|x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x\right)^{\chi-1}\\
&\leq C\int_{B_R}|\nabla u|^2|x'|^{\theta_3}|x|^{\theta_4}\,\mathrm{d}x\left(\int_{B_R}|u|^2|x'|^{\theta_1}|x|^{\theta_2}\,\mathrm{d}x\right)^{\chi-1}.
\end{aligned} \tag{4.1}$$

Then integrating Equation (4.1) from t_1 to t_2, it follows from Young's inequality that

$$\left(\int_{t_1}^{t_2}\int_{B_R}|u|^{2\chi}w_2\right)^{\frac{1}{\chi}} \leq C\left(\sup_{t\in(t_1,t_2)}\int_{B_R}|u|^2 w_1\,\mathrm{d}x\right)^{\frac{\chi-1}{\chi}}\left(\int_{B_R}|\nabla u|^2 w_2\,\mathrm{d}x\right)^{\frac{1}{\chi}}$$

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$$\leq C\left(\sup_{t\in(t_1,t_2)}\int_{B_R}|u|^2 w_1\,\mathrm{d}x+\int_{t_1}^{t_2}\int_{B_R}|\nabla u|^2 w_2\,\mathrm{d}x\,\mathrm{d}t\right).$$

The proof is complete. □

The Caccioppoli inequality for the truncated solution is given as follows.

Lemma 4.2. *Set* $\overline{m}\leq k\leq\overline{M}$*. Then for any* $B_R(x_0)\subset B_1$ *and non-negative* $\eta\in C^\infty(B_R(x_0)\times(-1,0))$*, which vanishes on* $\partial B_R(x_0)\times(-1,0)$*, we obtain that for* $-1<t_1<t_2<0$*,*

$$\begin{aligned}&\sup_{t\in(t_1,t_2)}\int_{B_R(x_0)}(v\eta)^2 w_1\,\mathrm{d}x+\int_{B_R(x_0)\times(t_1,t_2)}|\nabla(v\eta)|^2 w_2\,\mathrm{d}x\,\mathrm{d}t\\&\leq\int_{B_R(x_0)}(v^2+C_0v^3)\eta^2 w_1\,\mathrm{d}x\Big|_{t_1}+C_0\int_{B_R(x_0)\times(t_1,t_2)}(\eta|\partial_t\eta|w_1+|\nabla\eta|^2 w_2)v^2\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}&\sup_{t\in(t_1,t_2)}\int_{B_R(x_0)}(\tilde{v}^2-C_0\tilde{v}^3)\eta^2 w_1\,\mathrm{d}x+\int_{B_R(x_0)}|\nabla(\tilde{v}\eta)|^2 w_2\,\mathrm{d}x\,\mathrm{d}t\\&\leq\int_{B_R(x_0)}(\tilde{v}\eta)^2 w_1\,\mathrm{d}x\Big|_{t_1}+C_0\int_{B_R(x_0)\times(t_1,t_2)}\left(\eta|\partial_t\eta|w_1+|\nabla\eta|^2 w_2\right)\tilde{v}^2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where $C_0=C_0(n,p,\lambda,\overline{m},\overline{M})$*,* $v=(u-k)^+$*,* $\tilde{v}=(u-k)^-$*,* u *is the solution of problem (1.5).*

Proof. Choose test function $\varphi=v\eta^2$. By denseness, we obtain that for $t_1\leq s\leq t_2$,

$$\int_{t_1}^{s}\int_{B_R(x_0)}w_1\partial_t u^p v\eta^2\,\mathrm{d}x\,\mathrm{d}t+\int_{t_1}^{s}\int_{B_R(x_0)}Aw_2\nabla u\nabla(v\eta^2)\,\mathrm{d}x\,\mathrm{d}t=0.$$

On the one hand,

$$\begin{aligned}\int_{t_1}^{s}\int_{B_R(x_0)}w_1\partial_t u^p v\eta^2 dxdt&=p\int_{t_1}^{s}\int_{B_R(x_0)}w_1 v\eta^2 u^{p-1}\partial_t v\,dxdt\\&=p\int_{t_1}^{s}\int_{B_R(x_0)}w_1\eta^2[(v+k)^p-k(v+k)^{p-1}]\partial_t v\,dxdt\\&=p\int_{t_1}^{s}\int_{B_R(x_0)}w_1\eta^2\partial_t\mathcal{H},\end{aligned}$$

where

$$\mathcal{H}:=\frac{(v+k)^{p+1}}{p+1}-\frac{k(v+k)^p}{p}+\frac{k^{p+1}}{p(p+1)}.$$

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Remark that the last term $\frac{k^{p+1}}{p(p+1)}$ in $\mathcal{H}$ is added to keep it non-negative. In fact, by Taylor expansion, we obtain

$$\begin{aligned}\frac{(v+k)^{p+1}}{p+1} =& \frac{k^{p+1}}{p+1}\left(1+\frac{v}{k}\right)^{p+1}\\ =& \frac{k^{p+1}}{p+1}\left(1+(p+1)\frac{v}{k}+\frac{p(p+1)}{2}\frac{v^2}{k^2}+\frac{(p-1)p(p+1)}{6}\frac{v^3}{k^3}+O\left(\frac{v^4}{k^4}\right)\right),\end{aligned}$$

and

$$\begin{aligned}\frac{k(v+k)^{p}}{p} =& \frac{k^{p+1}}{p}\left(1+\frac{v}{k}\right)^{p}\\ =& \frac{k^{p+1}}{p}\left(1+p\frac{v}{k}+\frac{p(p-1)}{2}\frac{v^2}{k^2}+\frac{p(p-1)(p-2)}{6}\frac{v^3}{k^3}+O\left(\frac{v^4}{k^4}\right)\right).\end{aligned}$$

A consequence of these two relations shows that

$$\frac{(v+k)^{p+1}}{p+1}-\frac{k(v+k)^{p}}{p} = -\frac{k^{p+1}}{p(p+1)}+\frac{1}{2}k^{p-1}v^2+\frac{p-1}{3}k^{p-2}v^3+O\left(\frac{v^4}{k^4}\right),$$

which yields that

$$0 \le \mathcal{H}-\frac{1}{2}k^{p-1}v^2 \le C(p,\overline{m},\overline{M})v^3. \tag{4.2}$$

In light of Equation (4.2), it follows from integration by parts that

$$\begin{aligned}&\int_{t_1}^{s}\int_{B_R(x_0)} w_1\partial_t u^p v\eta^2\,\mathrm{d}x\,\mathrm{d}t\\ &\ge \frac{p}{2}k^{p-1}\int_{B_R(x_0)} w_1\eta^2(x,s)v^2(x,s)\,\mathrm{d}x-\frac{p}{2}k^{p-1}\int_{B_R(x_0)} w_1\eta^2(x,t_1)v^2(x,t_1)\,\mathrm{d}x\\ &\quad -C\int_{B_R(x_0)} w_1\eta^2(x,t_1)v^3(x,t_1)\,\mathrm{d}x-C\int_{t_1}^{s}\int_{B_R(x_0)} w_1\eta|\partial_t\eta|v^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

On the other hand, utilizing Young's inequality, we have

$$\begin{aligned}&\int_{B_R(x_0)\times(t_1,s)} Aw_2\nabla u\nabla(v\eta^2)\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_{B_R(x_0)\times(t_1,s)} Aw_2\nabla v(\eta^2\nabla v+2v\eta\nabla\eta)\,\mathrm{d}x\,\mathrm{d}t\\ &\ge \frac{1}{\lambda}\int_{B_R(x_0)\times(t_1,s)}|\eta\nabla v|^2 w_2\,\mathrm{d}x\,\mathrm{d}t-C\int_{B_R(x_0)\times(t_1,s)} v\eta|\nabla\eta||\nabla v|w_2\,\mathrm{d}x\,\mathrm{d}t\\ &\ge \frac{1}{2\lambda}\int_{B_R(x_0)\times(t_1,s)}|\eta\nabla v|^2 w_2\,\mathrm{d}x\,\mathrm{d}t-C\int_{B_R(x_0)\times(t_1,s)} v^2|\nabla\eta|^2 w_2\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

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$$\geq \frac{1}{4\lambda}\int_{B_R(x_0)\times(t_1,s)}|\nabla(\eta v)|^2 w_2\,\mathrm{d}x\,\mathrm{d}t - C\int_{B_R(x_0)\times(t_1,s)} v^2|\nabla\eta|^2 w_2\,\mathrm{d}x\,\mathrm{d}t,$$

where in the last inequality, we used the following elementary inequality:

$$|a-b|^2 \geq \frac{1}{2}|a|^2 - |b|^2 \quad \text{for any } a,b\in\mathbb{R}^n.$$

Therefore, the first inequality in Lemma 4.2 holds.

The proof of the second inequality in Lemma 4.2 is analogous by picking test function $\varphi = -\tilde{v}\eta^2$. Then we obtain

$$\begin{aligned}
-\int_{t_1}^{s}\int_{B_R(x_0)} w_1\partial_t u^p\tilde{v}\eta^2 &= p\int_{t_1}^{s}\int_{B_R(x_0)} w_1\tilde{v}\eta^2u^{p-1}\,\partial_t\tilde{v}\\
&= p\int_{t_1}^{s}\int_{B_R(x_0)} w_1\eta^2[-(k-\tilde{v})^p + k(k-\tilde{v})^{p-1}]\,\partial_t\tilde{v}\\
&= p\int_{t_1}^{s}\int_{B_R(x_0)} w_1\eta^2\,\partial_t\widetilde{\mathcal{H}},
\end{aligned}$$

where

$$\widetilde{\mathcal{H}} = \frac{(k-\tilde{v})^{p+1}}{p+1} - \frac{k(k-\tilde{v})^p}{p} + \frac{k^{p+1}}{p(p+1)}.$$

Similarly as before, it follows from Taylor expansion that

$$-C(p,\overline{m},\overline{M})\tilde{v}^3 \leq \widetilde{\mathcal{H}} - \frac{1}{2}k^{p-1}\tilde{v}^2 \leq 0,$$

which reads that

$$\begin{aligned}
&\int_{t_1}^{s}\int_{B_R(x_0)} w_1\partial_t u^p v\eta^2\,\mathrm{d}x\,\mathrm{d}t\\
&\geq \frac{p}{2}k^{p-1}\int_{B_R(x_0)} w_1\eta^2(x,s)\tilde{v}^2(x,s)\,\mathrm{d}x - C\int_{B_R(x_0)} w_1\eta^2(x,s)\tilde{v}^3(x,s)\,\mathrm{d}x\\
&\quad - \frac{p}{2}k^{q-1}\int_{B_R(x_0)} w_1\eta^2(x,t_1)\tilde{v}^2(x,t_1)\,\mathrm{d}x - C\int_{t_1}^{s}\int_{B_R(x_0)} w_1\eta|\partial_t\eta|\tilde{v}^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

By the same argument as before, we have

$$\begin{aligned}
&-\int_{B_R(x_0)\times(t_1,s)} Aw_2\nabla u\nabla(\tilde{v}\eta^2)\,\mathrm{d}x\,\mathrm{d}t = \int_{B_R(x_0)\times(t_1,s)} Aw_2\nabla\tilde{v}\nabla(\tilde{v}\eta^2)\,\mathrm{d}x\,\mathrm{d}t\\
&\geq \frac{1}{4\lambda}\int_{B_R(x_0)\times(t_1,s)}|\nabla(\eta\tilde{v})|^2 w_2\,\mathrm{d}x\,\mathrm{d}t - C\int_{B_R(x_0)\times(t_1,s)}\tilde{v}^2|\nabla\eta|^2 w_2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The proof is complete. □

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For $R>0$ and $(x_0,t_0)\in B_1\times[-1+R^{\theta_1+\theta_2},0]$, denote

$$Q_R(x_0,t_0):=B_R(x_0)\times(t_0-R^{\theta_1+\theta_2},t_0].$$

For brevity, we use Q_R to represent $Q_R(0,0)$ in the following. Introduce two Radon measures associated with the weights w_1 and w_2 as follows:

$$\mathrm{d}\mu_{w_i}=w_i\,\mathrm{d}x,\qquad \mathrm{d}\nu_{w_i}=w_i\,\mathrm{d}x\,\mathrm{d}t,\quad i=1,2,$$

satisfying that for $E\subset B_1$ and $\widetilde{E}\subset Q_1$,

$$|E|_{\mu_{w_i}}=\int_E w_i\,\mathrm{d}x,\qquad |\widetilde{E}|_{\nu_{w_i}}=\int_{\widetilde{E}} w_i\,\mathrm{d}x\,\mathrm{d}t.$$

Observe that by Hölder's inequality, we know that for $\widetilde{E}\subset Q_R$,

$$\frac{|\widetilde{E}|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}}\le\le\frac{C|\widetilde{E}|_{\nu_{w_1}}^{\frac{\theta_3}{\theta_1}}R^{\frac{(n+\theta_1+\theta_2)(\theta_1-\theta_3)}{\theta_1}}}{R^{n+\theta_1+\theta_2+\theta_3+\theta_4}}\le C\left(\frac{|\widetilde{E}|_{\nu_{w_1}}}{|Q_R|_{\nu_{w_1}}}\right)^{\frac{\theta_3}{\theta_1}},\tag{4.3}$$

where $C=C(n,\theta_1,\theta_2,\theta_3,\theta_4)$. Here we used the assumed condition that $\theta_1/\theta_3=\theta_2/\theta_4$, $\theta_3,\theta_4\neq 0$.

Similar to Lemma 3.2, we improve the oscillation of the solution u in a small region as follows.

Lemma 4.3. *Assume as in Theorem 1.6 or Theorem 1.9. For $R\in(0,1)$ and $t_0\in[-1+R^{\theta_1+\theta_2},0]$, let $0<\overline{m}\le m\le\inf_{Q_R(0,t_0)}u\le\sup_{Q_R(0,t_0)}u\le M\le\overline{M}$. Then*

(a) there exists a small constant $0<\tau_0=\tau_0(n,p,\theta_1,\theta_2,\theta_3,\lambda,\overline{m},\overline{M})<1$ such that for any $\varepsilon>0$ and $0<\tau<\tau_0$, if

$$|\{(x,t)\in Q_R(0,t_0):u(x,t)>M-\varepsilon\}|_{\nu_{w_1}}\le\tau|Q_R(0,t_0)|_{\nu_{w_1}},\tag{4.4}$$

then we have

$$u(x,t)\le M-\frac{\varepsilon}{2}\quad\textit{for}\quad(x,t)\in Q_{R/2}(0,t_0);\tag{4.5}$$

(b) there exist two small constant $0<\varepsilon_0=\varepsilon_0(n,p,\lambda,\overline{m},\overline{M})<1$ and $0<\tau_0=\tau_0(n,p,\theta_1,\theta_2,\theta_3,\lambda,\overline{m},\overline{M})<1$ such that for any $0<\varepsilon\le\varepsilon_0$ and $0<\tau<\tau_0$, if

$$|\{(x,t)\in Q_R(0,t_0):u(x,t)<m+\varepsilon\}|_{\nu_{w_1}}\le\tau|Q_R(0,t_0)|_{\nu_{w_1}},$$

then we have

$$u(x,t)\ge m-\frac{\varepsilon}{2}\quad\textit{for}\quad(x,t)\in Q_{R/2}(0,t_0).\tag{4.6}$$

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Remark 4.4. From the proof of Lemma 4.3 below, we see that the value of $\theta_3+\theta_4$ has to be restricted to 2 and thus affect our final regular index in Theorems 1.6 and 1.9.

Proof. Without loss of generality, let $t_0=0$.
Step 1. For $\varepsilon>0$ and $i=0,1,2,\ldots$, set

$$r_i=\frac{R}{2}+\frac{R}{2^{i+1}},\qquad k_i=M-\varepsilon+\frac{\varepsilon}{2}(1-2^{-i}).$$

Choose a cutoff function $\eta_i\in C_0^\infty(Q_{r_i})$ such that

$$\eta_i=1 \text{ in } Q_{r_{i+1}},\quad 0\le\eta_i\le 1,\ |\nabla\eta_i|\le\frac{C}{r_i-r_{i+1}},\ |\partial_t\eta_i|\le\frac{C}{r_i^{\theta_1+\theta_2}-r_{i+1}^{\theta_1+\theta_2}} \text{ in } Q_{r_i}.$$

Denote $v_i=(u-k_i)^+$ and $A(k,\rho)=\{(x,t)\in Q_\rho : u>k\}$ for $k\in[m,M]$ and $\rho\in(0,R]$. Then combining Proposition 4.1 and Lemma 4.2, we deduce

$$\left(\int_{Q_R}|\eta_i v_i|^{2\chi}w_2\right)^{\frac{1}{\chi}}\le C\int_{Q_R}\left(|\nabla\eta_i|^2+|\partial_t\eta_i||x'|^{\theta_1-\theta_3}|x|^{\theta_2-\theta_4}\right)v_i^2w_2, \tag{4.7}$$

where $\chi=(n+\theta_1+\theta_2+2)(n+\theta_1+\theta_2)^{-1}$. Note that

$$|\partial_t\eta_i|R^{\theta_1+\theta_2-\theta_3-\theta_4}\le\frac{CR^{2-\theta_3-\theta_4}}{(r_i-r_{i+1})^2}=\frac{C}{(r_i-r_{i+1})^2}. \tag{4.8}$$

Therefore, we have

$$\int_{Q_R}\left(|\nabla\eta_i|^2+|\partial_t\eta_i||x'|^{\theta_1-\theta_3}|x|^{\theta_2-\theta_4}\right)v_i^2w_2\le\frac{C(M-k_i)^2}{(r_i-r_{i+1})^2}|A(k_i,r_i)|_{\nu_{w_2}}$$

and

$$\left(\int_{Q_R}|\eta_i v_i|^{2\chi}w_2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{\chi}}\ge(k_{i+1}-k_i)^2|A(k_{i+1},r_{i+1})|_{\nu_{w_2}}^{\frac{1}{\chi}}.$$

Define

$$F_i:=\frac{|A(k_i,r_i)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}}.$$

Then we have

$$\begin{aligned}F_{i+1}\le&(C2^{4(i+2)})^\chi R^{(n+\theta_1+\theta_2+\theta_3+\theta_4)(\chi-1)-2\chi}F_i^\chi\\=&(C2^{4(i+2)})^\chi R^{\frac{2(\theta_3+\theta_4-2)}{n+\theta_1+\theta_2}}F_i^\chi=(C2^{4(i+2)})^\chi F_i^\chi\end{aligned}$$

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$$\leq \prod_{s=0}^{i} (C2^{4(i+2-s)})^{\chi^{s+1}} F_0^{\chi^{i+1}}. \tag{4.9}$$

Observe from Equations (4.8)–(4.9) that the value of $\theta_3 + \theta_4$ must be chosen to be 2. A consequence of Equations (4.3) and (4.9) shows that there exists a constant $i_0 = i_0(n, \theta_1, \theta_2, \theta_3) > 0$ such that if $i \geq i_0$,

$$F_{i+1} \leq (C^*)^{i+1} \left(\frac{|A(k_0, r_0)|_{\nu_{w_1}}}{|Q_R|_{\nu_{w_1}}} \right)^{\frac{\theta_3}{\theta_1}\chi^{i+1}} \leq (C^*)^{i+1} \left(\frac{|A(k_0, r_0)|_{\nu_{w_1}}}{|Q_R|_{\nu_{w_1}}} \right)^{\frac{\theta_3}{\theta_1}\chi(i+1)}.$$

By taking $\tau_0 = (C^*)^{-\frac{\theta_1}{\theta_3}\chi}$, we obtain that for any $\varepsilon > 0$ and $0 < \tau < \tau_0$, if Equation (4.4) holds, then

$$F_{i+1} \leq \left(C^* \tau^{\frac{\theta_3}{\theta_1}\chi} \right)^{i+1} = \left(\frac{\tau}{\tau_0} \right)^{\frac{\theta_3}{\theta_1}\chi(i+1)} \to 0 \quad \text{as } i \to \infty.$$

That is, Equation (4.5) holds.
Step 2. Analogously as before, pick

$$r_i = \frac{R}{2} + \frac{R}{2^{i+1}}, \quad \tilde{k}_i = m + \varepsilon - \frac{\varepsilon}{2}(1 - 2^{-i}), \quad i \geq 0.$$

Let $\varepsilon_0 = \frac{1}{C_0}$, where C_0 is given in Lemma 4.2. Denote $\tilde{v}_i = (u - \tilde{k}_i)^-$. Then we obtain that for any $0 < \varepsilon \leq \varepsilon_0$,

$$\tilde{v}_i^2 - C_0 \tilde{v}_i^3 \geq (1 - C_0\varepsilon)\tilde{v}_i^2 \geq 0,$$

which implies that Equation (4.7) holds with v_i replaced by $\tilde{v}_i$. Then following the left proof of Equation (4.5) above, we deduce that Equation (4.6) holds. The proof is complete. □

The decaying estimates for the distribution function of u are stated as follows.

Lemma 4.5. *Let the values of $n, p, q, \theta_i, i = 1, 2, 3, 4$ be assumed in Theorem 1.6 or Theorem 1.9 with $\theta_3 + \theta_4 = 2$ replaced by $0 \leq \theta_3 + \theta_4 \leq 2$. Suppose that $0 < \gamma < 1$, $0 < R < \frac{1}{2}$, $0 < a \leq 1$, $-\frac{1}{2} < t_0 \leq -aR^{\theta_1+\theta_2}$ and $\overline{m} \leq m_a \leq \inf_{B_{2R}\times[t_0, t_0+aR^{\theta_1+\theta_2}]} u \leq \sup_{B_{2R}\times[t_0, t_0+aR^{\theta_1+\theta_2}]} u \leq M_a \leq \overline{M}$. Then*

(a) for any $\varepsilon > 0$, if

$$\frac{|\{x \in B_R : u(x,t) > M_a - \varepsilon\}|_{\mu_{w_1}}}{|B_R|_{\mu_{w_1}}} \leq 1 - \gamma, \quad \forall t \in [t_0, t_0 + aR^{\theta_1+\theta_2}],$$

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then for any $j \geq 1$,

$$\frac{|\{(x,t) \in B_R \times [t_0, t_0 + aR^{\theta_1+\theta_2}] : u(x,t) > M_a - \frac{\varepsilon}{2^j}\}|_{\nu_{w_1}}}{|B_R \times [t_0, t_0 + aR^{\theta_1+\theta_2}]|_{\nu_{w_1}}} \leq \frac{C}{\sqrt[q]{\gamma}\sqrt{a} j^{\frac{2-q}{2q}}}; \quad (4.10)$$

(b) for any $0 < \varepsilon \leq \varepsilon_0 = C_0^{-1}$ *with* $C_0 = C_0(n, p, \lambda, \overline{m}, \overline{M})$ *given by Lemma 4.2, if*

$$\frac{|\{x \in B_R : u(x,t) < m_a + \varepsilon\}|_{\mu_{w_1}}}{|B_R|_{\mu_{w_1}}} \leq 1 - \gamma, \quad \forall t \in [t_0, t_0 + aR^{\theta_1+\theta_2}],$$

then for any $j \geq 1$,

$$\frac{|\{(x,t) \in B_R \times [t_0, t_0 + aR^{\theta_1+\theta_2}] : u(x,t) < m_a + \frac{\varepsilon}{2^j}\}|_{\nu_{w_1}}}{|B_R \times [t_0, t_0 + aR^{\theta_1+\theta_2}]|_{\nu_{w_1}}} \leq \frac{C}{\sqrt[q]{\gamma}\sqrt{a} j^{\frac{2-q}{2q}}},$$

where $C = C(n, p, q, \theta_1, \theta_2, \theta_3, \lambda, \overline{m}, \overline{M})$.

Remark 4.6. Since the proof of Lemma 4.5 only uses the aforementioned Proposition 2.10 and Lemma 4.2 instead of Lemma 4.3, we can obtain a larger range for the value of $\theta_3 + \theta_4$ than that in Lemma 4.3.

Proof. Step 1. For $i \geq 0$, denote $k_i = M_a - \frac{\varepsilon}{2^i}$ and

$$A(k_i, R; t) = B_R \cap \{u(\cdot, t) > k_i\}, \qquad A(k_i, R) = (B_R \times [t_0, t_0 + aR^{\theta_1+\theta_2}]) \cap \{u > k_i\}.$$

It then follows from Equation (2.13) that for $1 < q < 2$,

$$\begin{aligned}&(k_{i+1} - k_i)^q |A(k_{i+1}, R; t)|_{\mu_{w_1}}^q |B_R \setminus A(k_i, R; t)|_{\mu_{w_1}} \\ &\leq CR^{q(n+\theta_1+\theta_2+1)} \int_{A(k_i,R;t)\setminus A(k_{i+1},R;t)} |\nabla u|^q w_1 \, \mathrm{d}x. \end{aligned} \quad (4.11)$$

From the assumed condition, we have

$$|B_R \setminus A(k_i, R; t)|_{\mu_{w_1}} \geq \gamma |B_R|_{\mu_{w_1}} = C(n, \theta_1)\gamma R^{n+\theta_1+\theta_2}.$$

Substituting this into Equation (4.11) and integrating from t_0 to $t_0 + aR^{\theta_1+\theta_2}$, we deduce from Hölder's inequality that

$$\begin{aligned}&\int_{t_0}^{t_0+aR^{\theta_1+\theta_2}} |A(k_{i+1}, R; t)|_{\mu_{w_1}} \, \mathrm{d}t \\ &\leq \frac{C 2^{i+1}}{\varepsilon \sqrt[q]{\gamma}} R^{\frac{(n+\theta_1+\theta_2)(q-1)}{q}+1} \int_{t_0}^{t_0+aR^{\theta_1+\theta_2}} \left(\int_{A(k_i,R;t)\setminus A(k_{i+1},R;t)} |\nabla u|^q w_1 \, \mathrm{d}x \right)^{\frac{1}{q}} \mathrm{d}t\end{aligned}$$

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$$\leq \frac{C2^{i+1}a^{\frac{q-1}{q}}}{\varepsilon\sqrt[q]{\gamma}} R^{\frac{(n+2\theta_1+2\theta_2)(q-1)}{q}+1} \left(\int_{A(k_i,R)\setminus A(k_{i+1},R)} |\nabla u|^q w_1 \,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{q}}.$$

In light of $1 < q < 2$, it follows from Hölder's inequality again that

$$\begin{aligned}
&\left(\int_{A(k_i,R)\setminus A(k_{i+1},R)} |\nabla u|^q w_1 \right)^{\frac{1}{q}} \\
&\leq \left(\int_{A(k_i,R)\setminus A(k_{i+1},R)} |\nabla u|^2 w_2 \right)^{\frac{1}{2}} \left(\int_{A(k_i,R)\setminus A(k_{i+1},R)} |x'|^{\frac{2\theta_1-q\theta_3}{2-q}} |x|^{\frac{2\theta_2-q\theta_4}{2-q}} \right)^{\frac{2-q}{2q}} \\
&\leq R^{\frac{\theta_1+\theta_2-\theta_3-\theta_4}{2}} |A(k_i,R)\setminus A(k_{i+1},R)|_{\nu_{w_1}}^{\frac{2-q}{2q}} \left(\int_{B_R\times[t_0,t_0+aR^{\theta_1+\theta_2}]} |\nabla(u-k_i)^+|^2 w_2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Pick a cutoff function $\eta \in C_0^\infty(B_{2R})$ such that

$$\eta = 1 \text{ in } B_R, \quad 0 \leq \eta \leq 1,\ |\nabla \eta| \leq \frac{C(n)}{R} \text{ in } B_{2R}. \tag{4.12}$$

Then from Lemma 4.2, we deduce

$$\begin{aligned}
&\left(\int_{B_R\times[t_0,t_0+aR^{\theta_1+\theta_2}]} |\nabla(u-k_i)^+|^2 w_2 \right)^{\frac{1}{2}} \\
&\leq C \left(\int_{B_{2R}} |(u-k_i)^+(x,t_0)|^2 w_1 + \frac{1}{R^2} \int_{B_{2R}\times[t_0,t_0+aR^{\theta_1+\theta_2}]} |(u-k_i)^+|^2 w_2 \right)^{\frac{1}{2}} \\
&\leq \frac{C\varepsilon}{2^i} R^{\frac{n+\theta_1+\theta_2+\theta_3+\theta_4-2}{2}},
\end{aligned}$$

where in the last inequality, we used the assumed condition that $0 \leq \theta_3 + \theta_4 \leq 2$. Therefore, combining these above facts, we obtain

$$|A(k_{i+1},R)|_{\nu_{w_1}} \leq \frac{Ca^{\frac{q-1}{q}}}{\sqrt[q]{\gamma}} R^{\frac{(n+2\theta_1+2\theta_2)(3q-2)}{2q}} |A(k_i,R)\setminus A(k_{i+1},R)|_{\nu_{w_1}}^{\frac{2-q}{2q}},$$

which yields that for $j \geq 1$,

$$\begin{aligned}
j|A(k_j,R)|_{\nu_{w_1}}^{\frac{2q}{2-q}} &\leq \sum_{i=0}^{j-1} |A(k_{i+1},R)|_{\nu_{w_1}}^{\frac{2q}{2-q}} \\
&\leq \frac{Ca^{\frac{2(q-1)}{2-q}}}{\gamma^{\frac{2}{2-q}}} R^{\frac{(n+2\theta_1+2\theta_2)(3q-2)}{2-q}} |B_R\times[t_0,t_0+aR^{\theta_1+\theta_2}]|_{\nu_{w_1}}
\end{aligned}$$

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$$\leq \frac{C}{\gamma^{\frac{2}{2-q}} a^{\frac{q}{2-q}}} |B_R \times [t_0, t_0 + aR^{\theta_1+\theta_2}]|_{\nu w_1}^{\frac{2q}{2-q}}.$$

Then Equation (4.10) holds.
Step 2. For $i \geq 0$, write

$$\tilde{k}_i = m_a + \frac{\varepsilon}{2^i}$$

and

$$\tilde{A}(k_i, R; t) = B_R \cap \{u(\cdot, t) < k_i\}, \qquad \tilde{A}(k_i, R) = (B_R \times [t_0, t_0 + aR^{\theta_1+\theta_2}]) \cap \{u < k_i\}.$$

Using Equation (2.14), we obtain that for $1 < q < 2$,

$$\begin{aligned} &(\tilde{k}_i - \tilde{k}_{i+1})^q |\tilde{A}(\tilde{k}_{i+1}, R; t)|_{\mu w_1}^q |B_R \setminus \tilde{A}(\tilde{k}_i, R; t)|_{\mu w_1} \\ &\leq C R^{q(n+\theta_1+\theta_2+1)} \int_{\tilde{A}(\tilde{k}_i, R; t) \setminus \tilde{A}(\tilde{k}_{i+1}, R; t)} |\nabla u|^q w_1 \, \mathrm{d}x. \end{aligned}$$

Observe by the assumed condition that

$$|B_R \setminus \tilde{A}(\tilde{k}_i, R; t)|_{\mu w_1} \geq \gamma |B_R|_{\mu w_1} = C(n, \theta_1, \theta_2) \gamma R^{n+\theta_1+\theta_2}.$$

Analogously as above, integrating from t_0 to $t_0 + aR^{\theta_1+\theta_2}$ and using Hölder's inequality, we have

$$\begin{aligned} &\int_{t_0}^{t_0+aR^{\theta_1+\theta_2}} |\tilde{A}(\tilde{k}_{i+1}, R; t)|_{\mu w_1} \, \mathrm{d}t \\ &\leq \frac{C 2^{i+1} a^{\frac{q-1}{q}}}{\varepsilon \sqrt[q]{\gamma}} R^{\frac{(n+2\theta_1+2\theta_2)(q-1)}{q}+1} \left(\int_{\tilde{A}(\tilde{k}_i, R) \setminus \tilde{A}(\tilde{k}_{i+1}, R)} |\nabla u|^q w_1 \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{q}} \end{aligned}$$

and

$$\begin{aligned} &\int_{\tilde{A}(\tilde{k}_i, R) \setminus A(\tilde{k}_{i+1}, R)} |\nabla u|^q w_1 \\ &\leq \left(\int_{\tilde{A}(\tilde{k}_i, R) \setminus \tilde{A}(\tilde{k}_{i+1}, R)} |\nabla u|^2 w_2 \right)^{\frac{q}{2}} \left(\int_{\tilde{A}(\tilde{k}_i, R) \setminus \tilde{A}(\tilde{k}_{i+1}, R)} |x'|^{\frac{2\theta_1 - q\theta_3}{2-q}} |x|^{\frac{2\theta_2 - q\theta_4}{2-q}} \right)^{\frac{2-q}{2}} \\ &\leq R^{\frac{q(\theta_1+\theta_2-\theta_3-\theta_4)}{2}} |\tilde{A}(\tilde{k}_i, R) \setminus \tilde{A}(\tilde{k}_{i+1}, R)|_{\nu w_1}^{\frac{2-q}{2q}} \left(\int_{B_R \times [t_0, t_0+aR^{\theta_1+\theta_2}]} |\nabla (u - \tilde{k}_i)^-|^2 w_2 \right)^{\frac{q}{2}}. \end{aligned}$$

For any $0 < \varepsilon \leq \varepsilon_0 = C_0^{-1}$ with C_0 given in Lemma 4.2, we know that

$$[(u - \tilde{k}_i)^-]^2 - C_0[(u - \tilde{k}_i)^-]^3 \geq (1 - C_0\varepsilon)[(u - \tilde{k}_i)^-]^2 \geq 0.$$

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Therefore, in view of $0 \le \theta_3+\theta_4 \le 2$ and applying Lemma 4.2 with η defined by Equation (4.12), we derive

$$\begin{aligned}
&\int_{B_R\times[t_0,t_0+aR^{\theta_1+\theta_2}]} |\nabla(u-\tilde{k}_i)^-|^2 w_2 \\
&\le C\left(\int_{B_{2R}} |(u-\tilde{k}_i)^-(x,t_0)|^2 w_1 + \frac{1}{R^2}\int_{B_R\times[t_0,t_0+aR^{\theta_1+\theta_2}]} |(u-\tilde{k}_i)^-|^2 w_2\right) \\
&\le \frac{C\varepsilon^2}{4^i} R^{n+\theta_1+\theta_2+\theta_3+\theta_4-2}.
\end{aligned}$$

Then we deduce

$$|\tilde{A}(\tilde{k}_{i+1},R)|_{\nu_{w_1}}^{\frac{2q}{2-q}} \le \frac{Ca^{\frac{2(q-1)}{2-q}}}{\gamma^{\frac{2}{2-q}}} R^{\frac{(n+2\theta_1+2\theta_2)(3q-2)}{2-q}} |\tilde{A}(\tilde{k}_i,R)\setminus\tilde{A}(\tilde{k}_{i+1},R)|_{\nu_{w_1}}.$$

This leads to that for $j \ge 1$,

$$\begin{aligned}
j|\tilde{A}(\tilde{k}_j,R)|_{\nu_{w_1}}^{\frac{2q}{2-q}} &\le \sum_{i=0}^{j-1} |\tilde{A}(\tilde{k}_{i+1},R)|_{\nu_{w_1}}^{\frac{2q}{2-q}} \\
&\le \frac{Ca^{\frac{2(q-1)}{2-q}}}{\gamma^{\frac{2}{2-q}}} R^{\frac{(n+2\theta_1+2\theta_2)(3q-2)}{2-q}} |B_R\times[t_0,t_0+aR^{\theta_1+\theta_2}]|_{\nu_{w_1}} \\
&\le \frac{C}{\gamma^{\frac{2}{2-q}} a^{\frac{q}{2-q}}} |B_R\times[t_0,t_0+aR^{\theta_1+\theta_2}]|_{\nu_{w_1}}^{\frac{2q}{2-q}}.
\end{aligned}$$

The proof is complete. □

We now give explicit estimates for the distribution function of u at each time slice from the starting time.

Lemma 4.7. *Assume as in Theorem 1.6 or Theorem 1.9. Let* $0<\gamma<1$, $0<R<\frac{1}{2}$, $-\frac{1}{2}<t_0\le -R^{\theta_1+\theta_2}$ *and* $\overline{m}\le m_1\le \inf_{B_{2R}\times[t_0,t_0+R^{\theta_1+\theta_2}]} u \le \sup_{B_{2R}\times[t_0,t_0+R^{\theta_1+\theta_2}]} u \le M_1 \le \overline{M}$. *Then there exist a small constant* $\bar{\varepsilon}_0=\bar{\varepsilon}_0(n,p,\theta_1,\theta_2,\lambda,\gamma,\overline{m},\overline{M})>0$ *and a large constant* $\bar{l}_0=\bar{l}_0(n,p,q,\theta_1,\theta_2,\theta_3,\lambda,\gamma,\overline{m},\overline{M})>1$ *such that*

(i) *for every* $0<\varepsilon\le\bar{\varepsilon}_0$, *if*

$$\frac{|\{x\in B_R : u(x,t_0)>M_1-\varepsilon\}|_{\mu_{w_1}}}{|B_R|_{\mu_{w_1}}} \le 1-\gamma, \tag{4.13}$$

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then for any $t_0 \le t \le t_0 + R^{\theta_1+\theta_2}$,

$$\frac{|\{x \in B_R : u(x,t) > M_1 - 2^{-l_0}\varepsilon\}|_{\mu_{w_1}}}{|B_R|_{\mu_{w_1}}} \le 1 - \frac{\gamma}{2}; \tag{4.14}$$

(ii) for every $0 < \varepsilon \le \bar{\varepsilon}_0$, *if*

$$\frac{|\{x \in B_R : u(x,t_0) < m_1 + \varepsilon\}|_{\mu_{w_1}}}{|B_R|_{\mu_{w_1}}} \le 1 - \gamma, \tag{4.15}$$

then for any $t_0 \le t \le t_0 + R^{\theta_1+\theta_2}$,

$$\frac{|\{x \in B_R : u(x,t) < m_1 + 2^{-l_0}\varepsilon\}|_{\mu_{w_1}}}{|B_R|_{\mu_{w_1}}} \le 1 - \frac{\gamma}{2}. \tag{4.16}$$

Remark 4.8. It is worth emphasizing that the explicit values of $\bar{\varepsilon}_0$ and $\bar{l}_0$ are given by Equations (4.20) and (4.22) below.

Proof. Step 1. For $a \in (0,1]$ and $k \in [\overline{m}, \overline{M}]$, define

$$A^a(k,R) = (B_R \times [t_0, t_0 + aR^{\theta_1+\theta_2}]) \cap \{u > k\}.$$

Take a smooth cutoff function $\eta \in C_0^\infty(B_R)$ satisfying that $\eta = 1$ in $B_{\sigma R}$, where $\sigma \in (0,1)$ to be determined later. Set $k_1 > 1$. Denote $v = (u - (M_1 - \varepsilon))^+$. From Lemma 4.2, we obtain

$$\begin{aligned}&\sup_{t \in (t_0, t_0 + aR^{\theta_1+\theta_2})} \int_{B_R} v^2 \eta^2 w_1 \, \mathrm{d}x \\ &\le \int_{B_R} (v^2 + Cv^3)\eta^2 w_1 \, \mathrm{d}x\Big|_{t_0} + C \int_{B_R \times [t_0, t_0 + aR^{\theta_1+\theta_2}]} v^2 |\nabla \eta|^2 w_2 \, \mathrm{d}x \, \mathrm{d}t.\end{aligned} \tag{4.17}$$

Observe that for $t \in [t_0, t_0 + aR^{\theta_1+\theta_2}]$,

$$\int_{B_R} v^2 \eta^2 w_1 \, \mathrm{d}x\Big|_t \ge \varepsilon^2 (1 - 2^{-k_1})^2 |B_{\sigma R} \cap \{u(x,t) > M_1 - 2^{-k_1}\varepsilon\}|_{\mu_{w_1}},$$

and by Equation (4.13),

$$\begin{aligned}\int_{B_R} (v^2 + Cv^3)\eta^2 w_1 \, \mathrm{d}x\Big|_{t_0} &\le \varepsilon^2 (1 + C\varepsilon) |\{x \in B_R : u(x,t_0) > M_1 - \varepsilon\}|_{\mu_{w_1}} \\ &\le \varepsilon^2 (1 + C\varepsilon)(1 - \gamma)|B_R|_{\mu_{w_1}},\end{aligned}$$

and

$$\int_{t_0}^{t_0 + aR^{\theta_1+\theta_2}} \int_{B_R} v^2 |\nabla \eta|^2 w_2 \, \mathrm{d}x \, \mathrm{d}t \le \frac{C\varepsilon^2}{(1-\sigma)^2 R^2} |A^a(M_1 - \varepsilon, R)|_{\nu_{w_2}}$$

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$$\leq \frac{C\varepsilon^2 R^{\theta_3+\theta_4-2}}{(1-\sigma)^2}|B_R|_{\mu_{w_1}} \frac{|A^a(M_1-\varepsilon,R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}}$$
$$= \frac{C\varepsilon^2}{(1-\sigma)^2}|B_R|_{\mu_{w_1}} \frac{|A^a(M_1-\varepsilon,R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}},$$

where we utilized the assumed condition of $\theta_3+\theta_4=2$. A consequence of these facts gives that for $t\in[t_0,t_0+aR^{\theta_1+\theta_2}]$,

$$\begin{aligned}&|B_{\sigma R}\cap\{u(x,t)>M_1-2^{-k_1}\varepsilon\}|_{\mu_{w_1}}\\ &\leq |B_R|_{\mu_{w_1}}\left(\frac{(1+C\varepsilon)(1-\gamma)}{(1-2^{-k_1})^2}+\frac{C}{(1-\sigma)^2}\frac{|A^a(M_1-\varepsilon,R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}}\right),\end{aligned}$$

which, together with the fact that $|B_R\setminus B_{\sigma R}|_{\mu_{w_1}}\leq C(1-\sigma)|B_R|_{\mu_{w_1}}$, reads that

$$\begin{aligned}&\frac{|B_R\cap\{u(x,t)>M_1-2^{-k_1}\varepsilon\}|_{\mu_{w_1}}}{|B_R|_{\mu_{w_1}}}\\ &\leq \frac{(1+C\varepsilon)(1-\gamma)}{(1-2^{-k_1})^2}+\frac{C}{(1-\sigma)^2}\left((1-\sigma)^3+\frac{|A^a(M_1-\varepsilon,R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}}\right).\end{aligned}$$

Pick σ such that

$$(1-\sigma)^3=\frac{|A^a(M_1-\varepsilon,R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}},$$

which yields that for $t\in[t_0,t_0+aR^{\theta_1+\theta_2}]$,

$$\begin{aligned}&\frac{|B_R\cap\{u(x,t)>M_1-2^{-k_1}\varepsilon\}|_{\mu_{w_1}}}{|B_R|_{\mu_{w_1}}}\\ &\leq \frac{(1+\overline{C}\varepsilon)(1-\gamma)}{(1-2^{-k_1})^2}+\overline{C}\left(\frac{|A^a(M_1-\varepsilon,R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}}\right)^{\frac{1}{3}},\end{aligned} \tag{4.18}$$

where $\overline{C}=\overline{C}(n,p,\theta_1,\theta_2,\lambda,\overline{m},\overline{M})$. Note that

$$\frac{|A^a(M_1-\varepsilon,R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}}\leq a.$$

Take a small positive constant a such that a^{-1} is an integer and

$$\overline{C}a^{\frac{1}{3}}\leq\frac{\gamma}{8}.$$

By fixing the value of a, we now divide the time interval $[t_0,t_0+R^{\theta_1+\theta_2}]$ into finite small intervals. Denote $N=a^{-1}$ and $t_i=t_0+iaR^{\theta_1+\theta_2}$, $i=1,2,\ldots,N$.

Claim that there exist a small positive constant $\bar{\varepsilon}_0$ and a large positive constant $k_0 > 1$ depending only on $n, p, \theta_1, \theta_2, \lambda, \gamma, \overline{m}, \overline{M}$ such that for any $0 < \varepsilon \leq \bar{\varepsilon}_0$ and $k_1 \geq k_0$,

$$\frac{(1+\overline{C}\varepsilon)(1-\gamma)}{(1-2^{-k_1})^2} \leq 1-\gamma+\frac{\gamma}{8N}. \tag{4.19}$$

In fact, since $(1-t)^{-2} \leq (1+6t)$ for $t \in (0, \frac{1}{2})$, then

$$\frac{1+\overline{C}\varepsilon}{(1-2^{-k_1})^2} \leq (1+\overline{C}\varepsilon)(1+6\cdot 2^{-k_1}).$$

Let $\overline{C}\varepsilon = 6 \cdot 2^{-k_1}$. Then we have

$$\frac{1+\overline{C}\varepsilon}{(1-2^{-k_1})^2} \leq 1+2\overline{C}\varepsilon+\overline{C}^2\varepsilon^2.$$

Pick

$$\bar{\varepsilon}_0 = \frac{-2\overline{C}+\sqrt{4\overline{C}^2+\frac{\gamma}{2N(1-\gamma)}}}{2\overline{C}^2}, \qquad k_0 = -\frac{\ln 2}{\ln(\overline{C}\bar{\varepsilon}_0)-\ln 6}. \tag{4.20}$$

Then we obtain that for any $0 < \varepsilon \leq \bar{\varepsilon}_0$ and $k_1 \geq k_0$,

$$\frac{1+\overline{C}\varepsilon}{(1-2^{-k_1})^2} \leq 1+2\overline{C}\bar{\varepsilon}_0+\overline{C}^2\bar{\varepsilon}_0^2 = 1+\frac{\gamma}{8N(1-\gamma)}.$$

That is, Equation (4.19) holds.

Consequently, it follows from Equation (4.18) that for $0 < \varepsilon \leq \bar{\varepsilon}_0$, $k_1 \geq k_0$ and $t \in [t_0, t_1]$,

$$\frac{|B_R \cap \{u(x,t) > M_1 - 2^{-k_1}\varepsilon\}|_{\mu_{w_1}}}{|B_R|_{\mu_{w_1}}} \leq 1-\left(\frac{7}{8}-\frac{1}{8N}\right)\gamma.$$

Then applying Lemma 4.5, we deduce from Equation (4.3) that for any $k_2 > k_1 \geq k_0$,

$$\begin{aligned}\frac{|A^a(M_1-2^{-k_2}\varepsilon, R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}} \leq & C\left(\frac{|A^a(M_1-2^{-k_2}\varepsilon, R)|_{\nu_{w_1}}}{|Q_R|_{\nu_{w_1}}}\right)^{\frac{\theta_3}{\theta_1}} \\ \leq & \widehat{C}\left(\frac{\sqrt{a}}{\sqrt[q]{\gamma}(k_2-k_1)^{\frac{2-q}{2q}}}\right)^{\frac{\theta_3}{\theta_1}},\end{aligned}$$

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where $\widehat{C} = \widehat{C}(n, p, q, \theta_1, \theta_2, \theta_3, \lambda, \overline{m}, \overline{M})$. Pick

$$k_2 = k_1 + a^{\frac{q}{2-q}} \gamma^{-\frac{2}{2-q}} \left(\frac{\gamma}{8N\overline{C}\sqrt[3]{\widehat{C}}} \right)^{-\frac{6\theta_1 q}{\theta_3(2-q)}}.$$

Then we have

$$\overline{C} \left(\frac{|A^a(M_1 - 2^{-k_2}\varepsilon, R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}} \right)^{\frac{1}{3}} \leq \frac{\gamma}{8N}. \tag{4.21}$$

Choose $k_1 = k_0$ and $l_1 = k_1 + k_2$. By letting $2^{-k_2}\varepsilon$ substitute for ε in Equation (4.18), we have

$$\sup_{t \in [t_0, t_1]} |B_R \cap \{u(x,t) > M_1 - 2^{-l_1}\varepsilon\}|_{\mu_{w_1}} \leq \Big(1 - \gamma + \frac{\gamma}{4N}\Big) |B_R|_{\mu_{w_1}}.$$

Then it can be inductively proved that there exist a strictly increasing integer set $\{l_i\}_{i=1}^N$ such that for $i = 1, 2, \ldots, N$,

$$\sup_{t \in [t_{i-1}, t_i]} |B_R \cap \{u(x,t) > M_1 - 2^{-l_i}\varepsilon\}|_{\mu_{w_1}} \leq \Big(1 - \gamma + \frac{i\gamma}{4N}\Big) |B_R|_{\mu_{w_1}}.$$

In fact, let the above relation hold in interval $[t_{i-1}, t_i]$ and then prove that it also holds in the next interval $[t_i, t_{i+1}]$. For simplicity, denote $\varepsilon_i = 2^{-l_i}\varepsilon$ and $\gamma_i = \gamma(1 - \frac{i}{4N})$. Then the assumption implies that

$$|B_R \cap \{u(x, t_i) > M_1 - \varepsilon_i\}|_{\mu_{w_1}} \leq (1 - \gamma_i)|B_R|_{\mu_{w_1}}.$$

By the same argument as in Equation (4.18), it follows from Equations (4.19)–(4.21) that for $\bar{k}_1 \geq k_0$ and $t \in [t_i, t_{i+1}]$,

$$\begin{aligned}
&\frac{|B_R \cap \{u(x,t) > M_1 - 2^{-\bar{k}_1}\varepsilon_i\}|_{\mu_{w_1}}}{|B_R|_{\mu_{w_1}}} \\
&\leq \frac{(1 + \overline{C}\varepsilon_i)(1 - \gamma_i)}{(1 - 2^{-\bar{k}_1})^2} + \overline{C}\Big(\frac{|A^a(M_1 - \varepsilon_i, R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}} \Big)^{\frac{1}{3}} \\
&\leq 1 - \gamma_i + \frac{\gamma_i}{8N} + \overline{C}\Big(\frac{|A^a(M_1 - 2^{-l_i}\varepsilon, R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}} \Big)^{\frac{1}{3}} \\
&\leq 1 - \gamma_i + \frac{\gamma_i}{8N} + \frac{\gamma}{8N} \\
&< 1 - \gamma + \frac{i+1}{4N}\gamma,
\end{aligned}$$

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where $\overline{C} = \overline{C}(n, p, \theta_1, \theta_2, \lambda, \overline{m}, \overline{M})$ is defined above and in the third inequality, we used the fact that $l_i \geq l_1 > k_2$. By taking $\bar{k}_1 = k_0$ and $l_{i+1} = l_i + \bar{k}_1$, we obtain

$$\sup_{t\in[t_i,t_{i+1}]} |B_R \cap \{u(x,t) > M_1 - 2^{-l_{i+1}}\varepsilon\}|_{\mu_{w_1}} \leq \Big(1 - \gamma + \frac{(i+1)\gamma}{4N}\Big)|B_R|_{\mu_{w_1}}.$$

Then picking

$$\begin{aligned}\bar{l}_0 :=& l_N = l_1 + (N-1)k_0 \\ =&(N+1)k_0 + +a^{\frac{q}{2-q}}\gamma^{-\frac{2}{2-q}}\left(\frac{\gamma}{8N\overline{C}\sqrt[3]{\widehat{C}}}\right)^{-\frac{6\theta_1 q}{\theta_3(2-q)}},\end{aligned} \tag{4.22}$$

we obtain that Equation (4.14) holds.

Step 2. For $0 < a \leq 1$ and $\overline{m} \leq k \leq \overline{M}$, let

$$\tilde{A}^a(k, R) = (B_R \times [t_0, t_0 + aR^{\theta_1+\theta_2}]) \cap \{u < k\}.$$

Define $\tilde{v} = (u - (m_1 + \varepsilon))^-$. A direct application of Lemma 4.2 gives that

$$\begin{aligned}&\sup_{t\in(t_0,t_0+aR^{\theta_1+\theta_2})}\int_{B_R}(\tilde{v}^2 - C_0\tilde{v}^3)\eta^2 w_1\,\mathrm{d}x \\ &\leq \int_{B_R}\tilde{v}^2\eta^2 w_1\,\mathrm{d}x\Big|_{t_0} + C_0\int_{B_R\times[t_0,t_0+aR^{\theta_1+\theta_2}]}\tilde{v}^2|\nabla\eta|^2 w_2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where $C_0 = C_0(n, p, \lambda, \overline{m}, \overline{M})$ and η is defined in Equation (4.17). Pick a small constant $0 < \bar{\varepsilon}_1 \leq (2C_0)^{-1}$, which implies that $1 - C_0\bar{\varepsilon}_1 \geq \frac{1}{2}$. Then we obtain that for $t_0 < t < t_0 + aR^{\theta_1+\theta_2}$, $0 < \varepsilon \leq \bar{\varepsilon}_0$ and $k_1 > 1$,

$$\int_{B_R}(\tilde{v}^2 - C_0\tilde{v}^3)\eta^2 w_1\,\mathrm{d}x\Big|_t \geq (1 - C_0\varepsilon)\varepsilon^2(1 - 2^{-k_1})^2|B_{\sigma R} \cap \{u(x,t) < m_1 + 2^{-k_1}\varepsilon\}|_{\mu_{w_1}},$$

and in view of Equation (4.15),

$$\int_{B_R}\tilde{v}^2\eta^2 w_1\,\mathrm{d}x\Big|_{t_0} \leq \varepsilon^2|\{x \in B_R : u(x,t_0) < m_1 + \varepsilon\}|_{\mu_{w_1}} \leq \varepsilon^2(1-\gamma)|B_R|_{\mu_{w_1}},$$

and

$$\begin{aligned}\int_{B_R\times[t_0,t_0+aR^{\theta_1+\theta_2}]}\tilde{v}^2|\nabla\eta|^2 w_2\,\mathrm{d}x\,\mathrm{d}t \leq& \frac{C\varepsilon^2}{(1-\sigma)^2R^2}|\tilde{A}^a(m_1+\varepsilon, R)|_{\nu_{w_2}} \\ \leq& \frac{C\varepsilon^2 R^{\theta_3+\theta_4-2}}{(1-\sigma)^2}|B_R|_{\mu_{w_1}}\frac{|\tilde{A}^a(m_1+\varepsilon, R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}}\end{aligned}$$

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$$=\frac{C\varepsilon^2}{(1-\sigma)^2}|B_R|_{\mu_{w_1}}\frac{|\tilde{A}^a(m_1+\varepsilon,R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}}.$$

Therefore, we deduce that for $t_0 \leq t \leq t_0 + aR^{\theta_1+\theta_2}$,

$$\begin{aligned}
&|B_{\sigma R}\cap\{u(x,t)<m_1+2^{-k_1}\varepsilon\}|_{\mu_{w_1}}\\
&\leq |B_R|_{\mu_{w_1}}\left(\frac{1-\gamma}{(1-C\varepsilon)(1-2^{-k_1})^2}+\frac{C}{(1-\sigma)^2}\frac{|\tilde{A}^a(m_1+\varepsilon,R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}}\right)\\
&\leq |B_R|_{\mu_{w_1}}\left(\frac{(1+C\varepsilon)(1-\gamma)}{(1-2^{-k_1})^2}+\frac{C}{(1-\sigma)^2}\frac{|\tilde{A}^a(m_1+\varepsilon,R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}}\right),
\end{aligned}$$

and thus,

$$\begin{aligned}
&\frac{|B_R\cap\{u(x,t)<m_1+2^{-k_1}\varepsilon\}|_{\mu_{w_1}}}{|B_R|_{\mu_{w_1}}}\\
&\leq \frac{(1+C\varepsilon)(1-\gamma)}{(1-2^{-k_1})^2}+\frac{C}{(1-\sigma)^2}\left((1-\sigma)^3+\frac{|\tilde{A}^a(m_1+\varepsilon,R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}}\right).
\end{aligned}$$

Take σ such that

$$(1-\sigma)^3=\frac{|\tilde{A}^a(m_1+\varepsilon,R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}}.$$

Then we obtain that for $t_0 \leq t \leq t_0 + aR^{\theta_1+\theta_2}$,

$$\begin{aligned}
&\frac{|B_R\cap\{u(x,t)<m_1+2^{-k_1}\varepsilon\}|_{\mu_{w_1}}}{|B_R|_{\mu_{w_1}}}\\
&\leq \frac{(1+\overline{C}\varepsilon)(1-\gamma)}{(1-2^{-k_1})^2}+\overline{C}\left(\frac{|\tilde{A}^a(m_1+\varepsilon,R)|_{\nu_{w_2}}}{|Q_R|_{\nu_{w_2}}}\right)^{\frac{1}{3}},
\end{aligned}$$

where $\overline{C}=\overline{C}(n,p,\theta_1,\theta_2,\lambda,\overline{m},\overline{M})$. Consequently, by the same argument as in the left proof of Equation (4.14) above, we deduce that Equation (4.16) holds. The proof is complete. □

A consequence of Lemmas 4.3, 4.5 and 4.7 gives the improvement on oscillation of u in a small region.

Corollary 4.9. *Assume as in Theorem 1.6 or Theorem 1.9. Let* $0<\gamma<1$, $0<R<\frac{1}{2}$, $-\frac{1}{4}<t_0\leq 0$ *and* $\overline{m}\leq m\leq \inf_{B_{2R}\times[t_0-R^{\theta_1+\theta_2},t_0]} u \leq \sup_{B_{2R}\times[t_0-R^{\theta_1+\theta_2},t_0]} u\leq M\leq \overline{M}$. *Then there exist a small constant* $\tilde{\varepsilon}_0=\tilde{\varepsilon}_0(n,p,\theta_1,\theta_2,\lambda,\gamma,\overline{m},\overline{M})>0$ *and a large constant* $l_0=l_0(n,p,q,\theta_1,\theta_2,\theta_3,\lambda,\gamma,\overline{m},\overline{M})>1$ *such that for any* $0<\varepsilon\leq\tilde{\varepsilon}_0$,

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(i) if

$$\frac{|\{x \in B_R : u(x, t_0 - R^{\theta_1+\theta_2}) > M - \varepsilon\}|_{\mu_{w_1}}}{|B_R|_{\mu_{w_1}}} \le 1 - \gamma,$$

then

$$\sup_{Q_{R/2}(0,t_0)} u \le M - \frac{\varepsilon}{2^{l_0}};$$

(ii) if

$$\frac{|\{x \in B_R : u(x, t_0 - R^{\theta_1+\theta_2}) < m + \varepsilon\}|_{\mu_{w_1}}}{|B_R|_{\mu_{w_1}}} \le 1 - \gamma,$$

then

$$\inf_{Q_{R/2}(0,t_0)} u \ge m + \frac{\varepsilon}{2^{l_0}}.$$

Proof. Applying Lemma 4.3, Lemma 4.5 with $a=1$ and Lemma 4.7, we obtain that Corollary 4.9 holds. In particular, we fix $q = \frac{3}{2}$ under the assumed conditions in Theorem 1.9. □

Based on these above facts, we now give the proofs of Theorems 1.6 and 1.9, respectively.

Proof of Theorem 1.6. Pick a sufficiently large constant $\kappa_0 \ge 2$ such that

$$\frac{\overline{M} - \overline{m}}{\kappa_0} < \tilde{\varepsilon}_0,$$

where $\tilde{\varepsilon}_0$ is given by Corollary 4.9 with $\gamma = \frac{1}{2}$. For $0 < R \le \frac{1}{2}$ and $-\frac{1}{4} < t_0 < 0$, define

$$\overline{\mu}(R) = \sup_{(x,t)\in Q_R(0,t_0)} u(x,t), \quad \underline{\mu}(R) = \inf_{(x,t)\in Q_R(0,t_0)} u(x,t), \quad \omega(R) = \overline{\mu}(R) - \underline{\mu}(R).$$

Observe that there is at least one inequality holding in terms of the following two inequalities:

$$|\{x \in B_{R/2} : u(x, t_0 - (R/2)^{\theta_1+\theta_2}) > \overline{\mu}(R) - \kappa_0^{-1}\omega(R)\}|_{\mu_{w_1}} \le \frac{1}{2}|B_{R/2}|_{\mu_{w_1}}, \tag{4.23}$$

and

$$|\{x \in B_{R/2} : u(x, t_0 - (R/2)^{\theta_1+\theta_2}) < \underline{\mu}(R) + \kappa_0^{-1}\omega(R)\}|_{\mu_{w_1}} \le \frac{1}{2}|B_{R/2}|_{\mu_{w_1}}. \tag{4.24}$$

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From Corollary 4.9, it follows that there exists a large constant $l_0 > 1$ such that

$$\overline{\mu}(R/4) \le \overline{\mu}(R) - \frac{\omega(R)}{\kappa_0 2^{l_0}} \quad \text{if Equation (4.23) holds,}$$

and

$$\underline{\mu}(R/4) \ge \underline{\mu}(R) + \frac{\omega(R)}{\kappa_0 2^{l_0}} \quad \text{if Equation (4.24) holds.}$$

In both cases, we have

$$\omega(R/4) \le \left(1 - \frac{1}{\kappa_0 2^{l_0}}\right)\omega(R) = \frac{1}{4^\alpha}\omega(R), \quad \text{with } \alpha = -\frac{\ln\left(1 - \frac{1}{\kappa_0 2^{l_0}}\right)}{\ln 4}.$$

Note that for any $0 < R \le \frac{1}{2}$, there is an integer k such that $4^{-(k+1)} \cdot 2^{-1} < R \le 4^{-k} \cdot 2^{-1}$. In light of the fact that $\omega(R)$ is non-decreasing in R, it follows that

$$\omega(R) \le \omega(4^{-k} \cdot 2^{-1}) \le 4^{-k\alpha}\omega(2^{-1}) = 8^\alpha (4^{-(k+1)} \cdot 2^{-1})^\alpha \omega(2^{-1}) \le CR^\alpha,$$

where $C = C(n, p, q, \theta_1, \theta_2, \theta_3, \lambda, \overline{m}, \overline{M})$. Therefore, for any $(x, t) \in B_{1/2} \times (-1/4, t_0)$, we obtain that

(i) if $|t - t_0| \le 2^{-(\theta_1+\theta_2)}$, then

$$\begin{aligned} |u(x,t) - u(0,t_0)| \le& |u(x,t) - u(x,t_0)| + |u(x,t_0) - u(0,t_0)| \\ &\le C\left(|t - t_0|^{\frac{\alpha}{\theta_1+\theta_2}} + |x|^\alpha\right) \\ \le& C\left(|x| + |t - t_0|^{\frac{1}{\theta_1+\theta_2}}\right)^\alpha; \end{aligned}$$

(ii) if $|t - t_0| > 2^{-(\theta_1+\theta_2)}$, there exists a set $\{t_i\}_{i=1}^N$ such that $t < t_1 \le \cdots \le t_N < t_0$,

$$\begin{aligned} |u(x,t) - u(0,t_0)| \le& |u(x,t) - u(x,t_1)| + |u(x,t_1) - u(x,t_0)| + |u(x,t_0) - u(0,t_0)| \\ \le& C\left(|t - t_1|^{\frac{\alpha}{\theta_1+\theta_2}} + |t_1 - t_0|^{\frac{\alpha}{\theta_1+\theta_2}} + |x|^{\frac{\alpha}{\theta_1+\theta_2}}\right) \\ \le& C\left(|x| + |t - t_0|^{\frac{1}{\theta_1+\theta_2}}\right)^\alpha \quad \text{if } N = 1, \end{aligned}$$

and

$$\begin{aligned} &|u(x,t) - u(0,t_0)| \\ &\le |u(x,t) - u(x,t_1)| + \sum_{i=1}^{N-1} |u(x,t_i) - u(x,t_{i+1})| \end{aligned}$$

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$$
\begin{aligned}
&+|u(x,t_N)-u(x,t_0)|+|u(x,t_0)-u(0,t_0)|\\
&\le C\left(|t-t_1|^{\frac{\alpha}{\theta_1+\theta_2}}+\sum_{i=1}^{N-1}|t_i-t_{i+1}|^{\frac{\alpha}{\theta_1+\theta_2}}+|t_N-t_0|^{\frac{\alpha}{\theta_1+\theta_2}}+|x|^{\frac{\alpha}{\theta_1+\theta_2}}\right)\\
&\le C\left(|x|+|t-t_0|^{\frac{1}{\theta_1+\theta_2}}\right)^{\alpha}\quad \text{if } N\ge 2.
\end{aligned}
$$

The proof is complete.

□

Proof of Theorem 1.9. To begin with, applying the aforementioned proof of Theorem 1.6 with minor modification, we also obtain that there exists a small constant $0<\alpha<1$ and a large constant $C>0$, both depending only on $n,p,\theta_2,\lambda,\overline{m},\overline{M}$, such that for any $t_0\in(-1/4,0)$,

$$
|u(x,t)-u(0,t_0)|\le C\left(|x|+\sqrt[\theta_2]{|t-t_0|}\right)^{\alpha},\quad \forall (x,t)\in B_{1/2}\times(-1/4,t_0]. \tag{4.25}
$$

For $R\in(0,1/2)$, $(y,s)\in Q_{1/R}$, define

$$
u_R(y,s)=u(Ry,R^{\theta_2}s),\qquad A_R(y)=A(Ry).
$$

Therefore, u_R verifies

$$
|y|^{\theta_2}\partial_s u_R^q-\operatorname{div}\left(A_R|y|^2\nabla u_R\right)=0\quad \text{in } Q_{1/R}.
$$

By the change of variables, we obtain that this equation keeps uniformly parabolic in $B_{1/2}(\bar{y})\times(-R^{-\theta_2},0)$ for any $\bar{y}\in\partial B_1$.

For any $(x,t),(\tilde{x},\tilde{t})\in B_{1/2}\times(-1/4,0)$, let $|\tilde{x}|\le|x|$ without loss of generality. Write $R=|x|$. It then follows from the interior Hölder estimates for uniformly parabolic equations that there exist two constants $0<\beta=\beta(n,p,\theta_2,\lambda,\overline{m},\overline{M})<1$ and $0<C=C(n,p,\theta_2,\lambda,\overline{m},\overline{M})$ such that for any $\bar{y}\in\partial B_1$ and $\bar{s}\in(-4^{-1}R^{-\theta_2},0)$,

$$
|u_R(y,s)-u_R(\bar{y},\bar{s})|\le C(|y-\bar{y}|+\sqrt{|s-\bar{s}|})^{\beta}, \tag{4.26}
$$

for any (y,s) satisfying that $|y-\bar{y}|+\sqrt{|s-\bar{s}|}<1/2$.

Observe that for any $(x,t),(\tilde{x},\tilde{t})\in B_{1/2}\times(-1/4,0)$,

$$
|u(x,t)-u(\tilde{x},\tilde{t})|\le|u(x,t)-u(x,\tilde{t})|+|u(x,\tilde{t})-u(\tilde{x},\tilde{t})|.
$$

On the one hand, if $|t-\tilde{t}|\le R^{2\theta_2}$, then we deduce from Equation (4.26) that

$$
\begin{aligned}
|u(x,t)-u(x,\tilde{t})|\le&\left|u_R(x/R,t/R^{\theta_2})-u_R(x/R,\tilde{t}/R^{\theta_2})\right|\\
\le& C|(t-\tilde{t})/R^{\theta_2}|^{\beta/2}\le C|t-\tilde{t}|^{\beta/4},
\end{aligned}
$$

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while, if $|t-\tilde{t}| > R^{2\theta_2}$, then we have from Equation (4.25) that

$$\begin{aligned}
&|u(x,t)-u(x,\tilde{t})| \\
&\le |u(x,t)-u(0,t)| + |u(0,t)-u(0,\tilde{t})| + |u(0,\tilde{t})-u(x,\tilde{t})| \\
&\le C\big(R^\alpha + |t-\tilde{t}|^{\frac{\alpha}{\theta_2}}\big) \le C|t-\tilde{t}|^{\frac{\alpha}{2\theta_2}}.
\end{aligned}$$

On the other hand, if $|x-\tilde{x}| \le R^2$, then it follows from Equation (4.26) that

$$\begin{aligned}
|u(x,\tilde{t})-u(\tilde{x},\tilde{t})| =& \big|u_R(x/R,\tilde{t}/R^{\theta_2}) - u_R(\tilde{x}/R,\tilde{t}/R^{\theta_2})\big| \\
\le& C|(x-\tilde{x})/R|^\beta \le C|x-\tilde{x}|^{\beta/2},
\end{aligned}$$

while, if $|x-\tilde{x}| > R^2$, then we see from Equation (4.25) that

$$\begin{aligned}
|u(x,\tilde{t})-u(\tilde{x},\tilde{t})| \le& |u(x,\tilde{t})-u(0,\tilde{t})| + |u(0,\tilde{t})-u(\tilde{x},\tilde{t})| \\
\le& C\big(R^\alpha + |\tilde{x}|^\alpha\big) \le CR^\alpha \le C|x-\tilde{x}|^{\frac{\alpha}{2}}.
\end{aligned}$$

Consequently, we complete the proof of Theorem 1.9. □

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